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SUMMARY REPORT

The Cosserat Continuum *)

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Since about 1962, there has been a rapidly increasing number of publications on continuum mechanics that relate to the labors of the COSSERAT brothers that go back more than fifty years.

In order to understand the fate of these almost-forgotten works, one must know the résumés of the COSSERAT brothers [**1**]. EUGÈNE, the younger of the two, was a mathematician from the school of DARBOUX and in 1896, he was the successor to STIELTJES at the University of Toulouse at the age of thirty. At that time, his collaboration with his older brother FRANÇOIS on elasticity theory had already begun. FRANÇOIS, who was likewise highly gifted in mathematics, had attended the École Polytechnique and had the rank of "chef des Ponts et Chaussées" engaged by the eastern France railroad company. Here, one recalls CASTIGLIANO, who had occupied a similar position with the northern Italian railroads. The collaboration of the COSSERAT brothers extended over thirteen years and culminated in the book "Théorie des corps déformables" that was published in 1909 by Hermann in Paris. Already in 1908, EUGÈNE COSSERAT has assumed the leadership role for the Toulouse observatory, and this high office soon caused his mathematical output to die away. After the premature death of his brother FRANÇOIS in the year 1914 he was no longer answerable to elasticity theory. He did not want to face the pain of remembering their years of fruitful collaboration. EUGÈNE COSSERAT died in 1931 at the age of 65.

The first work of the COSSERAT brothers "Sur la théorie de l'élasticité" in the year 1896 [**2**] began with the sentences: "One knows what a powerful instrument the introduction of the moving triad (trièdre mobile) into surface theory was in the hands of RIBACOUR and DARBOUX. Based on the lectures of KOENIGS on kinematics, one recognizes that in the mechanics of rigid bodies the introduction of moving triads was not merely fortunate. We have resolved to extend the use of triads to the study of deformable bodies, and we were led by numerous important questions to results that recently came to us."

In the first chapter of their book, one reads "A deformable line is a continuous one-parameter manifold of triads, a deformable surface is a two-parameter one, and a deformable body is a three-parameter manifold." This illuminates the fact that the

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mathematical continuity that one assumes in such a definition associates each point of the trace with a rigid body. From the standpoint of mechanics, one can already assume that the well-known moments will appear that have been studied since EULER and BERNOULLI for lines and surfaces, and which LORD KELVIN and HELMHOLTZ attempted to rediscover in three-dimensional continua.

Any point of the deformable continuum will be associated with a rigid orthogonal dreibein. The point thus takes on an orientation (polar medium), and it makes sense to speak of the rotation of a point. Any material point of the COSSERAT continuum is an infinitesimal rigid body. Thus, in an elastic COSSERAT continuum, not only the usual stresses appear, but also moment stresses (moments of deformation). They are something completely new, and one should not confuse them with the moments of force stresses.

In Figure 1a, we consider a quadratic piece of a two-dimensional continuum (for the sake of simplicity). The measure of l is sufficiently small that one may regard the change in stresses with position as linear. We would like to consider the conditions of equilibrium. For this, one makes use of the so-called rigidification principle, as a result of which, any excised part of the continuum can be regarded as a rigid body [**3**]. We have the force geometry of a rigid body at our disposal. As a statically-equivalent force system in the cut in question x_2 = const., we choose a unit force in the middle of the sides and a force-couple. From this, we finally define the normal stress σ_{22} that is uniformly distributed over the cut surface and the statically-equivalent moment distribution (moment per unit surface area):

(1) $$m_{23} = \frac{l^2}{12}\frac{\partial \sigma_{22}}{\partial x_1}$$

that corresponds to the force-couple (Fig. 1b). One proceeds in that way with all four cut surfaces. (One refers to the conventional notation in Figure 1c.) We now write down the equilibrium conditions in the usual way. It is characteristic of the moment distribution m_{ik} that it vanishes like l^2 when one passes to the limit $l \to 0$ while the σ_{ik} remain constant. For that reason, the m_{ik} cannot appear in the equilibrium conditions for an infinitesimal element. The analogous consideration in a three-dimensional continuum, in which one also must define the moment distribution of the shear stresses, leads to the same result:

(2a), (2b) $$\partial_\alpha \sigma_{\alpha k} + X_k = 0, \qquad e_{k\alpha\beta}\, \sigma_{\alpha\beta} = 0.$$

These equilibrium conditions are not only necessary, but also sufficient. Then, when equilibrium exists at any infinitesimal element, it also exists for an arbitrary piece of the continuum. We thus confirm that the three equilibrium conditions (2b) for the moments of all forces at the element yield the symmetry of the stress tensor. ($e_{k\alpha\beta}\, \sigma_{\alpha\beta} = 0$ means that the skew-symmetric part of the stress tensor is zero.) It is remarkable that we can present six necessary and sufficient equilibrium conditions for a spatial continuum whose points have three functional degrees of freedom. Things are completely different when we present the equations of motion for the continuum. Here, we have the basic NEWTONIAN laws of mechanics at our disposal, and we have no more than three equations of motion for the material points:

(3) $$\partial_\alpha \sigma_{\alpha k} + X_k = \rho \dot{s}_k .$$

At first, it makes no sense to speak of the rotational motion of a material point. The definition of the impulse moment (of the spin) affords us a certain substitute. We expect a theorem of the form: "The change in the impulse moment per unit time (of an arbitrary volume) is equal to the moment sum of all external forces (volume and outer surface forces)." Thus, the volume integral over the skew-symmetric part of the stress tensor σ_{ik} appears on the right-hand side as an extra term:

(4) $$\frac{d}{dt}\int_V \rho\, e_{k\alpha\beta}\, x_\alpha\, s_\beta\, dV = \int_V e_{k\alpha\beta}\, x_\alpha\, X_\beta\, dV + \int_O e_{k\alpha\beta}\, x_\alpha\, \sigma_{\lambda\beta}\, n_k\, dO - \int_V e_{k\alpha\beta}\, \sigma_{\alpha\beta}\, dV .$$

However, the vanishing of this integral can only be asserted for the case of equilibrium; then, as we have shown, the stress tensor is symmetric. At this point, classical continuum mechanics poses an axiom: "The impulse moment theorem is valid in the aforementioned form: or the equivalent convention: "The stress tensor σ_{ik} is also symmetric under the motion of the continuum." G. HAMEL [**3**] calls this convention the BOLTZMANN axiom. The distinguished physicist and philosopher LUDWIG BOLTZMANN had expressly proved the axiomatic character [1]) of the assertion of the symmetry of the stress tensor in his lectures "Über die Grundprinzipien und Grundgleichungen der Mechanik" at Clark University in 1899 [**4**]. Continuum mechanics with an asymmetric stress tensor can be referred to as non-Boltzmannian mechanics (in analogy to non-Euclidian geometry). However, the COSSERAT continuum is such a theory of mechanics.

