Reprinted in Gesammelte mathematische Abhandlungen, pp. 555-574.

On the geometric foundations of the Lorentz group

By F. Klein

Translated by D. H. Delphenich

Presented to the Göttinger Mathematics Society on 10 May 1910
[Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 19 (1910)]

All of you have heard, in a more or less definite form, that the theory of the Lorentz group or – what amounts to the same thing – the modern principle of relativity, is associated with the *general study of projective metrics*, as it was developed in connection with Cayley's ground-breaking work in 1859. After meeting with our late friend Minkowski, it came to pass that I explored this more closely in my lectures on projective geometry in the last Winter semester, and indeed, it emerged as the ultimate result of my lectures. The study of projective metrics, which is already the basis for so many pages of work, takes on a new and surprising application here, whereas, on the other hand, the modern developments of the physicists, which, to the newcomer, easily create the impression of being paradoxes, prove to be corollaries, so to speak, to a long-established way of thinking. It is inevitable that this confluence of two completely separate schools of thought in history must act as a tonic for both schools, to a large extent; I hope that the individual results worked out here will present at least some interest on the part of both the physicists and the geometers, and might prove themselves to be welcome tools in the workshop of theoretical physics, moreover.

When I now undertake to carry out the statements of its essential foundations, I generally stand before a great obstacle: I would like to assume that the concept of *group* is familiar, as well as certain fundamental notions from projective geometry, such as *homogeneous point coordinates and plane coordinates*, that *collineations* correspond to linear substitutions of these coordinates, and ultimately, that for each group composed of collineations there is an associated *invariant theory* – this is always well-understood in the domain of arbitrarily many dimensions, – but I nonetheless know perfectly well that not only the numerous modern physicists, who I would especially like to welcome here as guests, but also the multitude of young technical mathematicians that belong to our society, have only been concerned with these things *per distans*, so to speak. Many of them have undoubtedly been of the opinion up to now that projective geometry, which had stood in the foreground of mathematical progress for so long, can only claim to be a specialized mathematical disciple nowadays. It is indeed very useful in itself that my present talk must express the opposite viewpoint, that, in fact, projective geometry, in the context of the desired mathematical presentation, must be regarded as equivalent to other fundamental studies, such as algebra or the theory of functions. However, this ideal situation can still create complexities that come from one's actual lack of adequate prior knowledge, not from the nature of the problem. I therefore cling to the method that is the most likely to promise results under circumstances of this sort: *I shall thus present things to you in their historical progression,* and must, by the way, request that you accept the enthusiasm with which I speak of the importance of projective ideas as a substitute for the missing thoroughness in the details.

Corresponding to my prior statements, I will thus begin by submitting to you *Cayley's original work of 1859,* which was the *sixth* in a series of treatises in which Cayley summarized his opinions and understanding in the realm of the theory of invariants of linear substitutions (A sixth memoir upon Quantics, v. 149 of the Phil. Trans. of the Roy. Soc. – v. 2 of his works, pp. 561, et seq.). Upon browsing it, you will first have no particular impression, because, before everything else, details about quadratic forms will be developed; it is, however, simple to single out the problem statement and its brilliant response. The development of geometry in the first half of the previous century has led to the splitting of the entire content of the study of space into different domains: the geometry of position (descriptive geometry), which treats such properties of figures that remain unchanged under arbitrary projections, and the geometry of measure, whose basic notions (distance, angle, etc.) in no way possess this invariance. This separation established itself in the consciousness of the mathematicians of the time, although Poncelet had already made the decisive remark that from a general standpoint the circles in the plane and the spheres in space – thus, the principal objects in the metric viewpoint – could be regarded as conic sections (surfaces, of second degree, resp.) that have a certain imaginary structure in common with the infinite breadth of the plane (space, resp.) – the so-called *circle point* of the plane (the *spherical circle* of space, resp.) – that is given by an equation of second degree. Now, it was Cayley's achievement to have recognized that this statement of Poncelet give one the means to reverse the aforementioned separation of geometry into two mutually distinct disciplines, or to replace it with a fundamentally new concept, moreover. His result is, like all fundamental thoughts in the mathematical sciences, extremely simple: All metrically-related geometrical figures can, with no further assumptions, be regarded as projective relations *as long as one adds the figures of circle points (sphere circles, resp.)* – according to whether they are planar or spatial; metric geometry then seems to be that piece of projective geometry that treats figures for which the pair of circle points (sphere circle, resp.) is involved.

This statement will become much clearer when I write down some simple formulas.

First, only in the plane: Let x, y be ordinary rectangular point coordinates. To make them homogeneous, we set: $x = x_1 / x_3$, $y = x_2 / x_3$; we further call u_1, u_2, u_3 the homogeneous coordinates of the straight line that is represented by the equation $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$. The circle point pair is then given in point coordinates by the combination of the two equations:

$$x_3 = 0, \qquad x_1^2 + x_2^2 = 0, \tag{1}$$

but in line coordinates it is the envelope of all lines that fulfill the *single* equation:

$$u_1^2 + u_2^2 = 0. \tag{2}$$

One now observes, in order to keep things simple, the formula for the distance between two points:

$$r = \sqrt{(x-\bar{x})^2 + (y-\bar{y})^2}\ .$$

To make this homogeneous, we write:

$$x = \frac{x_1}{x_3}, \qquad y = \frac{x_2}{x_3}; \qquad \bar{x} = \frac{y_1}{y_3}, \qquad \bar{y} = \frac{y_2}{y_3}$$

and obtain:

(3) $$r = \frac{\sqrt{(x_1y_3 - y_1x_3)^2 + (x_2y_3 - y_2x_3)^2}}{x_3y_3}.$$

The numerator vanishes in this when the two given points lie on a straight line with one of the circle points, and the denominator, when one of the given points lies on the connecting line between the two circle points. Both of these things are projective properties of the total figure that is defined by the two given points and the circle points! Algebraically, however, it follows from this (as I can specify in painful detail) that the expression r changes only by a constant factor when one simultaneously subjects our four points to an arbitrary collineation. For that reason, one calls r an *invariant* of our four points under the totality of all collineations, or also a "simultaneous invariant" of the two original points and the algebraic forms that were given in the left-hand sides of (1) or (2), resp. However, the content of the projective geometry of the plane is, algebraically speaking, nothing but the study of invariants that any plane figure possesses under the totality of all plane collineations, in particular, also the relations that such invariants may exhibit between themselves; it is therefore associated with all theorems in projective geometry that might exist between the distances between arbitrary points in the plane.

In space, things are complicated only by the increased number of coordinates. Let x, y, z be ordinary rectangular coordinates, and to make them homogeneous, we set $x = x_1 / x_4$, $y = x_2 / x_4$, $z = x_3 / x_4$. The "sphere circle" is then given in point coordinates by the pair of equations:

(4) $$x_4 = 0, \qquad x_1^2 + x_2^2 + x_3^2 = 0,$$

but in the associated plane coordinates (u_1, u_2 , u_3 , u_4) it is given by the *single* equation:

(5) $$u_1^2 + u_2^2 + u_3^2 = 0.$$

One again considers the expression for the distance between two points. When we give the latter the homogeneous coordinates $x_1 : x_2 : x_3 : x_4$ and $y_1 : y_2 : y_3 : y_4$, we obtain:

(6) $$r = \frac{\sqrt{(x_1y_4 - y_1x_4)^2 + (x_2y_4 - y_2x_4)^2 + (x_3y_4 - y_3x_4)^2}}{x_4y_4}$$

and link this formula with a discussion that is entirely similar to the one that was connected with (3).

The foregoing suggestions will suffice to make the general sense of Cayley's ground-breaking work somewhat more understandable. I may now speak for a moment of the concepts that I developed in my Erlanger Antrittsprogramm of 1872 [1]). For Cayley, one only speaks of invariants under the *totality* of collineations of the domain that actually comes into consideration. By contrast, I thus emphasize that one can just as well speak of invariants under a *subgroup* of collineations. This cast new light on the essence of metric geometry and Cayley's way of looking at it. It is a trivial remark that all statements of metric geometry are independent of the position and absolute magnitude of the figures, and can even be characterized by the statements about the individual volumes, as one establishes in topography. One expresses this in a modern mathematical way by saying that one first introduces two closely-related *groups* of collinear transformations: the group of *motions and transfers (Umlegungen)* and the more comprehensive group of *similarity transformations* (the group of "congruent" and "equiform" transformations, in the nomenclature that Heffter and Koehler introduced their text book [2]) and now says: The metric properties are characterized by the fact that they are invariant *relative to these groups.* We thus have: *Metric geometry and projective geometry both emerge from the study of an invariant theory, and their mutual relationship arises from the fact that the group of metric geometry is a subgroup of the group associated with projective geometry.*

A pair of simple formulas will clarify this situation and further organize things. We may remain in the plane and, for the sake of simplicity, use ordinary (non-homogeneous) rectangular coordinates x, y. If we then write:

$$\left\{\begin{aligned} x' &= \alpha_{11}x + \alpha_{12}y + \alpha_{13}, \\ y' &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}, \end{aligned}\right. \tag{7}$$

and consider the $\alpha_{11}, \ldots, \alpha_{23}$ here to be arbitrarily varying quantities then we have the six-parameter group of the so-called *affine* transformations before us. Among them, one finds the four-parameter group of *equiform* transformations when one demands that $dx'^2 + dy'^2$ agrees with $dx^2 + dy^2$ up to a factor. This is the case when and only when the following conditions are fulfilled:

$$\alpha_{11}^2 + \alpha_{21}^2 = \alpha_{12}^2 + \alpha_{22}^2, \qquad \alpha_{11}\,\alpha_{12} + \alpha_{21}\,\alpha_{22} = 0,$$

so the matrix:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix},$$

[1]) "Vergeichende Betrachtungen über neuere geometrische Forschungen," printed in Bd. 43 of the Math. Annalen and elsewhere. [See Abh. XXVII of this collection.]

[2]) Lehrbuch der analytischen Geometrie, Bd. 1, Leipzig 1905.

is, as one says, *orthogonal* [3]). However, the three-parameter group of congruence transformations arises when one sets the determinant $\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}$ equal to ± 1. One will then have: $dx'^2 + dy'^2 = dx^2 + dy^2$. Finally, we write down the most general collineation of the plane as:

$$\text{(8)} \qquad \begin{cases} x' = \dfrac{\alpha_{11}x + \alpha_{12}x + \alpha_{13}}{\alpha_{31}x + \alpha_{32}x + \alpha_{33}}, \\ y' = \dfrac{\alpha_{21}x + \alpha_{22}x + \alpha_{23}}{\alpha_{31}x + \alpha_{32}x + \alpha_{33}}. \end{cases}$$

One now effortlessly recognizes:

The group of affine transformations (7) *consists of those collineations that transform a certain straight line – namely, the infinitely distant line – into itself.*

The group of equiform transformations, however, consists of those collineations that leave a certain point pair on this line unchanged, which is precisely the circle point pair.

