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Invariant variational problems

(F. Klein on his fifty-year Doctoral Jubilee)

By

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(Presented by F. Klein at the session on 26 July 1918 [1])

Translated by D. H. Delphenich

We shall deal with variational problems that admit a continuous group (in the Lie sense); the results that this yields for the associated differential equations find their most general expression in the theorems that are formulated in § 1 and are proved in the following paragraphs. One can make more precise statements about these differential equations that arise from variational problems than one can about arbitrary differential equations that admit a group, which defines the context of Lie's investigations. Thus, what follows rests upon a coupling of the methods of the formal calculus of variations with those of the theory of Lie groups. For special groups and variational problems, this coupling is not new; I mention Hamel and Herglotz for special finite groups, and Lorentz and his school (e.g., Fokker), Weyl, and Klein for special infinite groups [2]). In particular, Klein's second note and the present efforts were mutually influenced by each other, so I will refer to the concluding remarks of Klein's note.

§ 1. Prefatory remarks and the formulation of the theorem.

All of the functions that enter into what follows shall be assumed to be analytic, or at least continuous and continuously differentiable finitely often, and single-valued in the domain in question.

As one knows, one understands the term "transformation group" to mean a system of transformations such that to every transformation included in the system there exists an inverse, and the composition of any two transformations of the systems again belongs to the system. The group is called a *finite, continuous group* $\mathfrak{G}_\rho$ when its transformations

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[1]) The final version of the manuscript was first submitted at the end of September.

[2]) Hamel: Math. Ann, Bd. 59 and Zeit. f. Math. u. Phys., Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, esp. § 9, pp. 511. Fokker, Verslag d. Amsterdamer Akad., 27/1, 1917. For further literature, cf., the second note of Klein: Göttinger Nachrichten, 19 July 1918.

In a recently-appearing paper of Kneser (Math. Zeit., Bd. 2), he treated the constructionof invariants by similar methods.

are included in the most general group that depends analytically upon ρ essential parameters s (i.e., the ρ parameters shall not be representable as ρ functions of fewer parameters). Correspondingly, one understands an *infinite continuous group* $\mathfrak{G}_{\infty\rho}$ to mean a group whose most general transformations depend analytically upon ρ essential, arbitrary functions $p(x)$ and their derivatives, or at least, one that are continuous and continuously differentiable finitely often. As an intermediate step between the two, one finds the groups that depend upon infinitely many parameters, but not on arbitrary functions. Finally, one refers to the *mixed groups* as the ones that depend upon arbitrary functions, as well as parameters [1]).

Let $x_1, \ldots, x_n$ be independent variables, and let $u_1(x), \ldots, u_\mu(x)$ be functions that depend upon them. If one subjects the x and u to the transformations of a group then, due to the assumed invertibility of the transformations, among the transformed quantities, there must again be found precisely n independent ones: $y_1, \ldots, y_n$; let the remaining ones that are independent of them be denoted by $v_1(y), \ldots, v_\mu(y)$. The derivatives of the u with respect to the x – viz., $\partial u / \partial x$, $\partial^2 u / \partial x^2$, … – can also enter into the transformations [2]). A function is called an *invariant* of the group when there exists a relation:

$$P\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) = P\left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \cdots\right).$$

In particular, an integral I becomes an invariant of the group when there exists a relation:

$$(1) \qquad \begin{aligned} I &= \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) dx \\ &= \int \cdots \int f\left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \cdots\right) dy \end{aligned} \qquad {}^{3})$$

when one integrates over an arbitrary real x-domain and the corresponding y-domain [4]).

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[1]) In “Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen” (Ber. d. K. Sächs. Ges. der Wiss. 1891) [cited as “Grundlagen”], Lie defined the infinite, continuous groups to be transformation groups whose transformations are given by the most general solutions of a system of partial differential equations, as long as these solutions do not depend upon only a finite number of parameters. In this way, one thus obtains one of the aforementioned types that are different from the finite groups, while, conversely, the limiting case of infinitely many parameters does not necessarily need to satisfy a system of differential equations.

[2]) To the greatest extent possible, I will omit indices, as well as summations; hence, one might have $\partial^2 u / \partial x^2$ for $\partial^2 u_\alpha / \partial x_\beta \partial x_\gamma$, etc.

[3]) To abbreviate, I write dx, dy for $dx_1, \ldots, dx_n$, $dy_1, \ldots, dy_n$.

[4]) All of the arguments x, u, ε, $p(x)$ that enter into the transformations shall be assumed to be real, while the coefficients might be complex. However, since one deals with identities in the x, u parameters and arbitrary functions in the final results, they are also true for complex values, as long as all of the functions that appear in them are assumed to be analytic. A greater part of the results can be established without integrals, moreover, such that the restriction to the reals is not necessary for the proof here either. On the other hand, the considerations at the conclusion of § 2 and the beginning of § 5 do not seem to be practicable without integrals.

On the other hand, I define the first variation δI for an arbitrary – not necessarily invariant – integral I, and convert it according to the rules of the calculus of variations by partial integration. As is known, as long as one assumes that δu, along with all of the derivatives that appear, vanish at the boundary, but are otherwise arbitrary, it becomes:

$$\delta I = \int \cdots \int \delta f \, dx = \int \cdots \int \left(\sum \psi_i \left(x, u, \frac{\partial u}{\partial x}, \cdots \right) \delta u_i \right) dx , \tag{2}$$

where ψ means the *Lagrangian expressions;* i.e., the left-hand sides of the Lagrange equations for the associated variational problem $\delta I = 0$. These integral relations correspond to an integral-free *identity* in du and its derivatives, which arises when one writes down the boundary terms. As partial integration shows, these boundary terms are integrals over *divergences* – i.e., over expressions:

$$\operatorname{Div} A = \frac{\partial A_1}{\partial x_1} + \cdots + \frac{\partial A_n}{\partial x_n},$$

where A is linear in δu and its derivatives. One thus comes to:

$$\sum \psi_i \, \delta u_i = \delta f + \operatorname{Div} A. \tag{3}$$

In particular, if f contains only first derivatives of u then in the case of a simple integral the identity (3) is identical with the one that Heun called the "central Lagrange equation":

$$\sum \psi_i \, \delta u_i = \delta f - \frac{d}{dx} \left(\sum \frac{\partial f}{\partial u_i'} \delta u_i \right) \qquad \left(u_i' = \frac{du_i}{dx} \right), \tag{4}$$

while for a n-fold integral, (3) goes to:

$$\sum \psi_i \, \delta u_i = \delta f - \frac{\partial}{\partial x_1} \left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_1}} \delta u_i \right) - \ldots - \frac{\partial}{\partial x_n} \left(\sum \frac{\partial f}{\partial \frac{\partial u_i}{\partial x_n}} \delta u_i \right). \tag{5}$$

