SELECTED PAPERS ON

TELEPARALLELISM

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Introduction

1. The unification of gravitation and electromagnetism. One of the articles of faith for many scientists (or at least the ones who regard science as a natural philosophy) is what one might call the principle of "maximal elegance." That is, if natural law is truly worthy of reverence then that would be because in its purest form it must admit a truly elegant statement. Whenever scientists are confronted with a disorganized set of natural phenomena, their natural instinct it to organize them, and ideally one hopes to find some classification scheme that begins with the smallest number of first principles.

For instance, consider the ongoing quest to understand the nature of matter. One of the earliest postulates about matter was the atomic hypothesis of Democritus, who felt that the reduction of matter to smaller pieces had to terminate after a finite number of steps. To him, the ultimate irreducible constituents would be "atoms," which would then essentially "generate" the more complex states of matter by some process of combination. Of course, this concept eventually led to the periodic table of the elements, which once more initially appeared to be a complex and disorganized set of rules for atoms until quantum mechanics managed to find a more fundamental first principle that made things simple again.

Similarly, there was once a time when electricity and magnetism seemed to be as distinct as lightning and lodestones. However, between the efforts of mostly Michael Faraday and James Clerk Maxwell, not only was it shown that electricity and magnetism were only two facets of a more general concept – viz., the electromagnetic field – but that one could also account for optical phenomena in the process by attributing electromagnetic waves to light.

Since there are many other examples of how progress in science has often been associated with reducing the complexity of first principles by unifying disparate natural phenomena, once Einstein had formulated his theory of gravitation as a manifestation of spacetime geometry, he became convinced that the story did not stop there. His final – albeit, unsuccessful – quest in theoretical physics was to unify his theory of gravitation with Maxwell's theory of electromagnetism by finding some more general – most likely, geometric – field on the spacetime manifold and a set of field equations for it that would subsume both the Einstein equations for gravitation and the Maxwell equations for electromagnetism in some limiting approximation. This problem is what we are calling the *Einstein-Maxwell unification problem.*

Here, one must clearly distinguish between the *unification* of two field theories and their mere *concatenation*. In the latter case, all that one has really accomplished is to construct essentially a Cartesian product of the two theories, and in particular, the resulting field equations for the fields that are to be unified say nothing new about either. What one hopes for in a unification of field theories is a general set of field equations for the unified field that contains some sort of coupling between the two fields that was not present in the individual field theories. The best example of this situation is the fact that in Maxwell's theory of electromagnetism, in addition to containing the equations of electrostatics and magnetostatics as special cases, the full set of Maxwell equations contains the far-reaching coupling of the electric and magnetic fields by electromagnetic induction. Consequently, one hopes that if gravitation and electromagnetism are unified in a similar manner then perhaps there might be some sort of induction process whereby electromagnetism (under some unspecified circumstances) might induce a gravitational field and vice versa. The fact that none of the attempts at solving the Einstein-Maxwell unification problem contained such a mechanism was always regarded as a symptom that the unification was still just a glorified concatenation.

Of course, one must objectively accept that there is nothing to say that a unification of two field theories exists, or at least in the form that one is expecting. Thus, one must treat the existence of unification as basically a conjecture that one is testing, and as such, a conjecture that might prove incorrect.

Einstein made many attempts to solve the Einstein-Maxwell unification problem, but one of the common features that they had was the belief that first one had to increase the degrees of freedom in the unified field to something that equaled at least the sum of the other two degrees of freedom. Now, the spacetime metric tensor field g has components $g_{\mu\nu}$ that are symmetric 4×4 real matrices, and therefore represents ten degrees of freedom. By comparison, the electromagnetic field strength 2-form F has components $F_{\mu\nu}$ that are anti-symmetric 4×4 matrices, and therefore represent six degrees of freedom. Thus, the unified field should probably include at least sixteen degrees of freedom.

Since the vector space $M(4; \mathbb{R})$ of 4×4 real matrices is itself sixteen-dimensional, one obvious first place to look for unification would be in the most general elements of $M(4; \mathbb{R})$, or since the components of g are invertible, perhaps just the invertible elements, which then define $GL(n)$. In fact, Einstein considered both possibilities, where the latter case of matrices in $GL(n)$ amounted to his theory of teleparallelism and the former case of more general matrices came later with the Einstein-Schrödinger approach to unification [**1, 2**]. In the latter theory, which is also discussed in Lichnerowicz [**3**], one basically replaces the symmetric, covariant, second-rank tensor field that the metric represents with one that has no specified symmetry, but can then be polarized into a symmetric and an anti-symmetric part.

Some of the earlier attempts to unify electromagnetism and gravitation involved increasing the dimension of the spacetime manifold. Notably, one has the theory of Kaluza [**4**] and Klein [**5**] (see also Lichnerowicz [**3**], who referred to it as the Jordan-Thiry theory), which looked at the Riemannian geometry of a five-dimensional manifold whose extra dimension was often ascribed a circular topology so the overall manifold was either cylindrical, in the sense of a Cartesian product of a circle with spacetime, or more generally, a $U(1)$-principal bundle over spacetime, which would not have to be trivial. The main defects of the Kaluza-Klein centered around the problem of interpreting the extra dimension and the absence of any coupling between the two fields being unified. Thus, in a sense, the theory achieved only the concatenation of gravitation and electromagnetism.

After Einstein and Mayer gave up on teleparallelism, in their 1931 theory [**6**] they returned to the problem of interpreting the fifth dimension. Cartan [**7**] also commented upon the geometric nature of their construction in a posthumously-published note from around 1934.

Another noble attempt to resolve the question of the interpretation of the fifth dimension took the form of projective relativity [**8**], which treated the extra dimension as coming about in the same way that one introduces homogeneous coordinates for projective spaces. This approach has the advantage that it is more in line with Felix Klein's Erlanger program, in which he proposed that geometries should be classified by the group of transformations of space that preserve some basic property, and that the ultimate geometry in that sense would be projective geometry, whose basic property is the incidence of subspaces.

Yet another five-dimensional theory was defined the theory of anholonomic spaces [**9-11**], which took the form of treating the four-dimensional spacetime as something that was defined by a non-integrable field of hyperplanes on a five-dimensional manifold. Thus, the approximation that gets one back to general relativity would be that of assuming the integrability of the differential system so that spacetime would constitute an integral submanifold of that system.

For a historical discussion of the various attempts at solving the Einstein-Maxwell unification problem, one might peruse Vizgin [**12**]. However, one must note that he does not devote much attention to teleparallelism in that discussion.

To return to the case at hand of teleparallelism, since the spacetime metric tensor field has a fundamental geometric significance, Einstein also believed that, ideally, the unified field should as well. In that sense, a global frame field seemed ideal, since it had the right number of degrees of freedom, generated a metric, and seemed to have a fundamental geometric significance. Of course, in 1929, when most of the following papers were published, Stiefel had yet to do his ground-breaking research on the topological aspects of teleparallelism, since that thesis was published in 1935, so Einstein and the others were not considering whether there might be something topologically over-simplistic about postulating the existence of a global frame field, and not just local ones.

Just as Einstein had the wisdom of Riemann, Christoffel, Levi-Civita, Bianchi, and others upon which to base his general theory of relativity, he also had a certain amount of accumulated wisdom that was due to Vitali [**13**], Bortolotti [**14**], Cartan and Schouten [**15**], and others upon which to base his theory of teleparallelism. To what extent he was aware of their work is debatable, since Einstein rarely cited references, even when he was implicitly using them.

Therefore, all that one can do is to follow the sequence of papers that Einstein published on his evolving theory and read the comments of the mathematicians and physicists that were following it in that era. One sees some of the details of the machinations of the theoretical mind in the successive refinements that Einstein made in response to the comments of the mathematicians and the failures of various attempts to formulate the unified field theory that he aspired to.

His first paper in 1928 [**16**] was purely geometric in character, and its primary intent was to introduce some of the fundamental tensor fields that related to the geometry of parallelizable spaces and show how they related to the more familiar context of Riemannian differential geometry. In the second paper [**17**], he then conjectures that the geometry of teleparallelism might serve as the basis for a unified theory of gravitation and electromagnetism.

The first two papers provoked a spate of responses from mathematicians and physicists, mostly throughout the year 1929. Still in 1928, the Austrian-Dutch geometer Roland Weitzenböck [**18**] summarized the mathematical work that had been done on the geometry of parallelizable manifolds and addressed the issue of finding differential invariants on parallelizable manifolds that were invariant under globally-constant Lorentz transformations, which could then be used for the construction of action functionals that were invariant under such transformations, as well. In 1929, the Italian geometer Ettore Bortolotti then commented on the geometric basis for Einstein's theory [**19**], and the Bulgarian physicist Raschko Zaycoff published his first [**20**] in a series of three successive papers (followed by [**23**]) on the physics of the theory. The German physicist and natural philosopher Hans Reichenbach then weighed in with his observations [**21**] on the place of teleparallel geometry as compared to Riemannian geometry and the geometries of more general metric connections. Einstein then took a different approach [**22**], by abandoning the Lagrangian formulation and concentrating on differential identities that would restrict the field equations. Zaycoff responded to that attempt in [**24**] and also began examining the way that one might approach the Dirac equation in the context of teleparallelism in [**25**]. Einstein returned with a Lagrangian formulation [**26**] and summarized the current state of the theory in a paper that was published in Mathematische Annalen [**27**] and was immediately followed by a historical outline of the geometry of teleparallelism by Cartan [**28**]. Zaycoff pursued the formulation of wave mechanics further in [**29**], while Einstein presented the theory to the Institut Henri Poincaré, resulting in a paper [**30**] that largely duplicated the Math. Ann. paper. Finally, Einstein returned in [**31**] to the solution of a problem regarding the compatibility of the field equations, which were over-determined.

This latter topic also defined the basis for an exchange of letters between Einstein and Cartan on absolute parallelism that was published in translation [**32**]. It is interesting that apparently Einstein did not seem to understand Cartan's comments regarding geometry, which is why the exchange drifted into the subject of the degree of determinism of the equations.

It was in 1930 that storm clouds began forming over teleparallelism as a physical theory. First, Einstein and Mayer computed some static solutions to the field equations [**33**] that suggested that the field equations admitted at least one unphysical solution, namely, a static configuration of uncharged, gravitating bodies. Despite that, Einstein published one last note on teleparallelism in [**34**]. This shadow of doubt was further reinforced by the calculations of the Scottish physicist G. C. McVittie [**35**] ([1]), who showed that the axially-symmetric solution of the field equations was inconsistent with the solution that he had obtained from using the equations of the general theory of relativity and did not seem to contain an electromagnetic field.

In 1931, Cartan made some further comments on the theory of teleparallelism in [**36**]. In 1932, the American mathematician and cosmologist H. P. Robertson published a further paper [**37**] on teleparallel spaces that admitted groups of motions as symmetries, but interest in the theory seems to have been otherwise largely disappearing by then.

([1]) Although the McVittie paper did not require translation into English, it is included here for the sake of conceptual completeness.

After all these subsequent decades, the original Einstein-Maxwell unification problem has been complicated by a number of advances in physics. Notably, quantum physics evolved into something that introduced two new forces of nature in the form of the weak and strong interactions of nuclear and elementary particle physics. This expanded the scope of the unification problem to something that might include all four fundamental interactions. Independently of the gravitational theorists, the particle theorists began posing other partial unification problems, the most successful of which was the unification of electromagnetism and the weak interaction into the theory of the electroweak interaction. There was also a Grand Unified Theory that is still evolving, and which proposed to include the strong interaction, along with the electroweak one, but not gravitation. Many then feel that the only hope for unifying gravitation with the other fundamental forces must then come from a "Theory of Everything" that would unify all four. In any event, the success of gauge field theories in bringing about that partial unification led many to suspect that any field unification should probably be defined by a gauge field theory.

Before quantum physics evolved into its obsession with gauge field theories, it first had to make sense of wave mechanics. In particular, the fact that the Dirac equation was regarded as a relativistic wave equation for the free electron or positron added a new aspect to the Einstein-Maxwell problem, since the Maxwell equations were regarded as a *classical* set of wave equations for the photon. Interestingly, to this day, although many researchers have observed that there is a close relationship between the Dirac equations and the Maxwell equations, nonetheless, the problem of finding a quantum wave equation for the photon is still regarded as open. Basically, the photon wave function seems to be easier to describe in momentum space than it is in configuration space. As a result of the success of the Dirac equation, Einstein himself, along with others, such as Zaycoff [**25, 29**], recognized that the Einstein-Maxwell unification problem was probably already incomplete, compared to what one might call the Einstein-Maxwell-Dirac unification problem. Indeed, at one point Einstein speculated that the unification of gravitation and electromagnetism might have to be based upon a more quantum-theoretical conception of electromagnetism.

Something else that changed the nature of the Einstein-Maxwell unification problem was the fairly recent experimental discovery of "gravito-magnetism" by satellite-based measurements. That is, just as a (relatively) moving electric charge generates a magnetic field, a moving mass generates a gravito-magnetic field. Hence, the analogy between Coulomb's law of electrostatics and Newton's law of universal gravitation becomes part of a more general analogy between electromagnetic fields and gravitational fields. This has the effect of suggesting that the Maxwell equations also describe weak-field gravitational phenomena. Hence, one then wonders if it is proper to unify them with a strong-field theory of gravitation, such as general relativity, and if not, then what would the corresponding "strong-field" equations of electromagnetism be that would properly replace Maxwell's. Again, one suspects that one would have to be describing quantum-electromagnetic phenomena, since that is the realm in which strong electric and magnetic fields seem to be unavoidable.

As far as that is concerned, a further obstacle to unifying gravitation and electromagnetism is then the fact that Einstein's theory of gravitation is rooted in a system of partial differential equations for a fundamental field – viz., the spacetime metric – while at no point does quantum electrodynamics ever pose such a system of field equations for the fundamental fields, such as electrons, positrons, and photons. Indeed, since the time of Heisenberg and Pauli, it has only aspired to be a theory of particle *interactions*, and one that usually treats them in the scattering approximation, which makes the initial and final times in the time evolution problem go to minus and plus infinity, respectively. Hence, in order to have any hope of unifying gravitation and quantum electrodynamics, one must first either recast general relativity as a theory of interactions that are treated in the scattering approximation, which seems naïve, considering the successes of the field equations, or find the "field equations" of quantum electrodynamics. By now, the latter problem seems to have been abandoned by most quantum physicists, who generally trivialize the problem as merely "classical" physics, and therefore an inappropriate problem for modern physics, although there is a continuing interest in "effective" field theories, which represent quantum corrections to the classical field theories, and thus give one a strongly-worded hint as to the nature of the quantum version of the theory.

The Einstein-Maxwell unification problem has also been complicated by the fact that mathematics now understands more about the topological nature of the parallelizability of manifolds than it did in 1929. Thus, a final translation of Stiefel's 1935 thesis [**38**] on the topological aspects of teleparallelism is included here to serve as a motivation to revisit the physical theory from that more topologically advanced standpoint. In particular, one suspects that topology can serve as the source of non-vanishing curvature for non-parallelizable manifolds, which suggests that perhaps one might consider essentially "singular teleparallelism" as an extension of the scope of the original theory. We thus include some speculations in that regard in a final section.

Finally, another hint that the Einstein-Maxwell unification is the wrong problem to be posing comes from the more modern theory of "pre-metric electromagnetism." Actually, its roots go back almost as far as the earliest work on the relativistic formulation of electromagnetism, with a paper by Friedrich Kottler [**39**] in which he observed, in effect, that the only place in Maxwell's equations where the presence of the spacetime metric was necessary was in the Hodge * isomorphism that relates to the definition of the codifferential operator on differential forms and that one could achieve the same objective by composing the Poincaré isomorphism that comes from a volume element on spacetime with the electromagnetic constitutive law that relates the electric and magnetic field strengths to the excitations that they induce in a polarizable medium. Cartan made a similar comment (without the part about the constitutive law) in [**40**], and David van Dantzig then expanded on the subject in a series of papers [**41**]. More recently, that approach was taken to defining the foundations of electromagnetism by Friedrich Hehl and Yurii Obukhov [**42**], as well as the author [**43**]. It has long been the view of the latter that the unification of electromagnetism and gravitation is already present implicitly in the sequence of papers that Einstein wrote on his theory of relativity, since he started out examining electromagnetism and ended up talking about gravitation. The connecting link is the fact that the light cones that define the basis for the existence of gravitation first arise in the context of the dispersion law for the propagation of electromagnetic waves. Thus, in a sense, gravity "emerges" from the electromagnetic structure of spacetime when the more general quartic dispersion law degenerates to the square of a quadratic one of Lorentzian type. This might happen, for instance, as one goes from the cloud of vacuum polarization that surrounds a "bare" electron into the space outside of it, assuming that the vacuum polarization is also associated with vacuum birefringence.

Thus, when one considers the early papers on teleparallelism nowadays, one must also consider the possibly inchoate nature of it as a theory. However, the issue of the parallelizability of the spacetime is topologically unavoidable (perhaps not necessarily in the context of the Einstein-Maxwell unification problem), so one suspects that the early discussion of the physics issues would serve as some guide in taking a more modern approach to the same subject.

2. The geometry of parallelizable manifolds. Since the study of parallelizable manifolds has both geometrical and topological aspects, in this section we will discuss the purely geometric aspects, and then treat the topological aspects in a later section. Of course, except for the last translation in this collection, none of the other papers addressed the topology of teleparallelism, so one can regard topology as perhaps something that was conspicuous by its absence all along. One of the classic texts on differential geometry that contains a discussion of the geometry of parallelizable manifolds is Bishop and Crittenden [**44**], although the discussion takes the form of numerous problem sets scattered throughout Chapter 6. For a more general reference on the Cartan approach to differential geometry, the standard reference is Kobayashi and Nomizu [**45**].

a. Parallelizable manifolds. An n-dimensional differentiable manifold M is said to be *parallelizable* if one can define a *global frame field* on it. This would be a set $\{\mathbf{e}_i(x), i = 1, \ldots, n\}$ of n vector fields on M that are linearly independent at each point; thus, they must also be globally non-zero. One can also regard such a global frame field as a global section of the principal fiber bundle $GL(M) \to M$ whose fibers consist of all linear frames at each point and whose structure group is $GL(n)$. Thus, one can treat a global frame field as a differentiable map $\mathbf{e}_i : M \to GL(M)$, $x \mapsto \mathbf{e}_i(x)$ such that $\mathbf{e}_i(x)$ defines a basis for the tangent vector space T_xM for every x.

Yet another way of characterizing the frame $\mathbf{e}_i(x)$ is to say that it represents a linear isomorphism $\mathbf{e} : \mathbb{R}^n \to T_xM$, $v^i \mapsto v^i\,\mathbf{e}_i(x)$. This simply says that if $\mathbf{v} = v^i\,\mathbf{e}_i(x)$ is a tangent vector at x then its *components* with respect to the frame $\mathbf{e}_i(x)$ would be v^i. The inverse isomorphism $\theta^i_x : T_xM \to \mathbb{R}^n$, $\mathbf{v} \mapsto v^i = \theta^i_x(\mathbf{v})$ then defines a *coframe* at x and a global section of the principal fiber bundle $GL^*(M) \to M$ whose fibers consist of all coframes at each point of M; that is, one has a *global coframe field* on M. The fact that the two linear isomorphisms are inverse to each other means that every frame field $\mathbf{e}_i$ has a unique *reciprocal coframe field* θ^i, which is defined by the property:

$$\theta^i(\mathbf{e}_j) = \delta^i_j\,. \tag{2.1}$$

Every differentiable manifold will admit local frame fields and local coframe fields; i.e., the bundles $GL(M) \to M$ and $GL^*(M) \to M$ are locally trivial. Indeed, every coordinate chart $\{U, x^i\}$ will define a *natural* local frame field and coframe field:

$$\mathbf{e}_i = \frac{\partial}{\partial x^i}, \qquad \theta^i = dx^i, \text{ resp.} \tag{2.2}$$

These local frame and coframe fields are characterized by the property that they are *holonomic*:

$$[\mathbf{e}_i, \mathbf{e}_j] = 0, \qquad d\theta^i = 0 \qquad \text{for all } i, j. \tag{2.3}$$

This amounts to an integrability condition on the local frame fields, since the fact that U is diffeomorphic to $\mathbb{R}^n$ means that it will be contractible – *a fortiori*, simply-connected. Thus, the vanishing of all n 1-forms $d\theta^i$ means that they are all exact, so there are n functions x^i on U such that $\theta^i = dx^i$. The fact that the map $x: U \to \mathbb{R}^n$, $p \mapsto x^i(p)$ is a diffeomorphism follows from the fact that the 1-forms θ^i are linearly independent and the use of the inverse function theorem.

Conversely, though, when a local frame field or coframe field is *anholonomic* – viz., $[\mathbf{e}_i, \mathbf{e}_j]$ and $d\theta^i$ are non-vanishing for some i, j – they cannot be directly integrable into the natural frame fields of any coordinate charts. Thus, in a sense, there are "more" local frame fields than coordinate charts.

However, one can say that the local frame fields on U are "integrable" in a more general sense of the word, namely, because both $\mathbf{e}_i$ and ∂_i define a basis for each tangent space, one can express the vectors of $\mathbf{e}_i$ in terms of the vectors ∂_i:

$$\mathbf{e}_i(x) = \partial_j \ \tilde{h}_i^j(x), \tag{2.4}$$

in which $\tilde{h}_i^j(x)$ defines a smooth function $h : U \to GL(n)$, $x \mapsto h_i^j(x)$ that one calls the *transition function* from ∂_i to $\mathbf{e}_i$, and the tilde over the h implies that we are using the inverse matrix, as it will from now on.