The definition of the group of congruent transformations is not geometrically as simple. We satisfy ourselves here with the algebraic characterization: They are the equiform transformations whose signed determinant is equal to ± 1. Naturally, the equiform transformations are affine, *eo ipso*.

Shall I add that one can now define – as an intermediary between projective geometry and metric geometry – an *affine* geometry, which treats all of those properties of plane figures that are invariant under the group (7)? We then have three geometries to compare, of which projective and metric geometric are the two extreme cases. The classification will be arrived at in this way, but the representation becomes unnecessarily tedious, because essentially the same things would have to be said several times. Thus, we shall speak of the main points only in terms of projective and metric geometry, while affine geometry, which generally will emerge at the conclusion of things, will only be mentioned parenthetically.

In this sense, I therefore distinguish between the *elementary* (direct) treatment of metric relationships and the *projective* one that Cayley initiated. This distinction is formulated by saying (in the sense of the Erlanger Programm): "The projective (higher) treatment seeks the invariant relations that the given figures possess under the *totality* of all collineations *after adding the circle point*. The elementary treatment seeks the invariant relations that the figures as such possess under the narrower group of those (equiform and congruent) collineations that take the circle point pair to itself."

Now, I have come to the end of these general preliminary considerations and I ask only that you retain the following thoughts in particular: Invariant theory is a relative concept; one can speak of the invariant theory associated with any group of transformations. This idea is so self-explanatory that it emerges spontaneously in the most diverse realms of application, and also everywhere in theoretical physics. The terminology by which it comes to be expressed is, like the realms themselves, naturally quite diverse. The various kinds of researchers, and therefore also physicists, then have

---

[3]) The term is used here in such a way that similarity transformations are included (so the numerical value of the determinant of α_{ik} is not given any importance).

neither the time nor the opportunity in the course of their work to find out whether any conceptual Ansätze that they require are found to be already completely constructed in the storeroom of pure mathematics, so they proceed – and it brings a certain freshness to their train of thought with it – on the assumption that the mathematical instruments that they require must be manufactured on a case-by-case basis. The eventual understanding with proper mathematicians, which generally seems to be an important thing to me, because they make the ideas precise and reveal all kinds of connections, then achieves, above all, a translation of the terminology that is used here and there from one language into the other. Thus, I will anticipate the theorem here by saying:

"What the modern physicists call *the theory of relativity* is the theory of invariants of the four-dimensional spacetime region x, y, z, t (the Minkowski "world") under a certain group of collineations, namely, the "Lorentz group"; or, more generally, and turned around:

"One can, if one finds such things important, replace the phrase 'invariant theory relative to a group of transformations' quite well with the phrase 'theory of relativity relative to a group."

I shall now treat some things concerning the purely mathematical examination that is connected with, e.g., Cayley's treatise. That is, in fact, the historical place of this eminent work, that it not only decisively answered the old problem of the relationship between metric geometry and projective geometry, but likewise brought a new question to the foreground that would prove momentous in following its various directions. Metric geometry emerges from projective geometry when one introduces the circle points, which are given by the equation $u_1^2 + u_2^2 = 0$ (or, in space, the sphere circle, which is given by the equation $u_1^2 + u_2^2 + u_3^2 = 0$). What will happen when one bases it on *any equation of second degree* $\sum\sum \alpha_{ik} u_i u_k = 0$ in some meaningful way, instead of them?

We remain in the plane, where our new equation represents any curve of second class. For the projective geometer, these curves decompose into five different types, which I will enumerate here by starting with a certain triangular coordinate system for the time being (whose "line coordinates" will likewise be called $u_1 : u_2 : u_3 : u_4$) instead of the rectangular parallel coordinate system used up to now. The list is the following one:

A. *Proper conic sections:*

1. $u_1^2 + u_2^2 + u_3^2 = 0$, imaginary conic section,
2. $u_1^2 + u_2^2 - u_3^2 = 0$, real conic section,

B. *Point-pairs:*

3. $u_1^2 + u_2^2 = 0$, imaginary point-pairs (corresponding to equation (2) for the circle point),
4. $u_1^2 - u_2^2 = 0$, real point-pair,

C. *Isolated point, counted twice:*

5, $u_1^2 = 0$.

The principle of this enumeration is so simple that anyone can write down the corresponding table for n variables $u_1, \ldots, u_n$ by analogy: one first writes down the equations with n squares, which will alternate + or – when joined together, then one with $(n - 1)$ squares, etc. The case of the first category shall be called the *general* one, the one that follows it, *simply specialized*, that of the third category, *doubly specialized*, etc.

For each of these cases, we now construct an analogy with formula (3) for the distance between points and obtain what Cayley called the associated *quasi-distance*. For the case of imaginary point-pairs, we will simply preserve formula (3) (except that now $x_1 : x_2 : x_3$ and $y_1 : y_2 : y_3$ will not necessarily be rectangular parallel coordinates, but generally speaking, the associated triangular coordinates). In the following two cases (real point-pairs and double points) small changes will be applied that we shall likewise return to later. What is more difficult, this yields the suitable Ansatz for the quasi-distance in the present two cases (proper conic sections); we would not like to go into it further here, because it would take up too much space, and its details will not enter into consideration in the present talk. In any event, the result is this: *We obtain five (and only five) types of metric geometries on the plane, of which, only the one that corresponds to imaginary point-pairs is known to us from the example of elementary metrics.* However, we call the archetype of all the theories that thus arise the *general study of projective metrics* (first for the plane, then for space, and ultimately for arbitrarily-extended manifolds).