The COSSERAT brothers, by their efforts, towered above the literature on continuum mechanics of the last 100 years. However, in order to climb this tower, one must accept unimaginable work. Their mechanics is nonlinear from the outset and their elaborate notation compels the modern reader to find an adequate vector and tensor representation for himself. Nowhere were the arguments that were carried out in full generality clarified by a single example.

Thus, it is not surprising that the work of the COSSERAT brothers remains practically unnoticed, to the extent that the surviving EUGÉNE did not concern himself with that work for the last twenty years of his life. The book "Théorie des corps déformables" has been thoroughly ignored and often untouched in the libraries. I can quickly procure a copy from a used bookstore that the library of the mathematical institute at Göttingen University had discontinued on 14 May 1964. My teachers, GEORG PRANGE of Hannover and MAX WINKELMANN of Jena, introduced me to the significance of the COSSERAT work thirty years ago. Around 1909 – the year of his Habilitation – WINKELMANN was an assistant to KARL HEUN of Karlsruhe (and, as such, a follower of GEORG HAMEL). HEUN had grasped the COSSERAT ideas immediately and made them the subject of a seminar. The encyclopedia article by HEUN [**5**] also paid tribute to the work of the COSSERAT brothers. Since about 1953, my colleague GÜNTHER and I have discussed stress functions and dislocation theory and

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[1]) Cf., on this, the footnote on pp. 546 of the article by C. TRUESDELL and R. TOUPIN: "The classical fields theories," *Handbuch der Physik*, Bd. III/1. Berlin/Göttingen/Heidelberg, 1960. "The German literature persists in attributing credit to BOLTZMANN here."

we eventually came to the conclusion that both circles of problems belong to the COSSERAT continuum. The "Théorie des corps déformables" gave us no information in that regard. Thus, GÜNTHER had to develop the statics and linear kinematics of the COSSERAT continuum independently in order to show that the model for dislocation theory is an incompatible COSSERAT continuum [**6**]. On the basis of this work, I could give a static interpretation of the stress functions of the three-dimensional continuum in 1959 [**7**]. I already made an attempt at a linear elasticity theory of the COSSERAT continuum in 1957 and presented it at the Lower Saxony Mechanics Colloquium. It seems appealing to me to develop non-Boltzmannian continuum mechanics by analogy to non-Euclidian geometry and to work out some simple examples of the special properties of this continuum. I dedicated the manuscript of this work [**8**] to W. TOLLMANN in 1960.

In the sequel, we would like to restrict ourselves – in complete contrast to the COSSERAT brothers – to the linear theory of the COSSERAT continuum. Thus, the things that are peculiar to continuum mechanics will not be lost to us. In a linear theory, the translation and rotation of the material points are infinitesimal, and, in particular, the infinitesimal rotation may be represented by a rotation vector $\boldsymbol{\varphi}$. Therefore, any point of the continuum is associated with a translation vector $\mathbf{u}$ and a rotation vector $\boldsymbol{\varphi}$. For GÜNTHER, the deformation state of the continuum will be described by an asymmetric deformation tensor:

(5) $$\gamma_{ik} = \partial_i u_k - e_{ik\alpha} \varphi_\alpha ,$$

and the likewise asymmetric tensor of the curvature (Verkrümmung):

(6) $$\chi_{ik} = \partial_i \varphi_k .$$

(We employ Cartesian coordinates x_1, x_2, x_3 throughout and the abbreviations $\partial/\partial x_i = \partial_i$; e_{ikl} is the alternating tensor of LEVI-CIVITÀ..) The symmetric part of the deformation tensor γ_{ik} is identical with the deformation tensor ε_{ik} of the classical continuum:

(7) $$\gamma_{(ik)} = \frac{1}{2}(\partial_i u_k + \partial_k u_i) = \varepsilon_{ik} .$$

$\boldsymbol{\omega}$ = (1/2) rot $\mathbf{u}$, the vector of the mean rotation in the displacement field, must be distinguished from the vector $\boldsymbol{\varphi}$, which describes the rotation of the position-dependent triad that is bound to the material point. This becomes particularly clear when one considers:

(8) $$\gamma_{[ik]} = \frac{1}{2}(\partial_i u_k - \partial_k u_i) - \varepsilon_{ik\,\alpha} \varphi_\alpha ,$$

which is the anti-symmetric part of the deformation tensor γ_{ik}. In vectorial notation:

(9) $$\gamma^A = \frac{1}{2} \text{ rot } \mathbf{u} - \boldsymbol{\varphi} = \boldsymbol{\omega} - \boldsymbol{\varphi} .$$

It describes the relative rotation of the position-dependent triad compared to the mean rotation of the displacement field.

χ_{ik} must be related to the tensors of twisting (Verdrillung) and curvature, since its three components with equal indices describe torsional deformations and the unequal indices, the curvature deformations. One convinces oneself that the two deformation tensors γ_{ik} and χ_{ik} vanish simultaneously when the continuum displaces like a rigid body.

We now assume that the elastic potential:

(10) $$\Phi = \Phi(\gamma_{ik}, \chi_{ik})$$

depends upon the 18 variables γ_{ik} and χ_{ik} , and define the force stresses σ_{ik} and the moment stresses [2]) μ_{ik} by:

(11) $$\sigma_{ik} = \frac{\partial\Phi}{\partial\gamma_{ik}}, \qquad \mu_{ik} = \frac{\partial\Phi}{\partial\chi_{ik}}.$$

We further define the kinetic energy per unit volume (ρ = density):

(12) $$T = \frac{1}{2}\rho\,\dot{u}_k\dot{u}_k + \frac{1}{2}\rho\,I\,\dot{\varphi}_k\dot{\varphi}_k\ .$$

In this, we have distributed a rotational energy $\rho\,I$ over the infinitesimal volume element of the material point – we assume kinetic isotropy (spherical symmetry) – but not a mass moment of inertia per unit volume (such a thing would go to zero with the square of the linear measure of a volume element), but a quantity that associates the material point with a "proper spin" $\rho\,I\,\dot{\boldsymbol{\varphi}}$ or "spin." We have defined kinetic and potential energy. With the help of HAMILTON's principle we then obtain the equations of motion for translation and rotation of the material point:

(13a), (13b) $$\partial_\alpha\sigma_{\alpha k} + X_k = \rho\,\ddot{u}_k\,, \qquad \partial_\alpha\sigma_{\alpha k} + e_{k\alpha\beta}\,\sigma_{\alpha\beta} + Y_k = \rho\,I\,\dot{\boldsymbol{\varphi}},$$

with the six boundary conditions:

(14a), (14b) $$n_\alpha\,\sigma_{\alpha k} = p_k\,, \quad n_\alpha\,\mu_{\alpha k} = q_k\,.$$

Obviously, not only is classical continuum theory generalized, but also NEWTONIAN mechanics. In detail: We have associated the volume moment Y_k with the volume forces X_k and the outer surface moment q_k with the outer surface force p_k ; the tensor of force stresses σ_{ik} is asymmetric, in general. Naturally, the appearance of the moment stresses μ_{ik} is also novel, which are, as we will show, completely distinct from the moment distribution m_{ik} that we considered above. We shall now once more formulate the impulse-momentum theorem (relative to the origin of our Cartesian coordinate system) by substituting the integrand in the volume integral on the right-hand side of (4)

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[2]) "Moments de déformation," "couple-stress," or (according to TRUESDELL) "conatus momentorum."

according to (13b), applying GAUSS's theorem, introducing the outer surface forces (14a) and the outer surface moments (14b):

(15) $$\frac{d}{dt}\int_V \rho(I\,\dot{\varphi}_k + e_{k\alpha\beta}x_\alpha \dot{u}_\beta)\,dV = \int_V (Y_k + e_{k\alpha\beta}x_\alpha X_\beta)\,dV + \int_O (q_k + e_{k\alpha\beta}x_\alpha p_\beta)\,dO\,.$$

In words: "The change in impulse-momentum per unit time is equal to the sum of the static moments of all external moments and forces that act on *V* and *O*." In our non-Boltzmannian mechanics, the spin theorem then becomes a simple consequence of the equations of motion (13) and the boundary condition (14).

In the case considered here of linear elasticity theory, the potential energy Φ is a homogeneous function of degree 2 in the γ_{ik} and χ_{ik}, such that one can write:

(16) $$\Phi = \frac{1}{2}\left(\frac{\partial\Phi}{\partial\gamma_{ik}}\gamma_{ik} + \frac{\partial\Phi}{\partial\chi_{ik}}\chi_{ik}\right) = \frac{1}{2}(\sigma_{ik}\,\gamma_{ik} + \mu_{ik}\,\chi_{ik}).$$

Here, the term $\sigma_{ik}\,\gamma_{ik}$ serves a special purpose. Namely, one has, on taking (7), (8), and (9) into account:

(17) $$\sigma_{ik}\,\gamma_{ik} = \sigma_{(ik)}\gamma_{(ik)} + \sigma_{[ik]}\gamma_{[ik]} = \sigma_{(ik)}\,\varepsilon_{ik} + \boldsymbol{\sigma}^A\,\boldsymbol{\gamma}^A\,.$$

The first summand on the right is well-known from classical elasticity. The second one describes the work that the anti-symmetric part of the stress tensor [3]) exerts on the relative rotational deformation:

(18) $$\sigma_{[ik]}\gamma_{[ik]} = \frac{1}{2}(\sigma_{ik} - \sigma_{ki})\left\{\frac{1}{2}(\partial_i u_k - \partial_k u_i) - e_{ik\alpha}\varphi_\alpha\right\} = \boldsymbol{\sigma}^A\left(\frac{1}{2}\operatorname{rot}\mathbf{u} - \boldsymbol{\varphi}\right).$$

One would search in vain through the entire book by the COSSERAT brothers for the linear material law of an elastic continuum. They were not concerned with such trivia. They addressed a much more general question as the main problem of their book, namely, how the integrand of the HAMILTONIAN variational problem (*la densité d'action*) would appear. For a continuum, the action is a scalar functional of geometric, kinematic, and kinetic variables, and the action must remain invariant when one subjects these variables to a transformation of the Euclidian group. The associated group of the infinitesimal transformations has seven parameters, such that the demand of invariance is equivalent to the existence of seven conservation laws for energy, impulse, and rotational impulse. On this main problem of COSSERAT, there is the monograph "L'action euclidienne de déformation et de mouvement," of I. SUDRIA [**9**]. A very beautiful presentation of this circle of problems for the elastic continuum in the nonlinear case was given by TOUPIN [**10**].

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[3]) In SOMMERFELD, *Mechanik der deformierbaren Medien*, 5th ed., pages 66 and 67, one finds a pseudo-proof of the symmetry of the stress tensor in fluids. One will deduce the logical fallacy in it on the basis of (18).

Along with the elastic bodies of one, two, and three dimensions, the COSSERAT brothers treated ideal fluids and the ether theories of McCULLAGH up to KELVIN. Here, one also naturally finds the gyrostatic bodies that KELVIN had imagined in order to conceive of the elastic properties of the light ether, along with the concept of kinetic anisotropy for the explanation of double refraction. Furthermore, it treated the circle of themes: "Étude de l'action euclidienne à distance, de l'action de contrainte et de l'action dissipative." In conclusion, there was a large chapter on the Euclidian action in EULER variables, in which they also sought to that to the work of POINCARÉ and LORENTZ on the dynamics of the electron. One may indeed say that here a truly grandiose quest was undertaken: to present mechanics, optics, and electrodynamics in a unified field theory under the fundamental principle of the Euclidian action. When EINSTEIN and MINKOWSKI soon showed that such a unified field theory is possible only under the LORENTZ group, this did not trivialize the work of the COSSERAT brothers, and all that one could do was to regret that they had not commenced their work from that new viewpoint. Their great work was completely ignored by theoretical physicists. The thought that the variational problem of a physical field theory, when postulated in the conservation laws, must remain invariant under a group of transformations, was first taken up again in 1918 in the works of FELIX KLEIN and EMMY NOETHER.