For a simple integral and κ derivatives with respect to u, (3) is given by:

$$\sum \psi_i \, \delta u_i = \delta f - \tag{6}$$

$$- \frac{d}{dx} \left\{ \sum \left(\binom{1}{1} \frac{\partial f}{\partial u_i^{(1)}} \delta u_i + \binom{2}{1} \frac{\partial f}{\partial u_i^{(2)}} \delta u_i^{(1)} + \cdots + \binom{\kappa}{1} \frac{\partial f}{\partial u_i^{(\kappa)}} \delta u_i^{(\kappa-1)} \right) \right\} +$$

$$+\ \frac{d^2}{dx^2}\left\{\sum\left(\binom{2}{2}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i+\binom{3}{2}\frac{\partial f}{\partial u_i^{(3)}}\delta u_i^{(1)}+\cdots+\binom{\kappa}{2}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-2)}\right)\right\}+\ldots$$

$$+\ (-1)^n\ \frac{d^\kappa}{dx^\kappa}\left\{\sum\binom{\kappa}{\kappa}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i\right\},$$

and a corresponding identity is true for n-fold integrals; in particular, A includes δu up to its $(k-1)^{\text{th}}$ derivative. The fact that the Lagrangian expressions ψ_i are, in fact, defined by (4), (5), (6) follows from the fact that all higher derivatives of the δu are eliminated by way of the combinations on the right-hand side, while, on the other hand, relation (2) is fulfilled, which leads to the partial integration *uniquely*.

One now deals with the two theorems in what follows:

I. *If the integral I is invariant under a* $\mathfrak{G}_\rho$ *then there will be* ρ *linearly independent couplings of the Lagrangian expressions with divergences; conversely, the invariance of I under a* $\mathfrak{G}_\rho$ *follows from the latter. The theorem is also true in the limiting case of infinitely many parameters.*

II. *If the integral I is invariant under a* $\mathfrak{G}_{\infty\rho}$, *in which the arbitrary functions appear up to their* σ^{th} *derivatives then there exist r identity relations between the Lagrangian expressions and their derivatives up to* σ^{th} *order; the converse is also true here* [1]).

The statement of both theorems is true for the mixed groups, so dependent, as well as independent divergence relations will appear.

If one goes from these identities to the associated variational problem – so one sets $\psi = 0$ [2]) – then in the one-dimensional case, for which the divergence goes to a total differential, Theorem I expresses the existence of ρ first integrals, between which, nonlinear dependencies can generally exist [3]); in the multi-dimensional case, one obtains divergence equations that are often referred to recently as “conservation theorems”; Theorem II expresses the idea that ρ of the Lagrangian equations are consequences of the remaining ones.

The simplest example of Theorem II – without the converse – is defined by the Weierstrass parametric representation. For it, the integral is known to be invariant due to its homogeneity of first order when one replaces the independent variable x by an arbitrary function of x that leaves u unchanged ($y = p(x)$; $v_i(y) = u_i(x)$). Thus, an arbitrary function enters in, but without any of its derivatives, and this corresponds to the well-known linear relation between the Lagrangian expressions themselves: $\sum \psi_i \frac{du_i}{dx} = 0$. The “general theory of relativity” of the physicists will serve as a further example. Here, one deals with the group of all transformations of the x: $y_i = p_i(x)$, while the u (which are

[1]) With certain trivial exceptions; cf., § 2, remark 2.
[2]) Somewhat more generally, one can also set $\psi_i = T_i$; cf., § 3, first remark.
[3]) Cf., the conclusion of § 3.

denoted by $g_{\mu\nu}$ and q) will be subjected to transformations that are induced by the coefficients of a quadratic and linear differential form that includes the first derivatives of the arbitrary functions $p(x)$. They correspond to the well-known n dependencies between the Lagrangian expressions and their first derivatives [1]).

In particular, if one specializes the group by saying that one allows no derivatives of the $u(x)$ in the transformations, and, in addition, the transformed independent quantities may depend upon only the x, but not the u, then that the relative invariance [2]) of $\sum \psi_i \; \delta u_i$ follows (as will be shown in § 5) from the invariance of I, and likewise for the divergences that appear in Theorem I, as long as the parameters are subjected to certain transformations. From this, it follows that the aforementioned first integrals also admit the group. For Theorem II, one likewise deduces the relative invariance of the left-hand sides of the dependencies that are composed by means of arbitrary functions, and as a consequence of this, one has a function whose divergence vanishes identically and admits the group that mediates the connection between dependencies and the energy theorem in the relativity theory of the physicists [3]). Theorem II ultimately gives a group-theoretic proof of an assertion of Hilbert that is connected with this concerning the breakdown of the proper energy theorems for "general relativity." With these extra remarks, Theorem I includes all of the known theorems on first integrals in mechanics, etc., while Theorem II can be regarded as the greatest possible group-theoretic generalization of "general relativity theory."

§ 2. Divergence relations and dependencies.

Let $\mathfrak{G}$ be a finite or infinite group; one may then always arrange that the identity transformation corresponds to the value zero for the parameter s (the arbitrary functions $p(x)$, resp.) [4]). The most general transformation then takes the form:

$$y_i \quad = A_i\left(x, u, \frac{\partial u}{\partial x}, \cdots\right) = x_i + \Delta x_i + \ldots$$

$$v_i(y) = B_i\left(x, u, \frac{\partial u}{\partial x}, \cdots\right) = u_i + \Delta u_i + \ldots,$$

where Δx_i , Δu_i mean the terms of lower dimension in ε ($p(x)$, resp.) and its derivatives; indeed, they shall be assumed to be linear in them. As we will show later, this is no loss of generality.

Now, let the integral I be an invariant under $\mathfrak{G}$, so relation (1) is fulfilled. In particular, I is then also invariant under the infinitesimal transformation:

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[1]) Cf., perhaps, Klein's presentation.

[2]) I. e, $\sum \psi_i \; \delta u_i$ takes on a factor under transformation.

[3]) Cf., Klein's second note.

[4]) Cf., perhaps, Lie: "Grundlagen," pp. 331. If one is dealing with an arbitrary function then the special values a^σ of the parameter must be replaced with fixed functions p^σ, $\partial p^\sigma/\partial x$, …, and correspondingly the values $a^\sigma + \varepsilon$ must be replaced with $p^\sigma + p(x)$, $\partial p^\sigma/\partial x + \partial p/\partial x$, etc.

$$y_i = x_i + \Delta x_i, \qquad v_i(y) = u_i + \Delta u_i$$

that is included in $\mathfrak{G}$, and under it relation (1) goes to:

$$(7) \qquad 0 = \Delta I = \int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy - \int \cdots \int f\left(x, u(x), \frac{\partial u}{\partial x}, \cdots\right) dx\,,$$

where the first integral is taken over the $x+\Delta x$-domain that corresponds to the x-domain. However, this integration can be converted into an integration over the x-domain by means of the conversion that is true for infinitesimal Δx:

$$(8) \qquad \int \cdots \int f\left(y, v(y), \frac{\partial v}{\partial y}, \cdots\right) dy = \int \cdots \int f\left(x, v(x), \frac{\partial v}{\partial x}, \cdots\right) dx + \int \cdots \int \mathrm{Div}(f \cdot \Delta x) \cdot dx\,.$$

If one then introduces the variation:

$$(9) \qquad \bar{\delta} u_i = v_i(x) - u_i(x) = \Delta u_i - \sum \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda$$

in place of the infinitesimal transformation Δx then (7) and (8) go to:

$$(10) \qquad 0 = \int \cdots \int \{\bar{\delta} f + \mathrm{Div}(f \cdot \Delta x)\} dx\,.$$

The right-hand side is the well-known formula for the simultaneous variation of the dependent and independent variables. Since the relation (10) is fulfilled under integration over an arbitrary domain, the integrand must vanish identically; Lie's differential equations for the invariance of I then go to the relation:

$$(11) \qquad \bar{\delta} f + \mathrm{Div}(f \cdot \Delta x) = 0.$$

If one expresses $\bar{\delta} f$ from (3) in this using the Lagrangian expressions then one gets:

$$(12) \qquad \sum \psi_i \bar{\delta} u_i = \mathrm{Div}\, B \qquad (B = A - f \cdot \Delta x),$$

and this relation thus represents an identity for any invariant integral I in all of the arguments that appear; this is the desired form for Lie's differential equations for I [1]).

