“Über die Mechanik des deformieren Körpers vom Standpunkt der Relativitätstheorie,” Ann. d. Phys. (1911), 493-533.

On the mechanics of deformable bodies from the standpoint of relativity theory

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The following attempt to modify the classical mechanics of deformable bodies in the sense of relativity theory is the extension of an – at the time occasional – activity that grew out of the definition of the equations of motion for a relativistic rigid body – itself far from completely achieved, by the way – whose development and publication I was first led to in this spring from considering the circumstances that emerge in the theory of relativistic rigid bodies through the fact that M. Laue presented some equations of motion in a general form from a different viewpoint [1]). Moreover, this comparison also leads to the extension of the model to non-adiabatic motion, such that I am obliged to extend my heartfelt thanks to M. Laue in both of these directions for his providing me with the means to do this at the time.

The assumption that comes out of the aforementioned considerations is, in following the direction that M. Planck took in his own work [2]) on the principle of least action, that there exists a kinetic potential for the motion of bodies, which is, first of all, invariant under Lorentz transformations (in homogeneous form), and second, reduces in the rest case to a given function of the deformations and entropy of a unit volume element, from which its general expression is determined immediately (§ 5). In particular, one finds that the rest deformations (§§ 1, 2) are definitive in the case of motion, i.e., any deformations that return the volume element to its normal form by reversing the Lorentz contraction that corresponds to its velocity. [3])

The equations of motion flow directly from the first variation of this kinetic potential, first in the Lagrangian form (§ 6), and then in the Eulerian form (§ 7). In the latter form, they are formally identical with the system that M. Abraham has presented in his investigations [4]) into the electrodynamics of moving bodies, from which comparison, the meaning of the 16-component matrix that he derived can be deduced. The ten relations (§ 7) that the symmetry properties of that matrix, and the combining of impulse, energy, and stress with each other bring to the expression, prove to be the complete system of partial differential equations (§ 3) that the kinetic potential must satisfy as a result of the form that it takes on from both of the aforementioned assumptions.

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[1] Which meanwhile appeared in the paper “Das Relativitätsprinzip,” Braunschweig 1911, and in the Ann. d. Phys. 35, pp. 524, 1911.

[2] M. Planck, Berl. Ber. 1907, pp. 642, Ann. d. Phys. 26, pp. 1, 1908.

[3] This remark has already been made by M. Born, Ann. d. Phys. 30, pp. 1, 1909.

[4] M. Abraham, Rendiconti del circolo mat. Di Palermo 28, pp. 1, 1909.

The 10-term group of "motions" in the corresponding space of (x, y, z, t) imply ten general integrals (§ 9) for the motion of the total body, and actually correspond to the four translations, the three impulse theorems and the energy theorem; the six rotations, however, correspond, in one case, to the three surface theorems, and in another, to three more that are – as a result of the equal status of the x, y, z, t – completely analogous to the equations that one derives in classical mechanics by once integrating the center of gravity theorems in a parallel manner. For force-free adiabatic motion, in particular, it follows from them that the center of energy – which appears here as the center of mass or gravity – moves in a uniform rectilinear fashion and that its velocity gives the total impulse when multiplied by the total energy.

If the rest potential depends only upon entropy and volume then one obtains the case of an ideal fluid with everywhere equal pressure (§ 10), and the Weber form of the hydrodynamic equations yields the Helmholtz theorem on vortex motion for the hydrodynamics of relativity theory (§ 11).

If one starts with the considerations of part one concerning the kinetic potential then those of the second part – on the inertial resistance and wave mechanics – depend upon the second variation itself. For the unit volume element, the components of the inertial resistance are connected with the components of the acceleration that it arouses by a linear transformation with a symmetric determinant whose six coefficients can be identified with the coefficients of inertia or mass densities at the location of the body in question. If one would now wish to not find any direction that leads to circumstances that are completely contrary to the usual situation [1]) then the postulate of positive mass in classical mechanics will be analogous to the requirement that the quadratic form Γ that is constructed from six coefficients of inertia – which is simply the second variation of the kinetic potential with respect to velocity – shall be positive definite, or that, intuitively, the inertial resistance shall always subtend an obtuse angle with the acceleration (§ 1).

The laws of wave mechanics are produced (§ 4) by means of another quadratic form W – which is simply the complete second variation of the kinetic potential – and it then emerges that the two forms Γ and W are mutually derivable from each other on the basis of their representations (§§ 2, 3, and 5). From this general connection, it follows that the requirement that is placed on inertial resistance of the impossibility of waves with velocity greater than light is implied; however, here it is actually necessary (§ 6).

If the six inertial coefficients reduce to only two – one longitudinal and one transversal – then only longitudinal and transversal waves with propagation velocities that are equal in all directions are possible, and conversely (§ 7). Thus, both inertial coefficients and both wave velocities are derivable from each other.

These special circumstances are realized for the ideal fluid (§ 8) – for which, however, the velocity of the transversal waves will be null – and for the isotropic elastic bodies (§ 9) with vanishing rest deformations. For the latter, the requirements that were placed on the inertial coefficients above come about as a result of the limits that are given by the rest mass density for the elastic coefficients.

[1] Say, e.g., the hyperbolic character of the equations of motion is not guaranteed in all cases (cf., remark 1, pp. 26).

PART ONE

The kinetic potential and the equations of motion.

§ 1. The rest deformations.

One thinks of a deformable body as itself being in a state of motion. Each particle, which takes on the coordinates ξ, η, ζ in the normal state of the body, is found at time t at the location x, y, z in space:

$$(1) \qquad \begin{cases} x = x(\xi, \eta, \zeta, t) \\ y = y(\xi, \eta, \zeta, t) \\ z = z(\xi, \eta, \zeta, t). \end{cases}$$

In order make the formulas homogeneous, one may somehow introduce a sort of time position:

$$(2) \qquad \tau = \tau(\xi, \eta, \zeta, t), \qquad \frac{\partial \tau}{\partial t} > 0$$

for the body, and then set:

$$(3) \qquad \begin{cases} x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = t, \\ \xi_1 = \xi, \quad \xi_2 = \eta, \quad \xi_3 = \zeta, \quad \xi_4 = \tau, \end{cases}$$

in which (1) may be written in the equivalent form:

$$(4) \qquad x_i = x_i(\xi_1, \xi_2, \xi_3, \xi_4), \quad i = 1, 2, 3, 4.$$

If one denotes the partial differential quotients of x_i with respect ξ_j by a_{ij}:

$$(5) \qquad a_{ij} = \frac{\partial x_i}{\partial \xi_j}, \qquad i, j = 1, 2, 3, 4,$$

such that:

$$(6) \qquad dx_i = \sum_{j=1}^{4} a_{ij} d\xi_j,$$

then one has, if one understands s, u, v, w to mean the components of the velocity of the particle:

$$(7) \qquad \frac{a_{14}}{a_{44}} = u, \qquad \frac{a_{24}}{a_{44}} = v, \qquad \frac{a_{34}}{a_{44}} = w, \qquad a_{44} = \frac{\partial \tau}{\partial t} > 0,$$

and under the ongoing assumption of always having subluminal velocity ($c = 1$):

(8) $$s = \sqrt{u^2 + v^2 + w^2} < 1.$$

Those Lorentz transformations that take the velocity of the particle to null – viz., the "rest transformations" – take x, y, z, t over to:

(9) $$\begin{cases} x^0 = x + \alpha u(ux + vy + wz) - \beta ut \\ y^0 = y + \alpha v(ux + vy + wz) - \beta vt \\ z^0 = z + \alpha w(ux + vy + wz) - \beta wt \\ t^0 = \quad - \quad \beta(ux + vy + wz) + \beta t. \end{cases}$$

(10) $$\alpha = \frac{1}{\sqrt{1-s^2}\,(1+\sqrt{1-s^2})}, \qquad \beta = \frac{1}{\sqrt{1-s^2}},$$

and the transformation that is inverse to it will be obtained simply by exchanging the x, y, z, t with x^0, y^0, z^0, t^0 with a simultaneous change of sign in the u, w, w.

If one goes from the dx^i to the dx_i^0 by way of the rest transformation:

(11) $$dx_i^0 = \sum_{j=1}^{4} a_{ij}^0 d\xi_j\,, \qquad i = 1, 2, 3, 4,$$

then one has:

(12) $$\begin{cases} a_{1i}^0 = a_{1i} + \alpha u(ua_{1i} + va_{2i} + wa_{3i}) - \beta ua_{4i} \\ a_{2i}^0 = a_{2i} + \alpha v(ua_{1i} + va_{2i} + wa_{3i}) - \beta va_{4i} \\ a_{3i}^0 = a_{3i} + \alpha w(ua_{1i} + va_{2i} + wa_{3i}) - \beta wa_{4i} \\ a_{4i}^0 = \quad - \quad \beta(ua_{1i} + va_{2i} + wa_{3i}) + \beta a_{4i} \end{cases}$$

and, in particular:

(13) $$a_{14}^0 = a_{24}^0 = a_{34}^0 = 0, \qquad a_{44}^0 = a_{44}\sqrt{1-s^2} = \sqrt{-A_{44}} > 0.$$

Moreover, let:

(14) $$(dx^0)^2 + (dy^0)^2 + (dz^0)^2 = d\xi^2 + d\eta^2 + d\zeta^2 + 2de^2,$$

(15) $$de^2 = e_{11}d\xi^2 + e_{22}d\eta^2 + e_{33}d\zeta^2 + 2e_{23}d\eta d\zeta + 2e_{31}d\zeta d\xi + 2e_{12}d\xi d\eta\,,$$

hence:

(16)
$$\begin{cases} 1+2e_{11} = \left(a_{11}^0\right)^2 + \left(a_{21}^0\right)^2 + \left(a_{31}^0\right)^2 \\ 1+2e_{22} = \left(a_{12}^0\right)^2 + \left(a_{22}^0\right)^2 + \left(a_{32}^0\right)^2 \\ 1+2e_{33} = \left(a_{13}^0\right)^2 + \left(a_{23}^0\right)^2 + \left(a_{33}^0\right)^2 \\ e_{23} = e_{32} = \tfrac{1}{2}(a_{12}^0 a_{13}^0 + a_{22}^0 a_{23}^0 + a_{32}^0 a_{33}^0) \\ e_{31} = e_{13} = \tfrac{1}{2}(a_{13}^0 a_{11}^0 + a_{23}^0 a_{21}^0 + a_{33}^0 a_{31}^0) \\ e_{12} = e_{21} = \tfrac{1}{2}(a_{11}^0 a_{12}^0 + a_{21}^0 a_{22}^0 + a_{31}^0 a_{32}^0), \end{cases}$$

then the e_{ij} – viz., the "rest deformations – produce those transformations that convert the "rest form" of a volume element into its normal form, or its actual deformation relative to the Lorentz contraction that corresponds to its velocity.

For a volume element at rest, the e_{ij} coincide with the actual deformations as they are usually defined.

Since the determinant of the rest transformation is + 1, the determinants of the a_{ij} and the a_{ij}^0 are equal to each other:

(16)
$$D = \frac{\partial(x, y, z, t)}{\partial(\xi, \eta, \zeta, \tau)} = |\, a_{ij} \,| = \left| a_{ij}^0 \right|, \qquad i, j = 1, 2, 3, 4.$$

Due to (13), however, one has $\left|a_{ij}^0\right| = \Delta a_{44}^0$, where:

(17)
$$\Delta = \left| a_{ij}^0 \right|, \qquad i, j = 1, 2, 3$$

denotes the ratio of the rest volume to the normal volume, such that:

(18)
$$D = \Delta\sqrt{-A_{44}}\ .$$

The ratio of the actual volume to the normal volume is, however, given by D / a_{44} .

One may further remark that the relation (14) gives the representation:

(17′)
$$\Delta^2 = \begin{vmatrix} 1+2e_{11}, & 2e_{12}, & 2e_{13} \\ 2e_{21}, & 1+2e_{22}, & 2e_{23} \\ 2e_{31}, & 2e_{32}, & 1+2e_{33} \end{vmatrix}.$$

§ 2. Second representation of rest deformations.