Now, it is in no way my purpose at the moment to go into the details of this theory; only its general meaning will be emphasized. First, I have to refute a prejudice that many people nurture: The layman will be initially disinclined to attach any meaning to a preoccupation with posing questions that arise from the subjective – so to speak, aesthetic – desire for knowledge of the mathematicians. However, the history of science shows that things are quite otherwise; I will say that everything that is mathematically sound takes on a far-reaching meaning outside its narrow domain. This is the case with the theory of conic sections, which was developed by the geometers of antiquity for its own sake and suddenly took on a great importance in our understanding of Nature with the discovery of Kepler's laws. Precisely the same thing is true of the door that is immediately opened by the study of projective metrics as it relates to conic sections. The first thing is that they take on a higher meaning in the pursuit of knowledge for its own sake by proving themselves to be the simplest basis for *non-Euclidian geometry*, which arose from the examination of the independence of the parallel axiom from the other axioms, and then proved to be much more esoteric than that [4]); I will then point out a few details related to this. The second thing was that they proved to be a useful *method* for the clarification of complex phenomena in the other realms of pure mathematics, such as the theory of automorphic functions or number theory [5]). Now, in latter years it has emerged that they just as well yield a rational foundation for the most modern speculations of physics; in particular, the difference between *classical* and *modern mechanics* becomes simple.

---

[4]) Cf., my papers "Über die sogenannte Nicht-Eucklidische Geometrie" in volumes 4 and 6 of Math. Annalen (1871 and 1873). [See Abh. XVI and XVIII of this collection.]

[5]) Cf., the general representation of Fricke-Klein, Vorlesungen über die Theorie der automorphen Funktionen (Part I, Leipzig 1897), and furthermore, my autographed lectures on a particular chapter of number theory (Leipzig 1897).

The relationship between the theory of projective metrics and parallels that I vaguely alluded to may be seen when we once more restrict ourselves to the plane, in order to focus on its exceptional features, that in case 1) of the list presented on pp. 6 (thus, on the basis of an imaginary conic section) we obtain the non-Euclidian geometry of Riemann, but in case 2) (i.e., based on a real conic section) we obtain the non-Euclidian geometry of Bolyai-Lobachevsky-Gauss. I would like to mention a particular point that is obvious as a result of the projective viewpoint with no further assumptions, while it seems to be surrounded by the aura of the mystics: The number of collineations by which a non-decomposable conic section is transformed into itself is ∞^3. It goes up to ∞^4 as long as the conic section degenerates to a point-pair. In this fact, one sees that the equiform transformations (similarlity transformations) of the Euclidian metric that are so familiar to us from the Elements become a special category that gets omitted from non-Euclidian geometry; all that remain are the ∞^3 congruent transformations (motions and transfers). The conclusion is that there is an absolute standard of length in non-Euclidian geometry, not just, as for Euclid, an absolute standard of angle measure. Moreover, the two groups – the G_3 of one or the other non-Euclidian geometry and the G_4 of Euclidian geometry – have little in common as far as their internal structures are concerned. That is why it is so difficult to understand non-Euclidian geometry from the standpoint of Euclidian geometry: A figure that moves in a non-Euclidian way experiences bizarre distortions compared to Euclidian geometry. However, all of the difficulties vanish when I avail myself of general projective ideas. In fact, the G_8 of projective geometry (i.e., the totality of all collineations of the plane) includes not only the G_3 of one or the other non-Euclidian geometry, but also the G_4 of Euclidian geometry. If I assume the projective viewpoint then I have the same advantage as a wanderer who stands on a mountaintop and surveys different valleys at once, while if he stands in a single valley then he finds it hard to describe the behavior of the other valleys. Here is one last point that is not unimportant! For all of the essential differences between cases 1), 2), and 3) it is as good as self-explanatory for the projective geometer that one can define a continuous transition between the three cases. For the fundamental equation, one chooses the projective metric:

$$u_1^2 + u_2^2 + \varepsilon u_3^2 = 0, \tag{9}$$

and lets the parameter ε in it range over positive values to negative ones while going through zero! Riemannian geometry will then turn into Euclidian geometry and then into the geometry of Bolyai, Lobachevsky, and Gauss. On closer inspection, things appear to be such that I can limit myself to an arbitrarily-extended region around the point $u_3 = 0$ (large enough that, when it suits one's purposes, it can include our entire solar system or even the entire universe of fixed stars) and then the ε, which can be positive or negative, can be made so small that inside of this region any distance, when measured in a non-Euclidian way, deviates from its Euclidian value less than some sufficiently small given amount

One may admit that I may carry these detailed considerations on the projective metrics in the plane somewhat further; this seems natural if I am to adequately prepare for certain considerations that I will make use of later on when comparing the new and classical mechanics. I further apply the aforementioned continuity principle to the cases

3), 4), and 5) of our list on pp. 6. Let the fundamental construction, relative to an ordinary rectangular coordinate system, be:

$$u_1^2 + \varepsilon u_2^2 = 0, \tag{10}$$

and then first give ε a very small positive value, then a very small negative value, and thirdly, a null value. Let the associated (ordinary, non-homogeneous) coordinates of a point be denoted by x, y. As the distance from this point to the coordinate origin, one then obtains an analogous alteration of formula (3):

$$r = \sqrt{\varepsilon x^2 + y^2}\,, \tag{11}$$

and here one would now like to ponder how the system is arranged around O as the center of the surrounding circle (i.e., the curves r = const.). Obviously, for positive ε we obtain elongated ellipses (whose major axis points in the direction of the X-axis), for negative ε we obtain hyperbolas whose asymptotes $y/x = \pm\sqrt{-\varepsilon}$ define a very small angle with the X-axis, and when it vanishes, we obtain pairs of straight lines $y = \pm\sqrt{\text{const.}}$ that run parallel to the X-axis. It is amusing to reflect upon how these pairs of parallel lines define the transition from ellipses to hyperbolas in the cases of positive (negative, resp.) ε!