In the linear elasticity theory of COSSERAT continua, the potential $\Phi(\gamma_{ik}, \chi_{ik})$ is trivially EUCLIDIAN invariant, because the deformations γ_{ik} and χ_{ik} vanish by definition when the continuum moves like a rigid body. Furthermore, Φ is a homogeneous function of degree 2 of the 18 tensor components γ_{ik} and χ_{ik}. The case of general anisotropy was discussed exhaustively in 1964 by KESSEL [**11**]. In the same year, the papers of NEUBER [**12**], MINDLIN [**13**], ERINGEN-SUHUBI [**14**] appeared, which likewise contained the material law (i.e., constitutive equations) for the isotropic, centrally-symmetric case:

(19a), (19b) $$\sigma_{(ik)} = 2G\left[\gamma_{(ik)} + \frac{\nu}{1-2\nu}\delta_{ik}\gamma_{\alpha\alpha}\right], \qquad \sigma_{[ik]} = 2G\,\eta_1\gamma_{[ik]},$$

(20a), (20b) $$\mu_{(ik)} = 2G\frac{L^2}{12}\left[\chi_{(ik)} + \eta_3\delta_{ik}\chi_{\alpha\alpha}\right], \qquad \mu_{[ik]} = 2G\,\frac{L^2}{12}\,\eta_2\,\chi_{[ik]}.$$

One adds four new constants to the two elastic constants of classical elasticity theory. η_1 and η_2 couple the anti-symmetric parts of the two deformation tensors with the anti-symmetric parts of the associated stress tensors. η_1, η_2, and η_3 are dimensionless. By contrast, the new constant L has the dimension of a length. The two-dimensional special case was already treated by H. SCHAEFER [**8**] in 1962.

We now have to express the force and moment stresses in the six equations of motion (13) in terms of the displacement vector **u** and the rotation vector $\boldsymbol{\varphi}$ with the help of the material laws (19) and (20). By restricting to the case of equilibrium and vanishing volume forces and moments, this produces the system of equations:

(21a) $$\mathcal{L}(\mathbf{u}) - 2\,\eta_1\,\boldsymbol{\gamma}^A = 0,$$

(21b) $$(1+\eta_2)\,\Delta\boldsymbol{\gamma}^A + (1-\eta_2+2\eta_3)\,\text{grad div}\,\boldsymbol{\gamma}^A - \frac{48}{L^2}\eta_1\boldsymbol{\gamma}^A - \frac{1+\eta_3}{2}\,\Delta\,\text{rot}\,\mathbf{u} = 0.$$

In this, we have – as is recommended – introduced the relative rotation $\boldsymbol{\gamma}^A$ in place of $\boldsymbol{\varphi}$. We put these two groups of equations for **u** and $\boldsymbol{\gamma}^A$ together with the NAVIER equation for the displacement vector **v** of the classical elastic isotropic continuum:

(22a), (22b) $$\mathcal{L}(\mathbf{u}) \equiv \Delta\mathbf{v} + \frac{1}{1-2\nu}\,\text{grad div}\,\mathbf{v} = 0, \qquad \Delta\,\text{rot}\,\mathbf{v} = 0,$$

One observes the appearance of the operator $\mathcal{L}$ in (21a). Δ is the three-dimensional LAPLACIAN operator in Cartesian coordinates. H. NEUBER [**12**] and R. D. MINDLIN [**15**] gave the general solution of these equations. They generalized the PAPKOWICH-NEUBER Ansatz for the classical continuum to the COSSERAT continuum.

In the classical continuum, rot **v** is harmonic, from (22b) and the fact that $\boldsymbol{\gamma}^A = 0$. One then recognizes that *any equilibrium state in the classical continuum with a displacement vector* **u** *whose components* u_k *possess continuous second derivatives is also a compatible equilibrium state of the Cosserat continuum.*

From this particular solution, one computes the moment stresses μ_{ik} from, in turn, $\boldsymbol{\varphi}$ = (1/2) rot **v**, $\chi_{ik} = \partial_i\,\varphi_k$, and the material law (20). Since $\boldsymbol{\gamma}^A = 0$, the tensor σ_{ik} of force stresses is now symmetric, and one has for the equilibrium of moment stresses:

(23) $$\partial_\alpha\,\mu_{\alpha\beta} = 0.$$

The moments and forces that belong to this particular solution and act on the outer surface preserve the equilibrium in themselves. In the case two-dimensional case, there exists a corresponding particular solution, as was already established in 1962 by H. SCHAEFER [**8**].

The general solution can be found in a relatively simple way with the use of this particular solution. For the vector $\boldsymbol{\gamma}^A$, use will be made of the well-known HELMHOLTZ Ansatz from continuum mechanics:

(24) $$\boldsymbol{\gamma}^A = \text{rot rot}\,\mathbf{H} + \text{grad}\,h.$$

the vector potential **H** and the scalar potential *h* satisfy an oscillation equation with an imaginary frequency:

(25a) $$\frac{L^2}{12}\frac{(1+\eta_1)(1+\eta_2)}{4\eta_1}\,\Delta\mathbf{H} - \mathbf{H} = 0.$$

(25b) $$\frac{L^2}{12}\frac{1+\eta_3}{2\eta_1}\,\Delta h - h = 0.$$

With (24), (21a) has the solution:

(26) $$\mathbf{u} = \mathbf{v} - 2\,\eta_1\,\mathrm{rot}\,\mathbf{H}, \qquad \mathcal{L}(\mathbf{v}) = 0.$$

Equations (25) are highly characteristic of a COSSERAT continuum. According to the tests of J. SCHIVE [**16**], the L for a metallic engineering material has the order of magnitude of 0.1 mm. For $L = 0$, it follows from (25) that $\mathbf{H} = 0$, $h = 0$, and one finds oneself in a classical continuum. Thus, **H** and h satisfy typical boundary layer equations. Only in such boundary layers can the stress tensor σ_{ik} be asymmetric. On the other hand, **H** and h will require six boundary conditions on the boundary surface of the continuum – for instance, one might let the displacements and rotations or the forces and moments be prescribed.