The quadratic differential form:

(19) $$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = \sum_{i,j=1}^{4} A_{ij}\, d\xi_i\, d\xi_j \,,$$

(20) $$\begin{cases} A_{ij} = a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} - a_{4i}a_{4j}, \quad i, j = 1, 2, 3, 4, \\ A_{44} = -a_{44}^2(1-s^2) < 0, \end{cases}$$

and the linear differential form:

(21) $$d\nu = a_{14}\, dx + a_{24}\, dy + a_{34}\, dz - a_{44}\, dt = \sum_{i=1}^{4} A_{i4}\, d\xi_i \,,$$

due to its invariance under Lorentz transformations, and with the introduction of dx_i^0 in place of the dx_i , goes to:

(19′) $$ds^2 = (dx^0)^2 + (dy^0)^2 + (dz^0)^2 - (dt^0)^2,$$

(21′) $$d\nu = -\, a_{44}^0 dt^0 \,,$$

such that:

(22) $$ds^2 = -\frac{1}{A_{44}}\, d\nu^2 = (dx^0)^2 + (dy^0)^2 + (dz^0)^2.$$

New expressions for the rest deformations then follow by means of (14):

(16′) $$\begin{cases} 1 + 2e_{11} = A_{11} - \dfrac{A_{14}^2}{A_{44}}, \quad e_{23} = e_{32} = \tfrac{1}{2}\left(A_{23} - \dfrac{A_{24}A_{34}}{A_{44}} \right), \\ 1 + 2e_{22} = A_{22} - \dfrac{A_{24}^2}{A_{44}}, \quad e_{31} = e_{13} = \tfrac{1}{2}\left(A_{31} - \dfrac{A_{34}A_{14}}{A_{44}} \right), \\ 1 + 2e_{33} = A_{33} - \dfrac{A_{34}^2}{A_{44}}, \quad e_{12} = e_{21} = \tfrac{1}{2}\left(A_{12} - \dfrac{A_{14}A_{24}}{A_{44}} \right). \end{cases}$$

§ 3. The complete system of ten partial differential equations for an arbitrary function of the rest deformations

What are the necessary and sufficient condition for a function $\Omega(a_{ij})$ of the 16 quantities a_{ij} to be expressed in terms of the 6 rest deformations e_{ij} alone? If one considers, for the moment, a fourfold extended manifold $R_4(X_1, X_2, X_3, X_4)$ with the metric:

(23) $$dS^2 = dX_1^2 + dX_2^2 + dX_3^2 - dX_4^2,$$

and at these five points M_0, M_1, M_2, M_3, M_4 with the coordinates:

(24) $$\begin{cases} M_0 & (X_1 = 0, \quad X_2 = 0, \quad X_3 = 0, \quad X_4 = 0), \\ M_i & (X_1 = a_{1i}, \quad X_2 = a_{2i}, \quad X_3 = a_{3i}, \quad X_4 = a_{4i}), \quad i = 1, 2, 3, 4, \end{cases}$$

then the expressions (20) for the A_{ij} show that they, and therefore also Ω, merely depend upon the relative positions of these five points to each other. Therefore, Ω is invariant under the "rotations" of M_0 and thus admits the six infinitesimal transformations:

(25) $$\begin{cases} X_2 \dfrac{\partial f}{\partial X_3} - X_3 \dfrac{\partial f}{\partial X_2}, \quad X_3 \dfrac{\partial f}{\partial X_1} - X_1 \dfrac{\partial f}{\partial X_3}, \quad X_1 \dfrac{\partial f}{\partial X_2} - X_2 \dfrac{\partial f}{\partial X_1}, \\ X_1 \dfrac{\partial f}{\partial X_4} + X_4 \dfrac{\partial f}{\partial X_1}, \quad X_2 \dfrac{\partial f}{\partial X_4} + X_4 \dfrac{\partial f}{\partial X_2}, \quad X_3 \dfrac{\partial f}{\partial X_4} + X_4 \dfrac{\partial f}{\partial X_3}, \end{cases}$$

From this fact, when one sets:

(26) $$\Omega_{ij} = \frac{\partial \Omega}{\partial a_{ij}}, \qquad i, j = 1, 2, 3, 4,$$

(27) $$\bar{\Omega}_{ij} = \sum_{h=1}^{4} a_{jh} \Omega_{ih}, \qquad i, j = 1, 2, 3, 4,$$

the six partial differential equations for Ω ensue:

(28) $$\begin{cases} \bar{\Omega}_{23} = \bar{\Omega}_{32}, \quad \bar{\Omega}_{31} = \bar{\Omega}_{13}, \quad \bar{\Omega}_{12} = \bar{\Omega}_{21}, \\ \bar{\Omega}_{14} + \bar{\Omega}_{41} = 0, \quad \bar{\Omega}_{24} + \bar{\Omega}_{42} = 0, \quad \bar{\Omega}_{34} + \bar{\Omega}_{43} = 0. \end{cases}$$

Furthermore, one must consider the fact that the e_{ij}, and therefore also Ω, are entirely independent of the choice of the time parameter. However, if one introduces τ' in place of τ:

(29) $$\begin{cases} \tau = \tau' + \lambda(\xi, \eta, \zeta, \tau'), \\ d\tau = d\tau' + \lambda_1 d\xi + \lambda_2 d\eta + \lambda_3 d\zeta + \lambda_4 d\tau', \end{cases}$$

then the a_{ij} go over to the transformation:

(30) $$a'_{ij} = a_{ij} + \lambda_j \, a_{i4}, \qquad i, j = 1, 2, 3, 4,$$

which must therefore leave Ω invariant.

From this fact, when the λ_j are chosen to be infinitely small the four partial differential equations for Ω then follow:

$$(31) \qquad \sum_{i=1}^{4} a_{i4}\Omega_{i1} = 0, \quad \sum_{i=1}^{4} a_{i4}\Omega_{i2} = 0, \quad \sum_{i=1}^{4} a_{i4}\Omega_{i3} = 0, \quad \sum_{i=1}^{4} a_{i4}\Omega_{i4} = 0,$$

or, by means of (27):

$$(31') \qquad \sum_{i=1}^{4} a_{i4}\bar{\Omega}_{i1} = 0, \quad \sum_{i=1}^{4} a_{i4}\bar{\Omega}_{i2} = 0, \quad \sum_{i=1}^{4} a_{i4}\bar{\Omega}_{i3} = 0, \quad \sum_{i=1}^{4} a_{i4}\bar{\Omega}_{i4} = 0.$$

Conversely, however, due to the group property of the transformations (28) and (31) define a complete system of partial differential equations with the six independent solutions e_{ij} such that we have the necessary and sufficient conditions for this before us, namely:

$$(32) \qquad \Omega(a_{ij}) = \Omega(e_{ij}).$$

§ 4. A general transformation formula

If the four functions $f_i(x_1, x_2, x_3, x_4)$ $(i = 1, 2, 3, 4)$ are related to the four functions $\varphi_j(\xi_1, \xi_2, \xi_3, \xi_4)$ $(j = 1, 2, 3, 4)$ by way of:

$$(33) \qquad Df_i = \sum_{j=1}^{4} a_{ij}\varphi_j\,, \qquad i = 1, 2, 3, 4$$

then one has the identity:

$$(34) \qquad D\sum_{j=1}^{4} \frac{\partial f_i}{\partial x_i} = \sum_{j=1}^{4} \frac{\partial \varphi_j}{\partial \xi_j}\,.$$

To prove this, one regards the equations (4) as the transformation of the point (ξ_i) in a Euclidian R_4 into the point (x_i).

Thus, if a surface element $d\omega$ with the projections $d\omega_j$ onto the four coordinate planes and a line element $d\sigma$ that goes through it, with the projections $d\xi_j$ onto the four coordinate axes, goes to a surface element do with the projections do_i and a line element ds through it that has the projections dx_i then one has:

$$(36) \qquad \sum_{i=1}^{4} do_i dx_i = D\sum_{j=1}^{4} d\omega_j d\xi_j$$

if the sum represents the fourfold volume of the infinitely small cone with do ($d\omega$, resp.) as its base surface and the endpoint of ds ($d\sigma$, resp.) as its vertex.

Since, from (6), one now has that the $Dd\xi_j$ transform into the dx_i in precisely the same way that, from (33), the φ_j transform into the f_i , one must therefore also have that:

$$\sum_{i=1}^{4} f_i do_i = \sum_{j=1}^{4} \varphi_j d\omega_j \ . \tag{37}$$

If one integrates here over a closed surface and likewise converts the hypersurface integral into a volume integral that is taken over the interior of the surface then it follows that:

$$\int \sum_{i=1}^{4} \frac{\partial f_i}{\partial x_i} dx_1 dx_2 dx_3 dx_4 = \int \sum_{j=1}^{4} \frac{\partial \varphi_j}{\partial x_j} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \tag{38}$$

and from this, by contracting the surface to a point one obtains the relation (34) that was to be proved.

As a special consequence of (34), for:

$$\begin{cases} \varphi_1 = 0, & \varphi_2 = 0, & \varphi_3 = 0, & \varphi_4 = Df, \\ f_1 = a_{14} f, & f_2 = a_{24} f, & f_3 = a_{34} f & f_4 = a_{44} f, \end{cases} \tag{39}$$

one must point out, in particular, the relation:

$$\frac{1}{D} \frac{\partial (Df)}{\partial \tau} = \frac{\partial (a_{14} f)}{\partial x} + \frac{\partial (a_{24} f)}{\partial y} + \frac{\partial (a_{34} f)}{\partial z} + \frac{\partial (a_{44} f)}{\partial t} \ . \tag{40}$$

From it, the differentiation symbols:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}, \tag{41}$$

$$D_t f = \frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial x} + \frac{\partial (vf)}{\partial y} + \frac{\partial (wf)}{\partial z}, \tag{42}$$

admit the perfectly intuitive representation:

$$\frac{df}{dt} = \frac{1}{a_{44}} \frac{\partial f}{\partial \tau}, \tag{41′}$$

$$D_t f = \frac{1}{D} \frac{\partial}{\partial \tau} \left(\frac{Df}{a_{44}} \right). \tag{42′}$$

§ 5. The kinetic potential

In order to go to the dynamics of bodies, we start by letting:

$$\varepsilon = \varepsilon(\xi, \eta, \zeta, \tau) \tag{4′}$$

denote the entropy per unit normal volume and remark that this is invariant [1]) under all Lorentz transformations and, in particular, all of the rest transformations.

Moreover, it will be assumed that there exists a kinetic potential of the form:

$$\int \Phi \, d\xi \, d\eta \, d\zeta \, d\tau \tag{43}$$

for the body. First, this potential shall, for the case of rest, assume the form:

$$\int \Omega(e_{ij}, \varepsilon) \, d\xi \, d\eta \, d\zeta \, dt \tag{43′}$$

of an ordinary kinetic potential that depends upon the deformation quantities e_{ij} and entropy ε; i.e., one must have:

$$\Phi = \Omega(e_{ij}, \varepsilon) \, a_{44} \tag{44}$$

for the case of rest.

Second, this potential shall be invariant under the Lorentz transformations; i.e., Φ shall exhibit the same invariance.

From this assumption, it follows that the general expression of Φ will be obtained when one subjects (44) to the inverse of the rest transformation. However, this happens simply when one understands the e_{ij} to mean the rest deformations (16) and replaces a_{44} with its rest value a_{44}^0 using (13). Thus, one will generally have:

$$\Phi(a_{ij}, \varepsilon) = \Omega(e_{ij}, \varepsilon) \, a_{44}^0 = \Omega(e_{ij}, \varepsilon) \sqrt{-A_{44}} \,. \tag{45}$$

The temperature θ is, from its connection to the kinetic potential in the rest case:

$$\theta = -\frac{\partial \Omega}{\partial \varepsilon} \,, \tag{46}$$

and since the rest transformation for it reads [2]):

$$\theta_0 = \beta\theta = \frac{a_{44}}{\sqrt{-A_{44}}} \theta \,, \tag{47}$$

it will generally be given by:

$$a_{44}\theta = -\frac{\partial \Phi}{\partial \varepsilon} \,. \tag{48}$$

The heat produced per unit time and normal volume is then represented by:

$$\theta \frac{d\varepsilon}{dt} = \frac{1}{a_{44}} \theta \frac{\partial \varepsilon}{\partial \tau} = -\frac{1}{a_{44}^2} \frac{\partial \Phi}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \tau} \,. \tag{49}$$

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[1]) M. Planck, Berl. Ber. 1907, pp. 542.
[2]) M. Planck, loc. cit.