This is essentially what the authors of the papers in this collection were using as the definition of the frame field itself.

Dually, one can say that:

$$\theta_x^i = h_j^i(x)\, dx^j, \tag{2.5}$$

One can then get some idea regarding the nature of global parallelizability, since certainly not every manifold admits a global coordinate system (unless it is diffeomorphic to $\mathbb{R}^n$), but just as there are more frame fields than coordinate charts, one also finds that there are more parallelizable manifolds than vector spaces. Some examples of parallelizable manifolds are: every compact, orientable 3-manifold, every Lie group, the products of parallelizable manifolds, and spheres of dimension 0, 1, 3, and 7, but none of the other ones. The last example shows that even homogeneous spaces do not generally have enough "symmetry" to make them parallelizable. We shall return to the topological nature of parallelizability in a later section, since for now we are only concerned with geometry. Thus, the constructions that we shall make either assume that M is parallelizable or that they are only local constructions.

The first thing that one usually wants to know about any 1-form is the nature of its exterior derivative. Since the coframe field θ^i also defines a basis for the vector spaces $\Lambda^2_x M$ of differential 2-forms by way of the set $\{\theta^i \wedge \theta^j, i < j\}$, one can always express its exterior derivatives in terms of that basis ([2]):

$$d_\wedge \theta^i = -\tfrac{1}{2} c^i_{jk}(x)\, \theta^i \wedge \theta^j, \tag{2.6}$$

in which the functions $c^i_{jk}(x)$ are called the *structure functions* of the coframe field θ^i.

The reason for the minus sign is that one also finds that if $\mathbf{e}_i$ represents the reciprocal frame field to θ^i then one has:

$$[\mathbf{e}_i(x), \mathbf{e}_j(x)] = c^k_{ij}(x)\, \mathbf{e}_k(x). \tag{2.7}$$

Since this looks strongly suggestive of the way that one gets the structure constants for any Lie group – or really, its Lie algebra – we point out that the reason for that is that since Lie groups are always parallelizable, one can always define a global frame field on them. The way that one defines such a global frame field is to take a frame at the identity and either left or right translate it to every other point, which then makes the global frame field either left or right invariant, respectively. That invariance manifests itself in the fact that the structure functions become constant functions, and the structure equations (2.6) then become the *Maurer-Cartan equations.*

One useful way of looking at parallelizable manifolds is to regard them as "almost Lie groups," and indeed, according to Singer and Sternberg [**46**], a compact parallelizable manifold is a group manifold iff there exists some global frame field on it whose structure functions are constant, although they only allude to the existence of a proof for that statement. As we shall see, what one does geometrically is to replace left or right translation with parallel translation.

The way that one shows the equivalence of (2.6) and (2.7) is to use the intrinsic formula for the exterior derivative of a 1-form α, namely, if $\mathbf{v}$ and $\mathbf{w}$ are vector fields on M then one has:

$$d_\wedge \alpha(\mathbf{v}, \mathbf{w}) = \mathbf{v}\, \alpha(\mathbf{w}) - \mathbf{w}\, \alpha(\mathbf{v}) - \alpha([\mathbf{v}, \mathbf{w}]). \tag{2.8}$$

One then applies this to the 1-forms θ^i and the vector fields $\mathbf{e}_i$, keeping in mind that $\theta^i(\mathbf{e}_j)$ are constant functions:

$$d_\wedge \theta^i(\mathbf{e}_j, \mathbf{e}_k) = -\,\theta^i([\mathbf{e}_j, \mathbf{e}_k]) = -\,c^i_{jk}\,. \tag{2.9}$$

([2]) Since we shall also have to consider the ordinary differential of the coframe field, we shall use a distinct notation for the exterior derivative.

b. Parallel vector fields. The way that one goes on to do geometry is to say that a vector field **v** on M is *parallel* (with respect to $\mathbf{e}_i$) iff its components v^i are constant functions. More generally, one says that a vector **v** that is tangent to x is *parallel* to a vector **v′** that is tangent to $y \neq x$ iff $v^i = v'^i$. Thus, a vector field is parallel iff all of its vectors are parallel to each other pair-wise.

In particular, one can consider a vector field $\mathbf{v}(x(t))$ along a curve $x(t)$ in M and say that it is parallel iff its component functions are constants. This then allows one to speak of parallel displacement along a curve, although the existence of a global frame field means that the parallelism of tangent vectors at finitely-separated points can be defined in a path-independent manner. Namely, for any pair of points x, y, one can define a linear isomorphism of T_xM with T_yM by taking any tangent vector $\mathbf{v}_x = v^i\,\mathbf{e}_i(x)$ at x to the tangent vector $\mathbf{v}_y = v^i\,\mathbf{e}_i(y)$.

Now that the notion of parallelism along curves has been defined, one can then define *geodesics* to be curves $x(t)$ whose velocity vectors $\dot{x}(t) = dx / dt$ are parallel for every t; thus their components $\dot{x}^i(t)$ with respect to $\mathbf{e}_i(t) = \mathbf{e}_i(x(t))$ will be constant functions of t, which then implies vanishing acceleration, in one sense of the word, namely:

$$\frac{d\dot{x}^i}{dt} = 0. \tag{2.10}$$

Now, any two global frame fields – say, $\mathbf{e}_i$ and $\mathbf{f}_i$ – can be related to each other by a unique transition function:

$$\mathbf{f}_i = g_i^j \mathbf{e}_j\,. \tag{2.11}$$

One might then ask under what conditions the two frame fields define the same notion of parallelism; that is, **v** is parallel with respect to one frame field iff it is parallel with respect to the other. One immediately finds that since **v** is parallel with respect to either frame field iff its components are constant functions, and the transformation of components from one frame field to the other takes v^i (with respect to $\mathbf{e}_i$) to $\tilde{g}^i_j v^j$, the only way that the components can be constant in both cases is if $\tilde{g}^i_j$ (and therefore g^i_j) is a constant function, as well. One thus defines an equivalence class of global frame fields that all define the same notion of parallelism by essentially the "orbit" of any one of them under the global action of the matrices in $GL(n)$. In effect, the equivalence class of frame fields, thus defined, becomes a coset of the group of constant functions from M to $GL(n)$ in the (infinite-dimensional) group of smooth functions from M to $GL(n)$. Note, that our definition of parallelism does not say that the components must all be *equal* in both frames, only that they are all *constant* in both frames.

c. The canonical connections defined by a global frame field. Since we have defined parallel translation, we naturally wish to know what sort of connection facilitates such a process. There are two basic ways to introduce a connection: One can use the fact that parallel vector fields have constant components with respect to the global frame field or one can use the fact that the frame field itself is assumed to be parallel. We shall use the former approach, since that is what Einstein followed. First, we note that geometrically it makes a difference whether one right-multiplies or left-multiplies the frame field by the transition function.

We start with right-multiplication. Suppose we have a vector field:

$$\mathbf{v} = v^i\,\partial_i = \bar{v}^i\mathbf{e}_i\,, \tag{2.12}$$

with:

$$\bar{v}^i = h^i_j v^j\,, \tag{2.13}$$

and we desire that it should be parallel by the aforementioned definition. Now:

$$d\bar{v}^i = dh^i_j\,v^j + h^i_j\,dv^j = h^i_j\,\nabla v^j\,, \tag{2.14}$$

in which we have defined the *covariant differential* of the holonomic components v^i:

$$\nabla v^i = dv^i + \Delta^i_j\,v^j\,, \tag{2.15}$$

with the 1-form of the connection, which takes its values in the Lie algebra $\mathfrak{gl}(n)$, being defined by

$$\Delta^i_j = \tilde{h}^\mu_\kappa\,dh^\kappa_\nu = \Delta^\mu_{\nu\kappa}dx^\kappa \qquad (\Delta^\mu_{\nu\kappa} = \tilde{h}^\mu_\lambda\frac{\partial h^\lambda_\nu}{\partial x^\kappa}). \tag{2.16}$$

It is useful to know that since $d(h^{-1}h) = dI = 0$, one can also say that:

$$\Delta^i_j = -\,d\tilde{h}^i_k\,h^k_j\,. \tag{2.17}$$

Note that if one subjects the frame field $\mathbf{e}_i$ to a globally-constant invertible linear transformation $\tilde{L}^i_j$ then h^i_j would go to $L^i_k h^k_j$, dh^i_j would go to $L^i_k dh^k_j$, and ultimately Δ^i_j would be unaffected by the change of frame. This is, of course, consistent with the notion that such a change of frame should not affect the definition of parallelism.

By contrast, if we had started with $d\mathbf{e}_i$ then the 1-form that resulted from that would have been $h^i_k\,d\tilde{h}^k_j = -\,dh^i_k\,\tilde{h}^k_j$, which takes its values in minus the transpose of the previous matrix. When one makes the frame field orthonormal, which we shall do later, this amounts to the same thing, but, for now, we use the form (2.16).

To say that $\mathbf{v}$ is parallel iff $\bar{v}^i$ is constant is to say that:

$$d\bar{v}^i = 0 \qquad \text{iff} \qquad \nabla v^i = 0. \tag{2.18}$$

If we define the covariant differential of $\mathbf{v}$ in general as:

$$\nabla\mathbf{v} = h^i_j\,\nabla v^j \otimes \partial_i = \nabla\bar{v}^i \otimes \mathbf{e}_i \tag{2.19}$$

then we see from (2.14) that we have effectively defined $\Delta^i_j = 0$ relative to the anholonomic frame field $\mathbf{e}_i$. Thus, it becomes precise to say, as some authors did, that geometry as observed in that frame field is "anholonomic Euclidian."

From the form of the connection 1-form, one can see that it is clearly analogous to the Maurer-Cartan 1-form on a Lie group. In fact, since $h^i_j : M \to GL(n)$, one finds that Δ^i_j is the pull-back of the Maurer-Cartan 1-form on $GL(n)$ by the transition function h^μ_ν. We shall then call Δ^i_j the 1-form of the *teleparallelism connection*. In Bishop and Crittenden [**44**], one finds it referred to as the *direct connection* that is defined by $\mathbf{e}_i$.

Dually, if we start with a covector field $\alpha = \alpha_i\, dx^i = \bar{\alpha}_i\, \theta^i$, $\bar{\alpha}_i = \alpha_j \tilde{h}^j_i$ then we get:

$$d\bar{\alpha}_i = d\alpha_j\, \tilde{h}^j_i + \alpha_j d\tilde{h}^j_i = \nabla\alpha_j\, \tilde{h}^j_i, \tag{2.20}$$

with:

$$\nabla\alpha_i = d\alpha_i - \Delta^j_i \alpha_j, \tag{2.21}$$

in which we have used (2.17).

One can then define the covariant differential of the covector field α accordingly:

$$\nabla\alpha = d\bar{\alpha}_i \otimes \theta^i = \nabla\alpha_i \otimes dx^i \tag{2.22}$$

and get:

$$d\bar{\alpha}_i = 0 \qquad \text{iff} \qquad \nabla\alpha_i = 0. \tag{2.23}$$

We could also look at the differential of the coframe field $\theta^i = h^i_j dx^j$:

$$d\theta^i = dh^i_j \otimes dx^j = -\,\Delta^i_j \otimes \theta^j = -\Delta^i_{kj}\theta^j \otimes \theta^k. \tag{2.24}$$

Here, we left-multiplied dx^i by h^i_j to be consistent with the fact that we right-multiplied ∂_i by $\tilde{h}^i_j$. Now, let us see what happens when we right-multiply dx^i by the transition function. One gets:

$$d\theta^i = dx^j \otimes dh^i_j = -\,\theta^j \otimes \Delta^i_j = -\Delta^i_{jk}\theta^j \otimes \theta^k. \tag{2.25}$$

One then sees that the components of the connection that is defined by right-translation are obtained from the components that one gets from left-translation by permuting the lower indices. The connection that results from right-multiplication is then the *opposite connection*, to use the terminology of [**44**], which one can then define by:

$$\bar{\Delta}^i_j = \bar{\Delta}^i_{jk} dx^k \qquad\qquad (\bar{\Delta}^i_{jk} = \Delta^i_{kj}). \tag{2.26}$$

As discussed in Cartan and Schouten [**15**], when the parallelizable manifold is a Lie group, the direct and opposite connections correspond to the aforementioned fact that one can define a global frame field on any Lie group by either left-translating or righttranslating a chosen frame at the identity to all of the other points. Furthermore, when the manifold is three-dimensional, one is essentially defining the two kinds of parallelism that Clifford had discussed previously in the context of projective geometry.

In the case of geodesics, the defining condition for such a curve $x(t)$ is that its velocity vector field:

$$\mathbf{v}(t) = \mathbf{v}(x(t)) = \bar{v}^i(t)\,\mathbf{e}_i = v^i(t)\,\partial_i \qquad (\bar{v}^i(t) = h^i_j(x(t))v^j(t)) \tag{2.27}$$

must be parallel along the curve. Thus, its components with respect to $\mathbf{e}_i$ must be constant. Since the velocity vector field is defined only along the curve, the partial derivatives that define $d\bar{v}^i$ are undefined, and one must use differentiation with respect to t. One gets:

$$\frac{d\bar{v}^i}{dt} = \frac{dx^k}{dt}\frac{\partial h^i_j}{\partial x^k}v^j + h^i_j\frac{dv^j}{dt} = h^i_j\nabla_{\mathbf{v}}v^j\,, \tag{2.28}$$

with

$$\nabla_{\mathbf{v}}v^i = \frac{dv^i}{dt} + \Delta^i_{(jk)}v^jv^k\,, \tag{2.29}$$

in which the parentheses on the lower indices of $\Delta^i_{(jk)}$ imply that one has symmetrized them. Thus, the curve is a geodesic iff $\bar{v}^i(t)$ is constant for all t iff $\nabla_{\mathbf{v}}v^i$ vanishes, which takes the form of the usual equations for geodesics. As a result, geodesics will look like straight lines with respect to the anholonomic frame field, as well as the integral curves to parallel vector fields.

We emphasize that since only the symmetric part of the connection Δ^i_j will enter into the geodesic equations, they will be indifferent to the torsion of that connection, which arises from the anti-symmetric part. Thus, although torsion is a fundamental aspect of teleparallelism connections, it is not a fundamental aspect of their geodesics.

In the case of a Lie group, there will be two-types of geodesics that correspond to the left and right translation of a group element by a one-parameter subgroup, since every one-parameter subgroup is defined by a tangent vector at the identity and therefore, the velocity vector field of any one-parameter subgroup will be the left or right translate of that tangent vector. Since the global frame field is also defined by translation, the components of the velocity vector field with respect to a right- (left-) invariant frame field will be constant.

One can now use the Cartan structure equations to compute the torsion and curvature 2-forms of the connection 1-form Δ^i_j. First, one starts with their formulation on the total space $GL(M)$, namely:

$$\Lambda^i = d_\wedge\theta^i + \Delta^i_j \wedge \theta^j, \qquad \Omega^i_j = d_\wedge\Delta^i_j + \Delta^i_k \wedge \Delta^k_j\,, \tag{2.30}$$

in which θ^i is the canonical 1-form on $GL(M)$ and Δ^i_j is defined on $GL(M)$, this time.

In order to relate these equations to the forms that they take in the classical papers, one pulls all of the differential forms involved down to M by way of a frame field. Of course, it is simplest to use the (global) anholonomic frame $\mathbf{e}_i$, since, as we pointed out above, the canonical 1-form pulls down to the reciprocal coframe field θ^i to $\mathbf{e}_i$ and the connection 1-form Δ^i_j pulls down to zero. The structure equations then become:

$$\Lambda^i = d_\wedge \theta^i, \qquad \Omega^i_j = 0. \tag{2.31}$$

Thus, one can say that the components of $\Lambda^i = \frac{1}{2} \Lambda^i_{jk} \theta^j \wedge \theta^k$ with respect to the anholonomic frame field are the structure functions of that frame field:

$$\Lambda^i_{jk} = c^i_{jk} . \tag{2.32}$$

Hence, one can see that torsion in this context is purely a manifestation of the almost-Lie-group structure that one gets on a parallelizable manifold. Indeed, when M is a Lie group, and the global frame field is left or right invariant, the torsion equation pulls down to the Maurer-Cartan equations, in which the structure functions c^i_{jk} are the structure constants, which implies that the connection has constant torsion.

The vanishing of curvature for the connection that we have defined is to be expected, since that is also the integrability condition that makes it possible for parallel translation to be path-independent or for parallel vector fields to exist.

One also finds that the *torsion translation vector field* on M that is associated with any two *parallel* vector fields $\mathbf{v}$ and $\mathbf{w}$ is:

$$T(\mathbf{v}, \mathbf{w}) \equiv \Lambda^i (\mathbf{v}, \mathbf{w})\, \mathbf{e}_i = -[\mathbf{v}, \mathbf{w}]. \tag{2.33}$$

This also follows from the intrinsic formula for $d_\wedge\theta^i$:

$$d_\wedge \theta^i(\mathbf{v}, \mathbf{w}) = \mathbf{v}w^i - \mathbf{w}v^i - \theta^i [\mathbf{v}, \mathbf{w}]. \tag{2.34}$$

If one uses (local) natural frame field ∂_μ then the canonical 1-form pulls down to dx^i and the connection form pulls down to (2.16). This makes the torsion take the form:

$$\Lambda^i = \Delta^i_j \wedge dx^j = \tfrac{1}{2}(\Delta^i_{jk} - \Delta^i_{kj})\; dx^j \wedge dx^k . \tag{2.35}$$

Thus, the components of Λ^i with respect to the natural frame field take the form:

$$\Lambda^i_{jk} = \Delta^i_{jk} - \Delta^i_{kj} , \tag{2.36}$$

which is what one finds in the classical papers, up to a factor of 1/2 that depends upon the author. One sees immediately that the torsion of the connection that comes from right-translation being used is minus the torsion of the connection that comes from left-translation. This would tend to justify referring to the former connection referred to as the opposite connection to the latter; i.e.:

$$\bar{\Lambda}^i_{jk} = - \Lambda^i_{jk} . \tag{2.37}$$

One can derive the vanishing of the connection 1-form Δ'^i_j with respect to the global coframe field θ^i by applying the transformation formula for the 1-form Δ^i_j from the natural coframe field to that coframe field:

$$\Delta' = h \, \Delta \, h^{-1} + h \, dh^{-1} = h \, (h^{-1}dh) \, h^{-1} + h \, dh^{-1} = d(hh^{-1}) = 0.$$

As for the curvature 2-form:

$$\Omega^i_j = d_\wedge \Delta^i_j + \Delta^i_k \wedge \Delta^k_j , \tag{2.38}$$

one also gets, by direct computation:

$$\Omega = dh \wedge d\tilde{h} + (dh\tilde{h}) \wedge (dh\tilde{h}) = dh \wedge d\tilde{h} + dh \wedge (\tilde{h} \, dh \, \tilde{h}) = 0. \tag{2.39}$$

The set $\mathcal{GL}(M)$ of all linear connections on $GL(M)$ is not a vector space, since the sum of two connections is not generally another connection, but it is an affine space. Thus, one can always define the difference of two connections, which then becomes a 1-form on $GL(M)$ with values in $\mathfrak{gl}(n)$, but not an actual connection. Hence, the vector space that $\mathcal{GL}(M)$ is modeled on is $\Lambda^1(GL(M)) \otimes \mathfrak{gl}(n)$, which is, of course, infinite-dimensional.

There is a third canonical connection that is defined by the frame field $\mathbf{e}_i$, in addition to the direct and opposite connections. It is what Bishop and Crittenden refer to as the *zero-torsion connection*, since that is its defining property. That is, one defines a difference 1-form C^i_j ([3]) that gives a connection 1-form:

$$\overset{0}{\Delta}{}^i_j = \Delta'^i_j + C^i_j = C^i_j , \tag{2.40}$$

such that the resulting torsion 2-form vanishes:

$$0 = d_\wedge \theta^i + C^i_j \wedge \theta^j , \tag{2.41}$$

which gives the defining equation for C^i_j in the form:

([3]) Our choice of the letter "*C*" for the difference form is based upon the fact that it is often referred to by the physicists who discuss spacetime torsion as the "contortion" tensor field.

$$C^i_j \wedge \theta^j = -\Lambda^j. \tag{2.42}$$

The equations (2.42) are easiest to solve for C^i_j when one introduces a metric, which we will do shortly. In the even that one is dealing with a metric connection, the zero-torsion connection would then become the Levi-Civita connection. One also notes that this implies that the zero-torsion connection will generally have non-zero curvature:

$$\overset{0}{\Omega}{}^i_j = d_\wedge C^i_j + C^i_k \wedge C^k_j . \tag{2.43}$$

From the form of the structure equations (2.31), one is clearly dealing with a situation that is complementary to Riemannian geometry, in which one has zero torsion and non-zero curvature, since one would see non-zero torsion and zero curvature for the teleparallelism connection.