Boundary layers appear when the continuity of the body is disturbed so there is an outer surface and an inner one, where forces or moments act, or where incompatibilities (dislocations) are present. The problem of shear-free bending also has an exact solution in a COSSERAT continuum that belongs to the class of particular solutions that was described above, moreover. The relative rotation of neighboring triads provokes moment stresses that are uniformly distributed over the cross-section. If one replaces the linearly distributed force stresses by a statically-equivalent uniform distribution of force-couples then one clearly recognizes that we are dealing with two moment distributions of completely different characters (Fig. 2). Indeed, both moments are proportional to the distortion, although they remain constant under a reduction of the measure of the COSSERAT moment stress, while the moments of the force stresses σ_{ik} go to zero with the square of the linear measurement. On surface or volume pieces that have linear measurements with the order of magnitude L, the moment stresses have the same order of magnitude as the moments of the force stresses. If the bending distance l of the beam has the order of magnitude $L/10$ then the bending moment will be practically defined by just the moment stresses. The assumption that is based on this that L has the same order of magnitude as the boundary-layer thickness will be confirmed by a series of worked examples. For holes or grooves whose measures have the same order of magnitude as L, the stress concentration deviates markedly from the corresponding one in a classical continuum.

One obtains the special case that was considered by the COSSERAT brothers of the "triédre caché" when the rotation of the position-dependent triad makes the mean rotation of the displacement field equal to:

(27) $$\boldsymbol{\gamma}^A = \frac{1}{2}\,\mathrm{rot}\,\mathbf{u} - \boldsymbol{\varphi} = 0.$$

This kinematic constraint condition is the root of the fact that the anti-symmetric part $\boldsymbol{\sigma}^A$ of the force stress tensor degenerates into a reaction force (in the sense of HAMEL [**3**]). With (27) and (19b), it then follows from (21b):

(28) $$\boldsymbol{\sigma}^A + \frac{GL^2}{48}(1 + \eta_2)\,\Delta\,\mathrm{rot}\,\mathbf{u} = 0,$$

and from (21a):

(29) $$\mathcal{L}(\mathbf{u}) + \frac{L^2}{48}(1 + \eta_2)\,\Delta\,\text{rot rot}\,\mathbf{u} = 0.$$

The displacement state:

(30) $$\mathbf{u} = \mathbf{v}, \qquad \mathcal{L}(\mathbf{u}) = 0, \qquad \Delta\,\text{rot}\,\mathbf{u} = 0$$

is also a specialization of (29) here. In the general solution:

(31) $$\mathbf{u} = \mathbf{v} - \text{rot}\,\boldsymbol{\chi}$$

the vector potential $\boldsymbol{\chi}$ must satisfy the equation:

(32) $$\left[\frac{L^2}{48}(1+\eta_2)\Delta - 1\right]\boldsymbol{\chi} = 0$$

Thus:

(33) $$\boldsymbol{\sigma}^A = G\,\text{rot rot}\,\boldsymbol{\chi}.$$

The first works on this pseudo-COSSERAT continuum of the triédres cachés [**17**, **18**, **19**] were written with no knowledge of the true COSSERAT theory, and the fact that one can fulfill only five boundary conditions on the outer surface in this reduced theory (instead of the six in the true COSSERAT theory) seems confusing at first. A lucid presentation of the elasticity theory of the pseudo-COSSERAT continuum was given later by W. T. KOTTER [**20**]. The assertion of a series of author that $\boldsymbol{\sigma}^A$ must remain undetermined in this theory because $\boldsymbol{\sigma}^A$ does no work is refuted by eq. (33).

For the problem of the stress concentration at the holes and grooves, there are a series of worked examples by MINDLIN and TIERSTEIN [**19**] for the pseudo-COSSERAT continuum and by NEUBER [**21**] for the true COSSERAT continuum. The characteristic length *L* in both theories is also the origin of the dispersion in the propagation of elastic waves (MINDLIN [**13**], PALMOV [**22**], ERINGEN and SUHUBI [**14**], ADOMEIT [**23**]). An experimental confirmation of the calculated effect is still lacking, up to now.

The continuum theory of the dislocations and proper stresses came about in the last fifteen years. For an introduction of this new domain of physics, we recommend the chapter "Plasticity and dislocation" by E. KRÖNER in volume two of the lectures of SOMMERFELD on theoretical physics, *Mechanics of Deformable Media*, 5th ed., 1964. Dislocations are defects in regular crystal structure. Empirically, they are so dense that a continuum theory is sensible. Continuously distributed dislocations can arise from a distortion or twisting of the crystal structure. Along with the classical deformation tensor, the distortion deformation tensor thus plays a role here. In 1958, W. GÜNTHER [**6**] showed that the kinematic model for the dislocation theory is a COSSERAT continuum with incompatible deformations. For given deformations χ_{ik} and γ_{ik}:

(34) $$\varphi_k(P) = \varphi_k(P_0) + \int_{P_0}^{P} \chi_{ik}\,dx_i$$

is a functional of the curve C that links the points P_0 and P. On the same curve C, one has:

(35) $$du_k = (\gamma_{ik} + \chi_{ik})\, dx_i\, .$$

If C is a double-point-free closed curve then:

(36) $$\oint_C \chi_{ik} dx_i = \Delta_C\, \varphi_k$$

gives a rotational jump and:

(37) $$\oint_C (\gamma_{ik} + e_{ik\alpha}\varphi_\alpha)\, dx_i = \Delta_C\, u_k$$

gives a displacement jump. From STOKES's theorem:

(38), (39) $$\overset{1}{\alpha}_{ik} = e_{ik\alpha}\,\partial_\alpha\,\chi_{\beta k}\,, \qquad \overset{2}{\alpha}_{ik} = e_{ik\alpha}\,\partial_\alpha\,\gamma_{\beta k} + \delta_{ik}\,\chi_{\alpha\alpha} - \chi_{ik}\,,$$

are different from each other one a surface that is bounded by C. With GÜNTHER [**6**], we call $\overset{1}{\alpha}_{ik}$ and $\overset{2}{\alpha}_{ik}$ the *incompatibilities*. The vanishing of these tensors is necessary and sufficient for the existence of a single-valued displacement and rotation field in a simply-connected body.