We may further consider the equiform and congruent transformations that are associated with our metric first in the cases of non-vanishing ε. Since the two points that are represented by (10) are different from each other for $\varepsilon < 0$ and $\varepsilon > 0$, they determine their connecting line – namely, the infinitely distant line – uniquely. Our transformations will thus be *affine* transformations, and can be presented in the form:

$$\begin{cases} x' = \alpha_{11} x + \alpha_{12} y + \alpha_{13}, \\ y' = \alpha_{21} x + \alpha_{22} y + \alpha_{23}. \end{cases} \tag{12}$$

Here, the coefficients on the right-hand side are to be measured in the equiform case in such a way that $\varepsilon(\alpha_{11} x + \alpha_{12} y)^2 + (\alpha_{21} x + \alpha_{22} y)^2$ agrees with $\varepsilon x^2 + y^2$ up to an arbitrary remaining factor. This gives two conditions for the coefficients α_{ik}, whose number increases to three when we set the determinant $\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}$ equal to ± 1 in order to go over to the congruent transformations. We thus have ∞^4 equiform and ∞^3 congruent transformations, in precise, self-explanatory, agreement with what we learned in the case $\varepsilon = 1$ of the Euclidian metric.

We now go on to the doubly-specialized case $\varepsilon = 0$, when we would like to establish the condition *that here, as well, only affine transformations* (12) *shall come under consideration* (this is a free abbreviation here, because the infinitely distant line is only one of the lines that include the point $u_1^2 = 0$ – i.e., the infinitely distant point of the X-axis – although there is initially no necessity for it to go into itself under the transformations that we are considering). We then obtain simply $\alpha_{31} = 0$ for the equiform transformations; any transformation:

$$(13) \qquad \begin{cases} x' = \alpha_{11}x + \alpha_{12}y + \alpha_{13}, \\ y' = \qquad\quad \alpha_{22}y + \alpha_{23}, \end{cases}$$

will act as an equiform one. *Despite our restricting assumptions, the equiform group still includes five parameters here.* As "motions" – i.e., congruent transformations with no transfer – one may then identify those elements (13) that are firstly unimodular, and secondly, leave the distance between two points *x*, *y* and $\bar{x}$, $\bar{y}$ unchanged – i.e., in the present case, $(y - \bar{y})$. This gives $\alpha_{11} = 1$, $\alpha_{22} = 1$ and *the three-parameter group of motions is given by the formula:*

$$(14) \qquad \begin{cases} x' = \alpha_{11}x + \alpha_{12}y + \alpha_{13}, \\ y' = \qquad\quad \alpha_{22}y + \alpha_{23}. \end{cases}$$

The equiform transformations thus include two parameters more than the congruent ones. We will say that we can now choose the units for the measures along the *X*-axis and *Y*-axis independently. In particular, we will have an invariant of motion for two arbitrary point in the form of $y - \bar{y}$; *however, if* $y - \bar{y} = 0$, *in particular, then* $x - \bar{x}$ *is also an invariant of motion.*

It now seems clear that all of these, certainly very simple, Ansätze can be carried over to a larger number of variables. In particular, we would like to go over to four variables *x*, *y*, *z*, *t* immediately (with Minkowski, we will refer to the totality of all values for these variables as the *world*, where *x*, *y*, *z* are the *space coordinates*, and *t* is the *time*). We thus pass over the systematic enumeration of the associated possible types of projective metrics, since this will ultimately be simple. Moreover, we restrict ourselves to showing that here, in the four-dimensional world, the *system of mechanics* is subordinate to the concept of projective metric, and indeed not only the system of *classical* mechanics, but also the *new* mechanics of Lorentz, Poincaré, Einstein, and Minkowski, where the essence of these two systems, and, in particular, their reciprocal position, may be brought to its greatest clarity.

First, we would like to temporarily set $x = x_1 / x_5$, $y = x_2 / x_5$, $z = x_3 / x_5$, $t = x_4 / x_5$. The general linear equation between *x*, *y*, *z*, *t* will be written accordingly: $u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 + u_5x_5 = 0$; in particular, $x_5 = 0$ will be the only thing that we shall call the "infinitely distant" points of the universe. We obtain our old acquaintance – the sphere circle – as before from the equation:

$$(15) \qquad u_1^2 + u_2^2 + u_3^2 = 0;$$

however, since we have five coordinates it must now be regarded as a *doubly-specialized* construction. Along with it, we define the *singly-specialized* construction:

$$u_1^2 + u_2^2 + u_3^2 - \frac{u_4^2}{c^2} = 0, \tag{16}$$

where c shall denote the "velocity of light", so $1/c^2$ (upon choosing a unit for it, as one generally allowed to do in mechanics) is a very small quantity. In point coordinates, this construction is given by the pair of equations:

$$x_5 = 0, \qquad x_1^2 + x_2^2 + x_3^2 - c^2 x_4^2 = 0, \tag{17}$$

which then uniquely determines the "infinite distance" of the universe. If one allows c to become infinite in order to arrive at the sphere circle then one will obtain *three* equations for it in point coordinates:

$$x_5 = 0, \qquad x_4 = 0, \qquad x_1^2 + x_2^2 + x_3^2 = 0. \tag{18}$$

Here, we have $x_1 / x_5 = t = 0/0$; the sphere circle can be thought of as being *timeless*, in a manner of speaking. The infinite distance of the world is now one of the linear manifolds that include the sphere circle. It then appears to be preferred among the linear manifolds when we let the sphere circle emerge from (16) ((17), resp.) by going to the limit. – *We would now like to analogously carry over all considerations regarding these constructions* (16, 17) ((15, 18), resp.), *which we just now linked to the equation* (10), *i.e.*, $u_1^2 + u_2^2 = 0$.