Of course, in order to compare the geometry to Riemannian, one must also introduce a metric on the tangent bundle, which we shall now do.

d. G-structures. A global frame field on a parallelizable manifold M is an example of a *G-structure*, namely, a reduction of the bundle $GL(M) \to M$ of linear frames on M to a bundle whose fibers are submanifolds of the fibers of $GL(M)$, and whose structure group is a subgroup of $GL(n)$. Examples of G-structures then include almost everything that is geometrically important, namely:

1. $G = GL^+(n)$ = matrices with positive determinants. A $GL^+(n)$ structure on M then comes from an orientation on $T(M)$ (if it admits one) and defines a bundle of oriented, linear frames.
2. $G = SL(n)$ = matrices with unit determinants. An $SL(n)$-structure on M then comes from a choice of (unit) volume element and defines a bundle of unit-volume frames.
3. $G = O(p¸ q)$, $SO(p, q)$. An $O(p¸ q)$-structure then comes from a metric of signature type $(p¸ q)$ – i.e., p negative signs and q positive ones – and defines a bundle of orthonormal frames, while an $SO(p¸ q)$-structure then reduces it to oriented, orthonormal frames.
4. $G = e$. Since there is only one element to G, there is only one frame in each fiber of an e-structure on M. This is then the case of a global frame field.

One can also describe symplectic structures, distinguished vector sub-bundles of $T(M)$ – i.e., differential systems – and almost-complex structures as G-structures, but we will have no use for that knowledge in the present discussion. For more details on the subject of G-structures, one might confer [**47-49**], as well as the author's observations on how they might apply to the spacetime manifold [**50**].

Note that we pointed out the fact that such reductions do not always exist. Indeed, it is topology that obstructs some of the reductions, and we will return to this in a later section, but for now, we shall focus on the geometric aspects of admitting the various reductions.

Furthermore, notice that in examples 1-4, we said "a bundle," instead of "the bundle," since, in fact, if one chooses any linear frame at a point of M then the orbit of that frame under the action of G will define the fiber of a bundle that becomes a G-structure. For instance, since $GL(n)$ has two connected components according to whether the determinant of a matrix is positive or negative, there will be two disconnected orbits to any linear frame, and a choice of one or the other is an orientation at that point. Thus, there is something arbitrary about calling a given frame oriented. Similarly, one can call any linear frame orthonormal and thereby define a metric by means of the orbit of that frame under the action of $O(p, q)$.

One should also note that some of the reductions are associated with a *fundamental tensor field* and the others are not. In particular, the reduction to $SL(n)$ is defined by a choice of unit-volume element, and the reduction to $O(p, q)$ is defined by a metric. Generally, the fibers of a G-structure that are defined by a fundamental tensor field will be level submanifolds of the tensor field, when one represents it as a G-equivariant map from $GL(M)$ into a vector space that carries a representation of G.

Now, the further one goes down in a chain of subgroups of $GL(n)$, the more fundamental tensor fields one can define by starting with the final reduction in the chain. When that final reduction is to $G = e$, as it is for parallelizable manifolds, one can then define the fundamental tensor fields that were necessary to reduce that far in terms of the global frame field or its reciprocal coframe field.

In particular, if θ^i is a global coframe field then one can define both a volume element:

$$V = \theta^1 \wedge \ldots \wedge \theta^n \tag{2.44}$$

and a metric (of any signature type) ([4]):

$$g = \eta_{ij}\, \theta^i \theta^j, \qquad n_{ij} = \text{diag}[-1, \ldots, -1, +1, \ldots, +1]. \tag{2.45}$$

What one has done in the former case is to arbitrarily specify that the frame or coframe at each point has unit volume, while in the latter case, one arbitrarily specifies it to be orthonormal of the desired signature type. If one thinks of g as a map from $GL(M)$ to $S(n)$ – viz., the symmetric, invertible $n\times n$ matrices – then an $O(p, q)$-structure is defined by all of the level submanifolds of the matrix η_{ij}. Since the action of $O(p, q)$ on linear frames preserves the metric g, these level submanifolds are clearly orbits of that action.

In the case of an $SL(n)$-structure, since the action of an invertible matrix A on any frame will take V to $\det(A)\, V$, one sees that defining a unit-volume element on M is equivalent to defining a "determinant" function on $GL(M)$. An $SL(n)$-structure is then the level submanifold of 1 under a choice of such a function. The fact that there is more than one possible choice of determinant function is based in the fact that any frame can potentially be described by any invertible matrix with respect to some reference frame.

([4]) We assume that the product of the two covector fields is the symmetrized tensor product, here.

When one starts with a linear connection on $GL(M)$, one can also speak of reducing the connection to a connection on a G-structure $G(M) \to M$ in some cases. Of course, there are conditions that must then be satisfied in order for the reduction to be possible. In order to be a G-connection, the 1-form ω^i_j of a connection must take its values in the Lie algebra $\mathfrak{g}$ and be Ad-equivariant under the action of G. When one has a fundamental tensor field, an equivalent condition to the former one is that the fundamental tensor must be parallel under the connection.

In the case of a volume element, this means the connection Γ^i_j must take its values in the Lie algebra $\mathfrak{sl}(n)$ and be Ad-equivariant under the action of $SL(n)$; hence, one must have:

$$\mathrm{Tr}(\Gamma^i_j) = \Gamma^i_i = 0. \tag{2.46}$$

One sees that this is, in fact, equivalent to the requirement that $\nabla V = 0$, since:

$$\begin{aligned}\nabla V &= \nabla\theta^1 \wedge \ldots \wedge \theta^n + \ldots + \theta^1 \wedge \ldots \wedge \nabla\theta^n \\ &= \Gamma^1_1\,\theta^1 \wedge \ldots \wedge \theta^n + \ldots + \theta^1 \wedge \ldots \wedge \Gamma^n_n\,\theta^n = \mathrm{Tr}(\Gamma^i_j)\; V.\end{aligned}$$

In order for a connection to reduce to $O(p, q)$, one must have that it takes its values in $\mathfrak{so}(p, q)$ and is Ad-equivariant under the action of that group on frames. Thus, one must have:

$$0 = \eta_{ik}\Gamma^k_j + \Gamma^k_i\eta_{kj} = \Gamma_{ij} + \Gamma_{ji}\,. \tag{2.47}$$

This is equivalent to the requirement that ∇g must vanish, since:

$$\nabla g = \eta_{ij}(\nabla\theta^i\theta^j + \theta^i\nabla\theta^j) = (\eta_{kj}\Gamma^k_i + \Gamma^k_j\,\eta_{kj})\theta^i\theta^j = (\Gamma_{ij} + \Gamma_{ji})\theta^i\theta^j. \tag{2.48}$$

If the coframe field θ^i is, by definition, parallel for the connection that it defines then one will have $\nabla\theta^i = 0$, and since ∇ is a derivation, any tensor field that is defined by finite linear combinations with constant coefficients of tensor products of the θ^i will also vanish. In particular:

$$\nabla V = 0, \qquad \nabla g = 0 \tag{2.49}$$

for the teleparallelism connection. Thus, it is, in particular, a metric connection, and for any choice of metric signature type. Of course, in the applications to physics, the metric used is usually the Lorentzian metric of spacetime, with the globally normal hyperbolic signature type of Minkowski space, which will be (+ 1, – 1, –1, –1) for our purposes.

Since any parallelizable manifold can be given a metric, it can also be given a Levi-Civita connection on its bundle of orthonormal frames. Such a connection is characterized uniquely by being a metric connection with vanishing torsion. That is, if the connection 1-form is denoted by $\overset{\circ}{\Delta}{}^i_j$ and its covariant derivative is denoted by $\overset{\circ}{\nabla}$, while its exterior covariant derivative is denoted by $\overset{\circ}{\nabla}_\wedge$ then one must have:

$$\overset{\circ}{\nabla} g = 0, \qquad \overset{\circ}{\nabla}_{\wedge} dx^i = \overset{\circ}{\Delta}{}^i_j \wedge dx^j = 0. \tag{2.50}$$

However, the Levi-Civita connection for a parallelizable manifold, when it is given a metric, does not have to have vanishing curvature:

$$\overset{\circ}{\Omega}{}^i_j = \overset{\circ}{\nabla}_{\wedge} \overset{\circ}{\Delta}{}^i_j = d_{\wedge} \overset{\circ}{\Delta}{}^i_j + \overset{\circ}{\Delta}{}^i_k \wedge \overset{\circ}{\Delta}{}^k_j . \tag{2.51}$$

For instance, a 3-sphere is parallelizable, but the Levi-Civita connection that comes from its metric has constant non-zero sectional curvature.

If g is the metric that is defined by a global frame field on a parallelizable manifold, as above, then the local components of $\overset{\circ}{\Delta}{}^i_j$ with respect to dx^i are, as usual:

$$\overset{\circ}{\Delta}{}^i_{jk} = \tfrac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) . \tag{2.52}$$

Note that this vanishes if dx^i is an orthonormal frame, and with it, the curvature. However, the question of whether there exist orthonormal frame fields that are also natural is one of the "integrability" of the G-structure in question, which is non-trivial. In particular, we now see that a necessary condition for integrability is that it must admit a flat connection.

Since the Levi-Civita connection is unique, one can use it as an "origin" for the affine space $\mathcal{O}(M)$ of all metric connections on $O(M)$ and uniquely characterize any other metric connection Γ^i_j by its difference 1-form:

$$C^i_j = \Delta^i_j - \overset{\circ}{\Delta}{}^i_j \tag{2.53}$$

relative to the Levi-Civita connection.

If the torsion of Δ^i_j is given by $\Lambda^i = \frac{1}{2}\Lambda^i_{jk}\theta^j \wedge \theta^k$, with θ^i orthonormal, and C^i_j = $C^i_{jk}\theta^k$ then the defining equation (2.42) for C^i_j can be solved by lowering the upper indices to the left-most lower index using η_{ij}, which makes:

$$C_{ijk} - C_{ikj} = -\Lambda_{ijk} . \tag{2.54}$$

since C^i_j is now an infinitesimal orthogonal matrix, one also has:

$$C_{kji} = -C_{ijk} . \tag{2.55}$$

Equations (2.54) can then be solved by anti-symmetrizing both sides, and one gets:

$$C_{ijk} = -\tfrac{1}{2}(\Lambda_{ijk} - \Lambda_{jki} + \Lambda_{kij}). \tag{2.56}$$

3. The field equations. Having settled upon a fundamental field, in the form of a global frame field, the next challenge to the solution of the Einstein-Maxwell unification problem was to find a set of field equations that the fundamental field would have to obey that might behave in a manner that was analogous to the way that the Einstein field equations of gravitation related to the Lorentzian metric tensor field. Since there was little physical intuition based in first principles to guide the formulation of the equations, except the ultimate goal that they should duplicate the Einstein and Maxwell systems of equations in some approximation, one gets a closer insight into the evolution of a field theory by examining the papers in chronological order. Of course, if one desires to simply start with the final form of the field equations then it is sufficient to read Einstein's papers in Mathematische Annalen [**27**] or the Annales de l'Institut Henri Poincaré [**30**] in order to get an idea of what Einstein was defining.

One sees that some of the recurring themes in that quest were:

1. The search for an appropriate field Lagrangian that would make the field equations take the form of the Euler-Lagrange equations for a standard variational problem that would be based upon Hamilton's principle.

2. The reduction of the possibilities by imposing symmetry constraints on the field.

3. The need to find identities that would ensure that the resulting system of equations was well-determined, when it tended to be over-determined.

In Einstein's first attempt at a unified field theory [**17**], he proposed a variational formulation that was based upon the field Lagrangian:

$$\mathcal{L}_1 = h\, g^{\mu\nu} \Lambda^{\alpha}_{\mu\beta} \Lambda^{\beta}_{\nu\alpha}\,, \tag{2.57}$$

if the volume element on spacetime is described by $dx^0 \wedge \ldots \wedge dx^3$ and $h = \det\,[h^{\mu}_{\nu}]$. Thus, if one uses the volume element that is defined by θ^{μ} then the factor of h disappears.

He did not pause to specify the resulting field equations in full generality, but simply went on to derive them in the "first approximation," in which the anholonomic frame field $\mathbf{e}_\mu = h^{\nu}_{\mu}\, \partial_\nu$ differs from the holonomic one ∂_μ by only small quantities:

$$h^{\nu}_{\mu} = \delta^{\nu}_{\mu} + k^{\nu}_{\mu}\ . \tag{2.58}$$

The resulting field equations for k^{ν}_{μ} took the form:

$$\frac{\partial^2 k_{\beta\alpha}}{\partial x_\mu^2} - \frac{\partial^2 k_{\mu\alpha}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\alpha\mu}}{\partial x_\mu \partial x_\beta} - \frac{\partial^2 k_{\beta\mu}}{\partial x_\mu \partial x_\alpha} = 0. \tag{2.59}$$

He then converted this system into separate systems for gravitation and electromagnetism by setting:

$$g_{\mu\nu} = \eta_{\mu\nu} + k_{\mu\nu} + k_{\mu\nu}, \qquad 2\phi_\mu = \partial_\nu k^{\nu}_{\mu} - \partial_\mu k^{\nu}_{\nu}\,. \tag{2.60}$$

The resulting field equations in the first approximation then took the form:

$$R_{\mu\nu} = \phi_{\mu,\,\nu} + \phi_{\nu,\,\mu}\,, \qquad \partial_\mu(h\,\phi^\mu) = 0, \quad \Box\phi_\mu = 0. \tag{2.61}$$

In the absence of an electromagnetic field, the first set agrees with the vacuum Einstein field equations for gravitation, while the second and third sets of equations are equivalent to the vacuum Maxwell equations, when they are expressed in terms of an electromagnetic potential 1-form $\phi = \phi_\mu\, dx^\mu$ that is constrained by the Lorentz gauge.

Since Einstein concluded the paper by pointing out that one got similar results by starting with the field Lagrangian:

$$\mathcal{L}_2 = h\, g_{\mu\nu} g^{\alpha\rho} g^{\beta\sigma} \Lambda^\mu_{\alpha\beta} \Lambda^\nu_{\rho\sigma}\,, \tag{2.62}$$

he admitted that one already had a certain degree of ambiguity in the proper foundations for the field theory.

As ,emtioned in the first section, the first two papers by Einstein provoked a number of responses from some of the dignitaries of mathematics and physics of the era. We shall briefly discuss the gist of some of them in regard to their mathematical and physical details.

The paper by Reichenbach [**21**] can be basically summarized in the diagram that he presented that shows how Riemannian geometry is complementary to teleparallel geometry, in that both of them are examples of the more general geometries of metric connections, in that the former geometry has vanishing torsion and non-vanishing curvature, while the opposite is the case for the latter geometry. In the current era it is probably incorrect to identify the general case of metric geometries with both non-vanishing torsion and curvature as Weyl-Eddington geometries, since nowadays that type of geometry is more commonly referred to as *Riemann-Cartan* geometry, while the geometry of Weyl and Eddington refers to the even more general case in which the connection is not metric, either.

The paper by Weitzenböck [**18**] addressed the fundamental issue of finding all possible field Lagrangians that would have the desired symmetry under globally-constant Lorentz transformations of the frame field. He first came up with the reduction theorem that in the absence of the specified symmetry, Lagrangians would have to be functions of θ^μ, $d_\wedge\theta^\mu$, $\nabla_\alpha\, d_\wedge\theta^\mu$, …, $\nabla_{\alpha_1\cdots\alpha_{m-1}}\, d_\wedge\theta^\mu$, and since $\Lambda^\mu = d_\wedge\theta^\mu$, one could also say that they were functions of θ^μ, Λ^μ, $\nabla_\alpha\Lambda^\mu$, $\nabla_{\alpha_1\cdots\alpha_{m-1}}\,\Lambda^\mu$. Upon imposing the invariance constraint, he then showed that acceptable Lagrangians would have to depend upon h, $g_{\mu\nu}$, Λ^μ, $\nabla_\alpha\Lambda^\mu$, $\nabla_{\alpha_1\cdots\alpha_{m-1}}\Lambda^\mu$; in particular, they would not depend upon the frame field, except by way of h and $g_{\mu\nu}$, and their dependence upon h would usually be based upon the fact that the volume element V_4 could be expressed in the form $\theta^0 \wedge \ldots \wedge \theta^3$ or $h\, dx^0 \wedge \ldots \wedge dx^3$.

He defined the *order* of the Lagrangian to mean the highest power of the covariant derivatives of θ^μ that appeared, and asserted that the only zero-order Lagrangian would be a constant, while there would be no Lagrangians of first-order in Λ^μ or $g_{\mu\nu}$ alone. As for second-order Lagrangians, one would have $\mathcal{L}_1$ and $\mathcal{L}_2$, as defined above, along with:

$$\mathcal{L}_3 = h\, g^{\mu\nu}\, \Lambda^{\alpha}_{\mu\alpha}\Lambda^{\beta}_{\nu\beta} = h\, g^{\mu\nu}\, \Phi_\mu\, \Phi_\nu\,. \tag{2.63}$$

One could also define invariants that were based upon just $g_{\mu\nu}$ in the form of the Riemann curvature tensor, Ricci curvature tensor, and scalar curvature that would be defined by its Levi-Civita connection $\Gamma^\mu_\nu = \Gamma^\mu_{\kappa\nu} dx^\kappa$ in the usual way. The way that one relates that connection back to the teleparallelism connection $\Delta^\mu_\nu = \Delta^\mu_{\kappa\nu}\, dx^\kappa$ is by way of:

$$\Gamma^\mu_\nu = \Delta^\mu_\nu + C^\mu_\nu\,, \tag{2.64}$$

in which C^μ_ν can be obtained from (2.56).

In particular, the usual Einstein-Hilbert Lagrangian takes the form:

$$\mathcal{L}_4 = h\, R. \tag{2.65}$$

Finally, one can consider Lagrangians of the form:

$$\mathcal{L}_5 = \nabla_\mu\, \Phi^\mu. \tag{2.66}$$

Zaycoff [**20**] then pointed out that the above five Lagrangians were related by the identity:

$$\mathcal{L}_1 + 2\mathcal{L}_2 - 4\mathcal{L}_3 - 8\mathcal{L}_4 + \mathcal{L}_5 = 0. \tag{2.67}$$

He further suggested that one might consider, more generally, linear combinations of the five with constant coefficients, and indeed subsequent discussion often looked for such a combination that would be "optimal," in some sense. In the rest of his first communication on Einstein's theory, he examined the various field equations that one would get from various Lagrangians in the first approximation.

Einstein [**22**] temporarily abandoned the variational approach to finding field equations and simply looked for identities that related to the basic field and its covariant derivatives that might look like physical conservation laws. He then came up with an over-determined system of equations – viz., 20 equations in 16 unknowns – and thus posed the compatibility problem that he would return to many times, namely finding the four supplementary identities that would restore the determinacy of the system. Zaycoff [**23**] then expanded his own analysis accordingly and claimed to have resolved the compatibility issue, although Einstein made not mention of that fact in later papers; then again, Einstein was notorious for never referring to anyone else's papers to begin with.

Einstein once more returned to the variational formulation in [**26**], in which he used the Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}\mathcal{L}_1 + \tfrac{1}{4}\mathcal{L}_2 - \mathcal{L}_3, \tag{2.68}$$

which had the property that it was the unique combination that made:

$$G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathcal{L}}{\partial \Lambda^\lambda_{\mu\nu}}\right) \tag{2.69}$$

symmetric. The resulting field equations were then:

$$G^{\mu\nu} = 0. \tag{2.70}$$

He then tried:

$$\bar{\mathcal{L}} = \mathcal{L} + \varepsilon_1 \mathcal{L}^* + \varepsilon_2\, \mathcal{L}_3 \qquad (\mathcal{L}^* = \tfrac{1}{2}\mathcal{L}_1 + \tfrac{1}{4}\mathcal{L}_2), \tag{2.71}$$

in which ε_1 and ε_2 are treated as infinitesimal quantities.

In Einstein's Mathematischen Annalen paper [**27**], which is similar in content to the one [**30**] in the Annales de l'Institut Henri Poincaré, the field equations that he settled upon, which are the ones that are generally used to this day, were ([5]):

$$\left.\begin{aligned} 0 = G^{\mu\nu} &\equiv \Lambda^{\nu}_{\underline{\mu}\,\underline{\lambda};\lambda} - \Lambda^{\sigma}_{\underline{\mu}\,\underline{\tau}}\Lambda^{\nu}_{\sigma\tau}, \\ 0 = F^{\mu\nu} &\equiv \Lambda^{\lambda}_{\underline{\mu}\,\underline{\nu};\lambda}. \end{aligned}\right\} \tag{2.72}$$

Since this system is also over-determined, being 22 equations in 16 unknowns, Einstein returned to solve that problem in [**31**], upon advice from Cartan, by finding the appropriate six identities.

However, it was when Einstein and Mayer [**33**], as well as McVittie [**35**], began looking at explicit solutions to the field equations (2.72) that the theory was dealt an essential death blow. Namely, in the Einstein and Mayer paper, they considered two static field configurations:

1. Spatial isotropy, which was seen as modeling the field of a charged, spherical mass.
2. A finite set of isolated uncharged mass points.

Since the last possibility is unphysical when one includes the mutual gravitational attraction of the masses, which would make a static configuration impossible in the absence of compensating forces of repulsion, the fact that the field equations admitted such a solution made them quite suspicious.

McVittie reached similar conclusions that were based upon his treatment of the static, axially-symmetric field. He found that the field equations admitted only a unique solution of that type, not a family of them, and it did not seem to have an electromagnetic aspect to it, nor did it agree as a purely gravitational field with the solution that he had previously found in the context of Einstein's general theory. Since he regarded the latter solution as more definitive, he reached the ultimate conclusion that teleparallelism, as it was formulated at that point in time, appeared to be unsatisfactory.