[1]) (12) goes to $0 = 0$ for the trivial case when $\mathrm{Div}(f \cdot \Delta x) = 0$, $\bar{\delta} u = 0$ – which can come about only when Δx, Δu also depend upon the derivatives of u; these infinitesimal transformations are therefore always separate from the group, and only the number of the remaining parameters or arbitrary functions are to be counted in the formulation of the theorems. Whether or not the remaining infinitesimal transformations still define a group must remain undecided.

Now let $\mathfrak{G}$ be assumed to be a *finite, continuous group*, to begin with. Since, by assumption, Δu and Δx are linear in the parameters $\varepsilon_1, \ldots, \varepsilon_\rho$, from (9), the same thing is true for $\bar{\delta} u$ and its derivatives; therefore, A and B are linear in the ε. I therefore set:

$$B = B^{(1)}\,\varepsilon_1 + \ldots + B^{(\rho)}\,\varepsilon_\rho, \qquad \bar{\delta} u = \bar{\delta} u^{(1)} \varepsilon_1 + \cdots + \bar{\delta} u^{(\rho)} \varepsilon_\rho,$$

where the $\bar{\delta} u^{(1)}, \ldots$ are thus functions of x, u, $\partial u / \partial x$, …, so the desired divergence relations follow from (12):

$$\sum \psi_i \bar{\delta} u_i^{(1)} = \operatorname{Div} B^{(1)}, \ \ldots, \ \sum \psi_i \bar{\delta} u_i^{(\rho)} = \operatorname{Div} B^{(\rho)}. \tag{13}$$

One thus has ρ linearly-independent couplings of the Lagrangian expressions with divergences; the linear independence follows from the fact that, from (9), it would follow that $\bar{\delta} u = 0$, $\Delta u = 0$, $\Delta x = 0$, so there would be a dependency between the infinitesimal transformations. However, by assumption, such a thing is not fulfilled for any parameter values, since otherwise the $\mathfrak{G}_\rho$ that further arises from the infinitesimal transformations by integration would depend upon less than ρ essential parameters. The further possibility that $\bar{\delta} u = 0$, $\operatorname{Div}(f\,\Delta x) = 0$ was, however, excluded. These conclusions are also still true in the limiting case of infinitely many parameters.

Now, let $\mathfrak{G}$ be *an infinite, continuous group;* $\bar{\delta} u$ and its derivatives, and therefore also B, will be linear in the arbitrary functions $p(x)$ and their derivatives [1]). By substituting the values of $\bar{\delta} u$, still independently of (12), let:

$$\sum \psi_i \bar{\delta} u_i =$$

$$\sum_{\lambda, i} \psi_i \left\{ a_i^{(\lambda)}(x, u, \cdots)\, p^{(\lambda)}(x) + b_i^{(\lambda)}(x, u, \cdots) \frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c_i^{(\lambda)}(x, u, \cdots) \frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma} \right\}.$$

One may now, analogously to the formula for partial integration, replace the derivatives of p with p itself and divergences that are linear in p and its derivatives using the identity:

$$\varphi(x, u, \ldots)\, \frac{\partial^\tau p(x)}{\partial x^\tau} = (-1)^\tau \cdot \frac{\partial^\tau \varphi}{\partial x^\tau} \cdot p(x) \qquad \text{mod divergences.}$$

One thus gets:

$$\sum \psi_i \delta u_i = \tag{14}$$

$$\sum_\lambda \left\{ (a_i^{(\lambda)} \psi_i) - \frac{\partial}{\partial x}(b_i^{(\lambda)} \psi_i) + \cdots + (-1)^\sigma \frac{\partial^\sigma}{\partial x^\sigma}(c_i^{(\lambda)} \psi_i) \right\} p^{(\lambda)} = \operatorname{Div}(B - \Gamma).$$

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[1]) The fact that it is no restriction to assume that the p are free of the u, $\partial u / \partial x$, … shows the converse.

I now construct the n-fold integral of (15), taken over any domain, and choose the $p(x)$ such that they vanish on the boundary of $(B - \Gamma)$, along with all of the derivatives that appear. Since the integral of a divergence reduces to a boundary integral, the integral of the left-hand side of (15) thus also vanishes for arbitrary $p(x)$ that only vanish on the boundary, along with sufficiently many derivatives, and from this, it follows, by a well-known argument, that the integrands vanish for any $p(x)$, so one has the ρ relations:

$$(16) \qquad \sum \left\{ (a_i^{(\lambda)} \psi_i) - \frac{\partial}{\partial x}(b_i^{(\lambda)} \psi_i) + \cdots + (-1)^\sigma \frac{\partial^\sigma}{\partial x^\sigma}(c_i^{(\lambda)} \psi_i) \right\} = 0 \qquad (\lambda = 1, 2, \ldots, \rho).$$

These are the desired dependencies between the Lagrangian expressions and their derivatives for the invariance of I under $\mathfrak{G}_{\infty\rho}$; the linear independence is clear, as above, since the inverse leads back to (12), and since one can again go from the infinitesimal transformations back to the finites ones, as will be done more thoroughly in § 4. Thus, ρ arbitrary transformations already appear in the infinitesimal transformations for a $\mathfrak{G}_{\infty\rho}$. From (15) and (16), it then follows that $\mathrm{Div}(B - \Gamma) = 0$.

If one correspondingly assumes a "mixed group" of Δx and Δu that are linear in the ε and the $p(x)$ then one sees, when one sets the $p(x)$ equal to zero and then the ε, that the divergence relations (13) exist, as well as the dependencies (16).

§ 3. Converse in the case of the finite group.

