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Quantum Theory in Hydrodynamical Form

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It is shown that the Schrödinger equation for one-electron problems can be transformed into the form of hydrodynamical equations.

According to E. Schrödinger ([1]), the quantum theory of one-electron problems follows from the "amplitude equation":

$$\Delta \psi_0 + \frac{8\pi^2 m}{h^2}(W - U)\psi_0 = 0, \qquad \psi = \psi_0 \, e^{i2\pi\frac{W}{h}t}. \tag{1}$$

Here, W means the energy of the system, U is the potential energy, as a function of the position of the electron, and m is its mass. One seeks a solution that is everywhere continuous and finite. This is possible only for certain values of W. These "eigenvalues" W_i shall be the energy that the system possesses in its "quantum state." They are – as you know – spectroscopically determined. The correspondence between theory and experience speaks well for the utility of the computational method described in what follows.

To each eigenvalue there belongs an "eigensolution," which is normalized, and shall be given the time factor $e^{i2\pi\frac{W}{h}t}$, and, according to Schrödinger, represents what happens in the system. Schrödinger gives Ansätze for an interpretation which, in principle, corresponds to the one given in what follows. I will pursue this interpretation and show that far-reaching analogies with hydrodynamics exist.

A second equation, also derived by Schrödinger, is obtained when one eliminates W from (1) upon including the time factor:

$$\Delta \psi - \frac{8\pi^2 m}{h^2} U\psi - i\frac{4\pi m}{h}\frac{\partial \psi}{\partial t} = 0. \tag{2}$$

He obtains as solutions those of the first equation, along with all linear combinations of the latter. That is very essential. Namely, if one sets $\psi = \alpha e^{i\beta}$, then, by (1), only β is considered linearly dependent on t, whereas, by (2), α, as well as β, can be time varying.

With, $\psi = \alpha e^{i\beta}$, (2) becomes:

$$\Delta\alpha - \alpha(\mathrm{grad}\,\beta)^2 - \frac{8\pi^2 m}{h^2} U + \frac{4\pi m}{h}\alpha\frac{\partial \beta}{\partial t} = 0, \tag{3}$$

and

$$\alpha\Delta\beta + 2(\mathrm{grad}\,\alpha\;\mathrm{grad}\,\beta)^2 - \frac{4\pi m}{h}\frac{\partial \alpha}{\partial t} = 0. \tag{4}$$

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[1] E. Schrödinger, Ann. d. Phys. **79**, 361, 489; **80**, 437; **81**, 109, 1926.

From (4), with $\varphi = -\beta h / 2\pi m$, it follows that:

$$\operatorname{div}(\alpha^2 \operatorname{grad}\varphi) + \frac{\partial \alpha^2}{\partial t} = 0. \tag{4'}$$

(4') has the character of a hydrodynamical equation of continuity when one regards α^2 as a density and φ as the velocity potential of a flow $u = \operatorname{grad}\varphi$.

(3) then gives:

$$\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}(\operatorname{grad}\varphi)^2 + \frac{U}{m} - \frac{\Delta\alpha}{\alpha}\frac{h^2}{8\pi^2 m^2} = 0. \tag{3'}$$

This equation also corresponds precisely to a hydrodynamical one, namely, that of an irrotational flow moving under the action of conservative forces ([2]).

Since rot $u = 0$, taking the gradient gives:

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\operatorname{grad}(u^2) = \frac{du}{dt} = -\frac{\operatorname{grad}(U)}{m} + \operatorname{grad}\frac{\Delta\alpha}{\alpha}\frac{h^2}{8\pi^2 m^2}\;. \tag{3''}$$

The term – grad(U)/m corresponds to the quantity f/σ (force density: mass density), and $(\Delta\alpha/\alpha)h^2/8\pi^2 m^2$ corresponds to the quantity $-\int(d\rho/\sigma)$, which one can interpret as the force function of an "internal" force of the continuum.

We therefore see that equation (2) is completely explainable in terms of hydrodynamics, and that a peculiarity appears only in one term, which represents the internal mechanism of the continuum.

In the case of equation (1), we get $\partial\alpha/\partial t = 0$ and $\partial\varphi/\partial t = -W/m$. Despite the time factors, the eigensolutions of (1) produce the picture of a stationary flow. In this interpretation, quantum states are considered to be stationary flow states, and in the case where grad $\beta = 0$ they are actually static structures.