After that, interest in the theory waned predictably. By 1931, Einstein and Mayer had moved on to a new unified field [**6**] that revisited Kaluza-Klein concepts from a different angle. In that same year, Cartan made some further observations [**36**] on

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([5]) The underbar on an index means that it has been raised.

absolute parallelism and unified field theory that went largely unnoticed. In 1932, the cosmologist H. P. Robertson made a study [**37**] of parallelizable manifolds that admitted groups of transformations as symmetries, which made no mention of the fact that the theory seemed unconvincing at the level of solutions, but emphasized that the study was mostly independent of an actual choice of field equations. If the theory of teleparallelism occasionally attracts attention to this day then it is usually treated as more a "toy model" with some interesting features, more than a definitive statement of spacetime structure.

What we will now consider in the rest of this introduction is the possibility that the flurry of research that was mostly done in 1929 and concluded around 1930 was mathematically "premature," in the sense that the topological question of what sort of differentiable manifolds would actually be parallelizable was not addressed by the mathematicians until Stiefel's landmark thesis in 1935 [**38**]. The question then arises of whether a topologically-enlightened approach to the problem of teleparallelism would have produced a different set of field equations with more satisfactory solutions. In particular, might the singular points of singular frame fields on non-parallelizable manifolds serve as the sources of non-vanishing curvature, which would not exist for the teleparallelism connection on a parallelizable manifold?

Thus, we shall first discuss the current understanding of the topology of parallelizable manifolds and the topological obstructions to parallelizability, then briefly discuss the nature of formulating the Dirac equation on a parallelizable manifold, and then pose some speculations on how singular frame fields on non-parallelizable manifolds might change the basic problem.

4. The topology of parallelizability. As mentioned above, not every differentiable manifold is parallelizable, although all of them are locally parallelizable. In fact, one begins to suspect that global parallelizability is usually hard to come by, since even such homogeneous spaces as 2-spheres will not have that property. Thus, it seems reasonable that one must be dealing with a manifold that is very closely related to a group manifold in order to expect global parallelizability.

a. A parallelizable covering manifold. Just as non-orientable manifolds always have an orientable covering manifold, one finds that non-parallelizable manifolds also always admit a parallelizable covering manifold, in a sense. This is because the total space $GL(M)$ of the bundle $GL(M) \to M$ on any M is itself always a parallelizable manifold. Of course, the reason that one should probably not think of this as a true covering manifold is that, whereas the dimension of the orientable covering manifold is the same as the non-orientable manifold, the dimension of $GL(M)$ is much larger than the dimension of M. That is, the fibers of most covering spaces are discrete, while the fibers of $GL(M)$ are continuous manifolds that behave like group manifolds.

The global frame field on any $GL(M)$ can be defined by the set of all fundamental vector fields $\tilde{\mathbf{E}}_A$ that are associated with a basis $\{\mathbf{E}_A, A = 1, \ldots, n^2\}$ for the Lie algebra $\mathfrak{gl}(n)$ by way of the (right) action of $GL(n)$ on frames and a set of n basic vector fields $\{\mathbf{E}_i, i = 1, \ldots, n\}$ that frame the horizontal subspaces $H_{\mathbf{e}}(GL(M))$ of $T(GL(M))$ that are defined by some choice of linear connection. Thus, the existence of a global frame field on $GL(M)$ is equivalent to the existence of a linear connection.

One can show (cf., e.g., [**44**] or Sternberg [**51**]) that a linear connection always exists, and, in fact, what one does is to start with a locally-finite covering of M by local frame fields, give each of them its teleparallelism connection, and piece the local connections together into a global one by means of partition of unity. Of course, since a partition of unity is not a canonically-defined construction on a manifold, neither is the resulting linear connection. Thus, its main usefulness is in proving the existence of a linear connection. One should notice that the process of piecing together the local connections with vanishing curvature will generally produce a global connection with non-vanishing curvature.

Dually, one can define a global coframe field on $GL(M)$ by combining the n^2 connection 1-forms ω^i_j with the n canonical 1-forms θ^i, $i = 1, \ldots, n$ that are defined on any frame bundle. These have the key property that if $\mathbf{e}_i : U \to GL(M)$ is a local frame field then the 1-forms on U that one gets by pulling θ^i down by way of $\mathbf{e}_i$ define the reciprocal coframe field to $\mathbf{e}_i$. The tangent subspaces that are annihilated by all θ^i are then the vertical subspaces $V_{\mathbf{e}}(GL(M))$, which are tangent to the fibers and thus project to zero, and the ones that are annihilated by all of the ω^i_j are, by definition, the horizontal subspaces $H_{\mathbf{e}}(GL(M))$; one thus has $T = H \oplus V$, which is true for any connection.

One finds that the teleparallelism connection Γ^μ_ν (μ, $\nu = 1, \ldots, n(n + 1)$) – which is referred to as a *Cartan connection* in this case – that is defined by the global coframe field $\{\theta^i, \omega^i_j\}$ actually contains all of the information that was in the connection ω^i_j. In particular, the curvature 2-form of ω^i_j becomes only one component of the torsion 2-form of Γ^μ_ν. This follows from the fact that:

$$[\mathbf{E}_\kappa, \mathbf{E}_\lambda] = -\Lambda^\mu(\mathbf{E}_\kappa, \mathbf{E}_\lambda)\, \mathbf{E}_\mu - \Omega^A(\mathbf{E}_\kappa, \mathbf{E}_\lambda)\, \tilde{\mathbf{E}}_A\,, \tag{3.1}$$

so the structure functions of the global frame field $\{\mathbf{E}_i\,, \tilde{\mathbf{E}}_A\}$ already include both the torsion and curvature of the original connection.

Thus, in a sense, teleparallelism connections can still embody non-vanishing curvature – at least, for a specialized class of manifolds that look like frame bundles.

To some extent, one can think of the horizontal sub-bundle $H(GL(M))$ of $T(GL(M))$ as an "unfolding" of the tangent bundle to M, in that its fibers $H_{\mathbf{e}}$ are all vector spaces of the same dimension as those of $T(M)$ and they project isomorphically onto tangent spaces under the bundle projection, but at each $x \in M$ the horizontal subspaces to $GL_x(M)$ represent a family that is parameterized by the points of an n^2-dimensional manifold, namely, $GL_x(M)$.

b. Obstruction theory and Stiefel-Whitney classes. The last translation in this collection is the doctoral dissertation of the Swiss mathematician Eduard Stiefel on the topological obstructions to the parallelizability of manifolds. Since his advisor at the Swiss Federal Institute of Technology was Heinz Hopf, it is not surprising that the stated objective of that research was to extend the Poincaré-Hopf theorem, which said that a compact manifold M admits a global non-zero vector field iff its Euler-Poincaré characteristic $\chi[M]$ vanishes, to the existence of a set of more than one linearly-independent vector fields. One can call such a set of k linearly-independent global vector fields $\{\mathbf{e}_a, a = 1, \ldots, k\}$ on a differentiable manifold M a *k-frame field* on M, and the maximum value of k for which such a field exists is then its *degree of parallelizability*. Thus, what we are calling parallelizability amounts to saying that the degree of parallelizability is equal to the dimension of the manifold.

What Stiefel defined were a set of $\mathbb{Z}_2$-homology classes that must vanish in order for a differentiable manifold to be parallelizable. Since they were subsequently given a more concise form by Hassler Whitney at Princeton [**52**], they – or rather, their Poincaré-dual $\mathbb{Z}_2$-cohomology classes – are now referred to as the *Stiefel-Whitney classes* of a manifold (really, of its bundle of linear frames).

More generally, one now considers characteristic cohomology classes that represent "obstructions" to the triviality of a principal fiber bundle, and which are then called *obstruction cocyles* [**53-55**]. Their vanishing is a necessary, but not generally sufficient, condition for the triviality of the bundle in question. Since the triviality of a principal fiber bundle is equivalent to the existence of a global section of the bundle, one sees that obstruction cocycles can also be regarded as obstructing the extension of local sections to global ones.

The general picture for obstruction theory, as it relates to parallelizability, starts by triangulating a compact differentiable manifold M; that is, by expressing it as a set composed of a 0-chain, a 1-chain, …, and an n-chain in some way, such as ones composed of polyhedral or singular simplexes. These chains are then referred to as the *k-skeletons* of M for each dimension k and they are related by the fact that the simplexes of the k-skeleton are the boundary simplexes of the simplexes of the k+1-skeleton.

One starts the dimensional recursion by defining an n-frame field on the 0-skeleton of M and looking at the obstruction to the extension of the frame field to the 1-skeleton. Now, as long as two 0-simplexes – i.e., vertices – are connected by a 1-simplex (i.e., they define its boundary) the extension of the frame field on the boundary to a frame field on the simplex itself would represent a path in the manifold $GL(n)$, which is the model space for linear frames on M. Thus, all that would be necessary for this extension to always be possible would be for $GL(n)$ to be path-connected. Of course, it is not, since it consists of two components that correspond to the two possible orientations for any linear frame. Thus, one is already looking at an obstruction to the orientability of M, namely, whether one can always restrict the frames on the 0-skeleton to lie in the same component of $GL(n)$.

So far, what we have defined is the association of an element of the homotopy set $\pi_0(GL(n)) = \mathbb{Z}_2$ (which is not a group in dimension zero) with a 1-simplex σ_1 by way of a map that is defined on $\partial\sigma_1$. That is, it is a 1-cochain $w_1[M]$ with values in $\pi_0(GL(n))$, which is also isomorphic to $\mathbb{Z}_2$, as a set; one can also prove that $w_1[M]$ is cocycle (see, e.g., [**53-55**]). The vanishing of $w_1[M]$ really amounts to saying that it takes a constant value, in this case, which depends upon whether M is orientable.

When one goes to next step, one starts with the frame field that is now defined on the 1-skeleton and examines the obstruction to extending it to a frame field on the 2-skeleton. Once again, if a 2-simplex σ_2 is bounded by a 1-boundary b_1 on which a frame field $\mathbf{e}_i(b_1)$ is defined then the extension from the boundary to the interior is possible iff the homotopy class of the map $\mathbf{e}_i : \partial\sigma_2 \to GL(n)$ is trivial. Since $\partial\sigma_2$ is homotopically equivalent to a 1-sphere, the homotopy class of this map defines an element of $\pi_1(GL(n))$, and thus, a 2-cochain $w_2[M]$ with values in $\pi_1(GL(n))$, which can also be shown to be a cocycle. The vanishing of $w_2[M]$ for every 2-simplex is necessary and sufficient for the extension of $\mathbf{e}_i$ to a frame field on the 2-skeleton.

One then proceeds analogously in each successive dimension k and defines a k-cocycle $w_k[M]$ that takes its values in the Abelian group $\pi_{k-1}(GL(n))$, which one calls the *obstruction cocycle* in dimension k. Its vanishing is necessary and sufficient for the extension from dimension $k - 1$ to dimension k. The first dimension in which a non-trivial obstruction occurs then defines the *primary obstruction cocycle.*

One sees that it is necessary to know the homotopy groups of $GL(n)$, in this case, or at least the first non-trivial one. By polar decomposition, one finds that the homotopy type of $GL(n)$ is carried by its maximal connected orthogonal subgroup, which would be $O(n)$. Of course, in order to reduce further to $SO(n)$, one would have to have orientability of $T(M)$, and thus, the vanishing of $w_1[M]$. Hence, we make that assumption in order to look at higher obstructions.

For $n = 2$, $SO(2)$ is S^1, up to homotopy. Thus, its first (and only) non-vanishing homotopy group is $\pi_1(SO(2)) = \mathbb{Z}$. This gives a 2-cocyle with values in $\mathbb{Z}$, and its $\mathbb{Z}_2$ reduction is then $w_2[M]$. Frame fields on closed, orientable surfaces are then potentially obstructed by an integer cohomology class in dimension two, which is basically the Euler class that also obstructs the existence of non-zero vector fields. This is related to the fact that Stiefel pointed out that if one has an $n-1$-frame field on an orientable manifold then one also has an n-frame field.

For $n = 3$, $SO(3)$ is $\mathbb{R}\mathrm{P}^3$, up to homotopy, which agrees with S^3, up to homotopy, except in dimension one, where $\pi_1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2$, due to the two-to-one covering of $\mathbb{R}\mathrm{P}^3$ by S^3. After that, the next non-trivial homotopy group is $\pi_3(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}$. The former homotopy group gives a 2-cocycle with values in $\mathbb{Z}_2$ that is again $w_2[M]$, and although the latter one gives a 4-cocycle with values in the integer, for a three-dimensional manifold, all 4-cocycles would vanish, to begin with, as would $w_3[M]$, since $\pi_2(\mathbb{R}\mathrm{P}^3) = 0$.

For $n = 4$, $SO(4)$ has the homotopy type of $S^3 \times \mathbb{R}\mathrm{P}^3$, so $\pi_n(SO(4)) \cong \pi_n(S^3) \oplus \pi_n(\mathbb{R}\mathrm{P}^3)$. This makes the first two non-zero homotopy groups $\pi_1(SO(4)) = \mathbb{Z}_2$ and $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. The former gives $w_2[M]$, and the latter gives a 4-cocycle with values in $\mathbb{Z} \oplus \mathbb{Z}$. Once again, $w_3[M]$ vanishes, and for the same reason as before.

Ultimately, the necessary, but not sufficient, condition for the complete parallelizability of M is the vanishing of all of its Stiefel-Whitney classes in each dimension. As we saw, the first Stiefel-Whitney class must vanish in order for M to be orientable. The second one relates to the existence of a spin structure for the orthonormal frames (of $SO(n)$, that is), and at the top dimension n, one finds that $w_n[M]$ is always the $\mathbb{Z}_2$ reduction of the Euler class $e[M]$, which gives the Euler-Poincaré characteristic of M when it is evaluated on the fundamental n-cycle. Thus, one sees that the Stiefel-Whitney classes do, indeed, extend the scope of the Poincaré-Hopf theorem.

Actually, in order to duplicate that theorem in the context of obstructions, one must consider the obstructions to the extension of a non-zero vector field – i.e., a 1-frame field – from the 0-skeleton of M on up, not the extension of an n-frame field. The space of non-zero tangent vectors at each point of M is homotopically equivalent to an n–1-sphere, so the homotopy group that one must consider in each dimension k is $\pi_k(S^{n-1})$. The first non-vanishing dimension is $k = n - 1$, with $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$, which corresponds to a n-cocycle with integer values. In fact, since one is looking at the degree of the map from an n–1-sphere to an n–1-sphere, one is indeed duplicating the basic construction of the Gauss map that gives one the Poincaré-Hopf theorem.

5. Teleparallelism and the Dirac equation. Although the inclusion of the Dirac equation into the unification problem means that one is going beyond the scope of the Einstein-Maxwell unification problem, nonetheless, we have seen that perhaps there was something flawed in that problem to begin with. In particular, one might wish to unify a strong-field theory of electromagnetism, such as the still-non-existent field equations of quantum electrodynamics, with the Einstein equations of gravitation, since they describe strong gravitational fields. Thus, we have included some early papers of Zaycoff [**25, 29**] on the subject of how teleparallelism might relate to the Dirac equation.

Actually, the usual way of introducing a spin structure into general relativity already makes use of the existence of 4-frames (i.e., vierbeins). That is because the most direct way of going from Minkowski space $\mathfrak{M}^4 = \{\mathbb{R}^4, \eta_{\mu\nu}\}$ to the Clifford algebra that it generates is by choosing a Lorentzian frame $\{\mathbf{e}_\mu, \mu = 0, 1, 2, 3\}$ in $\mathfrak{M}^4$ to serve as a set of generators for that algebra. That is, one forms all formal products $\mathbf{e}_\mu \mathbf{e}_\nu$, $\mathbf{e}_\lambda \mathbf{e}_\mu \mathbf{e}_\nu$, … of basis vectors and subjects them to the relation:

$$\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu = 2\eta_{\mu\nu} . \tag{3.2}$$

This has the effect of making all products of more than four basis vectors vanish, and one finds that only 16 independent products (including 1) survive. Thus, the Clifford algebra over $\mathfrak{M}^4$, which is spanned by all linear combinations of the basis vectors, becomes a 16-dimensional real algebra $\mathcal{C}(4, \eta_{\mu\nu})$ with unity that is associative, but neither commutative nor skew-commutative.

The four Dirac γ-matrices $\{\gamma_\mu, \mu = 0, 1, 2, 3\}$ then define a way of representing the algebra $\mathcal{C}(4, \eta_{\mu\nu})$ in a matrix algebra. Actually, they define a representation in a proper linear sub-algebra of the algebra $M(4; \mathbb{C})$ of complex 4×4 matrices, since the real dimension of that algebra is 32, which is twice as big as is needed in order to represent $\mathcal{C}(4, \eta_{\mu\nu})$ isomorphically. The representation is defined simply by associating the four generators $\mathbf{e}_\mu$ of $\mathcal{C}(4, \eta_{\mu\nu})$ with the four γ-matrices, so products of the former go to products of the latter.

So far, all of this construction pertains to a single tangent space to a four-dimensional Lorentzian manifold, such as, presumably, spacetime. In order to make it global, one has to define an $SL(2; \mathbb{C})$-principal bundle $Spin(M) \to M$ whose fibers consist of Lorentzian spin frames that map to the bundle $L(M)$ of Lorentzian frames in the same way that $SL(2; \mathbb{C})$ maps to $SO(3, 1)$. Here, one finds that there are further topological obstructions to such a global Lorentzian spin structure existing, and in fact, the primary one is the second Stiefel-Whitney class $w_2[M]$. Interestingly, Geroch [**56**] showed that a non-compact, orientable, Lorentzian manifold that admits a Lorentzian spin structure must be parallelizable. Thus, even the question of global Lorentzian spin structures is closely related to questions of teleparallelism.

6. Singular teleparallelism. Although one might easily take the position that teleparallelism seems to have been eliminated from serious consideration by the less-than-encouraging results of Einstein, Mayer, and McVittie, one can also say that their work was historically premature, in the sense that it did not take into account the topological nature of the problem that only began to emerge some years after they gave up. Similarly, one should admit that perhaps the Einstein-Maxwell unification problem itself was premature or perhaps even poorly posed, in its own right. Ultimately, one must admit that, regardless of its role in the Einstein-Maxwell unification problem, the topological issue of whether the spacetime manifold is parallelizable is as fundamental as asking whether it is compact, orientable, simply-connected, or any of the other topological issues that will bear upon the nature of solutions to systems of differential equations on it.

In that light, if one addresses the problem in a post-Stiefel-Whitney way then one first asserts that either the spacetime manifold is globally parallelizable or it is not, and that since parallelizability is, apparently, hard to come by, except locally, one should consider the possible contribution that the topological obstructions to parallelizability might make. In the context of field equations, one might speculate that they represent a sort of "topological defect" that can serve as the source of a field, just as the deleted point at the origin can serve as the source of a Coulomb field and dislocations in plastic media can serve as the sources of stress fields.

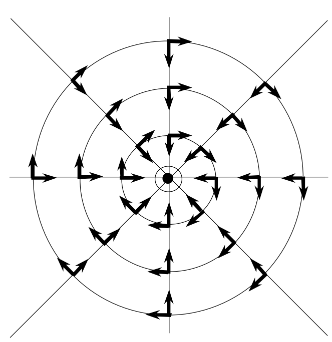
A possible basis for this sort of study is given by the notion of "singular" frame fields on non-parallelizable manifolds, which one sees discussed briefly towards the end of Stiefel's thesis. A *singular m-frame field* on an n-dimensional differentiable manifold M is defined by a set $\{\mathbf{e}_i(x), i = 1, \ldots, m\}$ of m vector fields. The *rank* $r(x)$ of $\mathbf{e}_i$ at x is the dimension of the linear subspace of T_xM that is spanned by $\mathbf{e}_i(x)$. Thus, M has degree of parallelizability m iff there is a singular m-frame field whose rank is a constant function that equals m, in which case, one could say that the m-frame field is *non-singular*.

One can imagine two possible ways that the rank of a singular frame field could be less-than-maximal at a point: Either one or more of the vectors fields has a zero at that point or some of the vectors coincide at that point. It is easy to see that the former scenario is "generic" up to homotopy, since one can choose one of the coincident vector fields arbitrarily and multiply the other ones by a smooth function that is defined on a neighborhood of the point and goes to zero at that point. Thus, one can treat the case in which the singularities are defined by zeroes as being the typical one.

Furthermore, the set of non-singular points, being the level set of the rank function for the value m, is open (when the set of non-negative integers is given the discrete topology), so its complement – viz., the set of singular points – is closed. One might ask what the maximal such non-singular subset of M might be; i.e., the minimal singular subset. This is essentially what Stiefel did by imagining m-frame fields for which the set of singularities defined a homology complex, in the form of a $\mathbb{Z}_2$-cycle. Actually, it becomes a $\mathbb{Z}_2$-cycle of mixed rank that amounts to the total Steifel-Whitney class of M, or rather, its Poincaré-Veblen dual in homology.

One finds that in many examples the singularity complex can be as simple as a finite set of isolated points. For instance, if one wishes to define a singular 2-frame field on S^2 that would have a minimal singularity subset then one can use vector fields that have a single zero at the North pole, although it is geometrically simpler to use ones that have zeroes at the North and South poles. For example, one can define a singular frame field on S^2 by using the fields of unit vectors that are tangent to the longitude and latitude circles everywhere except the poles. Topologically, one then represents S^2 as the *suspension* of a circle – say, its equator. That is, one first forms the cylinder $[-\pi, +\pi] \times S^1$, and then identifies each boundary circle – viz., $\{-\pi\}\times S^1$ and $\{+\pi\} \times S^1$ – with a point, which then become the two poles. Now, the cylinder $[-\pi, +\pi] \times S^1$ is parallelizable, so one can define the global frame field $\{\mathbf{e}_\phi, \mathbf{e}_\theta\}$ on it, where the vector fields are unit vector fields that point in the direction of increasing ϕ and θ, resp., assuming that the coordinate system for $[-\pi, +\pi] \times S^1$ takes the form $\{\phi, \theta\}$.