In order to show the converse, one must essentially follow through the foregoing argument in the opposite sequence. The validity of (12) follows from the validity of (13) upon multiplication by ε and addition, and by means of the identity (3), this implies a relation: $\bar{\delta} f + \mathrm{Div}(A - B) = 0$. If one then sets: $\Delta x = 1/f \cdot (A - B)$ then one arrives at (11) as a result of this. From this, (7) finally follows by integration: $\Delta I = 0$, and thus the invariance of I under the infinitesimal transformation that is determined by Δx, Δu, where the Δu is to be determined from Δx and $\bar{\delta} u$ by means of (9), and Δx and Δu become linear in the parameters. However, $\Delta I = 0$ implies, in a well-known way, the invariance of I under the finite transformations that arise by integrating the simultaneous system:

$$(17) \qquad \frac{dx_i}{dt} = \Delta x_i, \qquad \frac{du_i}{dt} = \Delta u_i \qquad \left(\begin{matrix} x_i = y \\ u_i = v_i \end{matrix} \text{ for } t = 0 \right).$$

These finite transformations include ρ parameters $a_1, \ldots, a_\rho$, namely, the couplings $t\varepsilon_1, \ldots, t\varepsilon_\rho$. From the assumption that there should be ρ and only ρ linearly independent divergence relations (13), it follows moreover that the finite transformations always define a group, as long as they do not include the derivatives $\partial u / \partial x$. In the opposite case – namely, at least one infinitesimal transformation arises from the Lie bracket process – there would be no linear coupling of the ρ remaining divergence relations, and since I also admits this transformation, there would be more than ρ linearly independent divergence relations, or else this infinitesimal transformation would be of the special form in which $\bar{\delta} u = 0$, Div$(f \cdot \Delta x) = 0$, but then Δx or Δu would depend upon derivatives, contrary to assumption. Whether or not this case can occur when derivatives appear in Δx or Δu must remain undecided. One then adds all functions Δx for which Div$(f \cdot \Delta x) = 0$ to the Δx that was determined above in order to once more preserve the group property. By convention, the parameters that are thus added shall not, however, be counted. The converse is thus proved.

From this converse, it then follows that, in fact, Δx and Δu can be assumed to be linear in the parameters. Namely, if Δu and Δx were of higher degree in ε then, due to the linear independence of the products of powers of ε, entirely analogous relations to (18) would follow, only in a greater number, from which, by the converse, one infers the invariance of I under a group whose infinitesimal transformations include the parameters linearly. Should this group contain precisely ρ parameters, then there would have to exist linear dependencies between the original divergence relations due to the terms of higher order in ε.

Let it be remarked that in the case where Δx and Δu also contain derivatives of the u the finite transformations can depend upon infinitely many derivatives of the u. In this case, the integration of (17) then leads from the determination of $\frac{d^2 x_i}{dt^2}$, $\frac{d^2 u_i}{dt^2}$ to $\Delta\left(\frac{\partial u}{\partial x_\kappa}\right) = \frac{\partial \Delta u}{\partial x_\kappa} - \sum_\kappa \frac{\partial u}{\partial x_\lambda}\frac{\partial \Delta u_\lambda}{\partial x_\kappa}$, such that the number of derivatives of u generally increases at each step. Perhaps the following will serve as an example:

$$f = \tfrac{1}{2} u'^2, \qquad \psi = -u'', \qquad \psi \cdot x = \frac{d}{dx}(u - u'x), \quad \bar{\delta} u = x \cdot \varepsilon,$$

$$\Delta x = \frac{-2u}{u'^{\lambda}}\varepsilon, \qquad \Delta u = \left(x - \frac{2u}{u'}\right)\cdot \varepsilon .$$

Since the Lagrangian expression of a divergence vanishes identically, the converse ultimately shows the following: If I admits a $\mathfrak{G}_\rho$ then any integral that differs from I only by a boundary integral – i.e., an integral of a divergence – will likewise admit a $\mathfrak{G}_\rho$ with the same $\bar{\delta} u$, whose infinitesimal transformation will generally contain derivatives of the u. Thus, perhaps referring to the example above, $f^* = \tfrac{1}{2}\left\{u'^2 - \frac{d}{dx}\left(\frac{u^2}{x}\right)\right\}$ admits the infinitesimal transformation $\Delta u = x\varepsilon$, $\Delta x = 0$, while derivatives of the u appear in the infinitesimal transformations that correspond to f.

If one goes on to the variational problem – i.e., if one sets $\psi_i = 0$ [1]) – then (18) goes to the equations: Div $B^{(\lambda)} = 0$, …, Div $B^{(\rho)} = 0$, which are often referred to as "conservation laws." In the one-dimensional case, it follows from this that $B^{(1)}$ = const., …, $B^{(\rho)}$ = const., and therefore the B contain at most $(2\kappa - 1)^{\text{th}}$ derivatives of the u (from (6)), as long as Δu and Δx include no higher derivatives than κ^{th} that appear in f. Since $2\kappa^{\text{th}}$ derivatives appear in ψ, in general [2]), one thus has the existence of ρ first integrals. The f above once more shows that nonlinear dependencies can exist between them. The linearly independent $\Delta u = \varepsilon_1$, $\Delta x = \varepsilon_2$ correspond to the linearly independent relations: $u'' = \frac{d}{dx}u'$, $u'' \cdot u' = \frac{1}{2}\frac{d}{dx}(u')^2$, while a nonlinear dependency exists between the first integrals u' = const., u'^2 = const. Thus, one is dealing with the elementary case in which Δu, Δx contain no derivatives of the u [3]).

§ 4. Converse in the case of infinite groups.

First, let us show that the assumption of the linearity of Δx and Δu presents no restriction, which one deduces here without the converse from the fact that $\mathfrak{G}_{\infty\rho}$ formally depends upon ρ and only ρ arbitrary functions. Namely, it shows that in the nonlinear case the number of arbitrary functions would increase under the composition of transformations in which the terms of lowest order would add together. In fact, let, say:

$$y = A\left(x, u, \frac{\partial u}{\partial x}, \cdots; p\right) = x + \sum a(x, u, \ldots)\, p^{\nu} + b(x, u, \ldots)\, p^{\nu-1}\frac{\partial p}{\partial x}$$

$$+ c p^{\nu-1}\left(\frac{\partial p}{\partial x}\right)^2 + \ldots + d\left(\frac{\partial p}{\partial x}\right)^{\nu} + \ldots \qquad \left(p^{\nu} = (p^{(1)})^{\nu_1}, \ldots, (p^{(\rho)})^{\nu_\rho}\right),$$

and analogously $v = B\left(x, u, \frac{\partial u}{\partial x}, \cdots; p\right)$, so under composition with $z = A\left(y, v, \frac{\partial v}{\partial y}, \cdots; q\right)$, one gets, for the terms of lowest order:

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[1]) $\psi_i = 0$, or, more generally, $\psi_i = T_i\,\psi_i$, where T_i are new functions that are to be added to the others, are referred to as "field equations" in physics. In the case $\psi_i = T_i$, the identities (13) go to identities: Div $B^{(\lambda)} = \sum T_i \delta u_i^{(\lambda)}$, which are also referred to as conservation laws in physics.

[2]) As long as f is nonlinear in the κ^{th} derivatives.