The solutions of the general equation (2) are now easy to obtain as linear combinations of the eigensolutions. For example, let us set $\psi = \alpha e^{i\beta} = \psi_1 + \psi_2 = c_1\alpha_1 e^{i\beta_1} + c_2\alpha_2 e^{i\beta_2}$, where ψ_1 and ψ_2 are eigensolutions of (1), with the time factor $e^{i2\pi\frac{W}{h}t}$ suppressed, so we have:

$$\alpha^2 = c_1^2\alpha_1^2 + c_2^2\alpha_2^2 + 2c_1c_2\alpha_1\alpha_2\cos(\beta_2 - \beta_1)$$

and

$$\alpha^2 \operatorname{grad}\beta = c_1^2\alpha_1^2\operatorname{grad}\beta_1 + c_2^2\alpha_2^2\operatorname{grad}\beta_2 + 2c_1c_2\alpha_1\alpha_2\operatorname{grad}(\beta_2 + \beta_1)\cos(\beta_2 - \beta_1),$$

$$\int \alpha^2 dV = c_1^2\int\alpha_1^2 dV + c_2^2\int\alpha_2^2 dV\;,$$

[2] Cf., e.g., Weber and Gans, Reportium d. Physik I, 1, pp. 304.

i.e, “density” as well as “flow strength.” One obtains a term $\nu = (W_1 - W_2)/h$ that is periodic in time. The “total set” remains, however, constant.

In the case of a stationary flow one finds from (3'):

$$W = \frac{m}{2}(\mathrm{grad}\varphi)^2 + U - \frac{\Delta\alpha}{\alpha}\frac{h^2}{8\pi^2 m^2}\,, \tag{5}$$

from which, one can also write, when one sets: $\alpha^2 = \rho$, $\rho m = \sigma$, corresponding to the normalization $\int \rho dV = 1$:

$$W = \int dV\left\{\frac{\sigma}{2}u^2 + \sigma U - \sqrt{\sigma}\cdot\Delta\sqrt{\sigma}\,\frac{h^2}{8\pi^2 m}\right\}. \tag{5'}$$

This form for the energy as the volume integral of kinetic and potential energy densities is immediately intuitive.

There is no obvious reason why this form, which one can also write as:

$$W = \frac{h}{2\pi}\int dV\alpha^2\frac{\partial\beta}{\partial t}\,,$$

is not obtained for the case of nonstationary flow. That the conservation law $dW/dt = 0$ is satisfied can be easily observed from the orthogonality of the eigensolutions.

It is interesting to ask the question: do equations (3'), (4') and (5') already contain all the known special cases? In particular, do they imply:

1. The existence of discrete stationary flow states with energy W_i.
2. The fact that all nonstationary states possess only periodicities of the form: $\nu_{ij} = \dfrac{W_i - W_j}{h}$.

Apparently, (2) follows uniquely from (3') and (4'), or, on the other hand, (1) and (5'). The hydrodynamical equations are thus identical with those of Schrödinger and deliver everything when they are given; i.e., they are sufficient in order to represent the essential elements of the quantum theory of the atom that can be modeled.

Since the foregoing quantum problem seems to be connected with the hydrodynamics of continuously distributed electricity, with the charge density proportional to the mass density, there remains a series of difficulties. On the one hand, the mass density is not of the type that one would expect in electrodynamics. On the other hand, one should expect that the interaction of the electrons with each other, which is represented by the term $\sqrt{\sigma}\cdot\Delta\sqrt{\sigma}\,h^2/(8\pi^2 m)$ depends not only on the density at a point and its derivatives, but also on the total distribution of the charge. I have no idea how to satisfy both of these expectations through a purely mathematical formulation.

How are we to treat the many-electron problem? Schrödinger gives no completely determined form. He claims only that the kinetic energy is to be computed from a representation of the motion in phase space, i.e., that one must define: $T = \sum_i m_i u_i^2 / 2$ to be the sum of the kinetic energies of the individual electrons, as when they are all pair-wise independent and do not, perhaps, constitute a single flow field.

In fact, this is an obvious possibility. We have to choose between the following alternatives:

a) Do more electrons assemble together into a bigger structure?
b) Do they annihilate themselves and pass into each other with a certain boundary condition?
c) Do they penetrate without amalgamation?

To me, *c*) seems the most likely. With these same solutions, *a*) leads to the one-electron problem – only with a different normalization – and this obviously leads to a false result. *b*) seems to be a bit like "jumping off the deep end ([3])," but is still conceivable.

From *c*), more vectors must be defined at each point of space, as well as their associated velocity potentials. The continuum then has the intuitive quality of a swarm whose parts possess an infinite free path length.

The form that the function *U* is to be given, insofar as it represents the interaction of the electrons with each other, as well as the "quantum term" of equation (3'), can first be determined from a successful calculation, at least in some particular cases.

There is thus a chance of erecting the quantum theory of the atom on this basis. The radiation process becomes, however, only piecewise explainable. Indeed, it appears to be clear that an atom does not radiate in a quantum state, and also that radiation of the correct frequency is correctly represented – without "jumps," moreover – by slowly going into a non-stationary state; however, other things – e.g., the fact of quantum absorption – remain completely unclear. I consider it premature to speculate about their nature.

3 Ed. Note: The German idiom was "eintauchende Bahnen."