For the differential quotients of Ω with respect to a_{ij} :

(26) $$\Omega_{ij} = \frac{\partial \Omega}{\partial a_{ij}}, \qquad i, j = 1, 2, 3, 4,$$

(for the quantities:

(27) $$\bar{\Omega}_{ij} = \sum_{h=1}^{4} a_{jh} \Omega_{ih}, \qquad i, j = 1, 2, 3, 4$$

that are derived from the, resp.), precisely the same ten relations (28) and (31′) that were derived in § 3 are true. The differential quotients of $\Phi(a_{ij}, \varepsilon)$ with respect to a_{ij} are expressed in terms of these Ω_{ij} :

(50) $$\Phi_{ij} = \frac{\partial \Phi}{\partial a_{ij}}, \qquad i, j = 1, 2, 3, 4$$

in the form:

(51) $$\Phi_{ij} = \sqrt{-A_{44}}\,\Omega_{ij} - \frac{1}{2}\frac{\Omega}{A_{44}}\frac{\partial A_{44}}{\partial a_{ij}}, \qquad i, j = 1, 2, 3, 4.$$

§ 6. **The Lagrangian equations of motion**

For the motion of the body, one shall now have:

(52) $$0 = \delta \int \Phi \, d\xi \, d\eta \, d\zeta \, d\tau + \int (\Xi \, \delta x + \mathrm{H} \, \delta y + \mathrm{Z} \, \delta z + \mathrm{T} \, \delta t + \mathrm{E} \, \delta\varepsilon) \, d\xi \, d\eta \, d\zeta \, d\tau .$$

The integration with respect to ξ, η, ζ shall be carried out over the entire finitely extended body, while the integration over τ shall, however, extend from τ_1 to τ_2 . Geometrically speaking, if:

(53) $$\varphi(\xi, \eta, \zeta) = 0$$

represents the bounding surface of the body then the integral in (ξ, η, ζ, τ)-space shall be taken over the volume that lies between the two planes $\tau = \tau_1$ and $\tau = \tau_2$ in the cylinder (53). The δx, δy, δz, δt, $\delta\varepsilon$ shall mean any variations of the five functions x, y, z, t, ε of the independent variables ξ, η, ζ, τ, of which only δx, δy, δz, δt vanish for $\tau = \tau_1$ and $\tau = \tau_2$ – i.e., the two base surfaces for the cylinder – while the $\delta\varepsilon$ shall be chosen arbitrarily.

The quantities:

$$\frac{1}{a_{44}}\Xi, \; \frac{1}{a_{44}}\mathrm{H}, \; \frac{1}{a_{44}}\mathrm{Z}$$

represent the external forces that act on a unit normal volume, whereas the meaning of T and E in the further course of motion can be derived from (52).

From this, and the rules of the variational calculus, it next follows that for each point of the body one has the Lagrangian equations of motion:

(54)
$$\begin{cases} \Xi = \dfrac{\partial\Phi_{11}}{\partial\xi} + \dfrac{\partial\Phi_{12}}{\partial\eta} + \dfrac{\partial\Phi_{13}}{\partial\zeta} + \dfrac{\partial\Phi_{14}}{\partial\tau}, \\ \mathrm{H} = \dfrac{\partial\Phi_{21}}{\partial\xi} + \dfrac{\partial\Phi_{22}}{\partial\eta} + \dfrac{\partial\Phi_{23}}{\partial\zeta} + \dfrac{\partial\Phi_{24}}{\partial\tau}, \\ \mathrm{Z} = \dfrac{\partial\Phi_{31}}{\partial\xi} + \dfrac{\partial\Phi_{32}}{\partial\eta} + \dfrac{\partial\Phi_{33}}{\partial\zeta} + \dfrac{\partial\Phi_{34}}{\partial\tau}, \\ \mathrm{T} = \dfrac{\partial\Phi_{41}}{\partial\xi} + \dfrac{\partial\Phi_{42}}{\partial\eta} + \dfrac{\partial\Phi_{43}}{\partial\zeta} + \dfrac{\partial\Phi_{44}}{\partial\tau}, \\ \mathrm{E} = -\dfrac{\partial\Phi}{\partial\varepsilon}. \end{cases}$$

These five equations are, however, independent of each other corresponding to the arbitrariness of the time parameter τ. Namely, if one sets in (52):

(55)
$$\delta x_i = a_{i4}\,\omega, \qquad \delta\varepsilon = \frac{\partial\varepsilon}{\partial\tau}\,\omega,$$

where $\omega = \omega(\xi, \eta, \zeta, \tau)$ vanishes for $\tau = \tau_1$ and $\tau = \tau_2$, but is otherwise arbitrary, then one has:

(56)
$$\delta a_{ij} = \frac{\partial a_{i4}}{\partial\xi_j} = \frac{\partial a_{i4}}{\partial\tau}\omega + \frac{\partial\omega}{\partial\xi_i}a_{i4}, \qquad i, j = 1, 2, 3, 4,$$

(57)
$$\delta\Phi = \sum_{i,j=1}^{4}\frac{\partial\Phi}{\partial a_{ij}}\delta a_{ij} + \frac{\partial\Phi}{\partial\varepsilon}\delta\varepsilon = \frac{\partial\Phi}{\partial\tau}\omega + \sum_{i,j=1}^{4}\frac{\partial\Phi}{\partial a_{ij}}\frac{\partial\omega}{\partial\xi_j}a_{i4}.$$

Here, the second term represents the variation of Φ under the transformation (30) for the differential values $\lambda_j = \partial\omega/\partial x_j$. From (45), this transformation generally takes Φ to:

(58)
$$\Phi(a'_{ij}, \varepsilon) = (1 + \lambda_4)\,\Phi(a_{ij}, \varepsilon),$$

such that any variation equals $\lambda_4\Phi = \partial\omega/\partial x_j$, and therefore:

(57′)
$$\delta\Phi = \frac{\partial\Phi\omega}{\partial\tau}.$$

Now, since ω shall vanish for $\tau = \tau_1$ and $\tau = \tau_2$ the first term in (52) drops out, and what results is the desired relation:

(59)
$$a_{14}\,\Xi + a_{24}\,\mathrm{H} + a_{34}\,\mathrm{Z} + a_{44}\mathrm{T} + \frac{\partial\varepsilon}{\partial\tau}\mathrm{E} = 0.$$

Since, from (48):

(60) $$\mathsf{E} = -\frac{\partial \Phi}{\partial \varepsilon} = a_{44}\theta,$$

then:

(61) $$-\mathsf{T} = u\Xi + v\mathsf{H} + w\mathsf{Z} + \theta\,\frac{\partial \varepsilon}{\partial \tau},$$

and therefore $-\mathsf{T}/a_{44}$ represents the sum of the work done and the heat produced per unit time and normal volume.

The rules of variational calculus next yield the boundary terms in the right-hand side of (52):

(62) $$\int \sum_{i,j=1}^{4} \Phi_{ij}\,\delta x_i\,d\omega_j\,,$$

in which the integral is taken over the entire boundary surface of the domain of integration in (ξ, η, ζ, τ)-space, and the $d\omega_j$ denotes the projections of an element $d\omega$ of this surface. Now, at the two base surfaces of the cylinders one has $\delta x_i = 0$, whereas for the element of the sleeve one has:

(63) $$d\omega_1 : d\omega_2 : d\omega_3 : d\omega_4 = \varphi_1 : \varphi_2 : \varphi_3 : \varphi_4\,,$$

in which we have set:

(64) $$d\varphi = \varphi_1\,d\xi + \varphi_2\,d\eta + \varphi_3\,d\zeta + \varphi_4\,d\tau, \quad \varphi_4 = 0.$$

Annulling the boundary terms then delivers the boundary conditions that are valid on the bounding surface of the body:

(65) $$\varphi_1\Phi_{i1} + \varphi_2\Phi_{i2} + \varphi_3\,\Phi_{i3} = 0, \quad i = 1, 2, 3, 4.$$

Now, since for the special variation (55) the boundary terms drop out, one must have:

(66) $$\sum_{i,j=1}^{4} a_{i4}\varphi_j\Phi_{ij} = 0,$$

along the sleeve of the cylinder, and therefore since $a_{44} \neq 0$ the fourth of equations (65) is a consequence of the remaining ones, and can therefore be omitted.

§ 7. The Eulerian equations of motion and the relations between impulse, energy, and stress

From the theorem of § 4, in order to obtain the Eulerian form of the equations of motion, one need only introduce the differential quotients with respect to x, y, z, t in place of ones with respect to ξ, η, ζ, τ.

When one sets:

(67) $DX = \Xi$, $DY = \mathrm{H}$, $DZ = \mathrm{Z}$, $Dt = \mathrm{T}$,

(68) $$DF_{ij} = \sum_{h=1}^{4} a_{jh}\Phi_{ih}\,, \qquad i, j = 1, 2, 3, 4,$$

(69) $$F = -\frac{\Omega}{\Delta} = -\frac{\Phi}{D},$$

one immediately obtains the Eulerian equations of motion:

(70) $$\begin{cases} X = \dfrac{\partial F_{11}}{\partial x} + \dfrac{\partial F_{12}}{\partial y} + \dfrac{\partial F_{13}}{\partial z} + \dfrac{\partial F_{14}}{\partial t}, \\ Y = \dfrac{\partial F_{21}}{\partial x} + \dfrac{\partial F_{22}}{\partial y} + \dfrac{\partial F_{23}}{\partial z} + \dfrac{\partial F_{24}}{\partial t}, \\ Z = \dfrac{\partial F_{31}}{\partial x} + \dfrac{\partial F_{32}}{\partial y} + \dfrac{\partial F_{33}}{\partial z} + \dfrac{\partial F_{34}}{\partial t}, \\ T = \dfrac{\partial F_{41}}{\partial x} + \dfrac{\partial F_{42}}{\partial y} + \dfrac{\partial F_{43}}{\partial z} + \dfrac{\partial F_{44}}{\partial t}, \\ \theta = \Delta\sqrt{1-s^2}\,\dfrac{\partial F}{\partial \varepsilon}. \end{cases}$$

Here, X, Y, Z are the external forces that act on the unit of actual volume, and since, from (61):

(71) $$-T = uX + vY + wZ + Q,$$

(72) $$Q = \frac{\theta}{\Delta\sqrt{1-s^2}}\frac{d\varepsilon}{dt} = \frac{\partial F}{\partial \varepsilon}\frac{d\varepsilon}{dt},$$

then $-T$ represents the work done and heat produced per unit of time and actual volume.

From (27) and (51), one further obtains for the F_{ij} :

(73) $$\Delta F_{ij} = \bar{\Omega}_{ij} + \frac{1}{2}\frac{\Omega}{A_{44}}\frac{\partial A_{44}}{\partial a_{i4}}a_{j4}\,, \qquad i, j = 1, 2, 3, 4,$$

and the relations (28) and (31′) between the $\bar{\Omega}_{ij}$ yield the 10 relations for the F_{ij} :

(74) $$\begin{cases} F_{23} = F_{32}, & F_{31} = F_{13}, & F_{12} = F_{21}, \\ F_{14} + F_{41} = 0, & F_{24} + F_{42} = 0, & F_{34} + F_{43} = 0. \end{cases}$$

(75) $$\sum_{i=1}^{4} a_{i4}F_{ij} + a_{j4}F = 0, \qquad j = 1, 2, 3, 4.$$

From the form [1]) of equations (70), one deduces that the F_{ij} (i, j = 1, 2, 3) represent the stresses, and that the impulse $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ and energy $\mathfrak{E}$ per unit of actual volume, which are computed by means of:

(76) $$\mathfrak{X} = F_{14}, \qquad \mathfrak{Y} = F_{24}, \qquad \mathfrak{Z} = F_{34}, \qquad \mathfrak{E} = -F_{44},$$

are given in such a way that one has, in particular, the energy equation before one in the fourth of these equations:

(70′) $$\frac{\partial \mathfrak{E}}{\partial t} + \frac{\partial \mathfrak{X}}{\partial x} + \frac{\partial \mathfrak{Y}}{\partial y} + \frac{\partial \mathfrak{Z}}{\partial z} = uX + vY + wZ + Q$$

Impulse, energy, and stress are, from (75), coupled with each other by the relations:

(77) $$\begin{cases} \mathfrak{X} = uF + uF_{11} + vF_{21} + wF_{31}, \\ \mathfrak{Y} = vF + uF_{12} + vF_{22} + wF_{32}, \\ \mathfrak{Z} = wF + uF_{13} + vF_{23} + wF_{33}, \\ \mathfrak{E} = \quad F + u\mathfrak{X} + v\mathfrak{Y} + w\mathfrak{Z}. \end{cases}$$

When computed per unit time and normal volume, impulse and energy:

(78) $$\bar{\mathfrak{X}} = \frac{D}{a_{44}}\mathfrak{X}, \quad \bar{\mathfrak{Y}} = \frac{D}{a_{44}}\mathfrak{Y}, \quad \bar{\mathfrak{Z}} = \frac{D}{a_{44}}\mathfrak{Z}, \quad \bar{\mathfrak{E}} = \frac{D}{a_{44}}\mathfrak{E},$$

or, by observing the connection between the F_{ij} and the Φ_{ij} :

(78′) $$\begin{cases} a_{44}\bar{\mathfrak{X}} = \sum_{i=1}^{4} a_{4i}\Phi_{1i}, \quad a_{44}\bar{\mathfrak{Y}} = \sum_{i=1}^{4} a_{4i}\Phi_{2i}, \quad a_{44}\bar{\mathfrak{X}} = \sum_{i=1}^{4} a_{4i}\Phi_{3i}, \\ \qquad\qquad -a_{44}\bar{\mathfrak{E}} = \sum_{i=1}^{4} a_{4i}\Phi_{4i}, \end{cases}$$

and especially, in the event that one chooses $t = \tau$:

(78") $$\begin{cases} \bar{\mathfrak{X}} = \Phi_{14} = \Phi_u, \qquad \bar{\mathfrak{Y}} = \Phi_{24} = \Phi_v, \qquad \bar{\mathfrak{Z}} = \Phi_{34} = \Phi_w, \\ \quad \bar{\mathfrak{E}} = -\Phi_{14} \qquad = u\Phi_u + v\Phi_v + w\Phi_w - \Phi. \end{cases}$$

In order to also ultimately express the boundary conditions (65) for the F_{ij}, one first writes them symmetrically:

(65) $$\varphi_1\Phi_{i1} + \varphi_2\Phi_{i2} + \varphi_3\Phi_{i3} + \varphi_4\Phi_{i4} = 0, \quad i = 1, 2, 3,$$

[1]) M. Abraham, Rendiconti del. circ. mat. d. Palermo **28**, pp. 1, 1909.

and then solves it for x, y, z, t :

(79)
$$\begin{cases} \varphi(\xi,\eta,\zeta) = f(x,y,z,t), \\ df = f_1 dx + f_2 dy + f_3 dz + f_4 dt. \end{cases}$$

If one then remarks that the transformation that takes φ_1, φ_2, φ_3, φ_4 to f_1, f_2, f_3 , f_4 – omitting the factor D – is precisely contragredient to the one that takes Φ_{i1}, Φ_{i2}, Φ_{i3}, Φ_{i1} into F_{i1}, F_{i2}, F_{i3}, F_{i4} then this illuminates the fact that the boundary conditions, when expressed in terms of F_{ij}, read:

(80)
$$f_1 F_{i1} + f_2 F_{i2} + f_3 F_{i3} + f_4 F_{i4} = 0, \qquad i = 1, 2, 3.$$

Now, if n_1 , n_2 , n_3 are the direction cosines of the normal to the bounding surface of the body and if s_n are the components of the velocity of a particle in the same frame as this normal then one has:

(81)
$$f_1 : f_2 : f_3 : f_4 = n_1 : n_2 : n_3 : - s_n .$$

Thus, in place of (80) one can also write:

(80′)
$$n_1 F_{i1} + n_2 F_{i2} + n_3 F_{i3} = s_n F_{i4}, \quad i = 1, 2, 3.$$

Under any Lorentz transformation the F_{ij} transform the x_j in exactly the same way that the product $u_i\, x_j$ transforms the x_j into the u_i that are contragredient to them.

The “rest values” F_{ij}^0 of the F_{ij} are obtained from the previous values for $u = v = w = 0$, in which $a_{ij} = a_{ij}^0$ $(i, j = 1, 2, 3)$:

(82)
$$\begin{cases} \Delta F_{ij}^0 = \sum_{h=1}^{3} a_{jh}^0 \dfrac{\partial\Omega}{\partial a_{ih}^0}, \quad i, j = 1, 2, 3, \\ F_{i4}^0 = F_{4i}^0 = 0, \quad i = 1, 2, 3, \\ F_{44}^0 = \dfrac{\Omega}{\Delta} = -F = -\mathfrak{E}^0. \end{cases}$$

The F_{ij} may be derived from them by the transformation that is inverse to the rest transformation.

§ 8. A third form for the equations of motion and the relative stresses

If one introduces the differential symbol D_i (cf., § 4) in place of the differential quotients $\partial/\partial t$ then one obtains the third form of the equations of motion:

(83)
$$\begin{cases} X = D_t\mathfrak{X} + \dfrac{\partial S_{11}}{\partial x} + \dfrac{\partial S_{12}}{\partial y} + \dfrac{\partial S_{13}}{\partial z}, \\ Y = D_t\mathfrak{Y} + \dfrac{\partial S_{21}}{\partial x} + \dfrac{\partial S_{22}}{\partial y} + \dfrac{\partial S_{23}}{\partial z}, \\ Z = D_t\mathfrak{Z} + \dfrac{\partial S_{31}}{\partial x} + \dfrac{\partial S_{32}}{\partial y} + \dfrac{\partial S_{33}}{\partial z}, \\ -T = D_t\mathfrak{E} + \dfrac{\partial(\mathfrak{X} - u\mathfrak{E})}{\partial x} + \dfrac{\partial(\mathfrak{Y} - v\mathfrak{E})}{\partial y} + \dfrac{\partial(\mathfrak{Z} - w\mathfrak{E})}{\partial z}, \end{cases}$$

in which the "relative" [1]) stresses S_{ij} (i, j = 1, 2, 3) that enter here:

(84)
$$\begin{cases} S_{11} = F_{11} - u\mathfrak{X}, & S_{12} = F_{12} - v\mathfrak{X}, & S_{13} = F_{13} - w\mathfrak{X}, \\ S_{21} = F_{21} - u\mathfrak{Y}, & S_{22} = F_{22} - v\mathfrak{Y}, & S_{23} = F_{23} - w\mathfrak{Y}, \\ S_{31} = F_{31} - u\mathfrak{Z}, & S_{32} = F_{32} - v\mathfrak{Z}, & S_{33} = F_{33} - w\mathfrak{Z}, \end{cases}$$

are coupled with the impulse and energy by:

(85)
$$\begin{cases} \mathfrak{X} = u\mathfrak{E} + uS_{11} + vS_{21} + wS_{31}, \\ \mathfrak{Y} = v\mathfrak{E} + uS_{12} + vS_{22} + wS_{32}, \\ \mathfrak{Z} = w\mathfrak{E} + uS_{13} + vS_{23} + wS_{33}. \end{cases}$$

The boundary conditions (80′) will be expressed in terms of the S_{ij} by way of:

(86)
$$s_n = u\, n_1 + v\, n_2 + w\, n_3 ,$$

and read:

(87)
$$n_1\, S_{i1} + n_2\, S_{i2} + n_3\, S_{i3} = 0, \qquad i = 1, 2, 3$$

and thus demand the vanishing of the relative stresses for each bounding surface element.

§ 9. The ten general integrals of the equations of motion

The ten-term group of "motions" in (x, y, z, t)-space with the corresponding metric:

(88)
$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2,$$

makes the 10 principles of the center of mass point, surfaces, and energy valid for the entire body, which are analogous to the theorems of ordinary mechanics.

Namely, if the components of an infinitely small motion were taken for δx, δy, δz, δt in the relation (52), and thus one were to choose $\delta s = 0$, then one would have $\delta\Phi = 0$ for

[1]) M. Abraham, loc. cit.

these variations, and therefore the first term on the right-hand side would vanish. However, since these variations do not satisfy the condition of vanishing for $\tau = \tau_1$ and $\tau = \tau_2$, the left-hand side of any relation will not, on the other hand, be null, but will be replaced by the boundary term (62):

$$(89) \qquad \sum_{i,j=1}^{4} \Phi_{ij}\,\delta x_i\, d\omega_j = \int (\Xi\ \delta x + \mathrm{H}\ \delta y + \mathrm{Z}\ \delta z + \mathrm{T}\ \delta t)\ d\xi\, d\eta\, d\zeta\, d\tau.$$

As a result of the boundary condition (65), the part of the bounding surface integral on the left that comes from the sleeve of the cylinder drops out, while for the base surfaces $\tau = \tau_1$ and $\tau = \tau_2$ of the cylinder one has: $d\omega_1 = d\omega_2 = d\omega_3 = 0$, $d\omega_4 = d\xi\, d\eta\, d\zeta$, such that:

$$(89') \qquad \int \sum_{i=1}^{4} \Phi_{i4}\,\delta x_i\, d\xi\, d\eta\, d\zeta \Big|_{\tau_1}^{\tau_2} = \int (\Xi\ \delta x + \mathrm{H}\ \delta y + \mathrm{Z}\ \delta z + \mathrm{T}\ \delta t)\ d\xi\, d\eta\, d\zeta\, d\tau.$$

If one now lets $\tau_1 = \tau_2 = \tau$ and chooses $t = \tau$ then it follows that:

$$(90) \qquad \frac{d}{dt}\int (\mathfrak{X}\,\delta x + \mathfrak{Y}\,\delta y + \mathfrak{Z}\,\delta z - \mathfrak{E}\,\delta t)\, dv = \int (X\ \delta x + Y\ \delta y + Z\ \delta z + T\ \delta t)\ dv,$$

where $dv = dx\ dy\ dz$ denotes the volume element of the body, and the integral is taken over the entire space swept out by the body up to time t.

From this, for each relation that is true for an infinitely small motion δx, δy, δz, δt there ensue 10 independent infinitesimal motions that correspond to the aforementioned theorem, and indeed the infinitesimal translations:

$$(91) \qquad \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial t}$$

correspond to the impulse and energy theorem:

$$(92) \qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int \mathfrak{X}\, dv = \int X\, dv, & \dfrac{d}{dt}\displaystyle\int \mathfrak{Y}\, dv = \int Y\, dv, \\[2ex] \dfrac{d}{dt}\displaystyle\int \mathfrak{Z}\, dv = \int Z\, dv, & -\dfrac{d}{dt}\displaystyle\int \mathfrak{E}\, dv = \int T\, dv, \end{cases}$$

the infinitesimal rotations:

$$(93) \qquad y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}, \quad z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}, \quad x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x},$$

correspond to the surface theorem:

$$(94)\qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int (y\mathfrak{Z} - z\mathfrak{Y})dv = \int (yZ - zY)dv, \\ \dfrac{d}{dt}\displaystyle\int (z\mathfrak{X} - x\mathfrak{Z})dv = \int (zX - xZ)dv, \\ \dfrac{d}{dt}\displaystyle\int (x\mathfrak{Y} - y\mathfrak{X})dv = \int (xY - yX)dv, \end{cases}$$

and the infinitesimal rotations:

$$(95)\qquad t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}, \quad t\frac{\partial f}{\partial y} + y\frac{\partial f}{\partial t}, \quad t\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial t},$$

correspond to the theorems:

$$(96)\qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int (t\mathfrak{X} - x\mathfrak{E})dv = \int (tX + xT)dv, \\ \dfrac{d}{dt}\displaystyle\int (t\mathfrak{Y} - y\mathfrak{E})dv = \int (tY + yT)dv, \\ \dfrac{d}{dt}\displaystyle\int (t\mathfrak{Z} - z\mathfrak{E})dv = \int (tZ + zT)dv. \end{cases}$$

If one were to subtract the corresponding equations (92), multiplied by t, from equations (96) then they would take on the form:

$$(96')\qquad \begin{cases} \displaystyle\int \mathfrak{X}\,dv = \frac{d}{dt}\int x\mathfrak{E}\,dv + \int xT\,dv, \\ \displaystyle\int \mathfrak{Y}\,dv = \frac{d}{dt}\int y\mathfrak{E}\,dv + \int yT\,dv, \\ \displaystyle\int \mathfrak{Z}\,dv = \frac{d}{dt}\int z\mathfrak{Z}\,dv + \int zT\,dv, \end{cases}$$

from which a certain parallel with the once-integrated center-of-mass theorem of ordinary mechanics emerges.

In particular, if the body moves adiabatically in the absence of forces then the impulse, impulse moment, and energy are constant, and moreover, the energy midpoint moves in a uniform, rectilinear manner, and its velocity, when multiplied by the energy, yields the impulse.