[3]) Otherwise, one would have u'^{λ} = const. for any λ, corresponding to:

$$u'' \cdot (u')^{\lambda-1} = \frac{1}{\lambda}\frac{d}{dx}(u')^{\lambda}.$$

$$z = x + \sum a(p^\nu + q^\nu) + b\left\{p^{\nu-1}\frac{\partial p}{\partial x} + q^{\nu-1}\frac{\partial q}{\partial x}\right\} + c\left\{p^{\nu-2}\left(\frac{\partial p}{\partial x}\right)^2 + q^{\nu-2}\left(\frac{\partial q}{\partial x}\right)^2\right\} + \dots$$

If a coefficient that is different from a and b here is different from zero then a term $p^{\nu-\sigma}\left(\frac{\partial p}{\partial x}\right)^\sigma + q^{\nu-\sigma}\left(\frac{\partial q}{\partial x}\right)^\sigma$ actually appears for $\sigma > 1$, so one cannot write this as the differential quotient of a single function or products of powers of them; the number of arbitrary functions thus has increased, contrary to assumption. If all of the coefficients that are different from a and b vanish then each of the values of the exponents $\nu_1, \dots, \nu_\rho$ will be the second term of the differential quotient of the first one (as is always the case for, e.g., a $\mathfrak{G}_{\infty 1}$), such that linearity actually enters in, or else the number of arbitrary functions would also increase here. Due to the linearity of the $p(x)$, the infinitesimal transformations thus satisfy a system of linear partial differential equations, and since the group property is fulfilled, they define an "infinite group of infinitesimal transformations," by Lie's definition (Grundlagen, § 10).

One deduces the converse now in a manner that is similar to the one in the case of finite groups. The existence of the dependencies (16) leads, upon multiplication by $p^{(\lambda)}(x)$ and addition, using the identity conversion (14), to $\sum \psi_i \bar{\delta} u_i$ = Div Γ, and from this, as in § 3, one infers the determination of Δx and Δu and the invariance of I under these infinitesimal transformations, which, in fact, depend linearly upon ρ arbitrary functions and their derivatives up to order σ. The fact that these infinitesimal transformations, when they include no derivatives $\partial u / \partial x$, …, certainly define a group follows, as in § 3, from the fact that otherwise more arbitrary functions would appear by composition, while, by assumption, there shall be only ρ dependencies (16); they thus define an "infinite group of infinitesimal transformations." However, such a thing consists (Grundlagen, Theorem VII, pp. 391) of the most general infinitesimal transformations of a certain "infinite group $\mathfrak{G}$ of finite transformations," in Lie's sense. Every finite transformation will then be generated by infinitesimal ones (Grundlagen, § 7) [1]), and thus arise from the integration of the simultaneous system:

$$\frac{dx_i}{dt} = \Delta x_i, \qquad \frac{du_i}{dt} = \Delta u_i, \qquad \begin{pmatrix} x_i = y_i \\ u_i = v \end{pmatrix} \text{for } t = 0,$$

in which, it can, however, be necessary to choose the arbitrary $p(x)$ to be independent of t. *$\mathfrak{G}$ thus depends, in fact, on ρ arbitrary functions;* in particular, if it suffices to choose $p(x)$ to be free of t then this dependency will be analytic in the arbitrary functions $q(x) = t$

[1]) From this, it follows, in particular, that the group $\mathfrak{G}$ that is generated by the infinitesimal transformations Δx, Δu of a $\mathfrak{G}_{\infty\rho}$ again leads back to $\mathfrak{G}_{\infty\rho}$. $\mathfrak{G}_{\infty\rho}$ then includes no infinitesimal transformations that are different from Δx, Δu that depend upon arbitrary functions, and can also contain none that are independent of them that depend upon parameters, since it would then be a mixed group. However, from the above, the finite transformations are determined by means of the infinitesimal ones.

$\cdot\, p(x)$ [1]). If derivatives $\partial u / \partial x, \ldots$ appear then it can be necessary to add infinitesimal transformations $\bar{\delta} u = 0$, $\text{Div}(f \cdot \Delta x) = 0$ before one can reach the same conclusion.

In connection with an example of Lie (Grundlagen, § 7), let a somewhat more general case be given, where one can advance to explicit formulas that likewise show that the derivatives of the arbitrary functions up to order σ appear, from which the converse is then complete. It is the example of those groups of infinitesimal transformations that correspond to the group of all transformations of the x and the transformations of the u that are "induced" by them; i.e., those transformations of the u for which Δu, and consequently u, depend upon only the arbitrary functions that appear in Δx, whereby let it be assumed that the derivatives $\partial u / \partial x, \ldots$ do not appear in Δu. One thus has:

$$\Delta x_i = p^{(i)}(x), \qquad \Delta u_i = \sum_{\lambda=1}^{n} \left\{ a^{(\lambda)}(x,u)\, p^{(\lambda)} + b^{(\lambda)} \frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c^{(\lambda)} \frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma} \right\}.$$

Since the infinitesimal transformation $\Delta x = p(x)$ generates any transformation $x = y + g(y)$ with arbitrary $g(y)$, in particular, $p(x)$ can be determined to be independent of t, such that the following one-parameter group will be generated:

$$x_i = y_i + t \cdot g_i(y), \tag{18}$$

which goes to the identity for $t = 0$ and to the desired $x = y + g(y)$ for $t = 1$. In fact, it follows by differentiation of (18) that:

$$\frac{dx_i}{dt} = g_i(y) = p^{(i)}(x, t), \tag{19}$$

where $p(x, t)$ is determined from $g(y)$ by inversion, and conversely, (18) arises from (19) by means of the auxiliary condition that $x_i = y_i$ for $t = 0$, by which, the integral is established uniquely. By means of (18), the x in Δu can be replaced with the "integration constants" y and t; thus, the $g(y)$ appear up to precisely the σ^{th} derivatives when one expresses the $\partial y / \partial x$ in terms of $\partial x / \partial y$ in $\frac{\partial p}{\partial x} = \sum \frac{\partial g}{\partial y_\kappa} \frac{\partial y_\kappa}{\partial x}$, and, in general, replaces $\frac{\partial^\sigma p}{\partial x^\sigma}$ with its values in terms of $\frac{\partial g}{\partial y}, \ldots, \frac{\partial x}{\partial y}, \ldots, \frac{\partial^\sigma x}{\partial y^\sigma}$. For the determination of the u, one then gets the system of equations:

$$\frac{du_i}{dt} = F_i\left(g(y), \frac{\partial g}{\partial y}, \cdots, \frac{\partial^\sigma g}{\partial y^\sigma}, u, t \right) \qquad (u_i = v_i \text{ for } t = 0)$$

[1]) The question of whether this latter case always occurs was posed by Lie in a different formulation (Grundlagen, § 7 and § 13, conclusion).

in which only t and u are variable, but the $g(y)$, … belong to the coefficient domain, such that the integration yields:

$$u_i = v_i + B_i\left(v, g(y), \frac{\partial g}{\partial y}, \cdots, \frac{\partial^\sigma g}{\partial y^\sigma}, t\right)_{t=1},$$

and therefore transformations that *depend upon precisely σ derivatives of the arbitrary functions*. From (18), the identity is included in this for $g(y) = 0$, and the group property follows from the fact that the chosen process produces any transformation $x = y + g(y)$, from which the one that is induced on the u is established uniquely, so the group $\mathfrak{G}$ will be exhausted.