If $\overset{1}{\alpha}_{ik} = 0$ then the distortion deformations χ_{ik} are compatible; one has $\chi_{ik} = \partial_i\, \varphi_k$, and there exists a single-valued rotation field φ_k. Naturally, since $\overset{2}{\alpha}_{ik} \neq 0$, the COSSERAT deformations γ_{ik} are incompatible, as before. Anyway, one can now give it the form:

(40) $$\gamma_{ik} = \beta_{ik} - e_{ik\alpha}\ \varphi_\alpha.$$

Since φ_k drops out of the compatibility condition (39), it can be written:

(41) $$e_{i\alpha\beta}\,\partial_\alpha\,\beta_{\beta k} = \overset{2}{\alpha}_{ik}$$

or:

(42) $$\mathrm{rot}\,\boldsymbol{\beta} = \overset{0}{\boldsymbol{\alpha}}\,.$$

We have thus found one of the basic equations of dislocation theory. The β_{ik} are rotation-free ($\varphi_k = 0$) COSSERAT deformations. One can interpret them as plastic deformations that arise from relative slips in adjacent crystal layers (Fig. 3), rot $\boldsymbol{\beta} = \overset{0}{\boldsymbol{\alpha}} \neq 0$ means the dislocations stay stuck in the volume element. In Fig. 4, we see such remaining dislocations, which are naturally the sources of proper stresses. The first index of $\overset{0}{\alpha}_{ik}$ gives the direction of the "dislocation line" (here, it is the x_3 direction), the second index gives the slip direction, and the absolute value of $\overset{0}{\alpha}_{ik}$ gives the magnitude of the relative slip. One must regard these dislocation lines as continuously distributed. Such dislocations are present in any crystal as lattice defects. If the externally-applied stress is sufficiently large then it is set in motion and thus generates a large number of new dislocations. With this, the crystal becomes plastic.

If one would like to calculate the proper stresses for given incompatibilities then one must next ascertain the incompatible deformations χ_{ik} and γ_{ik} in (38) and (39) by integration. This problem was solved in complete generality by H. SCHAEFER [**24**]. If one is not interested in the moment stresses then one can, by eliminating the distortion tensor χ_{ik}, directly arrive at a second-order differential equation for the symmetric part of the deformation tensor γ_{ik} that has the form of the linearized field equations of EINSTEIN's theory of gravitation:

$$e_{k\alpha\lambda}\, e_{l\beta\mu}\, \partial_\alpha \partial_\beta\, \gamma_{(\lambda\mu)} = \overset{1}{\alpha}_{kl} + e_{l\alpha\lambda}\partial_\alpha \overset{0}{\alpha}_{l\lambda} + \frac{1}{2} e_{kl\alpha}\partial_\alpha \overset{2}{\alpha}_{\lambda\lambda} = \overset{*}{\alpha}_{kl} . \tag{43}$$

As EINSTEIN has shown, by the introduction of the auxiliary condition on allowed divergence:

$$\partial_k \left(\gamma_{(kl)} - \frac{1}{2}\delta_{kl}\gamma_{\alpha\alpha} \right) = 0 \tag{44}$$

the integration problem reduces to the POISSON equation:

$$\Delta\gamma_{(kl)} = \overset{*}{\alpha}_{kl} - \delta_{kl}\, \overset{*}{\alpha}_{\lambda\lambda} . \tag{45}$$

From the $\gamma_{(ik)}$ thus obtained, one then calculates the symmetric part of the stress tensor σ_{ik} with the help of HOOKE's law and the equilibrium conditions of the classical continuum. E. KRÖNER [**25**] has very elegantly abbreviated this integration process by the introduction of stress functions.

Regarding the question of whether or when $\overset{1}{\alpha}_{ik}$ vanishes, dislocation theory seems to be somewhat unclear. If one calls $\overset{2}{\alpha}_{ik}$ the translation dislocation density then one must refer to $\overset{1}{\alpha}_{ik}$ as the rotational dislocation density. Both are analogous to the VOLTERRA distortions of type 1 and 2 in the classical continuum. These 6 distortions have still not been computed for the COSSERAT continuum up to now. On the other hand, the $\overset{1}{\alpha}_{ik}$ may always be determined such that $\overset{*}{\alpha}_{kl}$ – i.e., the "resultant incompatibility" – vanishes. Thus, the externally unloaded continuum is free of "macro-stresses" $\sigma_{(ik)}$, and as long as one sets the relative rotation $\boldsymbol{\gamma}^A$ to zero, the simple connection:

$$\sigma_{ik} = \frac{1}{2}\delta_{ik}\, \overset{2}{\alpha}_{\lambda\lambda} - \overset{2}{\alpha}_{ki} \tag{46}$$

exists between incompatible curvature and dislocation density, which is likewise one of the fundamental equations of dislocation theory. With these achievements, we must let it go at that in the context of our discussion.

A new dislocation theory that is apparently quite distinct from one the that is presented here was announced in 1965 by NOLL (Hdb. d. Physik, Bd. III, 3, 1965, pages 88-92) and propagated by TRUESDELL [**26**, **27**]. The presentation by TRUESDELL and NOLL in the Handbuch der Physik, which should be regarded as quite provisional, seems to me to place elevated demands on the comprehension of the reader.

In case the stresses σ_{ik} and μ_{ik} of the COSSERAT continuum satisfy the equilibrium equations:

(47a), (47b) $$\partial_i \sigma_{ik} = 0, \qquad \partial_i \mu_{ik} + e_{k\alpha\beta}\, \sigma_{\alpha\beta} = 0,$$

following W. GÜNTHER [**6**], there exists the covariant representation for them:

(48a), (48b) $$\sigma_{ik} = -\, e_{k\alpha\beta}\, \partial_\alpha S_{k\beta}, \qquad \mu_{ik} = -\, e_{\alpha\beta i}\, \partial_\alpha F_{\beta k} + S_{ik} - \delta_{ik}\, S_{\alpha\alpha}$$

in terms of the two tensors S_{ik}, F_{ik} with 18 components in all. In general, the representation (48) is not attained when the body in question possesses closed cavities in its interior. However, this case, which was examined in [**24**] and [**28**], shall now be left aside.

In the classical continuum, the moment stresses μ_{ik} are zero and the stress functions S_{ik} may be eliminated from the representation (48). One thus obtains the symmetric stress tensor $\sigma_{(ik)}$, expressed in terms of the second derivatives of the symmetric stress function tensors $F_{(ik)}$:

(49) $$\sigma_{(ik)} = e_{k\alpha\beta}\, e_{k\lambda\mu}\, \partial_\alpha \partial_\lambda\, F_{(\beta\mu)} .$$

This representation, which is degenerate from the standpoint of the COSSERAT continuum, has been known since BELTRAMI [**29**]. The problem of giving the symmetric tensor $F_{(ik)}$ an intuitive static interpretation like the AIRY stress functions of a two-dimensional stress state is certainly raised whenever one ponders the spatial stress functions. However, H. SCHAEFER [**7**] first arrived at the static interpretation in context of the COSSERAT continuum on the basis of the aforementioned representation of GÜNTHER; it is just as simple as that of the AIRY stress functions. One considers the outer surface of the continuum that is loaded with forces and moments as a thin shell (i.e., a crust). The S_{ik} are then the membrane forces and transverse forces, while the F_{ik} are the bending, torsional, and normal moments of this closed shell. One thus also has the boundary conditions for S_{ik} and F_{ik} on the loaded outer surface of the continuum.