§ 10. The hydrodynamic equations

In order to obtain the basic equations of hydrodynamics, one must let Ω depend only upon rest volume and entropy:

$$\Omega = \Omega(\Delta,\, \varepsilon) = \Omega\left(\frac{D}{\sqrt{-A_{44}}}, \varepsilon\right). \tag{97}$$

This Ansatz yields, when one sets:

$$p = \frac{\partial \Omega}{\partial \Delta}, \tag{98}$$

by a brief intermediate computation:

$$\bar{\Omega}_{ij} = p\Delta\left(\delta_{ij} - \frac{1}{2A_{44}}\frac{\partial A_{44}}{\partial a_{i4}} a_{j4}\right) \qquad i, j = 1, 2, 3, 4, \tag{99}$$

$$\delta_{ij} = 0 \quad \text{for } i \neq j, \qquad \delta_{ii} = 1,$$

and from this it then follows that:

$$F_{ij} = p\delta_{ij} - \frac{m}{a_{44}^2}\frac{\partial A_{44}}{\partial a_{i4}} a_{j4} \qquad i, j = 1, 2, 3, 4, \tag{100}$$

$$m = \frac{F+p}{1-s^2} = \frac{\Delta\Omega_\Delta - \Omega}{\Delta(1-s^2)} = -\frac{\Delta F_\Delta}{1-s^2}. \tag{101}$$

From this, the impulse and energy per unit actual volume are:

$$\mathfrak{X} = mu, \qquad \mathfrak{Y} = mv, \qquad \mathfrak{Z} = mw, \qquad \mathfrak{E} = m - p, \tag{102}$$

and the relative stresses take on the simple values:

$$\begin{cases} S_{11} = S_{22} = S_{33} = p, \\ S_{23} = S_{32} = S_{31} = S_{13} = S_{12} = S_{21} = 0. \end{cases} \tag{103}$$

From this, however, the third form of the equations of motion (81) goes over to the basic hydrodynamic equations:

$$\begin{cases} D_t(mu) + \dfrac{\partial p}{\partial x} = X, \\ D_t(mv) + \dfrac{\partial p}{\partial y} = Y, \\ D_t(mw) + \dfrac{\partial p}{\partial z} = Z, \\ -D_t(m) + \dfrac{\partial p}{\partial t} = T. \end{cases} \tag{104}$$

The impulse and energy, when computed per unit normal volume, are:

(105) $\bar{\mathfrak{X}} = \mu\, u, \qquad \bar{\mathfrak{Y}} = \mu\, v, \qquad \bar{\mathfrak{Z}} = \mu\, w, \qquad \bar{\mathfrak{E}} = \mu - p\Delta\sqrt{1-s^2}$,

(106) $$m = m\Delta\sqrt{1-s^2} = \beta(\Delta\Omega_\Delta - \Omega).$$

If one chooses $t = \tau$ and considers that one has:

(107) $$\Phi = \Omega\left(\frac{D}{\sqrt{1-s^2}}, \varepsilon\right)\sqrt{1-s^2}\,,$$

since $D = |\, a_{ij}\,|$ ($i, j = 1, 2, 3$) involves the u, v, w merely in the form $s = \sqrt{u^2+v^2+w^2}$, then (78") then teaches us that:

(108) $$\mu = \frac{1}{s}\Phi_s\,, \qquad \bar{\mathfrak{E}} = s\,\Phi_s - \Phi.$$

In particular, the expression (105) for the impulses shows us that the fluid takes on a longitudinal and a transversal inertia, which will be given per unit normal volume by:

(109) $$\mu_t = \mu = \frac{1}{s}\Phi_s\,, \qquad \mu_l = \Phi_{ss}\,,$$

where one computes:

(110) $$\begin{cases} \Phi_s = s\beta(\Delta\Omega_\Delta - \Omega), \\ \Phi_{ss} = \beta^3(\Delta\Omega_\Delta - \Omega) + s^2\beta^3\Delta^2\Omega_{\Delta\Delta}. \end{cases}$$

Since the independent variables are D, s, ε here, one thus has precisely specified the adiabatic-isochoric values of the coefficients of inertia.

§ 11. The Weber form of the hydrodynamical equations and the Helmholtz theorem on vortex motion

From equations (10), it follows in an obvious way that:

(111) $$\begin{cases} D[dx\, D_t(m\,u) + dy\, D_t(m\,v) + dz\, D_t(m\,w) - dt\, D_t(m)] \\ \qquad = -D\,dp + \Xi dx + \mathrm{H} dy + \mathrm{Z} dz + T dt. \end{cases}$$

If one now sets:

(112) $$M = \frac{m\,D}{a_{44}^2} = \frac{\mu}{a_{44}}$$

and employs the representation (42) for D_t then the left-hand side of (111) goes over to:

(113) $$\begin{cases} \dfrac{\partial M a_{14}}{d\tau}dx + \dfrac{\partial M a_{24}}{d\tau}dy + \dfrac{\partial M a_{34}}{d\tau}dz - \dfrac{\partial M a_{14}}{d\tau}dt \\ \qquad = \dfrac{\partial}{\partial \tau}(M\,dv) - \tfrac{1}{2}M\,dA_{44} \\ \qquad = \dfrac{\partial}{\partial \tau}(M\,dv) - M\,dA_{44} - \sqrt{-A_{44}}\,d(M\sqrt{-A_{44}}). \end{cases}$$

However, since one further has:

(114) $$M\sqrt{-A_{44}} = \mu\sqrt{1-s^2} = \Delta\Omega_\Delta - \Omega,$$

(111) is finally written, after a brief reduction:

(115) $$\begin{cases} \dfrac{\partial}{\partial \tau}(M\,dv) = d(M\,A_{44}) + d\Pi, \\ d\Pi = \Xi dx + \mathsf{H} dy + \mathsf{Z} dz + \mathsf{T} dt + \mathsf{E} d\varepsilon. \end{cases}$$

Therefore, if this is expressed in terms of $d\xi$, $d\eta$, $d\zeta$, $d\tau$:

(116) $$d\Pi = \mathsf{A}\,d\xi + \mathsf{B}\,d\eta + \Gamma\,d\zeta,$$

then a comparison of the coefficients of $d\xi$, $d\eta$, $d\zeta$ in (115) produces ordinary hydrodynamic equations that are analogous to those of H. Weber:

(117) $$\begin{cases} \dfrac{\partial M\,A_{14}}{\partial \tau} = \dfrac{\partial M\,A_{44}}{\partial \xi} + \mathsf{A}, \\ \dfrac{\partial M\,A_{24}}{\partial \tau} = \dfrac{\partial M\,A_{44}}{\partial \eta} + \mathsf{B}, \\ \dfrac{\partial M\,A_{34}}{\partial \tau} = \dfrac{\partial M\,A_{44}}{\partial \zeta} + \Gamma. \end{cases}$$

On the other hand, if one sets $d\tau = 0$ in (115) and chooses $t = \tau$ then it follows that:

(118) $$\frac{d}{dt}(\bar{\mathfrak{X}}dx + \bar{\mathfrak{Y}}dy + \bar{\mathfrak{Z}}dz) = d\mu\,(s^2 - 1) + \Xi\,dx + \mathsf{H}\,dy + \mathsf{Z}\,dz + \theta\,d\varepsilon.$$

Therefore, in the case of $dt = 0$:

(119) $$\Xi\,dx + \mathsf{H}\,dy + \mathsf{Z}\,dz + \theta\,d\varepsilon = d\omega(x, y, z, t),$$

for each closed integration path that is reducible to null, one has:

(120) $$\frac{d}{dt}\int(\bar{\mathfrak{X}}dx + \bar{\mathfrak{Y}}dy + \bar{\mathfrak{Z}}dz) = 0,$$

or: a line integral of the impulse that is taken over a closed curve that always represents the same particle has a constant value in time during the motion of the fluid.

However, from this the Helmholtz vortex theorem immediately comes into play, where the vorticial velocity p, q, r is defined by the curl of the impulse here:

(121) $$p = \frac{\partial\bar{\mathfrak{Z}}}{\partial y} - \frac{\partial\bar{\mathfrak{Y}}}{\partial z}, \qquad q = \frac{\partial\bar{\mathfrak{X}}}{\partial z} - \frac{\partial\bar{\mathfrak{Z}}}{\partial x}, \qquad r = \frac{\partial\bar{\mathfrak{Y}}}{\partial x} - \frac{\partial\bar{\mathfrak{X}}}{\partial y}.$$

§ 12. The kinetic potential of isotropic elastic bodies for small rest deformations

If one is dealing with an elastic body that is isotropic in the normal state then, other than entropy, Ω can depend upon only three principal dilatations that take the rest form of a volume element to its normal form. However, in their place one can introduce the three invariants J_1, J_2, J_3 of the rest deformation that are symmetrically constructed from them, and are determined from the identity in λ:

(122) $$\begin{vmatrix} e_{11}+\lambda, & e_{12}, & e_{13} \\ e_{21}, & e_{22}+\lambda, & e_{23} \\ e_{31}, & e_{32}, & e_{33}+\lambda \end{vmatrix} = \lambda^3 + J_1\lambda^2 + J_3\lambda + J_3,$$

by way of:

(123) $$\begin{cases} J_1 = e_{11} + e_{22} + e_{33}, \\ J_2 = (e_{23}e_{33} - e_{23}^2) + (e_{33}e_{11} - e_{31}^2) + (e_{11}e_{22} - e_{12}^2), \\ J_3 = e_{11}e_{22}e_{33} + 2e_{23}e_{31}e_{12} - e_{11}e_{23}^2 - e_{23}e_{31}^2 - e_{33}e_{12}^2, \\ \quad = |e_{ij}|, \end{cases}$$

and is connected with Δ by:

(124) $$\Delta^2 = 1 + 2J_1 + 4J_2 + 8J_3.$$

In particular, if the rest deformations e_{ij} are sufficiently small and the stresses vanish in the normal state then one can assume that Ω is a quadratic function of the e_{ij}; hence, it has the form:

(125) $$\Omega = -M - \tfrac{1}{2}AJ_1^2 + 2BJ_2,$$

where A, B, M can still depend upon ε.

PART TWO

Inertial resistance and wave mechanics

§ 1. The six inertial coefficients and the postulate of the positive-definite character of the form Γ.

If one chooses the special case $\tau = t$ in the sequel then one has:

$$\Phi = \Phi(a_{ij}, u, v, w, \varepsilon) \qquad i, j = 1, 2, 3 \tag{1}$$

and the impulse per unit normal volume is:

$$\bar{\mathfrak{X}} = \Phi_u, \qquad \bar{\mathfrak{Y}} = \Phi_v, \qquad \bar{\mathfrak{Z}} = \Phi_w. \tag{2}$$

If one now varies the velocity components u, v, w by the addition of $\gamma_1 dt$, $\gamma_2\, dt$, $\gamma_3\, dt$ and varies the impulse components by the addition of $\Gamma_1\, dt$, $\Gamma_2\, dt$, $\Gamma_3\, dt$, then $-\Gamma_1$, $-\Gamma_2$, $-\Gamma_3$ are called the inertial resistance per unit normal volume that is aroused by the computed components of the acceleration:

$$\gamma_1, \gamma_2, \gamma_3, \qquad \gamma = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}. \tag{3}$$

Obviously, from (2), if:

$$\Gamma(\gamma_1, \gamma_2, \gamma_3) = \sum_{i,j=1}^{3} \mu_{ij}\gamma_i\gamma_j = \delta^2\Phi \quad \text{for} \quad \delta u = \gamma_1,\ \delta v = \gamma_2,\ \delta w = \gamma_3 \tag{4}$$

then one will set:

$$\begin{cases} \Gamma_1 = \dfrac{1}{2}\dfrac{\partial\Gamma}{\partial\gamma_1} = \mu_{11}\gamma_1 + \mu_{12}\gamma_2 + \mu_{13}\gamma_3, \\ \Gamma_2 = \dfrac{1}{2}\dfrac{\partial\Gamma}{\partial\gamma_2} = \mu_{21}\gamma_1 + \mu_{22}\gamma_2 + \mu_{23}\gamma_3, \\ \Gamma_3 = \dfrac{1}{2}\dfrac{\partial\Gamma}{\partial\gamma_3} = \mu_{31}\gamma_1 + \mu_{32}\gamma_2 + \mu_{33}\gamma_3, \\ \qquad\qquad \mu_{ij} = \mu_{ji}. \end{cases} \tag{5}$$

The "inertial coefficients" μ_{ij} represent the masses per unit normal volume or densities of the body. [1]) Since a_{ij}, u, v, w, s are the independent variables, they specify the adiabatic-isochoric inertial coefficients precisely.

Hereafter there are always three mutually normal directions of the acceleration – the "principal inertial directions" – for which the inertial resistance possesses a direction that is equal or opposite to the acceleration. They are the principal axes of the second-degree surface: $\Gamma(x, y, z) = C$.