I will begin with the sphere circle immediately, by adopting the principle that, corresponding to the conceptual distinction of the linear manifold $x_5 = 0$, we shall seek the associated equiform and congruent transformations of the world only amongst the affine world transformations. Accordingly, there is no longer any point in maintaining the homogeneous notation. Moreover, we will immediately write down the general schema for the transformations that come under consideration corresponding to equations (13) in the following form:

$$\begin{cases} x' = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \alpha_{14}t + \alpha_{15}, \\ y' = \alpha_{21}x + \alpha_{22}y + \alpha_{23}z + \alpha_{24}t + \alpha_{25}, \\ z' = \alpha_{31}x + \alpha_{32}y + \alpha_{33}z + \alpha_{34}t + \alpha_{35}, \\ t' = \alpha_{41}x + \alpha_{42}y + \alpha_{43}z + \alpha_{44}t + \alpha_{45}. \end{cases} \tag{19}$$

We will call these transformations equiform when they take the system of equations (18) into itself. The only condition for this is that the matrix:

$$(20) \qquad \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}$$

be *orthogonal*. This delivers five equations for the nine coefficients α_{11}, …, α_{33} in a well-known way; in all, 12 of the 17 coefficients that appear in (19) thus remain arbitrary. – Among the equiform transformations thus determined, we will then, according to (14), refer to those transformations for which the determinant of the matrix (20) is equal to ± 1, while one also has $\alpha_{44} = 0$, as *congruent transformations*. *The group of congruent transformations thus defined still includes ten parameters.* If *x*, *y*, *z*, *t* and $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ are the coordinates of two world-points then, generally speaking, only the difference $t - \bar{t}$ remains unchanged by the group of congruent transformations; it is only when $t - \bar{t}$, in particular, is equal to zero that $(x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2$ is also an invariant. *Two world-points thus have a "purely geometrical" invariant only when their time-differences vanish.*

That we, in fact, come to the foundations of classical mechanics with these assumptions on the equiform and congruent world-transformations associated with the sphere-circle is demanded by things that have often been emphasized recently by other authors, rather than working through the problem. In fact, the foundations of classical mechanics remain unchanged when we:

1. Replace the arbitrarily-chosen rectangular space coordinate system *x*, *y*, *z* with any other similarly-oriented one.
2. Think of the rectangular system as allowing any uniform translation.
3. Let the origin from which we measure the time *t* vary arbitrarily.

This is precisely what finds its expression in the group of our congruent transformations. In particular, the uniform translations 2 correspond to the terms in our formula with $\alpha_{14}\, t$, $\alpha_{24}\, t$, $\alpha_{34}\, t$. However, the situation that our equiform transformation contains two more parameters than the congruent ones corresponds to the fact that in classical mechanics the unit of time and the unit of length can be chosen independently of each other (upon which, the study of "similarity" in classical mechanics is based).

Secondly, we consider the case of a basic construction that is only simply-specialized (17) (which still has no particular name, although it certainly deserves one):

$$x_5 = 0, \qquad x_1^2 + x_2^2 + x_3^2 - c^2 x_4^2 = 0.$$

The equiform transformations are necessarily affine here, which is all the more reason for us to go back to the non-homogeneous notation. The general schema for an affine transformation is then:

$$
(21) \qquad \begin{cases}
x' = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z + \alpha_{14}t + \alpha_{15}, \\
y' = \alpha_{21}x + \cdots\cdots\cdots\cdots\cdots\cdots + \alpha_{25}, \\
z' = \alpha_{31}x + \cdots\cdots\cdots\cdots\cdots\cdots + \alpha_{35}, \\
t' = \alpha_{41}x + \cdots\cdots\cdots\cdots\cdots\cdots + \alpha_{45}.
\end{cases}
$$

We have an equiform transformation, as long as the homogeneous substitution of the *x*, *y*, *z*, *t* that is given by the matrix:

$$
\begin{vmatrix}
\alpha_{11} & \cdots & \cdots & \alpha_{14} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{41} & \cdots & \cdots & \alpha_{44}
\end{vmatrix}
$$

takes the quadratic form $x^2 + y^2 + z^2 - c^2t^2$ into a multiple of itself. This reduces the 20 coefficients α_{ik} to nine; *the group of equiform transformations thus now includes eleven parameters*. Among them, the group of congruent transformations (as we have been defining them) emerges when we demand that the determinant:

$$
\begin{vmatrix}
\alpha_{11} & \cdots & \cdots & \alpha_{14} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{41} & \cdots & \cdots & \alpha_{44}
\end{vmatrix}
$$

has one of the values ± 1. *We thus have a group of ten parameters.* If *x*, *y*, *z*, *t* and $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ are the coordinates of two world-points then the square of their quasi-distance:

$$
(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 - c^2(t - \bar{t})^2
$$

proves to be unchanged.