When one deforms $[-\pi, +\pi] \times S^1$ continuously into S^2 (except at the last step, when one identifies the boundary circles to points), one finds that the frame field cannot be extended to the poles, since one has a situation at either pole like the one depicted in the following figure:



Clearly, one cannot expect to extend the frame field itself to the missing point, but one can imagine extending the *connection* on the bundle $SO(S^2) \to S^2$ of oriented, orthonormal 2-frames that it defines to one that is also defined on the fiber $SO_N(S^2)$ of the pole N. This is because the manifold $SO(S^2)$ is itself parallelizable, even though S^2 is not.

One might describe such an extension most easily by looking at the extension of the parallel translation of frames along longitude circles by continuity at the pole, where the parallel translation is defined everywhere except the pole by teleparallelism. One immediately sees that when a loop intersects the pole it is possible for the parallel translation of a frame around the loop to exhibit non-trivial holonomy at the pole; indeed, this will happen as long as the final velocity vector $\mathbf{v}(1)$ of the loop $x(s)$ is not coincident with the initial one $\mathbf{v}(0)$. If $\mathbf{v}(0)$ and $\mathbf{v}(1)$ form an angle α then one sees that the loop is associated with a non-zero rotation of the initial frame to the final one through the angle α. Thus, the extension of the teleparallelism connection to the singular point has introduced non-trivial curvature at the pole.

Of course, the connection and curvature that we have defined are not continuous or differentiable at the singularity, but behave like a step function and a delta function there, respectively. One can see how this is still consistent with the Gauss-Bonnet theorem if one represents the connection 2-form as:

$$\Omega = i(\delta(x, N) + \delta(x, S))\, V_2\,, \tag{4.1}$$

where V_2 is the volume element on the (unit) sphere and the delta functions produce 1 when either N or S are contained in the domain of integration. When that domain is all of S^2 and one divides by $4\pi i$, one gets:

$$\frac{1}{4\pi i}\int_{S^2}\Omega \;= 2 = \chi[S^2], \tag{4.2}$$

where $\chi[S^2] = 1 - 0 + 1$ is the Euler-Poincaré characteristic of the 2-sphere. One can think of the (closed) 2-form $1/4\pi i\ \Omega$ as representing the Euler class $e[S^2]$ of S^2 in the de Rham cohomology in dimension two. It also represent the first Chern class $c_1[SO(S^2)]$ of the $U(1)$-principal bundle $SO(S^2)$, and its $\mathbb{Z}_2$-reduction is the second Stiefel-Whitney class $w_2[S^2]$ of M, as well. Thus, we are clearly dealing with topological obstructions as the source of curvature, here.

However, although this example serves to illustrate the way that the topological obstructions can be the sources of non-zero curvature in their neighborhoods, it lacks a certain usefulness in geometric terms due to its non-differentiability. Hence, one might ponder the question of how to smooth out such a connection in those neighborhoods in a manner that has some basis in physical necessity.

One possibility is given by the example of topological defects in ordered media, which can serve as the sources of stress fields. If one then adds the extra information to the picture in the form of the mechanical constitutive law for the material then the stress field implies a corresponding infinitesimal strain field. Since the Cauchy-Green conception of strain involves essentially the deformation of a metric by a non-isometric diffeomorphism, one sees that this would put one back in the realm of differential geometry. However, one of the complicating factors is that going from an infinitesimal strain field to a finite one is a non-trivial problem in continuum mechanics, and our aforementioned process of suspension really amounts to a finite strain, not an infinitesimal one. One might also ponder Sakharov's [**57**] description of general relativity as a type of "metric elasticity" as a justification for the introduction of singular teleparallelism.

Nonetheless, one sees that it is entirely possible that what were missing from Einstein's theory were terms in the Lagrangian that would relate to curvature, since curvature always vanishes for non-singular teleparallelism. The question of whether the inclusion of such additional terms might change the character of the solutions to the unified field equations into something that had more physical justification is entirely worth considering.

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Riemannian geometry, while maintaining the notion of teleparallelism

By A. EINSTEIN

In general relativity, Riemannian geometry has led to a physical description of the gravitational field; however, it has produced no notion that can be attributed to the electromagnetic field. For that reason, the efforts of the theoretician are directed towards finding natural generalizations or extensions of RIEMANNIAN geometry that are richer in ideas that it is, in the hopes of arriving at a logical structure that unites all physical field concepts within a single viewpoint. Such efforts have led me to a theory, which will not be communicated along with any attempt to give it physical meaning, since it already commands a certain interest due to the naturality of the concepts that are introduced.

RIEMANNIAN geometry is characterized by the fact that the infinitesimal neighborhood of each point P possesses a Euclidian metric, in such a way that the magnitudes of two line elements that belong to the infinitesimal neighborhoods of two finitely-separated points P and Q are comparable. On the other hand, the notion of parallelism of two such line elements breaks down; the concept of direction does not exist for finite distances. The theory that is put forth in what follows is characterized by the fact that along with the RIEMANNIAN metric, the notion of *direction* (equality of directions, or *parallelism*, resp.) is introduced for finite distances. Correspondingly, new invariants and tensors appear, in addition to the invariants and tensors of RIEMMANNIAN geometry.

1. *n*-bein fields and the metric.

We imagine that an orthogonal n-bein that represents a local coordinate system is contructed from n unit vectors at the arbitrary point P of an n-dimensional continuum. Let A_a be the components of a line element – or another vector – relative to this local system (i.e., n-bein). Moreover, let the GAUSSIAN coordinate system x^ν be introduced for the description of a finite domain. Furthermore, let h_a^ν be the n-components of the units vectors that the n-bein is comprised of. We then have ([1]):

$$A^\nu = h_a^\nu A_a . \tag{1}$$

([1]) We denote coordinate indices by Greek characters and bein indices by Latin ones.

By inverting (1), one gets, when one introduces the normalized sub-determinant of the h_a^ν by $h_{\nu a}$:

(1*a*) $$A_a = h_{\nu a}\, A^\nu.$$

Due to the Euclidian character of the infinitesimal neighborhood, we have the following formula for the magnitude *A* of the vector (*A*):

(2) $$A^2 = \sum A_a^2 = h_{\mu a}\, h_{\nu a}\, A^\mu A^\nu.$$

The components of the metric tensor $g_{\mu\nu}$ can be represented in the form:

(3) $$g_{\mu\nu} = h_{\mu a}\, h_{\nu a}\,,$$

in which, naturally, *a* is to be summed over. For a fixed *a*, the h_a^ν are the components of a contravariant vector. Furthermore, we have the relations:

(4) $$h_{\mu a}\, h_a^\nu = \delta_\mu^\nu$$

(5) $$h_{\mu a}\, h_b^\mu = \delta_{ab}\,,$$

in which $\delta = 1$ ($\delta = 0$, resp.) whenever both indices are equal (different, resp.). The validity of (4) and (5) follows from the aforementioned definition of the $h_{\mu a}$ as the normalized sub-determinant of the h_a^μ. The vector character of the $h_{\mu a}$ follows easily from the fact that for each choice of vector (*A*) the left-hand, as well as the right-hand, side of (1*a*) is invariant with respect to an arbitrary coordinate transformation.

The *n*-bein field is determined by the n^2 functions h_a^μ, while the RIEMANN metric is determined by only the $n(n + 1)$ quantities $g_{\mu\nu}$. According to (3), the metric is determined by the *n*-bein field, but not, conversely, the latter by the former.

2. Teleparallelism and rotational invariance.

An expression for the existence of a RIEMANN metric and teleparallelism is given simultaneously by the construction of the *n*-field field. Namely, let (*A*) and (*B*) be two vectors at the points *P* and *Q* that have equal corresponding local coordinates when they are referred to the corresponding local *n*-beins (i.e., $A_a = B_a$), so they are to be regarded as equal (because of (2)) and *parallel.* If we regard only the metric and teleparallelism as essential – i.e., meaningful – then we must recognize that the *n*-bein field is not completely determined by these structures. The metric and parallelism remain intact when one replaces the *n*-beins at every point of the continuum by ones that are obtained from the original *n*-beins by the same rotation. We refer to this replaceability of the *n*-bein fields as *rotational invariance,* and establish that only those mathematical relations that are rotationally invariant are truly meaningful.

Thus, for a fixed coordinate system the h_a^μ are not completely determined by a given metric and parallel connection. The substitution of the h_a^μ that corresponds to rotational invariance – i.e., the equation:

$$A_a^* = d_{am} A_m , \tag{6}$$

in which the d_{am} are chosen to be orthogonal and independent of the coordinates, is possible. (A_a) is an arbitrary vector that is referred to the local system and (A_a^*) is referred to the rotated local system. According to (1*a*), it follows from (6) that:

$$h_{\mu a}^* A^\mu = d_{am}\, h_{\mu\, m}\, A^\mu,$$

or

$$h_{\mu a}^* = d_{am}\, h_{\mu\, m} , \tag{6a}$$

in which:

$$d_{an}\, d_{bn} = d_{ma}\, d_{mb} = \delta_{ab} , \tag{6b}$$

$$\frac{\partial d_{am}}{\partial x^\nu} = 0. \tag{6c}$$

The postulate of rotational invariance then says that the only relations in which the quantity h appear that are to be regarded as meaningful are the ones that remain valid when one passes over to the h^*, when one introduces the h^* by way of equations (6), etc. In other words: n-bein fields that go to each other by point-wise uniform rotations are equivalent.

The law of infinitesimal parallel translation of a vector when one goes from a point (x^ν) to a neighboring point $(x^\nu + dx^\nu)$ is obviously characterized by the equation:

$$dA_a = 0; \tag{7}$$

i.e., by the equation:

$$0 = d(h_{\mu a} A^\mu) = \frac{\partial h_{\mu a}}{\partial x^\tau} A^\mu\, dx^\tau + h_{\mu a}\, dA^\mu = 0.$$

Upon multiplying this by h_a^μ and taking (5) into account, this equation goes to:

$$dA^\nu = - \Delta_{\mu\tau}^\nu\, A^\mu\, dx^\tau,$$

in which (*):

$$\Delta_{\mu\sigma}^\nu = h_a^\nu \frac{\partial h_{\mu a}}{\partial x^\sigma} . \tag{7a}$$

(*) Trans. note: This equation was misprinted in the original article.

This law of parallel translation is rotationally-invariant and asymmetric with respect to the lower indices of the quantities $\Delta^{\nu}_{\mu\sigma}$. If one translates the vector (A) according to this law of translation around a closed path then it goes back to itself (*). This means that the RIEMANN tensor that is constructed from the translation coefficients $\Delta^{\nu}_{\mu\sigma}$ on the basis of (7a), namely:

$$R^{i}{}_{k,lm} = -\frac{\partial \Delta^{i}_{kl}}{\partial x^{m}} + \frac{\partial \Delta^{i}_{km}}{\partial x^{l}} + \Delta^{i}_{\alpha l}\Delta^{\alpha}_{km} - \Delta^{i}_{\alpha m}\Delta^{\alpha}_{kl} ,$$

vanishes identically, as one easily verifies.

However, in addition to this law of parallel translation, there exists a (non-inbtegrable) law of translation, which is symmetric this time, and which belongs to the RIEMANNIAN metric that comes from (2) and (3). As you know, it is given by the equations:

(8)
$$\bar{d}A^{\nu} = -\Gamma^{\nu}_{\mu\sigma} A^{\mu} dx^{\sigma},$$
$$\Gamma^{\nu}_{\mu\sigma} = \tfrac{1}{2} g^{\nu\alpha}\left[\frac{\partial g_{\mu\alpha}}{\partial x^{\sigma}} + \frac{\partial g_{\sigma\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial x^{\alpha}}\right].$$

The $\Gamma^{\nu}_{\mu\sigma}$ can be expressed in terms of the h quantities of the n-bein field by means of (3). In this, one must observe that:

(9)
$$g^{\mu\nu} = h^{\mu}_{a} h^{\nu}_{a} .$$

By this construction, and because of (4) and (5), the equation:

$$g^{\mu\lambda}\, g_{\nu\lambda} = \delta^{\mu}_{\nu} ,$$

which defines the $g^{\mu\nu}$ in terms of the $g_{\mu\nu}$, is then satisfied. Moreover, this translation law that is based upon only the metric is naturally rotationally-invariant, in the aforementioned sense.

3. Invariants and covariants.

There exist further tensors and invariants on the manifold considered by us, in addition to the tensors and invariants of RIEMANNIAN geometry, which involve the h quantities only in the combinations that are given by (3), and we will fix our attention upon only the simplest ones.

If one starts with a vector (A^{ν}) at a point (x^{ν}) then, corresponding to the translations d and $\bar{d}$, we have the two vectors:

$$A^{\nu} + dA^{\nu} \qquad \text{and} \qquad A^{\nu} + \bar{d}A^{\nu},$$

resp., at the neighboring point ($x^{\nu} + dx^{\nu}$). Likewise, the difference:

(*) Trans. note: He seems to be ignoring the translation that might come from non-vanishing torsion.

$$dA^\nu - \bar{d}A^\nu = (\Gamma^\nu_{\alpha\beta} - \Delta^\nu_{\alpha\beta})\, A^\alpha\, dx^\beta$$

has a vector character. Thus:

$$\Gamma^\nu_{\alpha\beta} - \Delta^\nu_{\alpha\beta}$$

is also a tensor, and so is its anti-symmetric component:

$$\tfrac{1}{2}(\Delta^\nu_{\alpha\beta} - \Delta^\nu_{\beta\alpha}) = \Lambda^\nu_{\alpha\beta}\,. \tag{10}$$

The fundamental meaning of this tensor in the theory that is developed here comes from the following fact: The continuum is Euclidian when this tensor vanishes. That is, if:

$$0 = 2\, A^\nu_{\alpha\beta} = h^{\nu a}\left[\frac{\partial h_{\alpha a}}{\partial x^\beta} - \frac{\partial h_{\beta a}}{\partial x^\alpha}\right]$$

then, upon multiplying this by $h_{\nu a}$, it follows that:

$$0 = \frac{\partial h_{\alpha a}}{\partial x^\beta} - \frac{\partial h_{\beta a}}{\partial x^\alpha}\,.$$

Thus, we may set:

$$h_{\alpha b} = \frac{\partial \Psi_b}{\partial x^\alpha}\,.$$

The field is therefore derivable from n scalars Ψ_b . We now choose the coordinates according to the equation:

$$\Psi_b = x^b\,.$$

According to (7*a*), all of the $\Delta^\nu_{\alpha\beta}$ then vanish, and the $h_{\mu a}$, as well as the $g_{\mu\nu}$, are constant. Moreover, since the tensor $\Lambda^\nu_{\alpha\beta}$ is obviously the simplest one that our theory allows, the simplest characterization of such a continuum is obtained from it, rather than from the RIEMANN curvature tensor. The simplest structures that come under consideration here are the vector:

$$\Lambda^\alpha_{\alpha\beta}\,,$$

as well as the invariants:

$$g_{\mu\nu}\,\Lambda^\alpha_{\mu\beta}\Lambda^\beta_{\nu\alpha} \quad \text{and} \quad g_{\mu\nu}\, g^{\alpha\sigma} g^{\beta\tau}\, \Lambda^\mu_{\alpha\beta}\Lambda^\nu_{\sigma\tau}\,.$$

One can construct an integral invariant J from one of the latter invariants (from a linear combination that is constructed from them, resp.) by multiplying it by the invariant volume element:

$$h\, d\tau,$$

in which h is the determinant $| h_{\mu a} |$, and $d\tau$ means the product $dx_1 \ldots dx_\mu$. By setting:

$$\delta J = 0,$$

we obtain 16 differential equations for the 16 quantities $h_{\mu a}$. Whether or not physically-meaningful laws can be obtained in this way shall be examined later on.

It is enlightening to contrast WEYL's modifications to RIEMANN's theory with the one that was developed here:

WEYL:	Distant equality of neither vector magnitude not direction
RIEMANN:	Distant equality of vector magnitudes, but not direction
Present theory:	Distant equality of vector magnitudes and directions

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A new possibility for a unified field theory of gravitation and electromagnetism

By A. EINSTEIN

Some time ago, I published a brief treatise in these Berichten in which, by the use of an *n*-bein field, a geometric theory was presented that rested upon the basic concepts of the RIEMANN metric and teleparallelism. I then left open the question of whether this theory could serve to represent of physical concepts. Since then, I have discovered that this theory yielded the field theories of gravitation and electromagnetism quite simply and naturally – at least, in the first approximation. It is therefore conceivable that this theory might supersede the original formulation of the general theory of relativity.

In order for the introduction of teleparallelism in the form that is employed here to be immediately applicable to field theory, one must only establish that:

1. The number of dimensions is 4 ($n = 4$).
2. The fourth local coordinate A_a ($a = 4$) of a vector is pure imaginary, and likewise for the components of the fourth leg of a vierbein; hence, the quantities h_4^ν and $h_{\nu 4}$ ([1]). The coefficients $g_{\mu\nu}$ ($= h_{\mu a}\, h_{\nu a}$) will all be naturally real then. We therefore choose the square of the magnitude of a time-like vector to be negative.

§ 1. The basic field law.

The variation of a HAMILTON integral:

$$\delta\left\{\int \mathfrak{H}\, d\tau\right\} = 0, \tag{1}$$

$$\mathfrak{H} = h\, g^{\mu\nu}\, \Lambda^{\alpha}_{\mu\beta}\, \Lambda^{\beta}_{\nu\alpha}\,, \tag{1a}$$

([1]) Instead of this, one could also define the square of the magnitude of the local vector to be $A_1^2 + A_2^2 + A_3^2 - A_4^2$, and in place of the rotations of local *n*-beins, one could introduce LORENTZ transformations. All of the *h* would then be real, but the immediate connection with the formulation of the general theory would be lost.

must vanish for variations of the field potentials $h_{\mu a}$ (h_a^μ, resp.) that vanish on the boundary of a region, where the quantities h (= | $h_{\mu a}$ |), $g^{\mu\nu}$, $\Lambda_{\mu\beta}^{\alpha}$ are defined by equations (9), (10) of *loc. cit.*

The h-field can simultaneously describe the electric and gravitational field. A "pure gravitational field" is then present when, along with fulfilling equation (1), the quantities:

$$\phi_\mu = \Lambda_{\mu\alpha}^{\alpha} \tag{2}$$

also vanish, which implies a covariant and rotationally invariant restriction ([1]).

§ 2. The field law in the first approximation.

If the manifold is the MINKOWSKI space of special relativity then one can choose the coordinate system in such a way that $h_{11} = h_{22} = h_{33} = 1$, $h_{44} = j$ (= $\sqrt{-1}$), and the remaining $h_{\mu a}$ vanish. This system of values for the $h_{\mu a}$ is somewhat inconvenient for the calculations. For that reason, we prefer to choose the x_4 coordinate to be pure imaginary for the calculations of this section; one can then, in fact, describe MINKOWSKI space (with no fields present, for some suitable choice of coordinates) by:

$$h_{\mu a} = \delta_{\mu a}\,. \tag{3}$$

The case of infinitely weak fields can be conveniently represented by:

$$h_{\mu a} = \delta_{\mu a} + k_{\mu a}\,, \tag{4}$$

where the $k_{\mu a}$ are small quantities of first order. By neglecting the terms of third and higher order, one then has to replace (1*a*), with consideration given to (10) and (7*a*) of *loc. cit.*, with:

$$\mathfrak{H} = \frac{1}{4}\left(\frac{\partial k_{\mu\alpha}}{\partial x_\beta} - \frac{\partial k_{\beta\alpha}}{\partial x_\mu}\right)\left(\frac{\partial k_{\mu\beta}}{\partial x_\alpha} - \frac{\partial k_{\alpha\beta}}{\partial x_\mu}\right). \tag{1b}$$

Upon performing the variation, one obtains the field equations that are valid in the first approximation:

$$\frac{\partial^2 k_{\beta\alpha}}{\partial x_\mu^2} - \frac{\partial^2 k_{\mu\alpha}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\alpha\mu}}{\partial x_\mu \partial x_\beta} - \frac{\partial^2 k_{\beta\mu}}{\partial x_\mu \partial x_\alpha} = 0. \tag{5}$$

([1]) Here, a certain indeterminacy of the interpretation is present, since one can also characterize the pure gravitational field by the vanishing of the $\partial\phi_\mu / \partial x_\nu - \partial\phi_\nu / \partial x_\mu$.