Incidentally, it then follows from the converse that it is no restriction to choose the arbitrary functions to depend upon only the x, but not on the u, $\partial u / \partial x$, … In the latter case, in fact, $\frac{\partial p^{(\lambda)}}{\partial u}$, $\frac{\partial p^{(\lambda)}}{\partial \frac{\partial u}{\partial x}}, \cdots$ enter into the identity transformation (14), as well as into (15), in addition to the $p^{(\lambda)}$. If one now chooses the $p^{(\lambda)}$ to be successively of degree zero, one, … in u, $\partial u / \partial x$, …, with arbitrary functions of x as coefficients, then the dependencies (16) emerge again, but in greater numbers, which, however, from the converse above, lead back to previous case under composition with arbitrary functions that depend upon only x. One likewise shows that the simultaneous appearance of dependencies and divergence relations that are independent of them corresponds to mixed groups [1]).

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[1]) As in § 3, it also follows from the converse here that, along with I, also any integral I^* that differs by a divergence likewise admits an infinite group with the same $\bar{\delta} u$, in which, however, Δx and Δu will generally include derivatives of the u. Einstein has introduced such an integral into the general theory of relativity in order to obtain a simpler statement of the energy theorem; I shall give the infinitesimal transformations that this I^* admits, for which I preserve the notation of Klein's second note precisely. The integral $I = \int\ldots\int K\, dw = \int\ldots\int \mathfrak{K}\, dS$ admits the group of all transformations of the w and the one that it induces on $g_{\mu\nu}$; they correspond to the dependencies ((30), in Klein):

$$\sum \mathfrak{K}_{\mu\nu} g_\tau^{\mu\nu} + 2\sum \frac{\partial g^{\mu\tau}\mathfrak{K}_{\mu\tau}}{\partial w^\sigma} = 0.$$

Now, one has: $I^* = \int\ldots\int \mathfrak{K}^*\, dS$, where $\mathfrak{K}^* = \mathfrak{K} + \text{Div}$, and consequently, one will have: $\mathfrak{K}^*_{\mu\nu} = \mathfrak{K}_{\mu\nu}$, where $\mathfrak{K}^*_{\mu\nu}$, $\mathfrak{K}_{\mu\nu}$ mean the Lagrangian expressions in each case. Therefore, the dependencies that were given are also true for $\mathfrak{K}^*_{\mu\nu}$, and after multiplying by p^τ and adding, one gets by the reverse conversion of the product differentiation:

$$\sum \mathfrak{K}_{\mu\nu}\, p^{\mu\nu} + 2\ \text{Div}\,(\sum g^{\mu\sigma}\mathfrak{K}_{\mu\nu}\ p^\tau) = 0,$$

$$\delta\mathfrak{K}^* + \text{Div}\left(\sum 2g^{\mu\sigma}\mathfrak{K}_{\mu\tau}p^\tau - \frac{\partial \mathfrak{K}^*}{\partial g_\sigma^{\mu\nu}} p^{\mu\nu}\right) = 0.$$

Comparing this with Lie's differential equation: $\delta\mathfrak{K}^* + \text{Div}(\mathfrak{K}^*\,\Delta w) = 0$, it then follows that:

§ 5. Invariance of the individual components of the relations.

If one specializes the group $\mathfrak{G}$ to the simplest case that is ordinarily considered by specifying that one allows no derivatives of the u in the transformations and that the transformed independent variables depend upon only the x, but not the u then one can deduce the invariance of the individual components in the formulas. First of all, this yields, from known reasons, the invariance of $\int\ldots\int(\sum\psi_i\,\delta u_i)\,dx$; thus, one infers the relative invariance of $\sum\psi_i\,\delta u_i$ [1]), where we understand δ to mean any variation. In fact, one has, on the one hand:

$$\delta I = \int\cdots\int \delta f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx = \int\cdots\int \delta f\left(y, v, \frac{\partial v}{\partial y}, \ldots\right) dy,$$

and, on the other hand, for δu, $\delta\frac{\partial u}{\partial x}$, … that vanish on the boundary, due to the linear, homogeneous nature of the transformation of the δu, $\delta\frac{\partial u}{\partial x}$, …, the δv, $\delta\frac{\partial v}{\partial y}$, … also vanish on the boundary, so one has, correspondingly:

$$\int\cdots\int \delta f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx = \int\cdots\int \left(\sum \psi_i(u, \ldots)\,\delta u_i\right) dx,$$

$$\int\cdots\int \delta f\left(y, v, \frac{\partial v}{\partial y}, \ldots\right) dy = \int\cdots\int \left(\sum \psi_i(v, \ldots)\,\delta v_i\right) dy,$$

and it follows that for δu, $\delta\frac{\partial u}{\partial x}$, … that vanish on the boundary:

$$\int\cdots\int \left(\sum \psi_i(u, \ldots)\,\delta u_i\right) dx = \int\cdots\int \left(\sum \psi_i(v, \ldots)\,\delta v_i\right) dy$$

$$= \int\cdots\int \left(\sum \psi_i(v, \ldots)\,\delta v_i\right) \left|\frac{\partial y_i}{\partial x_n}\right| dx.$$

If one expresses y, v, δv in the third integral in terms of x, u, δu and one sets it equal to the first one then one has a relation:

$$\Delta w^{\sigma} = \frac{1}{\mathfrak{K}^*}\left(\sum 2 g^{\mu\sigma}\mathfrak{K}_{\mu\tau} p^{\tau} - \frac{\partial \mathfrak{K}^*}{\partial g^{\mu\nu}_{\sigma}} p^{\mu\nu}\right), \Delta g^{\mu\nu} = p^{\mu\nu} + \sum g^{\mu\nu}_{\sigma}\Delta w^{\sigma}$$

are infinitesimal transformations that I^* admits. These infinitesimal transformations thus depend upon the first and second derivatives of the $g^{\mu\nu}$, and include the arbitrary p up to the first derivatives.

[1]) I.e., under transformation, $\sum\psi_i\,\delta u_i$ takes on a factor, which is always referred to as relative invariance in the algebraic theory of invariants.

$$\int \cdots \int \left(\sum \chi_i (u, \ldots) \delta u_i \right) dx = 0$$

for a *du* that vanishes on the boundary, but is otherwise arbitrary, and, as is known, the vanishing of the integrands for arbitrary δu follows from this; *one thus has the following relation identically in* δu*:*

$$\sum \psi_i (u, \ldots) \, \delta u_i = \left| \frac{\partial y_i}{\partial x_\kappa} \right| \left(\sum \psi_i (v, \ldots) \, \delta v_i \right),$$

which expresses the relative invariance of $\sum \psi_i \, \delta u_i$ *, and consequently, the invariance of* $\int \ldots \int (\sum \psi_i \, \delta u_i)$ [1]).