We now have yet a finer point to address that was already brought up in the discussion of the point-pair $u_1^2 + \varepsilon u_2^2 = 0$ as the fundamental construction of a plane metric. In order to single out the congruent transformations with no transfer from the totality of equiform transformations, one can restrict oneself to the ones for which one sets the determinant $\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 1$ in the substitutions (12). One thus defines, in fact, a Euclidian metric, which is based on the fundamental construction of an *imaginary* point-pair. However, this leads to motions only for the case of imaginary point-pairs (for the case of a positive ε). If the point-pair is real (ε negative) then closer geometric scrutiny shows that the unimodular, equiform transformations no longer define a continuum, as one would justifiably expect of the totality of all motions. Their totality decomposes, moreover, into four continua. *Only those transformations that leave the sign of the* *differential expression* $\varepsilon\, dx^2 + dy^2$ *unchanged and have* α_{22} *positive, moreover, will be referred to as motions in the strict sense, because they alone are continuously connected to the "identity" transformation* $x' = x$, $y' = y$. In order to eliminate motions for negative ε from the previously-given definition of congruent transformations one must therefore still worry about the two stated demands. This has no influence on the number of parameters that was given at that time. Moreover, in the limiting case $\varepsilon = 0$, when we set $\alpha_{22} = 1$ we have already treated the new arrangement accordingly. – Something similar is now going on with the case of the construction (17) to be treated now, as well (which, due to the negative sign on the term $c^2 x_4^2$ in its equation, is comparable, to a certain degree, to the case of a real point-pair in the plane). Now, a more precise geometric argument – which is not especially difficult, but it would take up more space here than we would like to give to it – shows that the group of congruent transformations, as we will next define them, still encompasses two continua, and *that for the group of motions of these two continua we can use only those motions that are characterized by a positive* α_{44}.

We may thus expressly add the requirement of a *positive* α_{44} to the definition of our ten-parameter group. We then have precisely the *Lorentz group* of the "new" mechanics before us. Generally, one says most commonly that the Lorentz group has six (not ten) parameters. This is, however, only a consequence of the fact that in mathematical physics one ordinarily does not consider the transformations (21) of the coordinates x, y, z, t, but only the corresponding transformations of the differentials dx, dy, dz, dt, by which the additive constants α_{15}, α_{25}, α_{35}, α_{45} are omitted from formulas (21) for self-explanatory reasons. However, the situation that the group of equiform transformations now contains only one more parameter than that of the congruent ones finds its counterpart in the fact when one is given c (the speed of light) in the new mechanics the unit of space and the unit of time are linked together (such that only of them is arbitrary).

Thus, the old mechanics and the new mechanics are consistently incorporated into the schema of projective metrics for four variables – the goal that I brought into my view at the beginning of this talk has thus been achieved. Everything that I said at the outset on the relationship of metric geometry to projective geometry has been carried over analogously. I confine myself to adding two brief remarks:

First: According to the terminology that I occasionally touched upon above, we may say that classical mechanics, like the new mechanics, is a "theory of relativity" relative to a group of ten parameters. One may ask: Why then is the term "theory of relativity" used exclusively as an attribute of the new mechanics in the physical literature? This seems to be the answer: Because historically the new mechanics originated in the context of electrodynamics. In order to clarify matters, it suffices to set down the Maxwell equations – for the pure ether, say – in the Hertzian notation:

$$\frac{1}{c}\cdot\frac{\partial L}{\partial t} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \qquad \frac{1}{c}\cdot\frac{\partial X}{\partial t} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y},$$
$$\frac{1}{c}\cdot\frac{\partial M}{\partial t} = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \qquad \frac{1}{c}\cdot\frac{\partial Y}{\partial t} = \frac{\partial N}{\partial x} - \frac{\partial L}{\partial z},$$

$$\frac{1}{c}\cdot\frac{\partial N}{\partial t} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}, \qquad \frac{1}{c}\cdot\frac{\partial Z}{\partial t} = \frac{\partial L}{\partial y} - \frac{\partial M}{\partial x},$$

$$\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0, \qquad \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

These equations obviously remain unchanged when one replaces the x, y, z-system with any other (similarly-oriented) rectangular coordinate system, or when one displaces the origin of time arbitrarily; together, that defines a group of seven parameters. However, they no longer remain unchanged when one subjects the coordinate system to a uniform translation, and thus sets:

$$x' = x + \alpha_{14}\, t, \qquad y' = y + \alpha_{24}\, t, \qquad z' = z + \alpha_{34}\, t.$$

In this, lies the reason for one who is subject to Maxwell's equations for the electrodynamic ether to regard them as residing in space, such that the concept of *absolute* space again finds a position of honor. What remains is the seven-parameter group of changes that correspond to the purely external transition from a coordinate system x, y, z, t to another one that is just as valid. – Then came the discovery that this seven-parameter group is contained in a ten-parameter one that keeps the Maxwell equation themselves unchanged, namely, the Lorentz group. Again, absolute space (or perhaps it is better to say: the absolute world) disappears – the world again becomes a relative concept, as it was before – and one imagines, without thinking, that one recovers only the prior state of affairs *mutatis mutandis*, while the phrase "theory of relativity" is a new term that relates exclusively to the Lorentz group.

However, as concluding remarks, I will choose these: It was suggested above that the difficulties that anyone perceives when one begins by trying to adapt Euclidian geometry fall away when using the non-Euclidian doctrines with no further assumptions when one takes the more general standpoint of projective thinking as the starting point. Analogous statements are true for the study of the new phenomena that emerge in the mechanics that is based upon the Lorentz group. It seems inappropriate to always begin this study with ideas that are true in classical mechanics and then to ponder how one may cleverly deform them in order to pass over to the new mechanics. Moreover, it seems more proper to first rise from the standpoint of old mechanics to an enveloping one, which then includes the old and new mechanics together as special cases. Accordingly, what was suggested above is not necessary for this – viz., to think in terms of *projective* concepts – when it would suffice to think in terms of *affine* ones. It would come down to a question of writing down a systematic invariant theory of the "affine" world, in which generally all of the elements of the many-dimensional investigations of the mathematicians are already present, and in which the two types of mechanics – old and new – can be treated together. The manner by which the old mechanics can be regarded as a limiting case of the new one, to the extent that it can be regarded as an approximation to the latter, then emerges in its own right. Who will bring this program to its realization?