These are sixteen equations ([1]) for the sixteen quantities $k_{\alpha\beta}$. Our problem is now to check whether this system of equations includes the known laws of the gravitational and electromagnetic fields. To that end, we must introduce the $g_{\alpha\beta}$ and ϕ_α into (5) instead of $k_{\alpha\beta}$. We must set:

$$g_{\alpha\beta} = h_{\alpha a}\, h_{\beta a} = (\delta_{\alpha a} + k_{\alpha a})\,(\delta_{\beta a} + k_{\beta a}),$$

or, in quantities that are precise to first order:

$$g_{\alpha\beta} - \delta_{\alpha\beta} = \overline{g}_{\alpha\beta} = k_{\alpha\beta} + k_{\beta\alpha}. \tag{6}$$

From (2), one further obtains the quantities that are precise to first order:

$$2\phi_\alpha = \frac{\partial k_{\alpha\mu}}{\partial x_\mu} - \frac{\partial k_{\mu\mu}}{\partial x_\alpha}. \tag{2a}$$

By permuting α and β in (5) and adding the terms thus obtained to (5), one now gets:

$$\frac{\partial^2 \overline{g_{\alpha\beta}}}{\partial x_\mu^2} - \frac{\partial^2 k_{\mu\alpha}}{\partial x_\mu \partial x_\beta} - \frac{\partial^2 k_{\mu\beta}}{\partial x_\mu \partial x_\alpha} = 0.$$

If one adds both of the equations that follow from (2a), namely:

$$\begin{aligned} -\frac{\partial^2 k_{\alpha\mu}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\mu\mu}}{\partial x_\alpha \partial x_\beta} &= -2\,\frac{\partial \phi_\alpha}{\partial x_\beta}, \\ -\frac{\partial^2 k_{\beta\mu}}{\partial x_\mu \partial x_\alpha} + \frac{\partial^2 k_{\mu\mu}}{\partial x_\alpha \partial x_\beta} &= -2\,\frac{\partial \phi_\beta}{\partial x_\alpha} \end{aligned}$$

to this equation then one obtains, with consideration given to (6):

$$\frac{1}{2}\left(-\frac{\partial^2 \overline{g_{\alpha\beta}}}{\partial x_\mu^2} + \frac{\partial^2 \overline{g_{\mu\alpha}}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 \overline{g_{\mu\beta}}}{\partial x_\mu \partial x_\alpha} - \frac{\partial^2 \overline{g_{\mu\mu}}}{\partial x_\alpha \partial x_\beta}\right) = \frac{\partial \phi_\alpha}{\partial x_\beta} + \frac{\partial \phi_\beta}{\partial x_\alpha}. \tag{7}$$

Let the case in which an electromagnetic field is absent be characterized by the vanishing of the ϕ_μ. In this case, (7) agrees with the equation in first-order quantities:

$$R_{\alpha\beta} = 0$$

([1]) Naturally, there exist four identities between the field equations that are due to general covariance. In the first approximation that is used here, one expresses this by saying that the divergence of the left-hand side of (5), when taken over the index α, vanishes identically.

($R_{\alpha\beta}$ = once-contracted RIEMANN tensor) that was previously established in general relativity. With that, we have proved: *Our new theory yields the law of the pure gravitational field that is correct to first order.*

By differentiating (2*a*) with respect to x_α , one obtains, upon consideration of the equation that results from (5) by contracting over α and β:

$$\frac{\partial \phi_\alpha}{\partial x_\alpha} = 0. \tag{8}$$

In light of the fact that the left-hand side of (7) fulfills the identity:

$$\frac{\partial}{\partial x_\beta}(L_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta}L_{\sigma\sigma}) = 0,$$

it follows from (7) that:

$$\frac{\partial^2 \phi_\alpha}{\partial x_\beta^2} + \frac{\partial^2 \phi_\beta}{\partial x_\alpha \partial x_\beta} - \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \phi_\sigma}{\partial x_\sigma}\right) = 0,$$

or

$$\frac{\partial^2 \phi_\alpha}{\partial x_\beta^2} = 0. \tag{9}$$

Equations (8) and (9) collectively are equivalent to the well-known MAXWELL equations for empty space. *Therefore, the new theory also delivers MAXWELL's equations in the first approximation.*

However, the separation into the gravitational and electromagnetic field in this theory seems artificial. It is also clear that equations (5) say more than equations (7), (8), and (9) do together. It is further remarkable that the electric field does not enter into the field equations quadratically in this theory.

Supplementary remark: One obtains entirely similar results when one starts with the HAMILTON function:

$$\mathfrak{H} = h\, g_{\mu\nu}\, g^{\alpha\sigma}\, g^{\beta\tau}\, \Lambda^{\mu}_{\alpha\beta}\, \Lambda^{\nu}_{\sigma\tau}\,.$$

Thus, for the time being, a certain indeterminacy exists that relates to the choice of $\mathfrak{H}$.

Differential invariants in EINSTEIN's theory of teleparallelism

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(Submitted by EINSTEIN on 18 October 1928 [cf., *supra*, pp. 449].)

In two short notes to these proceedings (which are referred as [**13**] and [**14**] in the following bibliography), EINSTEIN gave an extension of RIEMANNIAN geometry that allows us to compare the directions of two line elements that emanate from two points that are separated by a finite distance. This theory is based upon knowing the differential invariants that are obtained when one begins with linearly-independent vectors and considers only such differential invariants that exhibit a particular structure (viz., "rotational invariance").

In what follows, I will develop the theory of these structures, establish the simplest invariants, and compute the associated field equations that arise when these invariants are taken to be action functions. In the last section, I will finally give a short summary of further results that can be used as the starting point for field physics.

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§ 1.

Let n linearly-independent vectors in n variables be given, such that the k^{th} vector has the components ${}^k h_1, {}^k h_2, \ldots, {}^k h_n$, the determinant $h = | {}^a h_\nu | \neq 0$, and the formula ${}^a \overline{h}_\mu = {}^a h_\nu \cdot \dfrac{\partial x_\nu}{\partial \overline{x}_\mu}$ is valid under a transformation $x \to \overline{x}$.

If ${}_a h^\nu$ are the algebraic complements of the elements ${}^a h_\nu$ in h, divided by this determinant, then these ${}_a h^\nu$ represent n linearly-independent contravariant vectors that are defined by the ${}^a h_\nu$ uniquely. If we sum over two equal (i.e., Latin or Greek) indices – one upper one, one lower one – then we get, in the usual way:

$$(1) \qquad {}^a h_\nu \cdot {}_b h^\nu = {}^a_b \delta, \qquad {}^a h_\nu \cdot {}_a h^\mu = \delta^\mu_\nu .$$

A covariant differential is derived from the n vectors ${}^a h_\nu$ (or ${}_a h^\nu$) [**1-4**] whose connection components are given by:

$$(2) \qquad \Delta^\lambda_{\alpha\beta} = {}_a h^\mu \cdot \frac{\partial\, {}^a h_\alpha}{\partial x_\beta} \qquad \left(\frac{\partial\, {}^a h_\alpha}{\partial x_\beta} = {}^a h_\nu \cdot \Delta^\nu_{\alpha\beta} \right),$$

so the covariant derivative of the vector v_ρ (w^ρ, resp.) is given by the tensor:

$$(3) \qquad v_{\rho[\sigma]} = \frac{\partial v_\rho}{\partial x_\sigma} - \Delta^\lambda_{\rho\sigma} v_\lambda , \qquad \left(w^\rho{}_{[\sigma]} = \frac{\partial w^\rho}{\partial x_\sigma} + \Delta^\rho_{\lambda\sigma} w^\lambda , \text{ resp.}\right)$$

The covariant derivatives of the ${}^a h_\rho$ vanish: ${}^a h_{\rho[\sigma]} = 0$, and likewise the curvature tensor that is derived from the $\Delta^\lambda_{\alpha\beta}$. The linear displacement that is defined by the $\Delta^\lambda_{\alpha\beta}$ is examined in [**3**, **4**, **9**, **10**].

The theory of differential invariants of the vectors ${}^a h_\nu$ is developed in [**5**]. We derive the following fact from it: The n vectors ${}^a h_\nu$ lead to just as many rotations:

(4) $${}^a p_{\lambda\mu} = \frac{\partial\, {}^a h_\lambda}{\partial x_\mu} - \frac{\partial\, {}^a h_\mu}{\partial x_\lambda} = \text{rot}\ {}^a h_\nu .$$

One gets the following tensor from the $\Delta^\lambda_{\alpha\beta}$:

(5) $$\Lambda^\lambda_{\alpha\beta} = \tfrac{1}{2}(\Delta^\lambda_{\alpha\beta} - \Delta^\lambda_{\beta\alpha}),$$

which is alternating with respect to α and β. (Using the notation of EINSTEIN [**13**]. In [**5**], GRISS writes $S^\nu_{\alpha\beta}$ for $2\Delta^\nu_{\alpha\beta}$) We then have the **Reduction Theorem 1** ([**5**], pp. 12):

The m^{th}-order differential invariants that are determined from the linearly-independent vectors ${}^a h_\nu$ refer to the affine invariants that are constructed from ${}^a h_\nu$, ${}^a p_{\lambda\mu}$, and the covariant derivatives ${}^a p_{\lambda\mu\,[\rho]}$, … of the ${}^a p_{\lambda\mu}$, up to order $(m - 1)$.

Now, it is easy to show, moreover, that the ${}^a p_{\lambda\mu}$ can be expressed in terms of the $\Lambda^\nu_{\alpha\beta}$, and conversely ([**5**], pp. 10):

(6) $${}^a p_{\lambda\mu} = 2 \cdot {}^a h_\tau \cdot \Lambda^\tau_{\lambda\mu}, \qquad \Lambda^\tau_{\lambda\mu} = \tfrac{1}{2} \cdot {}_a h^\nu \cdot {}^a p_{\lambda\mu} .$$

For that reason, Theorem 1 can also be formulated ([**5**], pp. 14) as **Theorem 2:**

The m^{th}-order differential invariants that are determined from the vectors ${}^a h_\nu$ refer to the affine invariants that are constructed from the ${}^a h_\nu$, $\Lambda^\nu_{\alpha\beta}$, and the covariant derivatives $\Lambda^\nu_{\alpha\beta[\rho]}$, … of the tensor $\Lambda^\nu_{\alpha\beta}$ up to order $(m - 1)$.

§ 2.

The tensors and invariants that appeared in EINSTEIN [**13**] were assumed to have "rotational invariance." This means the following: If we think of the n vectors ${}^a h_\nu$ as emanating from a point G, and take O to be the origin of an orthogonal Cartesian frame to which we refer the n vectors ${}^a h_\nu$ then the tensors and invariants that are used in what follow must be absolutely invariant under the rotations:

(7) $${}^a h^*_\mu = {}^a_b D \cdot {}^b h_\mu ,$$

where the n^2 constants ${}^a_b D$ thus define a real-orthogonal matrix.

From (7), it now follows that $h^* = h$ and $({}_a h^\mu)^* = {}^b_a D \cdot {}_b h^\mu$; hence, from (2), the $\Delta^\lambda_{\alpha\beta}$ are rotationally invariant. Thus, from (5), the same thing is true for the $\Lambda^\nu_{\alpha\beta}$ and all of the covariant derivatives $\Delta^\nu_{\alpha\beta[\rho]}$, … of these $\Delta^\nu_{\alpha\beta}$.

It is now easy to give the general structure of the desired invariants W. From Theorem 2, we have:

$$W = W({}^a h_\nu, \Lambda^\nu_{\alpha\beta}, \Lambda^\nu_{\alpha\beta[\rho]}, \cdots), \tag{8}$$

$W^* = W$. Since the $\Lambda^\nu_{\alpha\beta}$, $\Lambda^\nu_{\alpha\beta[\rho]}$, … are to be regarded as constants under the rotations (7), everything comes down to the ${}^a h_\nu$. If we hold n fixed and regard the ${}^1 h_\nu$, ${}^2 h_\nu$, …, ${}^n h_\nu$ as the components of a contravariant vector then we obtain precisely n such vectors for ν = 1, 2, …, n, and they have no other invariants under (7) other than the inner product:

$$g_{\lambda\mu} = \sum_a {}^a h_\lambda \cdot {}^a h_\mu . \tag{9}$$

These $g_{\lambda\mu}$ are the components of a tensor that can be used for the metric tensor at the point O. From (9), one gets:

$$g = |\, g_{\lambda\mu} \,| = |\, {}^a h_\nu \,|^2 = h^2, \tag{10}$$

$$g^{\lambda\mu} = \sum_a {}_a h^\lambda \cdot {}_a h^\mu , \tag{11}$$

and instead of (8), we get:

$$W = F(h, g_{\lambda\mu}, g^{\lambda\mu}, \Lambda^\nu_{\alpha\beta}, \Lambda^\nu_{\alpha\beta[\rho]}, \ldots). \tag{12}$$

If we further consider that the covariant derivatives of the $g_{\lambda\mu}$ and $g^{\lambda\mu}$ vanish identically then this gives **Theorem 3:**

All of the differential invariants of the vectors ${}^a h_\nu$ that also remain invariant under rotations are constructed from h, the tensors $g_{\lambda\mu}, \Lambda^\nu_{\alpha\beta}$, and the covariant derivatives of $\Lambda^\nu_{\alpha\beta}$.

At this point, we remark that, just as in (3), everything is to be covariant differentiated using the $\Delta^\nu_{\alpha\beta}$, a process that we previously denoted by [ρ]; e.g., $\Delta^\nu_{\alpha\beta[\rho]}$, etc. The $g_{\lambda\mu}$ yield a second type of covariant derivative that is computed by means of the connection components:

$$\Gamma^\nu_{\alpha\beta} = \tfrac{1}{2} g^{\nu\tau} \left(\frac{\partial g_{\tau\alpha}}{\partial x_\beta} + \frac{\partial g_{\tau\beta}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\tau} \right);$$

we denote this type of covariant derivative by (ρ):

$$v_{\alpha(\rho)} = \frac{\partial v_\alpha}{\partial x_\rho} - \Gamma^\lambda_{\alpha\rho} v_\lambda .$$

From (9) and (10), one computes:

(13) $$\Gamma^\nu_{\alpha\beta} = \Delta^\nu_{\alpha\beta} - \Lambda^\nu_{\alpha\beta} + \Theta^\nu_{\alpha\beta} ,$$

in which $\Theta^\nu_{\alpha\beta}$ is a tensor that is symmetric in α and β:

(14) $$\Theta^\nu_{\alpha\beta} = g^{\nu\rho} (g_{\alpha\sigma}\Lambda^\sigma_{\rho\beta} + g_{\beta\sigma}\Lambda^\sigma_{\rho\alpha}) .$$

With the help of (13), one now finds that:

(15) $$v_{\rho(\sigma)} = v_{\rho[\sigma]} + \Lambda^\nu_{\rho\sigma} v_\nu - \Theta^\nu_{\rho\sigma} v_\nu .$$

We cite some additional formulas that will be used later:

(16) $$g_{\lambda\mu} \cdot {}_a h^\mu = {}^a h_\lambda , \qquad g^{\lambda\mu} \cdot {}^a h_\mu = {}_a h^\lambda ,$$

(17) $$\frac{\partial h}{\partial x_\rho} = h \cdot \Delta^\alpha_{\alpha\rho} .$$

We further set [**14**]:

(18) $$\Lambda^\alpha_{\rho\alpha} = - \Lambda^\alpha_{\alpha\rho} = \Phi_\rho = \tfrac{1}{2}(\Delta^\alpha_{\rho\alpha} - \Delta^\alpha_{\alpha\rho}) , \qquad g^{\lambda\mu} \Phi_\mu = \Phi^\lambda.$$

§ 3.

We now pose the question of whether the action function:

(19) $$\mathfrak{W} = h \cdot W$$

can be used for the derivation of field equations. The W in (19) is an absolute invariant and from Theorem 3 it can be constructed from:

$$g_{\lambda\mu} , \ \Lambda^\nu_{\alpha\beta} , \ \Lambda^\nu_{\alpha\beta[\rho]} , \ \ldots$$

The order of W is the highest order of differentiation of the arguments that are present in W. For the single action function of order zero, we have $\mathfrak{W}_0 = h$.

Every W_1 of first order is constructed from the $g_{\lambda\mu}$ and the $\Lambda^\nu_{\alpha\beta}$. The determination of all W_1 thus leads (when we consider $g_{\lambda\mu}$ to be the metric tensor) to the hitherto-unsolved problem of tensor algebra: Determine the orthogonal invariants of the mixed tensor $\Lambda^{\nu}_{\alpha\beta}$ ([1]), which is alternating with respect to α and β.

It is easy to prove (on the basis of the so-called "first fundamental theorem for rotational invariants") that there is no W_1 of first degree in the $\Lambda^{\nu}_{\alpha\beta}$, and those of second degree are completely specified by:

$$
(20) \qquad \begin{cases} A = g^{\mu\nu}\Lambda^{\alpha}_{\mu\beta}\Lambda^{\beta}_{\nu\alpha} & [\mathbf{13},\mathbf{14}], \\ B = g_{\mu\nu}g^{\alpha\rho}g^{\beta\gamma}\Lambda^{\mu}_{\alpha\beta}\Lambda^{\nu}_{\sigma\gamma} & [\mathbf{13},\mathbf{14}], \\ \Phi = g^{\mu\nu}\Lambda^{\alpha}_{\mu\alpha}\Lambda^{\beta}_{\nu\beta} = g^{\mu\nu}\Phi_{\mu}\Phi_{\nu} = \Phi^{\nu}\Phi_{\nu}. \end{cases}
$$

As you know, no first-order invariants W_1 can be constructed from the $g_{\lambda\mu}$ alone; it is only in second order that the curvature tensors $R^{i}_{km,l}$, R_{km}, and the invariant R appear. From Theorem 3, these must all be expressible in terms of $g_{\lambda\mu}$, $\Lambda^{\nu}_{\alpha\beta}$, and $\Lambda^{\nu}_{\alpha\beta[\rho]}$. In fact, after some computations that start with:

$$
R^{i}_{km,l} = \frac{\partial\Gamma^{i}_{km}}{\partial x_l} - \frac{\partial\Gamma^{i}_{kl}}{\partial x_m} + \Gamma^{i}_{lr}\Gamma^{r}_{km} - \Gamma^{i}_{mr}\Gamma^{r}_{kl}
$$

along with (13), (14), then one gets:

$$
(21) \quad \begin{aligned} R^{i}_{km,l} = &-\Lambda^{i}_{km[l]} + \Lambda^{i}_{kl[m]} + \Theta^{i}_{km[l]} - \Theta^{i}_{kl[m]} + 2\Lambda^{\rho}_{lm}\Lambda^{i}_{k\rho} - \Lambda^{\rho}_{km}\Lambda^{i}_{l\rho} + \Lambda^{\rho}_{kl}\Lambda^{i}_{m\rho} \\ &+ 2\Lambda^{\rho}_{ml}\Theta^{i}_{k\rho} - \Lambda^{\rho}_{km}\Theta^{i}_{l\rho} + \Lambda^{\rho}_{kl}\Theta^{i}_{m\rho} + \Lambda^{i}_{l\rho}\Theta^{\rho}_{km} - \Lambda^{i}_{m\rho}\Theta^{\rho}_{kl} + \Theta^{i}_{l\rho}\Theta^{\rho}_{km} - \Theta^{i}_{m\rho}\Theta^{\rho}_{kl}, \end{aligned}
$$

$$
(22) \quad \begin{aligned} R_{km} = &-\Lambda^{i}_{km[i]} + 2\Phi_{k[m]} + \Theta^{i}_{km[i]} + \Lambda^{\beta}_{\alpha m}\Lambda^{\alpha}_{k\beta} + 2\Phi_{\rho}\Lambda^{\rho}_{km} - 2\Phi_{\rho}\Theta^{\rho}_{km} \\ &+ \Lambda^{\beta}_{m\alpha}\Theta^{\alpha}_{k\beta} + \Lambda^{\beta}_{k\alpha}\Theta^{\alpha}_{m\beta} - \Theta^{\alpha}_{k\beta}\Theta^{\beta}_{k\alpha}, \end{aligned}
$$

$$
(23) \quad R = 4(\Psi - \Phi) - 2A - B,
$$

in which we have set:

$$
(24) \qquad \Psi = \Phi^{\alpha}_{[\alpha]} = g^{\alpha\nu}\Phi_{\nu[\alpha]} = g^{\alpha\nu}\Lambda^{\beta}_{\nu\beta[\alpha]}.
$$

Ψ is the simplest second-order invariant.

([1]) The determination of the affine invariants of such a tensor has still not been realized, either. The furthest advance was for $n = 4$: J. C. CHOUFOER, "Het bilineare Punt-Lijn Connex in de driedimensionale ruimte," Dissertation, Amsterdam (1927).

§ 4.

We shall now treat the field equations that the action functions that were found in the previous paragraphs yield. We think of the variations:

$$\delta\, {}^a h_\nu = {}^a v_\nu \tag{25}$$

as having been chosen in the usual way such that they, along with their derivatives, vanish on the boundary of the integration domain.