In order to apply this to the derived divergence relations and the dependencies, one must first confirm that the $\bar{\delta} u$ that is derived from the Δu, Δx actually satisfies the transformation laws for the variation δu, as long as only the parameter (arbitrary functions, resp.) in $\bar{\delta} v$ are determined in a way that corresponds to the way that they are determined for the similar group of infinitesimal transformations in y, v. Let $\mathfrak{T}_q$ denote the transformation that takes x, u to y, v; since $\mathfrak{T}_q$ is an infinitesimal in x, u, the one that is similar to it in y, v is given by $\mathfrak{T} = \mathfrak{T}_q \mathfrak{T}_p \mathfrak{T}_q^{-1}$, where the parameters (arbitrary functions r, resp.) are therefore determined from p and q. One expresses this in formulas as:

$$\mathfrak{T}_p: \quad \xi = x + \Delta x(x, p), \qquad u^* = u + \Delta u(x, u, p),$$

$$\mathfrak{T}_q: \quad y = A(x, q), \qquad v = B(x, u, q),$$

$$\mathfrak{T}_q \mathfrak{T}_p: \quad \eta = A(x + \Delta x(x, p), q), \qquad v^* = B(x + \Delta x(p), u + \Delta u(p), q).$$

From this, one has, however, $\mathfrak{T}_r = \mathfrak{T}_q \mathfrak{T}_p \mathfrak{T}_q^{-1}$, so:

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[1]) This conclusion breaks down when y also depends upon the u, since then $\delta f \left(y, v, \frac{\partial v}{\partial y}, \ldots \right)$ also includes terms like $\sum \frac{\partial f}{\partial y} \delta y$, so the divergence conversion does not lead to the Lagrangian expressions, just as when one allows derivatives of the u; then, in fact, the δv will lead to linear combinations of δu, $\delta \frac{\partial u}{\partial x}$, …, so after a further divergence conversion this will lead to an identity $\int \ldots \int (\sum \chi_i (u, \ldots) \, \delta u_i) \, dx = 0$, such that the Lagrangian expressions once again do not appear on the right-hand side.

The question of whether one can also already conclude the existence of divergence relations from the invariance of $\int \ldots \int (\sum \psi_i \, \delta u_i) \, dx$ is, from the converse, equivalent to the question of whether one can conclude that from the invariance of I under a group that does not necessarily lead to the same Δu, Δx, but still leads to the same $\bar{\delta} u$. In the special case of simple integrals and only first derivatives in f, one can deduce the existence of first integrals from the invariance of the Lagrangian expressions for finite groups (cf., e.g., Engel, Gött. Nachr. (1916), pp. 270.).

$$\eta = y + \Delta y(r) = y + \sum \frac{\partial A(x,q)}{\partial x} \Delta x(p),$$

$$v^* = v + \Delta v(r) = v + \sum \frac{\partial B(x,u,q)}{\partial x} \Delta x(p) + \sum \frac{\partial B(x,u,q)}{\partial u} \Delta u(p)\,.$$

One replaces $x = x + \Delta x$ in this with $\xi - \Delta\xi$, from which, x again goes to x, so Δx vanishes; thus, from the first formula in (20), η also again goes to $y = \eta - \Delta\eta$. If $\Delta u(p)$ goes to $\bar{\delta} u(p)$ then $\Delta v(r)$ also goes to $\bar{\delta} v(r)$, and the second formula in (20) gives:

$$v + \bar{\delta} v(y, v, \ldots, r) = v + \sum \frac{\partial B(x,u,q)}{\partial u} \bar{\delta} u(p)\,,$$

$$\bar{\delta} v(y, v, \ldots, r) = \sum \frac{\partial B}{\partial u_\kappa} \bar{\delta} u_\kappa(x, u, p)\,,$$

such that the transformation formulas for variations are, in fact, therefore fulfilled, as long as $\bar{\delta} v$ *is assumed to depend only on the parameters (arbitrary functions r, resp.)* [1]).

In particular, the relative invariance of $\sum \psi_i \bar{\delta} u_i$ then follows; thus, the relative invariance of Div B also follows, since the divergence relations are also fulfilled in y, v, and furthermore, from (14) and (13), one also has the relative invariance of Div Γ and that of the left-hand side of the dependencies, when composed with the $p^{(\lambda)}$, where the arbitrary $p(x)$ (the parameters, resp.) are always replaced with the r in the transformation formulas. This then yields the relative invariance of Div($B - \Gamma$), and therefore that of a divergence of a non-vanishing system of functions $B - \Gamma$ whose divergence vanishes identically.

From the relative invariance of Div B, one may, in the one-dimensional case and for finite groups, draw a conclusion about the invariance of the first integrals. The parameter transformation that corresponds to the infinitesimal transformation will, from (20), be linear and homogeneous, and due to the invertibility of all transformations, the ε will also be linear and homogeneous in the transformed parameters ε^*. This invertibility certainly remains preserved when one sets $\psi = 0$, since no derivatives of the u enter into (20). By equating the coefficients of the ε^* in:

$$\text{Div } B(x, u, \ldots, \varepsilon) = \frac{dy}{dx} \cdot \text{Div } B(y, v, \ldots, \varepsilon^*),$$

[1]) This again shows that y must be assumed to independent of u, etc., in order for the conclusion to be valid. As an example, let us, perhaps, mention the $\delta g^{\mu\nu}$ and δq_ρ that were given by Klein, which satisfy the transformations for variations, as long as p is subject to a vector transformation.

the $\frac{d}{dy}B^{(\lambda)}(y, v, \ldots)$ will then be linear, homogeneous functions of the $\frac{d}{dx}B^{(\lambda)}(x, u, \ldots)$, such that from $\frac{d}{dx}B^{(\lambda)}(x, u, \ldots) = 0$ or $B^{(\lambda)}(x, u) =$ const. it also follows that: $\frac{d}{dy}B^{(\lambda)}(y, v, \ldots) = 0$ or $B^{(\lambda)}(y, v) =$ const. *The first ρ integrals that correspond to a $\mathfrak{G}_\rho$ thus admit the group in any case,* such that the further integration is also simplified. The simplest example of this is the one in which f is free of x or one u, which corresponds to the transformation $\Delta x = \varepsilon$, $\Delta u = 0$ ($\Delta x = 0$, $\Delta u = \varepsilon$, resp.). One has $\bar{\delta} u = -\varepsilon \frac{du}{dx}$ (ε, resp.), and since B can be derived from f and $\bar{\delta} u$ by differentiation and rational couplings, it is then also free of x (u, resp.) and admits the corresponding groups [1]).

§ 6. An assertion of Hilbert.

From the foregoing, one ultimately finds the proof of an assertion of Hilbert about the connection between the break-down of the proper energy theorem and "general relativity" (Klein's first note, Göttinger Nachr. (1917), answer, first passage), and indeed, in a generalized group-theoretic context.

Let the integral I admit a $\mathfrak{G}_{\infty\rho}$, and let $\mathfrak{G}_\sigma$ be any finite group that arises from specializing the arbitrary functions, so it is a subgroup of $\mathfrak{G}_{\infty\rho}$. The infinite group $\mathfrak{G}_{\infty\rho}$ then corresponds to dependencies (16), and the finite one $\mathfrak{G}_\sigma$, to divergence relations (13), and conversely, it follows from the existence of any sort of divergence relations that I is invariant under a finite group that is identical to $\mathfrak{G}_\sigma$ when and only when the $\bar{\delta} u$ are linear combinations of the ones obtained from $\mathfrak{G}_\sigma$. The invariance under $\mathfrak{G}_\sigma$ can thus lead to no divergence relations that differ from (13). However, since the invariance of I under the infinitesimal transformations Δu, Δx of $\mathfrak{G}_{\infty\rho}$ for arbitrary $p(x)$ follows from the validity of (16), it already follows from this, in particular, that it is invariant under the infinitesimal transformations of $\mathfrak{G}_\sigma$ that arise by specialization, and consequently, under $\mathfrak{G}_\sigma$. The divergence relations $\sum \psi_i^* \bar{\delta} u_i^{(\lambda)} =$ Div $B^{(\lambda)}$ must then must then be consequences of the dependencies (16), which can also be written: $\sum \psi_i a_i^{(\lambda)} =$ Div $\chi^{(\lambda)}$, where the $\chi^{(\lambda)}$ are linear couplings of the Lagrangian expressions and their derivatives. Since the ψ enter into (13), as well as (16), linearly, the divergence relations must then be linear combinations of the dependencies (16), in particular, and the $B^{(\lambda)}$ themselves are