In Fig. 5a and and 5b, these static connections are elucidated on a body outer surface x_3 = const. The crust degenerates to a plate (lamina, resp.) here. In Fig. 5a, one reads off the equilibrium conditions for the plate:

(50) $$\begin{cases} \partial_1 S_{21} - \partial_1 S_{11} + \sigma_{31} = 0, \\ \partial_1 S_{22} - \partial_1 S_{12} + \sigma_{32} = 0, \\ \partial_1 F_{23} - \partial_2 F_{13} + S_{11} + S_{22} + \mu_{33} = 0, \end{cases}$$

and in Fig. 5b, the equilibrium conditions for the lamina:

(51)
$$\begin{cases} \partial_1 F_{21} - \partial_1 F_{11} - S_{13} + \mu_{31} = 0, \\ \partial_1 F_{22} - \partial_1 F_{12} - S_{23} + \mu_{32} = 0, \\ \partial_1 S_{23} - \partial_2 S_{13} + \sigma_{33} = 0, \end{cases}$$

and one infers from this that (50) and (51) are included in (48). One finds a thorough discussion of the boundary conditions for curved outer surfaces in G. RIEDER [**30**].

In the elastic continuum, the stress functions S_{ik}, F_{ik} must satisfy compatibility conditions. Meanwhile, their determination for the COSSERAT continuum, which threatens to lead one into inextricable systems of equations, was attained by S. KESSEL [**31**].

Let us make a comment that is peripheral to the stress functions. The scalar product of stress functions times dislocation density:

(52)
$$F_{ik} \overset{1}{\alpha}_{ik} + S_{ik} \overset{2}{\alpha}_{ik}$$

has the interpretation of the energy density of a proper stress state. Here, the stress functions seem to be "impressed forces," while in a continuum with no dislocation density, they degenerate to "reaction forces;" here, they seem to hinder the existence of dislocations. The concepts "impressed force" and "reaction force" will also be employed in the sense of HAMEL [**3**]. One would like to suppose that the still-unknown connection between fluid hypotheses and dislocation densities in plastic bodies must be approached with the help of stress functions.

I shall now speak of the role that is played by the COSSERAT continuum as an engineering model for approximate theories of rods, plates, and shells.

First, I shall cite the COSSERAT brothers (page 5 of their book): "If one lets one or more dimensions of an elastic body become infinitely small then one must consider the so-called thin body (corps mince). This concept was developed in 1828 by POISSON and somewhat later by CAUCHY. Their objective, like that of all others that were later concerned by this very difficult problem of elasticity theory, was to find a transition between the otherwise distinct theories of one, two, and three-dimensional bodies. As is known, a good number of the papers of BARRÉ DE SAINT-VENANT and KIRCHHOFF were concerned with discussions of the investigations of POISSON and CAUCHY. These teachers, as well as their students, have thus misjudged the actual difficulty of the problem. Namely, the difficulty consists in the fact that in general the zero value of the parameters in question is not an ordinary point, as CAUCHY and POISSON had assumed, nor even a pole, but an essential singular point. This important fact justifies the separate consideration of lines, surfaces, and continua in this book.

As an example of the COSSERAT picture of a two-dimensional continuum that is simultaneously an example of an approximate theory, I consider the bending of plates with the formation of transverse forces. Fig. 6 shows the plane plate as a two-dimensional manifold of triads. From Fig. 7, one extracts the equilibrium conditions:

(53)
$$\begin{cases} \partial_1 Q_{13} + \partial_2 Q_{23} + p = 0, \\ \partial_1 M_{11} + \partial_2 M_{21} + Q_{23} = 0, \\ \partial_1 M_{12} + \partial_2 M_{22} - Q_{13} = 0, \end{cases}$$

which correspond to be the equilibrium conditions (47) for the three-dimensional continuum with that indexing and also appeared in BIEZANO-GRAMMEL [**32**], moreover. The direction of the displacement u_3 of the origin of the triad is perpendicular to the plane of the plate, so one can infer the direction of the absolute triad rotations φ_1 and φ_2 from Fig. 8. As the deformations of the continuum, we have called up, on the one hand, the relative triad rotations:

(54)
$$\gamma_{13} = \partial_1 u_3 + \varphi_2 , \qquad \gamma_{23} = \partial_2 u_3 - \varphi_1 ,$$

and, on the other hand, the curvature χ_{ik} by means of the absolute triad rotations:

(55)
$$\chi_{ik} = \partial_i \varphi_k , \qquad (i, k = 1, 2).$$

Let the elastic potential of the isotropic continuum be:

(56)
$$\Phi = G(C_{ijkl}\, \chi_{ij}\, \chi_{kl} + c\, \gamma_{ij}\, \gamma_{kl})$$

with

(57), (58)
$$C_{ijkl} = \frac{h^3}{12}(\alpha_1\, \delta_{ij}\, \delta_{kl} + \alpha_2\, \delta_{ik}\, \delta_{jl} + \alpha_3\, \delta_{il}\, \delta_{jk}), \quad c = \frac{h}{2}\alpha_2 .$$

Since Φ is positive definite, the constants α must satisfy the inequalities:

(59), (60), (61)
$$\alpha_1 + \alpha_2 + \alpha_3 > 0, \qquad \alpha_2 > |\, \alpha_3 \,|, \qquad \alpha_4 > 0.$$

On dimensional grounds, h must be a length. It then follows in the usual way that:

(62)
$$Q_{13} = \frac{\partial \Phi}{\partial \gamma_{13}} = G\, h\, \alpha_4\, \gamma_{13} ,$$

(63)
$$M_{11} = \frac{\partial \Phi}{\partial \chi_{11}} = \frac{Gh^3}{6}[\alpha_4(\chi_{11} + \chi_{22}) + (\alpha_2 + \alpha_3)\chi_{11}],$$

(64)
$$M_{12} = \frac{\partial \Phi}{\partial \chi_{12}} = \frac{Gh^3}{6}(\alpha_4\, \chi_{11} + \alpha_3 \chi_{21}),$$

(65)
$$M_{11} + M_{22} = \frac{Gh^3}{6}(2\alpha_1 + \alpha_2 + \alpha_3)(\chi_{11} + \chi_{22}).$$

The further calculation is unavoidable, but simple: The three equilibrium conditions (53) yield three equations for u_3, φ_1, φ_2 . Their coupling is achieved by the Ansatz:

(66) $$\varphi_1 = \partial_2 W + \partial_1 H, \qquad \varphi_2 = -\partial_1 W + \partial_2 H,$$

and the displacement functions W and H must satisfy the differential equations:

(67), (68) $$\frac{Gh^3}{6}\alpha_2 \Delta\Delta W = p, \qquad \Delta u_3 = \Delta W - \frac{p}{Gh\alpha_4},$$

(69) $$\left[\frac{h^2}{6}(\alpha_1 + \alpha_2 + \alpha_3)\Delta - \alpha_4\right]\Delta H = 0.$$

Here, $\Delta = \partial_1\partial_1 + \partial_2\partial_2$.