From (25), one then computes:

$$\delta_b h^\sigma = - {}_b h^\nu \cdot {}_a h^\sigma \cdot {}^a v_\nu , \tag{26}$$

$$\delta h = h \cdot {}_a h^\nu \cdot {}^a v_\nu , \tag{27}$$

$$\delta g_{\mu\nu} = \sum_a {}^a h_\mu \cdot {}^a v_\lambda + \sum_a {}^a h_\lambda \cdot {}^a v_\mu , \tag{28}$$

$$\delta g^{\mu\nu} = - (g^{\mu\nu} \cdot {}_a h^\lambda + g^{\lambda\nu} \cdot {}_a h^\mu) \cdot {}^a v_\nu , \tag{29}$$

$$\delta \Delta^\sigma_{\alpha\beta} = - \Delta^\nu_{\alpha\beta} \cdot {}_a h^\sigma \cdot {}^a v_\nu + {}_a h^\sigma \cdot \frac{\partial\, {}^a v_\alpha}{\partial x_\beta} , \tag{30}$$

$$\delta \Lambda^\sigma_{\alpha\beta} = - \Lambda^\nu_{\alpha\beta} \cdot {}_a h^\sigma \cdot {}^a v_\nu + \tfrac{1}{2}\, {}_a h^\sigma \left(\frac{\partial\, {}^a v_\alpha}{\partial x_\beta} - \frac{\partial\, {}^a v_\beta}{\partial x_\alpha} \right), \tag{31}$$

$$\delta \Phi_\alpha = - \Lambda^\nu_{\alpha\rho} \cdot {}_a h^\rho \cdot {}^a v_\nu + \tfrac{1}{2}\, {}_a h^\rho \left(\frac{\partial\, {}^a v_\alpha}{\partial x_\rho} - \frac{\partial\, {}^a v_\rho}{\partial x_\alpha} \right). \tag{32}$$

If $\mathfrak{W} = h \cdot W$ is the action function then we set:

$$\delta \int h\, W\, dx = \int \delta(h\, W)\, dx = \int {}_a[h\, W]^\nu \cdot {}^a v_\nu\, dx , \tag{33}$$

so the contravariant vector densities:

$${}_a[\mathfrak{W}]^\nu = {}_a[h\, W]^\nu \tag{34}$$

are the "variational derivatives" of $\mathfrak{W}$, and when we set them equal to zero they give the field equations that are associated with W.

We begin with the second-order invariant $W = \Psi = \Phi^\alpha_{[\alpha]}$:

$$\delta(h\,\Psi) = \delta h \cdot \Psi + h \cdot \delta\Psi = h\Psi \cdot {}_a h^\nu \cdot {}^a v_\nu + h \cdot \delta\left(\frac{\partial \Phi^\alpha}{\partial x_\alpha} + \Delta^\alpha_{\lambda\alpha}\Phi^\lambda\right).$$

We then treat the second term further, in which the derivatives $\partial\, {}^a v_\nu \;/\; \partial x_\alpha$ will be removed by means of partial integration. One gets:

(35) $$_a[h\,W]^\nu = 2h \cdot \left\{(\Psi - \Phi)\cdot {}_a h^\nu - \Phi^\nu_{[\alpha]} \cdot {}_a h^\alpha\right\}.$$

If one then chooses $\Psi = \Phi^\alpha_{[\alpha]}$ to be the action function then the n^2 field equations amount to the determination of the n^2 vector components ${}^a h_\nu$:

(36) $$(\Psi - \Phi) \cdot {}_a h^\nu - \Phi^\nu_{[\alpha]} \cdot {}_a h^\alpha = 0.$$

If one multiplies this by ${}^a h_\nu$ then one gets $n(\Psi - \Phi) - \Psi = 0$, so $\Psi = \dfrac{n}{n-1}\Phi$ $(n > 1)$, and instead of (36), we have:

(37) $$\Phi^\alpha_{[\beta]} = \frac{1}{n-1}\Phi\,\delta^\alpha_\beta, \qquad \Phi_{\alpha[\beta]} = \frac{1}{n-1}\Phi \cdot g_{\alpha\beta}.$$

The integrability conditions for these equations are obtained from the generalized RICCI equation ([**5**], pp. 14) for an arbitrary tensor, viz.:

$$T_{\alpha[\beta]} - T_{\beta[\alpha]} = -2\Lambda^\mu_{\alpha\beta}\,T_{[\mu]},$$

when one considers (37) and assumes that $\Phi \neq 0$, and they take the form:

(38) $$(n-1)\,\Lambda^\nu_{\alpha\beta} = \Phi_\alpha\delta^\nu_\beta - \Phi_\beta\delta^\nu_\alpha \qquad (\Lambda^\nu_{\alpha\beta}\,\Phi_\nu = 0).$$

With the help of rot ${}^a h_\nu = {}^a p_{\alpha\beta}$, this can be written in a particularly simple way:

(39) $$(n-1)\,p_{\alpha\beta} = \Phi_\alpha\,h_\beta - \Phi_\beta\,h_\alpha.$$

The integrability conditions for these $\frac{1}{2}n(n-1)$ first-order equations are satisfied ([1]) due to (37).

([1]) From (37_2), it further follows that $\Phi_{\alpha[\beta]} - \Phi_{\beta[\alpha]} = 0$, and from this, due to (38_2): rot $\Phi_\nu = \dfrac{\partial\Phi_\alpha}{\partial x_\beta} - \dfrac{\partial\Phi_\beta}{\partial x_\alpha} = 0$. Thus, if these last equations are to be used to characterize the pure gravitational field (cf., EINSTEIN [14], remark pp. 225) then Ψ is the action function that will accomplish this.

The other invariants Φ, A and B, when taken to be action functions yield the following variational derivatives:

(40) $${}_a[h\,\Phi]^\nu = -\,h\cdot\Phi^\nu_{[\alpha]}\cdot{}_a h^\alpha\,.$$

(When ${}^a h_\nu$ is contracted, this gives: $-\,h\,\Psi$.)

(41) $${}_a[h\,A]^\nu = h\,A\cdot{}_a h^\nu - 2h\,g^{\lambda\nu}\cdot{}_a h^\nu\,\Lambda^\alpha_{\lambda\beta}\Lambda^\beta_{\mu\alpha} + h\,g^{\lambda\mu}\cdot{}_a h^\beta\,\Lambda^\nu_{\lambda\beta[\mu]} - h\,g^{\lambda\nu}\cdot{}_a h^\beta\,\Lambda^\mu_{\lambda\beta[\mu]}$$
$$-\,2h\,\Phi^\lambda\cdot{}_a h^\beta\cdot\,\Lambda^\nu_{\lambda\beta} + 2h\,\Phi_\alpha\cdot{}_a h^\beta\cdot g^{\lambda\nu}\,\Lambda^\alpha_{\lambda\beta}\,.$$

(when ${}^a h_\nu$ is contracted this gives: $h\{(n-2)\,A + \Psi - 2\Phi\}$.)

(42) $${}_a[h\,B]^\nu = h\,B\cdot{}_a h^\nu - 4h\,g^{\lambda\mu}\,g^{\alpha\nu}\,g_{\rho\sigma}\cdot{}_a h^\beta\,\Lambda^\rho_{\lambda\alpha}\Lambda^\sigma_{\mu\beta} + 2h\,g^{\lambda\mu}\,g^{\alpha\nu}\cdot{}^a h_\rho\,\Lambda^\rho_{\lambda\alpha[\nu]}$$
$$+\,2h\,g^{\lambda\sigma}\,g^{\alpha\mu}\cdot{}^\sigma h_\rho\,\Lambda^\rho_{\lambda\alpha}\Lambda^\nu_{\sigma\mu} - 4h\,g^{\alpha\varpi}\cdot{}^a h_\rho\cdot\Phi^\lambda\,\Lambda^\rho_{\lambda\alpha}\,.$$

(when ${}^a h_\nu$ is contracted this gives: $h\{(n-2)\,B + 2\Psi - 4\Phi\}$.)

${}_a[h\;R]^\nu$ is computable using this, with the help of (23), but one does not arrive at a simpler expression.

§ 5.

For the sake of completeness, we shall briefly state how one obtains connection components of a different sort from simpler tensors for $n = 4$.

According to RIEMANN, we have ten functions g_{ik} that are the coefficients of a quadratic differential form, from which the $\Gamma_{i,kl}$ and Γ^i_{kl} are derived in a well-known way. In the above, we employed 16 functions ${}^a h_\nu$ instead of the ten g_{ik}, which are the components of four independent vectors, and we then derived the $\Delta^\nu_{\alpha\beta}$ from these [equation (2)]. In both cases, differential invariants were used as action functions, and from this the “field equations” were ascertained by varying the ten g_{ik} (16 ${}^a h_\nu$, resp.).

As we would like to explain shortly, one can now also manage with less than 10, namely, 8, 6, and 5 functions, and in doing so, to be sure, the order of differentiation will be higher (on this, cf. [**6**]).

1. Namely, if we first begin (for $n = 4$) with two covariant vectors a_i and α_i that have the rotations $f_{ik} = \text{rot}\;a_i$, $\phi_{ik} = \text{rot}\;\alpha_i$ then we will get five first-order scalar densities:

(43) $$\mathfrak{A}_{11} = \tfrac{1}{2}\sum f_{12}f_{34}\,,\qquad \mathfrak{A}_{12} = \tfrac{1}{2}\sum f_{12}\phi_{34}\,,\qquad \mathfrak{A}_{22} = \tfrac{1}{2}\sum \phi_{12}\phi_{34}\,,$$
$$\mathfrak{B}_1 = \sum f_{12}(a_3\alpha_4 - a_4\alpha_3)\,,\qquad \mathfrak{B}_2 = \sum \phi_{12}(\alpha_3 a_4 - \alpha_4 a_3)\,.$$

If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are two arbitrary scalar densities then:

(44) $$b_i = \frac{1}{\mathfrak{R}_1}\sum f_{ik}\,\phi^{kl} a_l\,, \qquad \beta_i = \frac{1}{\mathfrak{R}_2}\sum \phi_{ik}\, f^{kl}\alpha_l$$

define two new covariant vectors that, together with a_i and α_i , constitute four linearly-independent vectors when $\mathfrak{A}_{11}\mathfrak{B}_2^2 + 2\mathfrak{A}_{12}\mathfrak{B}_1\mathfrak{B}_2 + \mathfrak{A}_{22}\mathfrak{B}_1^2 \neq 0$.

We can then use these four vectors in place of the previous ${}^a h_\nu$. The type of covariant derivative that was given here, as well as the simplest action function and field equations, are examined in the dissertation of the Dutch chess master MAX EUWE [**8**].

2. Secondly, for $n = 4$, we can start with a second-rank alternating covariant tensor p_{ik} , for which the invariant $\mathfrak{R} = \frac{1}{2}\sum p_{12}\,p_{34} \neq 0$, and which has zero rotation [**12**]. Here:

$$\xi^m = \frac{1}{\mathfrak{R}}\sum_{\text{cycl.}}\frac{\partial p_{ik}}{\partial x_l}$$

is a contravariant vector; therefore, $\xi_i = p_{i\lambda}\,\xi^\lambda$ is a (first-rank) covariant vector and π_{ik} = rot ξ_ν is a (second-rank) alternating covariant tensor. Thus, one can easily derive a (third-rank) covariant vector η_i from p_{ik}, ξ_i, and π_{ik} , namely

$$\eta_i = \text{grad}\ \frac{\sum p_{12}\pi_{34}}{\sum p_{12}\,p_{34}}\,,$$

and we can proceed as we did in case 1 with ξ_i and η_i .

3. If we have $\mathfrak{R} = \frac{1}{2}\sum p_{12}\,p_{34} = 0$ then we have five independent components p_{ik} for a special alternating tensor. Moreover, if p_{ik} is a rotation then it gives us no differential invariant; on the other hand, if p_{ik} is not a rotation then a third-rank scalar density can be recognized ([**7**] and [**12**], pp. 18, *et seq*.), and differential invariants of fourth and higher rank can be computed with it, as in case 2.

Received on 28 November

Stars of congruences and absolute parallelism: Geometric basis for a recent theory of Einstein

Note by E. BORTOLOTTI
presented ([1]) to the Society by T. LEVI-CIVITA

1. In this last year, the literature of relativity has been expanded by, among other things, numerous researches that attempt to construct an "einheitliche Feldtheorie von Gravitation und Elektrizität" (unified theory of gravitation and electricity). Einstein was led to make some very tentative and substantially diverse contributions to this study in 1923 ([2]), 1925 ([3]), and, more recently, in 1928 ([4]). The last effort – whose physical justification is perhaps not completely obvious, and which I will refer to the author ([5]) – has, however, an advantage over the preceding one in that it has a much simpler mathematical formulation. It is precisely upon the geometric basis of this new theory that I will now expound. Einstein has constructed this geometric basis [**5**], and has recovered, among other things, many results of the preceding research that he had not shown to be known. It will therefore not be pointless to treat this research, which is, indeed, little known, since (from a viewpoint that is a little more general), as a result of new results and observations, as well as the more noteworthy results of the preceding papers, by its exposition one more or less arrives at a link to Einstein's new theory, and defines its mathematical basis. I will then limit myself to results that relate to the theory of *Euclidian* connection with absolute parallelism, in particular; one might confer another recent note ([6]) that was dedicated to the more general study of *affine* connections with absolute parallelism for all that remains (except for a brief hint that will do for now) as regards this argument and for the bibliography on relativity.

([1]) At the session on 17 March 1929.

([2]) **1**. "Zur allgemeinen Relativitätstheorie," Sitz. Preuss. Akad. der Wiss. (1923), 32-38.
2. "Bemerkung zu meiner Arbeit 'Zur allgemeinen Relativitätstheorie,'" *ibid.*, pp. 76-77.
3. "Zur affinen Feldtheorie," *ibid.*, pp. 137-140.

([3]) **4.** "Einheitliche Feldtheorie von Gravitation und Elektrizität," *ibid.* (1925), 414-419.

([4]) **5.** "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus," *ibid.* (1928), 217-222.
6. "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektizität," *ibid.*, pp. 224-227.

([5]) "It is therefore conceivable that this theory will supersede the original conception of the general theory of relativity."

([6]) **7.** "Parallelismo assoluto nelle varietà a connessione affine, e nuove vedute sulla relatività," presented on 27 January 1929 at the Academy of Science in Bologna.

2. In an n-dimensional manifold (X_n), in which the u^λ ($\lambda, \mu, \nu, \tau, \omega = 1, 2, \ldots, n$) are curvilinear coordinates, consider n fields of (independent) contravariant vectors $\underset{i}{X}^\lambda$ ($i, j, h, k, l = 1, 2, \ldots, n$), namely, n^2 quantities $\underset{i}{X}^\lambda$ such that $1 / h = \left| \underset{i}{X}^\lambda \right| \neq 0$, and which that are functions of the points in X_n . Such an n-tuple of vector fields, which one can also suppose to subjected *to an arbitrary linear* (affine) *transformation* with *constant* coefficients $c^i_{\cdot\, j}$ of the n fields:

$$(1) \qquad \underset{i}{X}'^\lambda = c^i_{\cdot\, j} \underset{i}{X}^\lambda \qquad \left| c^i_{\cdot\, j} \right| \neq 0,$$

determine an *affine connection with zero curvature* ([1]) in X_n, namely, one with absolute parallelism, that has the parameters:

$$(2) \qquad \Gamma^\lambda_{\mu\nu} = \underset{i}{X}^\lambda \frac{\partial \overset{i}{X}_\mu}{\partial u^\nu},$$

where $\overset{i}{X}_\mu$ are the reciprocal elements to $\underset{i}{X}^\mu$ in $\left| \underset{i}{X}^\mu \right|$, so they are n fields of covariant vectors that are uniquely determined by the fields $\underset{i}{X}^\mu$. The n fields $\underset{i}{X}^\mu$ (or $\overset{i}{X}_\mu$) are n *parallel vector fields* that are also *equipollent* for such a connection. Others ([2]) have called this connection the *Weitzenböck-Vitali affine connection*; in effect, the covariant derivative that corresponds to it, namely:

$$(3) \qquad \nabla_\nu \xi^\lambda = \frac{\partial \xi^\lambda}{\partial u^\nu} + \Gamma^\lambda_{\mu\nu} \xi^\mu, \qquad \nabla_\nu \eta_\mu = \frac{\partial \eta_\mu}{\partial u^\nu} - \Gamma^\lambda_{\mu\nu} \eta_\lambda,$$

was introduced (in relation to an n-tuple of vector fields) in 1921 by Weitzenböck (*loc. cit.*, in [**7**]) and then, independently, in 1924 by Vitali ([3]).

In particular, when an n-tuple $\underset{i}{X}^\lambda$ is subjected to *an orthogonal substitution* (namely, a *rotation*) with constant coefficients it also defines a symmetric tensor $a_{\lambda\mu}$ that can be assumed to represent the fundamental tensor of a metric in X_n, which then makes it a V_n ([4]):

([1]) See the cited paper [**7**].
([2]) In the paper **8**. "Reti di Cebiceff e sistemi conjugati nelle V_n riemanniane," Rend. Acc. dei Lincei," (6) **5** (1927), 741-747 on pp. 745.
([3]) **9.** "Una derivazione covariante formata coll'ausilio di n sistemi covarianti del 1° ordine," Atti Soc. Liguistica **2** (1924), 248-253.
([4]) See [**9**], pp. 250. This is, in another form, the well-known result of the research of Ricci on n-tuples of congruences (1895). In particular, cf.:

10. A. CARPANESE, "Parallelismo e curvature in una varietà qualunque," Annali di Matem. (3) **28** (1918), 147-168.

(4) $$a_{\lambda\mu} = \overset{i}{X}_{\lambda}\,\overset{i}{X}_{\mu}\,, \qquad a^{\lambda}_{\mu} = \underset{i}{X}^{\lambda}\,\overset{i}{X}_{\mu}\,, \qquad a^{\lambda\mu} = \underset{i}{X}^{\lambda}\,\underset{i}{X}^{\mu}\,.$$

The n fields $\underset{i}{X}^{\lambda}$ (or $\overset{i}{X}_{\lambda}$) are *unitary* and *orthogonal* with respect to this metric. One obviously has:

(5) $$\overset{i}{X}^{\lambda} = a^{\lambda\mu}\,\overset{i}{X}_{\mu} = \underset{i}{X}^{\lambda}\,;$$

hence, we will no longer distinguish indices i, j, h, k, l, … as upper or lower. The Weitzenböck-Vitali affine connection then becomes a *Euclidian connection* (*with absolute parallelism*) in relation to this metric ([1]). However, the tensor $a_{\lambda\mu}$ is also given another Euclidian connection, namely, the usual (torsion-less) or (following Cartan) *Levi-Civita connection:* It has the parameters $\left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$ (viz., the Christoffel symbols that are constructed from $a_{\lambda\mu}$). The Weitzenböck-Vitali Euclidian connection can be represented by means of its *relative components:*

(6) $$T_{\lambda\mu}^{\cdot\cdot\cdot\nu} = \Gamma^{\nu}_{\lambda\mu} - \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\}$$

with respect to those of Levi-Civita ([2]). If one indicates the elements that refer to this latter connection with the index $^{\mathrm{o}}$ then we have [**11**]:

(7) $$T_{\lambda\mu}^{\cdot\cdot\cdot\nu} = \overset{\mathrm{o}}{\nabla}_{\mu}\,\overset{i}{X}_{\lambda}\,\overset{i}{X}^{\nu}\,,$$

(8) $$\nabla_{\nu}\,\xi^{\lambda} = \overset{\mathrm{o}}{\nabla}_{\nu}\xi^{\lambda} + T_{\mu\nu}^{\cdot\cdot\cdot\lambda}\xi^{\mu}\,, \qquad \nabla_{\nu}\,\eta_{\mu} = \overset{\mathrm{o}}{\nabla}_{\nu}\eta_{\mu} - T_{\mu\nu}^{\cdot\cdot\cdot\lambda}\eta_{\lambda}\,.$$

The *Cartesian components* of the tensor $T_{\lambda\mu}^{\cdot\cdot\cdot\nu}$ with respect to the generic n-tuple $\overset{i}{X}_{\lambda}$ are precisely ([**11**], pp. 458) the *rotation coefficients* of the n-tuple:

(9) $$T_{\lambda\mu}^{\cdot\cdot\cdot\nu} = \gamma_{ijl}\,\overset{j}{X}_{\lambda}\,\overset{l}{X}_{\mu}\,\overset{i}{X}^{\nu}\,, \qquad \gamma_{ijl} = T_{\lambda\mu}^{\cdot\cdot\cdot\nu}\,\overset{j}{X}^{\lambda}\,\overset{l}{X}^{\mu}\,\overset{i}{X}_{\nu}\,.$$

There, the author defined a metric by means of n Pfaffians $\omega^{j} = \overset{i}{X}_{\lambda}\,dx^{\lambda}$ by setting $ds^2 = \sum_i (\omega^{j})^2$.

([1]) For *Weitzenböck-Vitali Euclidian connections*, see my note:

11. "Parallelismi assoluti nelle V_n riemanniane," Atti Istituto Veneto **86** (1926/27), 455-465, and

12. "On metric connections with absolute parallelism," Proc. Kon. Akad. Amsterdam **30** (1927), 216-218.

([2]) The tensor $T_{\lambda\mu}^{\cdot\cdot\cdot\nu}$ was introduced with its expressions (7) and (9) by me ([**11**], [**12**]) and then rediscovered, from another viewpoint, by:

13. A. TONOLO, "Stelle di ennuple ortogonali di congruenze di curve in una V_n," Rend. Ist. Lombardo (2) **60** (1927), 253-263 on page 256 (T_{lmn} and φ_{pql} in that paper).

3. We go on to the geometric interpretation of the elements that were introduced. As in [**7**], we will call the totality of vector fields that have *constant Cartesian components* with respect to an *n*-tuple $\overset{i}{\mathbf{X}}$ the *star* $S_{\mathbf{X}}$ of vector fields that are derived from $\overset{i}{\mathbf{X}}$. We say that $S_{\mathbf{X}}$ is an *affine star* – or *angular star*, respectively – (or simply a *star*, as in the more interesting case) when the *n*-tuple $\overset{i}{\mathbf{X}}$ is arbitrary or *unitary orthogonal*, respectively. Therefore, an (angular) star of vector fields is constructed by starting with an *n*-tuple $\overset{i}{\mathbf{X}}$ and then including all of the ones that are obtained by means of arbitrary rotations (with constant coefficients).