If one takes the components of the acceleration and the inertial resistance with respect to these three principal axes then one has:

$$\Gamma = \mu_1\gamma_1^2 + \mu_2\gamma_2^2 + \mu_3\gamma_3^2, \tag{4′}$$

$$\Gamma_1 = \mu_1\gamma_1, \qquad \Gamma_2 = \mu_2\gamma_2, \qquad \Gamma_3 = \mu_3\gamma_3. \tag{5′}$$

Therefore, if one excludes the possibility that an acceleration provokes an inertial resistance that is in the same direction then the three principal inertial coefficients μ_1, μ_2, μ_3 are positive and thus the quadratic form $\Gamma(\gamma_1, \gamma_2, \gamma_3)$ is positive-definite.

Since:

$$\Gamma(\gamma_1, \gamma_2, \gamma_3) = \mu_1\Gamma_1 + \mu_2\Gamma_2 + \mu_3\Gamma_3 \tag{6}$$

one can also express this assumption as: The inertial resistance shall always define an obtuse angle with the acceleration.

If the direction of the velocity and each of its normals is a principal direction then one obtains the well-known case of purely longitudinal and transversal inertial coefficients u_l and u_t. One then has:

$$\Gamma = \mu_l\gamma_l^2 + \mu_t\gamma_t^2, \tag{4"}$$

in which:

$$\gamma_l = \frac{1}{s}(u\gamma_1 + v\gamma_2 + w\gamma_3), \qquad \gamma_t = \sqrt{\gamma^2 - \gamma_l^2} \tag{7}$$

denote the longitudinal and transversal components of the acceleration; i.e., the ones that are parallel and normal to the velocity.

For a rest element the inertial coefficients μ_i^0 are given immediately. From I (77), it immediately follows that:

$$\begin{cases} \mu_{ij}^0 = \Delta F_{ij}^0, & i \neq j, \\ \mu_{ii}^0 = \bar{\mathfrak{E}}^0 + \Delta F_{ii}^0, \end{cases} \tag{8}$$

hence:

$$\Gamma_0(\gamma_1 + \gamma_2 + \gamma_3) = \bar{\mathfrak{E}}^0(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) + \Delta\sum_{i,j=1}^{3} F_{ij}^0\gamma_i\gamma_j\,. \tag{9}$$

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[1]) In general, equations I (54) are linear in the second derivatives of the functions x, y, z, t with respect to ξ, η, ζ, τ. If one now chooses in particular: $t = \tau$ then the μ_{ij} are the coefficients by which the three derivatives of second order of x, y, z, with respect to t are multiplied, hence, the accelerations enter into the first three of equations I (54).

The principal inertial directions are thus simply the principal stress directions for a rest element.

For the case of pure longitudinal and transversal mass one must obviously set:

(10) $$F_{ij}^0 = 0, \qquad i \neq j, \qquad F_{ii}^0 = p,$$

hence, one has:

(11) $$\mu_l^0 = \mu_t^0 = \overline{\mathfrak{E}}^0 + p\Delta.$$

§ 2. First representation of the form Γ

In order to compute the inertial coefficients for a moving element one must now only define the form $\Gamma(\gamma_1, \gamma_2, \gamma_3)$, hence, the second variation $\delta^2\Phi$ of:

(12) $$\Phi = \Omega(e_{ij}, \varepsilon)\sqrt{1-s^2}$$

for:

(13) $$\delta u = \gamma_1, \qquad \delta u = \gamma_2, \qquad \delta u = \gamma_3\,.$$

One next has:

(14) $$\delta^2\Phi = \sqrt{1-s^2}\;\delta^2\Omega + 2\delta\Omega\;\delta\sqrt{1-s^2} + \Omega\,\delta^2\sqrt{1-s^2}\,,$$

(15) $$\begin{cases} \delta\Omega = \displaystyle\sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}}\,\delta e_{ij}\,, \\ \delta^2\Omega = \displaystyle\sum_{ijhk=1}^{3} \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{hk}}\,\delta e_{ij}\delta e_{ij} + \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}}\,\delta^2 e_{ij}. \end{cases}$$

For the definition of the δe_{ij}, $\delta^2 e_{ij}$, one starts by assuming that $dt = d\tau = 0$, so:

(16) $$d\sigma^2 = dx^2 + dy^2 + dz^2 + \frac{(u\,dx + v\,dy + w\,dz)^2}{1-s^2}\,,$$

when expressed in terms of $d\xi$, $d\eta$, $d\zeta$, reads:

(16′) $$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 + 2de^2,$$

(17) $$de^2 = \sum_{i,j=1}^{3} e_{ij}\,d\xi_i\,d\xi_j\;.$$

Thus, if one lets the u, v, w in $d\sigma^2$ go to $u + \gamma_1$, $v + \gamma_2$, $w + \gamma_3$ and then develops it in powers of the γ_1, γ_2, γ_3 then $d\sigma^2$ will become:

(18) $$d\sigma^{*2} = d\sigma^2 + 2\delta\,de^2 + \delta^2\,de^2 + \ldots$$

(19) $$\begin{cases} \delta de^2 = \sum_{i,j=1}^{3} \delta e_{ij} d\xi_i d\xi_j, \\ \delta^2 de = \sum_{i,j=1}^{3} \delta^2 e_{ij} d\xi_i d\xi_j. \end{cases}$$

The directly produced development of $d\sigma^{*2}$ in the γ_i from (16) will thus be furnished by its linear terms in the δe_{ij}, and its quadratic terms in $\delta^2 e_{ij}$.

In order to carry out any truncated development, one first remarks that the γ_1, γ_2, γ_3 , expressed in terms of the components of the "rest acceleration:"

(20) $$\gamma_0^1,\ \gamma_0^2,\ \gamma_0^3,\ \gamma^0 = \sqrt{(\gamma_1^0)^2 + (\gamma_2^0)^2 + (\gamma_3^0)^2}$$

by means of the rest transformation, are:

(21) $$\begin{cases} \beta^{-2}\gamma_1^0 = \gamma_1 + \alpha u(u\gamma_1 + v\gamma_2 + w\gamma_3), \\ \beta^{-2}\gamma_2^0 = \gamma_2 + \alpha v(u\gamma_1 + v\gamma_2 + w\gamma_3), \\ \beta^{-2}\gamma_3^0 = \gamma_3 + \alpha w(u\gamma_1 + v\gamma_2 + w\gamma_3), \end{cases}$$

and especially for the longitudinal and transversal components one has:

(22) $$\gamma_l^0 = \beta^3 \gamma_l\,, \qquad \gamma_t^0 = \beta^3 \gamma_t\,.$$

Second, one replaces:

(23) $$\begin{cases} dv = udx + vdy + wdz, \\ dv^0 = udx^0 + vdy^0 + wdz^0, \end{cases}$$

(24) $$\begin{cases} d\gamma = \gamma_1 dx + \gamma_2 dy + \gamma_3 dz, \\ d\gamma^0 = \gamma_1^0 dx^0 + \gamma_2^0 dy^0 + \gamma_3^0 dz^0, \end{cases}$$

and establishes, on the basis of the equations that couple the dx, dy, dz with the dx^0, dy^0, dz^0 for $dt = 0$ (I. § 1), that:

(25) $$\begin{cases} dv = \beta^{-1} dv^0, \\ d\gamma = \beta^{-2}(d\gamma^0 - s\gamma_t^0 dv^0). \end{cases}$$

Having made this assumption, one now has, with no further assumptions:

(26) $$\begin{cases} d\sigma^{*2} = dx^2 + dy^2 + dz^2 + \dfrac{(dv + d\gamma)^2}{1 - s^2 - 2s\gamma_l - \gamma^2} \\ \qquad = d\sigma^2 + \dfrac{2}{\beta} dv^0 d\gamma^0 + \dfrac{1}{\beta^2}[(d\gamma^0)^2 + 2s\gamma_l^0 dv^0 d\gamma^0 + (\gamma^0)^2 (dv^0)^2 + \cdots, \end{cases}$$

hence:

(27)
$$\begin{cases} \delta de^2 = \dfrac{1}{\beta} dv^0 d\gamma^0, \\ \delta^2 de^2 = \dfrac{1}{\beta^2}[(d\gamma^0)^2 + 2s\gamma_l^0 dv^0 d\gamma^0 + (\gamma^0)^2 (dv^0)^2 + \cdots, \end{cases}$$

and ultimately one finds that:

(28)
$$\begin{cases} \delta\sqrt{1-s^2} = -\beta s \gamma_l = -\beta^2 s \gamma_l^0, \\ \delta^2 \sqrt{1-s^2} = -\beta\gamma^2 - \beta^3 s^2 \gamma_l^2 = -\beta^{-3} s^2 (\gamma^0)^2. \end{cases}$$

Substituting everything in (14) finally yields the result:

(29)
$$\beta^3 \Gamma = \sum_{ijhk=1}^{3} \frac{\partial^2 \Omega}{\partial e_{ij} \partial e_{hk}} \varepsilon_{ij} \varepsilon_{hk} + \sum_{ij=1}^{3} \frac{\partial \Omega}{\partial e_{ij}} \varepsilon_{ij}^* - \Omega (\gamma^0)^2 ,$$

in which ε_{ij} and are ε_{ij}^* defined by:

(30)
$$\begin{cases} dv^0 d\gamma^0 = \displaystyle\sum_{i,j=1}^{3} \varepsilon_{ij} d\xi_i d\xi_j \\ (d\gamma^0)^2 + (\gamma^0)^2 (dv^0)^2 = \displaystyle\sum_{i,j=1}^{3} \varepsilon_{ij}^* d\xi_i d\xi_j . \end{cases}$$

Thus, if one sets:

(31)
$$\begin{cases} dv^0 = \chi_1 d\xi + \chi_2 d\eta + \chi_3 d\zeta, \\ d\gamma^0 = \pi_1 d\xi + \pi_2 d\eta + \pi_3 d\zeta, \end{cases}$$

in which obviously:

(32)
$$\begin{cases} \chi_i = a_{1i}^0 u + a_{2i}^0 v + a_{3i}^0 w, \\ \pi_i = a_{1i}^0 \gamma_1^0 + a_{2i}^0 \gamma_2^0 + a_{3i}^0 \gamma_3^0, \end{cases}$$

then one obtains:

(33)
$$\begin{cases} \varepsilon_{ij} = \frac{1}{2}(\pi_i \chi_j + \pi_j \chi_i), \qquad i, j = 1, 2, 3, \\ \varepsilon_{ij}^* = \pi_i \pi_j + (\gamma^0)^2 \chi_j \chi_i . \end{cases}$$

Thus, Γ has the form:

(34) $$\beta^3\Gamma = P(\gamma_1^0, \gamma_2^0, \gamma_3^0, u, v, w) + Q(\gamma_1^0, \gamma_2^0, \gamma_3^0),$$

where:

(35) $$P = \sum_{ijhk=1}^{3} \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{hk}} \varepsilon_{ij}\varepsilon_{hk} + (\gamma^0)\sum_{ij=1}^{3} \frac{\partial\Omega}{\partial e_{ij}} \chi_i \chi_j$$

is a quadratic form in $\gamma_1^0, \gamma_2^0, \gamma_3^0$, as well as in u, v, w, while:

(36) $$Q = \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}} \pi_i \pi_j - \Omega(\gamma^0)^2$$

is a quadratic form in $\gamma_1^0, \gamma_2^0, \gamma_3^0$ alone. The coefficients of both forms depend upon only the a_{ij}^0 (i, j = 1, 2, 3) and ε.

For a rest element $u = v = w = 0$, one has:

(37) $$\Gamma_0 = Q(\gamma_1, \gamma_2, \gamma_3),$$

and a comparison with (9) shows that in general, one has:

(38) $$Q = \overline{\mathfrak{E}}^0(\gamma^0)^2 + \Delta\sum_{i,j=1}^{3} F_{ij}^0 \gamma_i^0 \gamma_j^0 .$$

§ 3. Second representation of the form Γ and the character of the forms *P* and *Q*

The assumption that was made in § 1 relative to the inertial resistance yields:

(39) $$\begin{cases} P + Q \geq 0 \text{ for all } \gamma_1^0, \gamma_2^0, \gamma_3^0 \text{ and } u^2 + v^2 < 1, \\ \quad Q \geq 0 \text{ for all } \gamma_1^0, \gamma_2^0, \gamma_3^0. \end{cases}$$

The character of the form P alone gives us information about a second representation of P and Q in which, by means of I (16), Ω is thought of as a function of the a_{ij}^0 (i, j = 1, 2, 3) and ε.