Minkowski has undoubtedly gone into the matters demanded here quite precisely in their own right. However, since he was writing for the broader circle of physically-interest readers, in the interests of understandability he expounded upon his developments more conveniently, not in terms of his own personal way of thinking, but only after it was crystallized into the form of an algorithm that led him into the case of the Lorentz group. That was Minkowski's four-dimensional vector calculus, which, in the absence of a more rigorous basis, placed his electrodynamic developments at the pinnacle of a certain system of rigidly-defined algebraic processes [6]).

P.S. of August 1910. In my paper on 10 May, I also spoke of the elegant representation of the coefficients of the Lorentz group in terms of ten independent parameters on the basis of a famous quaternion formula that was first presented by Cayley.

The ultimate formula is the following one: I understand i to mean the ordinary imaginary unit, and i_1, i_2, i_3 to mean the units specific to the quaternion calculus. Let A, A', …, D, D' be eight parameters that shall be linked by the bilinear equation:

$$A\,A' + B\,B' + C\,C' + D\,D' = 0,$$

along with the inequality:

$$A^2 + B^2 + C^2 + D^2 > A'^{\,2} + B'^{\,2} + C'^{\,2} + D'^{\,2}.$$

Likewise, let x_0, y_0, z_0, t_0 be four parameters. The substitutions of the Lorentz group are then given by the following formula:

$$(i_1\,x' + i_2\,y' + i_3\,z' + i\,ct') - (i_1\,x_0 + i_2\,y_0 + i_3\,z_0 + i\,ct_0)$$

$$= \frac{\begin{bmatrix} (i_1(A+iA') + i_2(B+iB') + i_3(C+iC') + (D+iD') \\ \cdot(i_1 x + i_2 y + i_3 z + ict) \\ \cdot(i_1(A-iA') + i_2(B-iB') + i_3(C-iC') + (D-iD') \end{bmatrix}}{(A'^2 + B'^2 + C'^2 + D'^2) - (A^2 + B^2 + C^2 + D^2)}.$$

Since the multiplication of A, A', …, D, D' by an arbitrary common factor does not change the formula, but, on the other hand, the A, A', … are subject to the aforementioned bilinear relation, we have, in fact, a ten-fold infinitude of substitutions before us.

For the finer details and literary reference, one can confer, say, the "Zusätze und Ergänzungen" that Fritz Nöther appended to the recently-appearing final volume of Sommerfeld and my own "Theorie des Kreisels" (Leipzig, Teubner 1910).

---

[6]) [These remarks on the manner of representation chosen by Minkowski relate definitively to his publications in 1910 and also to Minkowski's collected works (Leipzig, 1911). In the meantime, in 1915 his estate found the manuscripts of some publications by him – namely, ones that appeared on 5 Nov. 1907 in the Göttinger Mathematischen Gesellschaft – in which he presented his undisguised mathematical thoughts. This paper was printed shortly thereafter under the title of "Das Relativitätsprinzip" by Sommerfeld in v. 47 of the 4th series of Annalen der Physik and is also found, by the way, in the 24th volume of the Jahresberichte of the Deutschen Mathematischen Vereinigung (1916). I would like to make some entirely distinct remarks on this subject shortly. K.]

[Cunningham and Bateman have already remarked in 1909 that the Maxwell equations remain invariant under not only the linear transformations of the Lorentz group, but also under the extended G_{15} , which arises from the Lorentz group when one adds a precise number of transformations of the following type (which correspond to a conversion of the world through "reciprocal radii") [7]):

$$x' = \frac{x}{x^2+y^2+z^2-c^2t^2}, \qquad y' = \frac{y}{-}, \qquad z' = \frac{z}{-}, \qquad t' = \frac{t}{-}.$$

Bateman made an interesting application to the theory of Maxwell's equations in the Proceedings of the London Mathematical Society (2) 8 (1910).

Bateman, *loc. cit.*, went further than this by interpreting the value system x, y, z, t in terms of a sphere in three-dimensional space with midpoint coordinates x, y, z and a radius ct (this is the same idea that Timerding developed independently in v. 21 of the Jahresberichts der Deutschen Mathematiker-Vereinigung, 1912). The transformations of the four-dimensional "world" that we just mentioned then behave, as Bateman says, like "spherical wave transformations." *These are precisely the transformations of Lie's sphere geometry.* Among them, G_{10} single out the Lorentz transformations that convert the planes into planes.

Obviously, these developments are intimately linked with the ones that Lie and I carried out in 1871, and which I must refer one, in particular, to no. VIII of the present collection ("Über Liniengeometrie und metrische Geometrie").

For physics, this G_{15} generally does not have the same meaning as its subgroup, the G_{10} Lorentz group. This is based upon the fact that the latter is only a generalization of the G_{10} of classical mechanics (which it turns into when one lets the speed of light go infinite), but a generalization of physics must encompass both mechanics and electrodynamics. Einstein expressed this situation to me casually as follows: The transformation through reciprocal radii indeed preserves the form of the Maxwell equations, but not the connection between coordinates and the results of measuring yardsticks and clocks. K.]

[7]) The individual transformations of this type would alter the Maxwell equations like a change of sign of t, or, what amounts to the same thing, the transition from a left-handed coordinate system x, y, z, as Hertz used, to a right-handed coordinate system.