The questions remain of finding the magnitudes of the constants α_1 to α_4 and the meaning of h. If we take the position that the theory that is presented here shall correct the established classical theory of thin plates only in regard to the transverse deformations then we may set:

$$h = \text{plate thickness}, \qquad \alpha_2 = \frac{1}{1-\nu}, \qquad \alpha_3 = \frac{-\nu}{1-\nu}.$$

Due to the equality of the torsion moments $M_{11} = -M_{22}$, it follows from (65) that $\alpha_1 = -1/2$. Thus, from (69), one has:

(70) $$\left(\frac{h^2}{12}\Delta - \alpha_4\right)\Delta H = 0.$$

From (62), α_4 subsumes the uniform distribution of the shear stresses over a cross-section of the plate. The gross approximation $\alpha_4 = 1$ brings complete agreement with the theory of HENCKY [**33**], which, although it was concealed and unexpressed, employed the same model, so it reached the goal in a roundabout way. The theory of E. REISSNER [**34**] is generally not associated with the COSSERAT theory at all. However, his results agreed, except for minor deviations, with the (chronologically later) theory of HENCKY.

Eq. (69) for the displacement function H, which – *cum grano salis* [*] – represents the transverse deformations, has the type of the COSSERAT boundary-layer equations: The transverse force deformations are meaningful only in a the boundary zone of width h at the boundary of the plate. Thus, whoever commences a shell theory that writes down six equilibrium equations for the shell element next finds himself in a COSSERAT continuum with its six displacement quantities. In the further development of an engineering shell theory the kinematic constraint equations for a COSSERAT continuum will be imposed when one expresses the three rotations in terms of the derivatives of the three displacements; the position-dependent triad will emerge from the displacement field. Once again, this is the special case of the "triédres cachés" that the COSSERAT brothers studied. The first complete linear shell theory in which the shell was, in turn,

[*] DHD: i.e., with a grain of salt!

regarded as a two-dimensional COSSERAT continuum was presented by W. GÜNTHER in 1961 [**35**].

In concluding my lecture, I would like to briefly mention some other continuum theories that extend the concept of the COSSERAT continuum. Whoever is interested in them might confer the beautiful paper of TOUPIN in the year 1964 [**10**]. In 1958, ERICKSEN and TRUESDELL [**36**] has already considered the COSSERAT continuum as a special case of a continuum with "directors." In a COSSERAT continuum, the triad that is attached to a material point is rigid and orthogonal, and we would like to suggest that this triad is spanned by three mutually orthogonal unit vectors. In the theory of ERICKSEN and TRUESDELL, the length and directions of these three unit vectors – called *directors* – are independent of each other under the deformation of the continuum. They therefore define a homogeneous deformation of the material point – i.e., a *micro-deformation* of the continuum.

It is worth mentioning that recently C. MØLLER associated every point of the space-time continuum with a "Vierbein" (i.e., tetrad) in general relativity theory. Numerous authors have already commented on the formulation of this extended gravitation theory [**37**].

In Fig. 9a, the directors are carried along by the displacement field, while in Fig. 9b they define a position-dependent incompatible deformation. In the year 1964, many papers appeared almost simultaneously on the elasticity theory of such a continuum with micro-structure by the authors MINDLIN, ERINGEN and SUHUBI, GREEN and RIVLIN. In the linear theory, one has the deformations:

(71), (72), (73) $$\varepsilon_{ik} = \frac{1}{2}(\partial_i u_k + \partial_k u_i), \qquad \gamma_{ik} = \partial_i u_k - \psi_{ik}, \qquad \chi_{lik} = \partial_l \psi_{ik}.$$

The division of γ_{ik} into symmetric and anti-symmetric parts yields the relative deformation:

(74) $$\gamma_{(ik)} = \varepsilon_{ik} - \psi_{(ik)}$$

and the relative rotation:

(75) $$\gamma_{[ik]} = \frac{1}{2}(\partial_i u_k - \partial_k u_i) - \psi_{[ik]}.$$

ε_{ik} will be referred to as the *macro-deformations*, while $\gamma_{(ik)}$ and χ_{lik} are the *micro-deformations*. Here, the COSSERAT rotation vector takes the form of the anti-symmetric part of ψ_{ik} . The gradient of the tensor ψ_{ik} defines deformations that do not consist of just twists and curves – as in the COSSERAT continuum. For this reason, along with the moment stresses that known for the COSSERAT continuum, one also obtains couple stresses without lever arms that are called "self-equilibrated hyper-stress." One finds illustrations of these micro-deformations and stresses in MINDLIN [**13**]. GREEN and RIVLIN [**38**] gave still more far-reaching generalizations of the continuum model ("multipolar media"), in which presumably the *ne plus ultra* is attained. The innovation of this theory is the appearance of self-equilibrated stresses. One believes that outer surface stresses on elastic bodies can be regarded in this way. Up to now, not one example of these extensions of the COSSERAT theory has been worked out. For that reason, one must regard such theories with a healthy skepticism. It is common to all of these theories that they deviate from the classical continuum theory noticeably only for boundary layers.

In the last years, I often found the opinion expressed that the revival of the already-forgotten COSSERAT theory is a passing fad. It would please me if I were to succeed in making it clear in this lecture that anyone who reflects upon the foundations of continuum mechanics will eventually find themselves in the realm that the COSSERAT brothers once imagined.

I am obliged to extend my deepest thanks to my colleagues W. GÜNTHER, S. KESSEL, and E. KRÖNER for their continual willingness to afford me their written discussions.

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