Namely, also due to eq. II (14), one has:

(40) $$d\sigma^2 = (dx^0)^2 + (dy^0)^2 + (dz^0)^2,$$

and thus, if – under the assumption that α_i, β_i (i = 1, 2, 3) are arbitrary numbers and one sets:

(41) $$\begin{cases} d\alpha^0 = \alpha_1 dx^0 + \alpha_2 dy^0 + \alpha_3 dz^0, \\ d\beta^0 = \beta_1 dx^0 + \beta_2 dy^0 + \beta_3 dz^0, \end{cases}$$

the a_{ij}^0 are given the variations:

(42) $$\delta a_{ij}^0 = \alpha_i (a_{1j}^0 \beta_1 + a_{2j}^0 \beta_2 + a_{3j}^0 \beta_3)\,,$$

in which one likewise sets:

(43) $$\delta dx^0 = \alpha_1\, d\beta^0, \qquad \delta dy^0 = \alpha_2\, d\beta^0, \qquad \delta dz^0 = \alpha_3\, d\beta^0,$$

then it follows from the same argument as above that:

(44) $$\begin{cases} \delta de^2 = d\alpha^0 d\beta^0 \\ \delta^2 de^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(d\beta^0)^2. \end{cases}$$

A comparison of (35) and (36) immediately shows that one can also write:

(35′) $$P = \sum_{ijhk=1}^{3} \frac{\partial^2 \Omega}{\partial a_{ij}^0 \partial a_{hk}^0} \gamma_i^0 \gamma_h^0 \chi_j \chi_k\,,$$

(36′) $$Q = \sum_{i,j=1}^{3} \frac{\partial \Omega}{\partial a_{ij}^0} \gamma_i^0 \pi_j^0 - \Omega(\gamma^0)^2,$$

such that P is simply the second variation of Ω:

(35″) $$P = \delta^2\Omega \qquad \text{for } \delta a_{ij}^0 = \gamma_i^0 \chi_j\,, \qquad i, j = 1, 2, 3.$$

However, for the stability of equilibrium it is necessary [1]) that $\delta^2\Omega$ must be negative-definite for all variations of the a_{ij}^0 of the form $\delta a_{ij}^0 = A_i\, B_j$. Therefore, if this stability condition is satisfied then one will have:

(39′) $$P \leq 0 \quad \text{for all} \quad \gamma_1^0,\ \gamma_2^0,\ \gamma_3^0,\ u, v, w\,.$$

[1]) J. Hadamard, Propagation des Ondes, Paris 1903, art. 270.

§ 4. The discontinuous solutions of the equations of motion and the form *W*.

The examination of the possible waves in a body necessarily raises some issues from the theory [1]) of discontinuous solutions of differential equations that arise from a variational problem.

We will then direct our attention to the general form I (54) of the equations of motion, in which ε is assumed to be continuous, along with its first differential quotients; i.e., restrict ourselves the consideration of adiabatic waves.

Now, should the second differential quotients of *x*, *y*, *z*, *t* with respect on the "wave surface:"

$$\text{(45)}\qquad \begin{cases} \varphi(\xi,\eta,\zeta,\tau)=0 \\ d\varphi=\varphi_1 d\xi+\varphi_2 d\eta+\varphi_3 d\zeta+\varphi_4 d\tau \end{cases}$$

be discontinuous then, by the aid of four quantities λ_1, λ_2, λ_3, λ_4 , the resulting variations of the values of these differential quotients when one crosses the wave surface can be represented in the form:

$$\text{(46)}\qquad \left[\frac{\partial^2 x_i}{\partial\xi_h\partial\xi_k}\right]=\lambda_i\,\varphi_h\,\varphi_k,\qquad i, h, k = 1, 2, 3, 4,$$

which culminate in the requirement of the so-called compatibility conditions.

If one further defines for:

$$\text{(47)}\qquad \delta a_{ij}=\lambda_i\,\varphi_j,\qquad i, j = 1, 2, 3, 4$$

the second variation of the kinetic potential Φ:

$$\text{(48)}\qquad W=\delta^2\Phi=\sum_{ijhk=1}^{4}\frac{\partial^2\Phi}{\partial a_{ij}\partial a_{hk}}\lambda_i\lambda_j\varphi_h\varphi_k$$

then λ_i , φ_j must satisfy the conditions:

$$\text{(49)}\qquad \frac{\partial W}{\partial\lambda_1}=\frac{\partial W}{\partial\lambda_2}=\frac{\partial W}{\partial\lambda_3}=\frac{\partial W}{\partial\lambda_4}=0.$$

However, since *W* is a quadratic form in the λ_i, as well as in the φ_j, these are linear, homogeneous equations in the λ_i, and thus their determinant – i.e., the discriminant of the form *W* relative to the λ_i – must vanish:

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[1]) J. Hadamard, loc. cit., Chap. VII.

(50) $$\left|\frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}\right| = 0,$$

an equation that represents a relation between just the φ_j – viz., the partial differential equation of the wave surface.

In order to better comprehend its meaning, one considers the equation (45) for the wave surface when expressed in terms of x, y, z, t:

(51) $$\begin{cases} f(x,y,z,t) = 0, \\ df = f_1 dx + f_2 dy + f_3 dz + f_4 dt, \end{cases}$$

in which the f_i will be connected with the φ_j:

(52) $$\varphi_j = \sum_{i=1}^{4} a_{ij} f_i\,, \qquad j = 1, 2, 3, 4.$$

If one then denotes the direction cosines of the wave normal by n_1, n_2, n_3 and Θ denotes the normal velocity of the wave then one has:

(53) $$\begin{cases} f_1 : f_2 : f_3 : f_4 = n_1 : n_2 : n_3 : -\Theta, \\ \Theta = \pm \dfrac{f_4}{\sqrt{f_1^2 + f_2^2 + f_3^2}}, \end{cases}$$

and it therefore equation (50), which is homogeneous in φ_j, represents a relation between n_1, n_2, n_3 and Θ.

For a given wave normal n_1, n_2, n_3, the first things that follow from (50) are the possible values of the wave velocity Θ and then (49) gives the associated possible directions of the "wave vectors" λ_1, λ_2, λ_3, λ_4.

Without proof, because the fact will not be used in what follows, let us finally remark that that the direction cosines of the ray s_1, s_2, s_3 and the reciprocal ray velocity s_4 will be given by:

(49′) $$s_1 : s_2 : s_3 : s_4 = \frac{\partial W}{\partial f_1} : \frac{\partial W}{\partial f_2} : \frac{\partial W}{\partial f_3} : \frac{\partial W}{\partial f_4}.$$

§ 5. Representation of the form *W* and its connection with the form Γ.

The form $W(\lambda_i, \varphi_j)$, which thus completely delivers the laws of wave propagation, shall now be constructed for a rest element in particular, since one may in fact arrive at the laws that are valid for a moving element from those of a rest element by a Lorentz transformation.

Moreover, corresponding to the arbitrariness of the time parameter, which will always be assumed, it can happen that at the spacetime point in question, one has indeed:

$$a_{41} = a_{42} = a_{43} = 0, \qquad a_{43} = 1$$

such that one must therefore define the second variation of Φ for $\delta a_{ij} = \lambda_i\ \varphi_j$ with the initial values:

(54)
$$\begin{cases} a_{ij} = a_{ij}^0, \quad i, j = 1, 2, 3, \\ a_{14} = a_{41} = a_{24} = a_{42} = a_{34} = a_{43} = 0, \\ a_{44} = 1. \end{cases}$$

One then has:

(55)
$$\delta^2\Phi = \sqrt{-A_{44}}\,\delta^2\Omega + 2\delta\Omega\,\delta\sqrt{-A_{44}} + \Omega\,\delta^2\sqrt{-A_{44}}$$

(56)
$$\begin{cases} \delta\Omega = \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}}\delta e_{ij}, \\ \delta^2\Omega = \sum_{ijhk=1}^{3} \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{hk}}\delta e_{ij}\delta e_{hk} + \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}}\delta^2 e_{ij}. \end{cases}$$

However, in order to define the δe_{ij}, $\delta^2 e_{ij}$ one must – using an argument that is analogous to the one in § 2 – start with the fact that when:

(57)
$$\begin{cases} d\sigma^2 = dx^2 + dy^2 + dz^2 - \frac{1}{A_{44}}dv^2, \\ dv = a_{14}dx + a_{24}dy + a_{34}dz - a_{44}dt \end{cases}$$

is expressed in terms of $d\xi$, $d\eta$, $d\zeta$, $d\tau$, it looks like:

(57′)
$$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 + 2de^2.$$

In order to once more carry out the truncated development of the form $d\sigma^{*2}$, in which the a_{ij} in $d\sigma^2$ are replaced by $a_{ij} + \lambda_i\ \varphi_j$, one sets:

(58)
$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

(59)
$$\begin{cases} d\lambda = \lambda_1 dx + \lambda_2 dx + \lambda_3 dx, \\ d\varphi = \varphi_1 d\xi + \varphi_2 d\eta + \varphi_3 d\zeta \\ \quad = f_1 dx + f_2 dy + f_3 dz = \frac{f_4}{\Theta}dn, \\ dn = n_1 dx + n_2 dy + n_3 dz. \end{cases}$$

If one then remarks that generally $d\tau$ falls out of $d\sigma^2$ – hence, from now on we will set $d\tau = 0$ – then, considering the special nature of the initial values (54) of the a_{ij}, when the a_{ij} are replaced with $a_{ij} + \lambda_i\,\varphi_j$ the dx, dy, dz, dt, dv, A_{44} go to:

$$\text{(60)} \qquad \begin{cases} dx^* = dx + \lambda_1 d\varphi, \quad dy^* = dy + \lambda_2 d\varphi, \\ dz^* = dz + \lambda_3 d\varphi, \quad dt^* = \lambda_4 d\varphi, \end{cases}$$

$$\text{(61)} \qquad \begin{cases} dv^* = \varphi_4 d\lambda - \lambda_4 d\varphi + (\lambda^2 - \lambda_4^2)\varphi_4 d\varphi, \\ A_{44}^* = -(1+\lambda_4\varphi_4)^2 + \lambda^2\varphi_4^2. \end{cases}$$

However, it then follows, with no further assumptions, that:

$$\text{(62)} \qquad d\sigma^{*2} = d\sigma^2 + 2\, d\lambda\, d\varphi + [\lambda^2\, d\varphi^2 - 2\lambda_4\varphi_4\, d\lambda\, d\varphi + \varphi_4^2 d\lambda^2] + \ldots;$$

hence:

$$\text{(63)} \qquad \begin{cases} \delta de^2 = d\lambda\, d\varphi, \\ \delta^2 de^2 = \lambda^2 d\varphi^2 - 2\lambda_4\varphi_4 d\lambda d\varphi + \varphi_4^2 d\lambda^2, \end{cases}$$

which finally gives A_{44}^* from:

$$\text{(64)} \qquad \begin{cases} \delta\sqrt{-A_{44}} = \lambda_4\varphi_4, \\ \delta^2\sqrt{-A_{44}} = -\lambda^2\varphi_4^2. \end{cases}$$

When everything is substituted in (55), one ultimately derives the result:

$$\text{(65)} \qquad \Theta^2 f_4^{-2}\, W = \sum_{ijhk=1}^{3} \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{hk}} \varepsilon_{ij}\varepsilon_{hk} + \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}} \varepsilon_{ij}^* - \Theta^2\Omega\lambda^2,$$

in which the ε_{ij}, ε_{ij}^* are defined by:

$$\text{(66)} \qquad \begin{cases} d\lambda\, dn = \sum_{i,j=1}^{3} \varepsilon_{ij} d\xi_i d\xi_j \\ \lambda^2 dn^2 + \Theta^2 d\lambda^2 = \sum_{i,j=1}^{3} \varepsilon_{ij}^* d\xi_i d\xi_j. \end{cases}$$

A comparison with the representation (29) of Γ gives:

$$\text{(67)} \qquad f_4^{-2}\Theta^2 W = P(\lambda_1, \lambda_2, \lambda_3, n_1, n_2, n_3) + \Theta^2 Q(\lambda_1, \lambda_2, \lambda_3).$$

The quantity λ_4 drops out of W – as would correspond to the arbitrariness of the time parameter – and therefore one may drop the fourth of equations (49), and use the discriminant of W with respect to λ_1, λ_2, λ_3 in (50).