[1]) In the case where the existence of first integrals already follows from the invariance of $\int (\sum \psi_i \, \delta u_i) \, dx$, they do not admit the complete group $\mathfrak{G}_\rho$; e.g., $\int (u'' \, \delta u) \, dx$ admits the infinitesimal transformation: $\Delta x = \varepsilon_2$, $\Delta u = \varepsilon_1 + x\varepsilon_2$, while the first integral $u - u'x =$ const., which corresponds to $\Delta x = 0$, $\Delta u = x\varepsilon_3$, does not admit the other two infinitesimal transformations, since it includes u, as well as x, explicitly. This first integral corresponds simply to infinitesimal transformations of f that include derivatives. One then sees that, in any case, the invariance of $\int\ldots\int (\sum \psi_i \, \delta u_i) \, dx$ is achieved less often than the invariance of I, which responds to a question that was posed in a previous remark.

thus composed linearly from the χ – i.e., from the Lagrangian expressions and their derivatives, and from functions whose divergences vanish identically, like perhaps the $B - \Gamma$ that appeared in the conclusion to § 2, for which Div $(B - \Gamma) = 0$ and the divergence likewise has the invariant property. I will refer to divergence relations for which $B^{()}$ of the given kind can be composed from the Lagrangian expressions and their derivatives as "unreal," and all others as "real."

Conversely, if the divergence relations are linear couplings of the dependencies (16) – hence, "unreal" – then the invariance under $\mathfrak{G}_\sigma$ follows from the invariance under $\mathfrak{G}_{\infty\rho}$; $\mathfrak{G}_\sigma$ becomes a subgroup of $\mathfrak{G}_{\infty\rho}$. *The divergence relations that correspond to a finite group $\mathfrak{G}_\sigma$ will then be unreal when and only when $\mathfrak{G}_\sigma$ is a subgroup of an infinite group that I is invariant under.*

The original Hilbert assertion is obtained from this by specializing the group. Let the term "translation group" mean the finite group:

$$y_i = x_i + \varepsilon_i\,, \qquad v_i(y) = u_i(x),$$

so

$$\Delta x_i = \varepsilon_i, \qquad \Delta u_i = 0, \qquad \bar{\delta} u_i = -\sum_\lambda \frac{\partial u_i}{\partial x_\lambda}\varepsilon_\lambda\,.$$

As is known, invariance under the translation group expresses the idea that the x do not enter into $I = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx$ explicitly. Let the associated n divergence relations:

$$\sum \psi_i \frac{\partial u_i}{\partial x_\lambda} = \text{Div } B^{(\lambda)} \qquad (\lambda = 1, 2, \ldots, n)$$

be referred to as "energy relations," since the "conservation law" Div $B^{(\lambda)} = 0$ that corresponds to the variational problem corresponds to the "energy law," while the $B^{(\lambda)}$ correspond to the "energy components." One then has: *If I admits the translation group then the energy relations become unreal when and only when I is invariant under an infinite group that includes the translation group as a subgroup* [1]).

An example of such an infinite group is given by the group of all transformations of the x, along with those induced transformations of the $u(x)$ in which only derivatives of the arbitrary functions $p(x)$ appear; the translation group then arises by specializing $p^{(i)}(x) = \varepsilon_i$. Therefore, it must remain undecided whether the most general of these groups is therefore already given – along with the groups that arise from altering I by a boundary integral. Induced transformations of the given sort arise perhaps when one subjects the u to the coefficient transformations of a "total differential form;" i.e., a form $\sum a\, d^\lambda x_i + \sum b\, d^{\lambda-1} x_i\, dx_\kappa + \ldots$ that includes higher differentials, in addition to the dx. Special induced transformations for which the $p(x)$ only appear in the first derivatives are given by the

[1]) The energy law in classical mechanics, and likewise in the older "relativity theory" (where $\sum dx^2$ goes to itself), are "unreal," since no infinite groups appear there.

coefficient transformations of ordinary differential forms $\sum c\ dx_{i_1} \dots dx_{i_\lambda}$, and ordinarily one has considered only these.

Another group of the given kind that cannot be a coefficient transformation, due to the appearance of logarithmic terms, is perhaps the following one:

$$y = x + p(x), \qquad v_i = u_i + \ln(1 + p'(x)) = u_i + \ln \frac{dy}{dx},$$

$$\Delta x = p(x), \qquad \Delta u_i = p'(x) \quad {}^{1}), \qquad \bar{\delta} u_i = p'(x) - u_i' p(x).$$

Here, the dependencies (16) become:

$$\sum_i \left(\psi_i u_i' + \frac{d\psi_i}{dx} \right) = 0,$$

while the unreal energy relations become:

$$\sum \left(\psi_i u_i' + \frac{d(\psi_i + \text{const.})}{dx} \right) = 0.$$

The simplest invariant integral for the group is:

$$I = \int \frac{e^{-2u_1}}{u_1' - u_2'} dx.$$

The most general I is determined by integrating Lie's differential equation (11):

$$\bar{\delta} f + \frac{d}{dx}(f \cdot \Delta x) = 0,$$

which goes to:

$$\frac{\partial f}{\partial x} p(x) + \left\{ \sum \frac{\partial f}{\partial u_i} - \frac{\partial f}{\partial u_i'} u_i' + f \right\} p'(x) + \left\{ \sum \frac{\partial f}{\partial u_i''} \right\} p''(x) = 0$$

(identically in $p(x)$, $p'(x)$, $p''(x)$) by substituting the values of Δx and $\bar{\delta} u$, as long as one assumes that f depends upon only first derivatives of the u. This system of equations already possesses solutions that actually include the derivatives for two functions $u(x)$, namely:

$$f = (u_1' - u_2') \Phi \left(u_1 - u_2, \frac{e^{-u_1}}{u_1' - u_2'} \right),$$

[1]) One computes the finite transformations from these infinitesimal ones backwards from the method that was given in the conclusion of § 4.

where Φ means an arbitrary function of the given arguments.

Hilbert expressed his assertion in such a way that the break-down of the proper energy law was a characteristic feature of the "general theory of relativity." In order for this assertion to be literally true, the term "general relativity" must then be further regarded as it usually is, and also extended to the previous groups that depend upon n arbitrary functions [1]).

[1]) With this, the validity is again confirmed of a remark of Klein that the usual terminology "relativity" in physics should be replaced with "invariance under a group." ("Über die geometrischen Grundlagen der Lorentzgruppe," Jber. d. deutsch. Math. Verein. **19** (1910), pp. 287; printed in Phys. Zeit.)