§ 6. The adiabatic waves

Each of the wave normals n_1, n_2, n_3 are then associated with the three possible wave velocities through the third order equation in Θ^2:

(68) $$\left|\frac{\partial^2 P}{\partial\lambda_i\,\partial\lambda_j}+\Theta^2\,\frac{\partial^2 Q}{\partial\lambda_i\,\partial\lambda_j}\right| = 0$$

which then correspond to the three possible directions of the wave vectors λ_1, λ_2, λ_3 by way of:

(69) $$\frac{\partial P}{\partial\lambda_i}+\Theta^2\,\frac{\partial Q}{\partial\lambda_i} = 0, \qquad i = 1, 2, 3.$$

In particular, from (69), there exists the relation between n_i, λ_i, Θ^2:

(70) $$P(\lambda_1, \lambda_2, \lambda_3, n_1, n_2, n_3) + \Theta^2 Q(\lambda_1, \lambda_2, \lambda_3) = 0.$$

Now, if the assumption that was stated in § 1 relative to the inertial resistance is made and the stability condition that was stated in § 3 is satisfied then one may assert the following about the roots Θ^2 of (68):

Since the form Q in the pencil of linear forms $P + \Theta^2 Q$ is definite the three roots Θ^2 are certainly real and finite, and since one always has $P < 0$, $Q > 0$, because (70) is never negative, the value of Θ itself is always real.

Furthermore, from (39), for $0 \leq s < 1$ one always has:

(71) $$s^2 P + Q \geq 0.$$

Hence, by means of (70):

(72) $$(1 - s^2\Theta^2)Q \geq 0,$$

and thus, since $Q > 0$:

(73) $$1 - s^2\Theta^2 \geq 0 \quad \text{or} \quad \Theta^2 \leq \frac{1}{s^2}, \qquad \text{i.e.} \qquad \Theta \leq 1.$$

Each wave normal is thus always associated with three possible wave velocities, which never exceed unity – i.e., the speed of light – and they correspond to the three possible directions of the wave vectors.

These three directions may be defined geometrically as the common triple of mutually conjugate intersectors of the two ellipsoids:

(74) $$\begin{cases} P(x,y,z,n_1,n_2,n_3) = -1, \\ Q(x,y,z) = +1. \end{cases}$$

If l_1, l_2, l_3 denote the components of the wave vectors λ_1, λ_2, λ_3 in these three directions and Θ_1, Θ_2, Θ_3 those of the corresponding wave velocities then one has:

(75) $$\begin{cases} -P(\lambda_i, n_i) = m_1\Theta_1^2 l_1^2 + m_2\Theta_2^2 l_2^2 + m_3\Theta_3^2 l_3^2, \\ Q(\lambda_i, n_i) = m_1 l_1^2 + m_2 l_2^2 + m_3 l_3^2, \\ m_i > 0. \end{cases}$$

Therefore, if $\gamma_1^0, \gamma_2^0, \gamma_3^0$ are the components of the rest acceleration relative to the velocity direction in the same way that the wave normal corresponds to the three directions of the wave vectors then Γ assumes the form:

(76) $$\beta^3\Gamma = m_1(1 - s^2\Theta_1^2)(\gamma_1^0)^2 + m_2(1 - s^2\Theta_2^2)(\gamma_2^0)^2 + m_3(1 - s^2\Theta_3^2)(\gamma_3^0)^2 .$$

§ 7. Connections between the longitudinal and transversal waves with the longitudinal and transversal inertia

If the body has simply a transversal and longitudinal inertial coefficient μ_t and μ_l and μ_0 is their common rest value that depends only upon the a_{ij}^0 $(i, j = 1, 2, 3)$ then one has:

(77) $$\Gamma = \mu_l\gamma_l^2 + \mu_t\gamma_t^2 = \mu_l\beta^{-6}(\gamma_l^0)^2 + \mu_t\beta^{-4}(\gamma_t^0)^2,$$

and from this:

(78) $$\begin{cases} Q = \Gamma_0 = \mu^0(\gamma_l^0)^2 + \mu^0(\gamma_t^0)^2, \\ P = \beta^3\Gamma - \Gamma_0 = (\mu_l\beta^{-3} - \mu^0)(\gamma_l^0)^2 + (\mu_t\beta^{-1} - \mu^0)(\gamma_t^0)^2. \end{cases}$$

Now, since P is a quadratic form in the $\gamma_1^0, \gamma_2^0, \gamma_3^0$ as well as in the u, v, w, with coefficients that depend only upon the a_{ij}^0 $(i, j = 1, 2, 3)$, due to the fact that:

(79) $$s\gamma_l^0 = u\gamma_1^0 + v\gamma_2^0 + w\gamma_3^0, \qquad (\gamma_l^0)^2 + (\gamma_t^0)^2 = (\gamma_1^0)^2 + (\gamma_2^0)^2 + (\gamma_3^0)^2,$$

one must have:

(80) $$\begin{cases} \mu_l\beta^{-3} - \mu^0 = -a\mu^0 s^2, \\ \mu_t\beta^{-1} - \mu^0 = -b\mu^0 s^2, \end{cases} \quad \text{or} \quad \begin{aligned} \mu_l &= \mu^0(1 - as^2)\beta^2, \\ \mu_t &= \mu^0(1 - bs^2)\beta, \end{aligned}$$

where a, b, as well as as μ^0, merely depend upon the a_{ij}^0 $(i, j = 1, 2, 3)$, from which, one has:

$$(81) \qquad \begin{cases} -P = \mu^0 s^2 \left(a(\gamma_l^0)^2 + b(\gamma_t^0)^2 \right), \\ \ \ Q = \mu^0 \left((\gamma_l^0)^2 + (\gamma_t^0)^2 \right). \end{cases}$$

Hence, if λ_l and λ_t are taken to be the longitudinal and transversal components of the wave vector λ_1, λ_2, λ_3 – i.e., parallel and normal to the wave normal – then one has:

$$(82) \qquad \begin{cases} -P(\lambda_i, n_i) = \mu^0 s^2 \left(a\lambda_l^2 + b\lambda_t^2 \right), \\ \quad Q(\lambda_i) = \mu^0 \left((\gamma_l^0)^2 + (\gamma_t^0)^2 \right). \end{cases}$$

Comparing this with (75) shows that the wave vector must be either parallel (longitudinal waves) or normal (transversal waves) to the wave normal, and that the propagation velocities of both types of waves are:

$$(83) \qquad \Theta_l = \sqrt{a}\,, \qquad \Theta_t = \sqrt{b}\,.$$

Conversely, if one assumes the possibility of pure longitudinal and pure transversal waves of velocities Θ_l and Θ_t then any two mutually normal directions will be conjugate intersectors of the second ellipsoid (74), which is then a sphere:

$$(84) \qquad Q(\lambda_i) = \mu^0 \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right).$$

The common triple of conjugate intersectors will be linked with the wave normal and any two directions that are normal to it and each other; i.e., from (75):

$$(85) \qquad -P(\lambda_i, n_i) = \mu^0 \left(\Theta_l^2 \lambda_l^2 + \Theta_t^2 \lambda_t^2 \right).$$

Hence:

$$(86) \qquad \beta^3 \Gamma = \mu^0 \left(1 - s^2 \Theta_l^2\right)(\gamma_l^0)^2 + \mu^0 \left(1 - s^2 \Theta_t^2\right)(\gamma_t^0)^2,$$

$$(87) \qquad \Gamma = \mu^0 \beta^3 \left(1 - s^2 \Theta_l^2\right)(\gamma_l^0)^2 + \mu^0 \beta \left(1 - s^2 \Theta_t^2\right)(\gamma_t^0)^2,$$

and the body therefore possesses only a longitudinal and a transversal inertial coefficient:

$$(88) \qquad \mu_l = \mu^0 \left(1 - s^2 \Theta_l^2\right) \beta^3, \qquad \mu_t = \mu^0 \left(1 - s^2 \Theta_t^2\right) \beta.$$

Transversal and longitudinal waves, on the one hand, and transversal and longitudinal inertia, on the other, thus cause each other, and the inertial coefficients are always coupled with the wave velocities by (88).

§ 8. Adiabatic gas waves

That a fluid possesses merely a longitudinal and a transversal inertial coefficient was established already in I, § 10, and we found that:

$$(89)\qquad \begin{cases} \mu_l = (\Delta\Omega_\Delta - \Omega + s^2\Omega_{\Delta\Delta})\beta^3, \\ \mu_t = (\Delta\Omega_\Delta - \Omega)\beta. \end{cases}$$

Also, there are thus merely longitudinal and transversal waves in them with the velocities:

$$(90)\qquad \Theta_l = \sqrt{\frac{\Delta^2\Omega_{\Delta\Delta}}{\Omega - \Delta\Omega_\Delta}} \qquad \Theta_l = 0;$$

i.e., only longitudinal waves are possible.

By the introduction of pressure and rest energy:

$$(91)\qquad p = \Omega_\Delta, \qquad \mathfrak{E}^0 = -\frac{\Omega}{\Delta}$$

one simply has:

$$(90')\qquad \Theta_l = \sqrt{\frac{dp}{d\mathfrak{E}^0}}\,.$$

§ 9. Adiabatic, elastic waves with vanishing rest deformations

In the case of elastic, isotropic bodies (cf., I § 12) with:

$$(92)\qquad \Omega = -M - \tfrac{1}{2}AJ_1^2 + 2B\,J_2$$

the form Γ may be computed merely for vanishing rest deformations; i.e., $e_{ij} = 0$. In its representation (29), one must set:

$$(93)\qquad \Omega = -M, \qquad \sum_{i,j=1}^{3} \frac{\partial\Omega}{\partial e_{ij}} \varepsilon_{ij}^{*} = 0,$$

whereas, since J_1^2, J_2 are quadratic forms in the e_{ij}, one has:

$$(94)\qquad \sum_{ijhk=1}^{3} \frac{\partial^2\Omega}{\partial e_{ij}\partial e_{hk}} \varepsilon_{ij}\varepsilon_{hk} = -\,A\bar{J}_1^2 + 4B\bar{J}_2,$$

if $\bar{J}_1$, $\bar{J}_2$ mean the expressions I (123) for J_1, J_2 defined with ε_{ij} instead of e_{ij}. The values (33) for the ε_{ij} make them equal to:

$$(95)\quad \begin{cases} \bar{J}_1 = \pi_1\chi_1 + \pi_2\chi_2 + \pi_3\chi_3, \\ -4\bar{J}_2 = (\pi_2\chi_3 - \pi_3\chi_2)^2 + (\pi_3\chi_1 - \pi_1\chi_3)^2 + (\pi_1\chi_2 - \pi_2\chi_1)^2, \\ \qquad = (\pi_1^2 + \pi_2^2 + \pi_3^2)(\chi_1^2 + \chi_2^2 + \chi_3^2) - \bar{J}_1^2. \end{cases}$$

However, since for $e_{ij} = 0$, one has:

$$(dx^0)^2 + (dy^0)^2 + (dz^0)^2 = d\xi^2 + d\eta^2 + d\zeta^2,$$

hence, the a_{ij}^0 $(i, j = 1, 2, 3)$ are then the coefficients of an orthogonal transformation, the substitution of the values (33) for π_i, χ_i further yields:

$$(95')\quad \begin{cases} \bar{J}_1 = u\gamma_1^0 + v\gamma_2^0 + w\gamma_3^0 = s\gamma_l^0, \\ -4\bar{J}_2 = s^2(\gamma^0)^2 - s^2(\gamma_l^0)^2 = s^2(\gamma_t^0)^2. \end{cases}$$

Thus, one finally has:

$$(96)\qquad \beta^3\Gamma = M(\gamma^0)^2 - As^2(\gamma_l^0)^2 - Bs^2(\gamma_t^0)^2,$$

$$(97)\qquad \Gamma = (M - As^2)\beta^3\gamma_l^2 - (M - Bs^2)\beta\,\gamma_t^2.$$

The body thus possesses just a transversal inertial coefficient and a longitudinal one:

$$(98)\qquad \mu_l = (M - As^2)\beta^3, \qquad \mu_t = (M - Bs^2)\beta, \qquad \mu_t^0 = \mu_l^0 = M,$$

and purely longitudinal and purely transversal waves propagate in it with the velocities:

$$(99)\qquad \Theta_l = \sqrt{\frac{A}{M}}\,, \qquad \Theta_t = \sqrt{\frac{B}{M}}\,.$$

The elasticity coefficients A, B are linked by the condition:

$$(100)\qquad 0 \le A \le M, \qquad 0 \le B \le M.$$

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