Just as an *n*-tuple of vector fields determines an *n-tuple of line congruences*, similarly, a star of vector fields will determine a totality of $\infty^{n(n-1)/2}$ congruences that one calls *stars of congruences.* More precisely: An (angular) *star of congruences* will be a totality of line congruences such that *every line of any arbitrary one of them is the isogonal trajectory to all of the remaining congruences*; we agree to say an *affine star* of congruences in the general case. For $n = 2$, one has a *sheaf of congruences*, a notion that was introduced by Ricci and applied by him systematically, ending in 1898, in his *Teoria delle superficie* (1). It was also studied recently by Delens (2), who referred to such sheaves as *réseau angulaire*. Given this, it is obvious that for the Weitzenböck-Vitali Euclidian connection that is determined from the *n*-tuple $\overset{i}{\mathbf{X}}$ *the lines of the star* $S_{\mathbf{X}}$ *of congruences are the geodetic (i.e., auto-parallel) lines of the connection.* If one assigns a metric to V_n and, arbitrarily, a *star of congruences* (or, what amounts to the same thing, an *n*-tuple of orthogonal congruences) then the corresponding connection is well-defined. The *absolute parallelism* that corresponds to it consists of the invariance of the angles that the directions (which vary with the parallelism) make with the lines of the star of congruences, or (more simply) of the *n* congruences of an (arbitrary) orthogonal *n*-tuple that belongs to the star (3).

The *curvature* of the connection in question is zero: By contrast, the *torsion* is not zero, as long as V_n is not a Euclidian R_n ([**11**], pp. 459; [**5**], pp. 220; [**7**]). More precisley, the torsion tensor is [**11**]:

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \tfrac{1}{2}(\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda}) = \tfrac{1}{2}(T_{\lambda\mu}^{\cdot\cdot\nu} - T_{\mu\lambda}^{\cdot\cdot\nu}) = \tfrac{1}{2}(\gamma_{ijl} - \gamma_{ilj})\overset{j}{X}_{\lambda}\overset{l}{X}_{\mu}\overset{i}{X}^{\nu}. \tag{10}$$

(1) **14.** "Lezioni sulla teoria delle superficie," Padua, Drucker, 1898, pp. 163-223. See also:
15. RICCI and LEVI-CIVTA, "Méthodes de calcul différentiel absolu et leurs applications," Math. Ann. **54** (1900), 125-201, esp. pp. 165-168.

(2) **16.** P. C. DELENS, *Méthodes et problèmes des géométries différentielles euclidienne et conformé*, Paris, Gauthier-Villars, 1927, pp. 79.
Two recent notes are dedicated to the general case (viz., *n* arbitrary):
17. G. POATO, "Stelle di ennuple ortogonali in una varietà V_n a metrica qualunque," Bolletino Un. Matem. Italiana **5** (1926), 125-127 and A. TONOLO, [**13**] , cited above.

(3) This interpretation is the one that was given by VITALI ([**9**], pp. 253) *in relation to an n-tuple.*

The *torsion vector* relative to the planar face that is defined by the directions $\left(\overset{i}{\mathbf{X}}\right)$, $\left(\overset{j}{\mathbf{X}}\right)$ is ([**8**], pp. 746):

(11) $$\overset{ij}{S}{}^{\nu} = 2S_{\lambda\mu}^{\cdot\cdot\nu}\,\overset{i}{X}{}^{\lambda}\,\overset{j}{X}{}^{\mu} = \frac{\bar{d}^{\circ}\,\overset{j}{X}{}^{\nu}}{ds_i} - \frac{\bar{d}^{\circ}\,\overset{i}{X}{}^{\nu}}{ds_j},$$

where $\dfrac{\bar{d}^{\circ}}{ds_i} = \overset{i}{X}{}^{\lambda}\,\overset{\circ}{\nabla}_{\lambda}$ is the cogredient derivative in the direction $\left(\overset{i}{\mathbf{X}}\right)$.

4. The vanishing of curvature is a *characteristic* property of the Weitzenböck-Vitali connection. In fact, it has been proved ([**1**], pp. 461; [**12**], pp. 217) that *any Euclidian connection with zero curvature can always be interpreted as a Weitzenböck-Vitali connection with respect to* $\infty^{n(n-1)/2}$ *orthogonal n-tuples of congruences.* Each of them is determined in the direction of the line that emanates from an initial point – namely, *with respect to an (angular) star of congruences.*

We thus have that *the differential geometry of Euclidian connections with absolute parallelism coincides with the theory of differential invariants for an (angular) star of vector fields; namely, of a Riemannian metric and a star of congruences.* It is precisely this theory, in substance, upon which Einstein based his ultimate formulation of relativity, insofar as he introduced the $n^2 = 16$ components $\overset{i}{X}{}^{\lambda}$ in order to define the geometry of the universe, while postulating ([**5**], pp. 218) *Drehungsinvarianz* (i.e., rotational invariance), namely, taking into consideration only those elements that are invariant under orthogonal substitution with constant coefficients of the vectors $\overset{i}{\mathbf{X}}$.

Weitzenböck has proved ([1]) that the differential invariants of order *m* in the vectors $\overset{i}{\mathbf{X}}$ of the orthogonal group (namely: of the (angular) star of vector fields that the *n*-tuple $\overset{i}{\mathbf{X}}$ defines) *are the algebraic invariants of the tensors* $a_{\lambda\mu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$, *and the covariant derivatives* (for the derivative ∇_λ) *of* $S_{\lambda\mu}^{\cdot\cdot\nu}$ *up to order* $m - 1$.

5. In particular, the invariants of first order are the algebraic invariants of $a_{\lambda\mu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$. More simply, they are: The symmetric tensors:

(12) $$b_{\lambda\mu} = T_{\tau\lambda\omega}T^{\tau\cdot\omega}_{\cdot\mu} = a_{\nu\tau}\overset{\circ}{\nabla}_{\lambda}\overset{i}{X}{}^{\nu}\cdot\overset{\circ}{\nabla}_{\mu}\overset{i}{X}{}^{\tau} = \gamma_{ijl}\gamma_{ijh}\overset{l}{X}_{\lambda}\overset{h}{X}_{\mu},$$

([1]) **18.** "Differentialinvarianten in der Einsteinschen Theorie des Fernparallelismus," Sitz. Preuss. Akad. Berlin (1928), 466-474, on pp. 469.

(13) $$g_{\lambda\mu} = S_{\nu\lambda}^{\cdot\cdot\cdot\tau} S_{\tau\mu}^{\cdot\cdot\cdot\nu},$$

the vector [1]:

(14) $$\Phi_\mu = S_{\mu\lambda}^{\cdot\cdot\cdot\lambda} = \tfrac{1}{2} T_{\mu\lambda}^{\cdot\cdot\cdot\lambda} = \tfrac{1}{2}\gamma_{hjh} \overset{j}{X}_\mu = \frac{1}{2h}\nabla_\mu h,$$

and the (absolute invariant) scalars:

(15) $$\Phi = \Phi_\mu \Phi^\mu = a_{\lambda\mu} \Phi^\lambda \Phi^\mu = \tfrac{1}{2} \gamma_{hih} \gamma_{kik},$$

(16) $$B = T_{\lambda\mu\nu} T^{\lambda\mu\nu} = b_{\lambda\mu} a^{\lambda\mu} = \sum_{ijl} (\gamma_{ijl})^2,$$

(17) $$C = T_{\lambda\mu\nu} T^{\mu\lambda\nu} = \gamma_{ijl} \gamma_{ilj},$$

(18) $$S = S_{\lambda\mu\nu} S^{\lambda\nu\mu} = g_{\lambda\mu} a^{\lambda\mu} = \frac{3C - B}{4},$$

(19) $$T = S_{\lambda\mu\nu} S^{\lambda\mu\nu} = \sum_{ij} (\mathrm{mod}\,\overset{ij}{S})^2 = \frac{B - C}{2}.$$

The character of invariance under rotations of the n-tuple $b_{\lambda\mu}$ (i.e., *Drehungsinvarianz*) had been noticed by G. Poato ([**17**], pp. 127); that of the scalar B was noted by Ricci for $n = 2$ ([**14**], pp. 186; [**15**], pp. 167), and in the general case, by A. Tonolo ([**13**], pp. 263). The invariants Φ_μ, S, T were introduced by Einstein ([**5**], pp. 221; [**6**], pp. 225), who expressed the hypothesis that the vanishing of the vector Φ_μ can characterize a "pure gravitational field," and that the scalar S can take on the role of a *universal function* by which one deduces the field equations of the theory by a variational process. Weitzenböck also took into consideration [**18**] the cases in which one chooses T to be a universal function, and then Φ, and then the second-order scalar invariant:

(20) $$\Psi = a^{\lambda\mu} \nabla_\lambda \Phi_\mu,$$

whose vanishing for $n = 2$ (cf., Ricci, [**14**], pp. 205) expresses the idea that the sheaf of congruences is *isothermal*.

6. The preceding expressions (12) to (19) for the invariant tensors and scalars of first order (or at least some of them) can also exhibit a simple geometric significance: The vector Φ_μ is *the sum of the curvature vectors* of the lines of an (arbitrary) orthogonal n-

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(1) Where $\nabla_\mu h = \dfrac{\partial h}{\partial u^\mu} - \Gamma_{\mu\nu}^{\nu} h$ is the *covariant derivative* of the relative (invariant) scalar (i.e., scalar density) $h = |\overset{i}{X}_\lambda| = \sqrt{a}$ $(a = |a_{\lambda\mu}|)$. See [**7**].

tuple of the star (at the point in question), In the case of $n = 2$, the fact that this vector is *the same* for all pairs of orthogonal congruences that intersect a given congruence isogonally is contained implicitly in an observation of Ricci ([**14**], pp. 193) and stated explicitly by C. G. Weatherburn ([1]). For $n = 2$, one has:

$$\Phi_\mu = \tfrac{1}{2}\bar{\varphi}_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu}\,\varphi^\nu, \tag{21}$$

where the vector φ_μ was introduced by Ricci ([**14**], pp. 110, 168-170; [**15**], pp. 166 ([2])), and was called the *deduced system* of the system $\overset{i}{X}_\nu$ - or *covariant coordinate system* – of the sheaf that is defined by the pair of congruences $(\overset{i}{X}_\nu)$ $(i = 1, 2)$. One has (for $n = 2$):

$$\varphi_\mu = \gamma_{21j}\,\overset{i}{X}_\mu = T_{\tau\mu\nu}\,\overset{1}{X}{}^\tau\,\overset{2}{X}{}^\nu = \frac{\varepsilon^{\tau\nu}}{2}\,T_{\tau\mu\nu}, \tag{22}$$

$$\bar{\varphi}_\mu = \varepsilon_{\mu\nu}\,\varphi^\nu = T_{\mu\nu}^{\cdot\cdot\,\nu} = \gamma_{212}\,\overset{1}{X}_\mu + \gamma_{121}\,\overset{2}{X}_\mu. \tag{23}$$

We then have, in any case:

$$\overset{\circ}{\nabla}_\mu \overset{i}{X}_\nu \cdot \Phi^\mu = \tfrac{1}{2}\gamma_{ijl}\gamma_{hlh}\,\overset{j}{X}_\nu, \tag{24}$$

where for $n = 2$:

$$\overset{\circ}{\nabla}_\mu \overset{i}{X}_\nu \cdot \Phi^\mu = 0. \tag{25}$$

Therefore: For $n = 2$, *the lines of the sheaf of congruences admit the flow lines of the vector field Φ_μ as transversals for the Levi-Civita parallelism*, a property that is described in an equivalent form and proved in another way by Delens ([**16**], pp. 79).

If one desires (in the case $n = 4$) that the vector Φ_μ must represent the *electromagnetic potential* then it is suitable that it is not determined completely by the geometry of the universe, but *only up to an additive gradient*. For this, it is enough [**7**] to suppose that the vectors $\overset{i}{X}{}^\lambda$ are determined only *up to a factor ρ*, which is a function of the points in V_n, and also that the metric of V_n is defined *up to a conformal transformation.* If one supposes this then one part of the new theory of Einstein needs to be modified: One could utilize the results of Weyl geometry, but that would diminish the simplicity of the present formulation. The "pure gravitational field" will then be characterized (cf., Einstein [**6**], pp. 225) by:

([1]) **19.** "Some new theorems in the geometry of a surface," The Mathematical Gazette **13** (1926), 1-6, on pp. 6.

([2]) The vectors (which are mutually *supplemental*) φ, $\bar{\varphi}$ are denoted by – $\boldsymbol{f}$, $\boldsymbol{g}$ in DELENS, ([**16**], pp. 78).

(26) $$\mathrm{rot}^{\circ}(\Phi_\mu) = \overset{\circ}{\nabla}_\nu \Phi_\mu - \overset{\circ}{\nabla}_\mu \Phi_\nu = \frac{\partial \Phi_\mu}{\partial u^\nu} - \frac{\partial \Phi_\nu}{\partial u^\mu} = 0.$$

The significance of the tensor $b_{\lambda\mu}$ seems obvious if one observes that if ξ is an arbitrary vector then:

(27) $$b_{\lambda\mu}\,\xi^\lambda\,\xi^\mu = \sum_i \left(\mathrm{mod}\,\frac{\bar{d}^{\circ}\overset{i}{\mathbf{X}}}{ds} \right)^2 ,$$

where $\bar{d}^{\circ}/ds = \xi^\lambda \overset{\circ}{\nabla}_\lambda$. Therefore: *The form $b_{\lambda\mu}\,\xi^\lambda\,\xi^\mu$ is the sum of the squares of the curvatures that are associated with the n directions of the n-tuple* $\overset{i}{\mathbf{X}}$ (any one from the star S_{X}) *in the direction* (ξ)*, and the invariant B is the sum of the values that the preceding form takes when one chooses the direction* (ξ) *to be along one of the n directions of the n-tuple.*

A *necessary* condition for there to exist a congruence of transversals for Levi-Civita parallelism of the lines of a star of congruences is that the tensor $b_{\lambda\mu}$ have rank $< n$. There is a particular case in which this condition is certainly *not satisfied:* The case of (Riemannian) *group spaces*, according to Cartan (1). Such a space is characterized by admitting *two* Euclidian connections with absolute parallelism (of the first and second kind), and correspondingly, two classes of *translations* (of the first and second kind) such that for a translation of the first (second, resp.) kind any vector is moved by parallelism of the second (first, resp.) kind (Cartan). Among the more noteworthy properties of these spaces, we emphasize the following ones:

1. The geodetics (i.e., auto-parallels) of the two connections with absolute parallelism coincide with the geodetics of the Levi-Civita connection (namely, the $a_{\lambda\mu}$), and then *the geodetic lines* (of $a_{\lambda\mu}$) *form a star of congruences* (2). In particular, they can be distributed into $\infty^{n(n-1)/2}$ orthogonal n-tuples.

2. The tensor $S_{\lambda\mu\nu}$ is semi-symmetric, and thus coincides with $T_{\lambda\mu\nu}$, and $S = C = - B = - T$, while $\Phi_\mu = 0$.

3. The tensors $a_{\lambda\mu}$, $g_{\lambda\mu}$, $b_{\lambda\mu}$ differ from each other only by constant factors: $g_{\lambda\mu} = - b_{\lambda\mu} = \overset{\circ}{R}{}_{\tau\lambda\mu}^{\cdots\tau} = \overset{\circ}{R}_{\lambda\mu} = c\,a_{\lambda\mu}$, where $c / n = a^{\lambda\mu}\,\overset{\circ}{R}_{\lambda\mu}$ is the *constant* mean Riemannian curvature. In particular, if $n > 2$ then $b_{\lambda\mu}$ *has rank n*, if one excludes the trivial case in which $c = 0$, for which the space is Euclidian.

4. One has $\overset{\circ}{\nabla}_\omega \overset{\circ}{R}{}_{\lambda\mu\nu}^{\cdots\tau} = 0$, $\overset{\circ}{\nabla}_\omega S_{\lambda\mu}^{\cdot\cdot\nu} = 0$, so the transport by Levi-Civita parallelism will preserve the Riemannian curvature and the torsion. This property shows how this space, with its connection with absolute parallelism, must play a noteworthy role in the new Einsteinian theory: It poses the interesting question of its physical interpretation, although I will limit myself to only pointing that out.

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(1) See my paper [**7**], as well, for the bibliography.

(2) This property, which is quite expressive and *characteristic*, was pointed out by RICCI ([**14**], pp. 192-193) in the case of $n = 2$ (a case that gives only developable surfaces).

On the foundations of a new field theory of A. Einstein

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(Received on 13 January 1929)

The geometric foundations of Einstein's new theory will be discussed briefly, and some related identities will be derived. Various Ansätze for Hamilton's principle in the first, and for a simple special case, the second, approximation will then be calculated, and the corresponding fundamental equations will be presented. The rigorous form of the fundamental equations will be given for the aforementioned case. Finally, it will be remarked that the theory has advanced very far by now.

In all of the known speculations on world geometry, one usually prefers to exclude the distant comparison of geometric concepts. *Hessenberg*, *Schouten*, et al., have shown that one can construct very different local comparison geometries when one subjects the various geometric quantities to specific functions of a system of non-integrable equations. *Riemann* has already constructed a geometry in which direction depends upon the path. *A. Einstein* employed this for the interpretation of the relativity postulate. *H. Weyl* and *A. S. Eddington* went further. They extended this non-integrable path in world geometry to the remaining notions, such as length, angle, and volume. Furthermore, geometries with torsion (e.g., *Cartan, L. Infeld, K. Hattori*) come under consideration, and finally, one goes over to the non-integrability of the covariant metric (the pure 1 and 0), such that, independently of the purely geometric structure, a further arithmetic structure was introduced (e.g., *Schouten, H. J. Gramatzki*). The five-dimensional theories (e.g., *Th. Kaluza, O. Klein, H. Mandel, E. Reichenbächer, the author*) also exclude distant comparison.

In contrast to all of these conceptions, A. Einstein recently (*) took a completely unexpected position. He made the simplest possible Ansatz for the world geometry: the integrability of its fundamental notions. In this way, one arrives at non-linear second-order equations for the world geometry functions from which the physical situation can be computed.

In the first approximation, they yield:

a) Maxwell's equations.

b) The form of the gravitational laws that K. Lanczos considered, which was subject to Mach's principle.

(*) A. Einstein, Sitzungsber. d. Preuss. Akad. **17/18** (1929).

The analysis of the second approximation leads to substantial deviations from the superposition principle of the fields that are responsible for the appearance of matter.

Part one. Geometric foundations.

§ 1. We refer the points of space-time to a Gaussian coordinate system S in a continuous and one-to-one manner. The structural arrangement of points will not change when the coordinates are transformed according to:

$$\bar{x}^{\alpha} = \bar{x}^{\alpha}(x^1, x^2, x^3, x^4). \tag{1}$$

We can further postulate that a Euclidian tangent space-time E_T is constructed at each point. Now, let E_T be rigidly coupled with a system of four unit vectors (h_m) (viz., the *vierbein*). $h_{\alpha m}$ is the projection of (h_m) onto the imagined extension of dx^{α}. Let A^{α} be the α^{th} component of an arbitrary vector (A) that lies in E_T, which are referred to S in a similar way. We let A_m denote the projection of (A) onto the imagined extension of (h_m). It is then obvious that:

$$\left.\begin{array}{c} A_m = h_{\alpha m} A^{\alpha}, \ A^{\alpha} = h^{\alpha m} A_m, \ h_{\alpha m} h^{\beta m} = \varepsilon_{\alpha}^{\cdot \beta}, \ h_{\nu m} h^{\nu n} = \varepsilon_{m}^{\cdot n}, \\ \text{where } \varepsilon_{x}^{\cdot y} = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases} \end{array}\right\} \tag{2}$$

The $h_{\alpha m}$ are coordinate functions. For the displacement (ds) that the point $P(x^{\alpha})$ undergoes in order to reach the neighboring point $P'(x^{\alpha} + dx^{\alpha})$, one has, from (2):

$$ds_m = h_{\alpha m}\, dx^{\alpha}. \tag{3}$$

A Cartesian system will be represented in E_T by means of the vierbein. With that, one has:

$$(A)^2 = \sum_m A_m^2 = g_{\alpha\beta} A^{\alpha} A^{\beta}, \tag{4}$$

with

$$g_{\alpha\beta} = h_{\alpha m}\, h_{\beta m},$$

from which, one also has:

$$h_{\alpha m} = g_{\alpha\nu}\; h^{\nu m}, \qquad |\, h_{\alpha m}\,|^2 = |\, g_{\alpha\beta}\,| = g.$$

If we set $A_{\alpha} = g_{\alpha\nu} A^{\nu}$ then it follows from (2) and (4) that:

$$A_m = A_{\alpha}\, h^{\alpha m}. \tag{5}$$

One has the transformation rules (*):

$$A^{\alpha} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\nu}} \bar{A}^{\nu}, \qquad h_{\alpha m} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}} \bar{h}_{\nu m}. \tag{6}$$

§ 2. One now has:

$$dA_m = h_{\alpha m} \nabla_{\mu} A^{\alpha} dx^{\mu}, \tag{7}$$

where

$$\nabla_{\mu} A^{\alpha} = \frac{\partial A^{\alpha}}{\partial x_{\mu}} + \Delta^{\alpha}_{\lambda\mu} A^{\lambda}$$

and

$$\Delta^{\alpha}_{\lambda\mu} = h^{\alpha m} \frac{\partial h_{\lambda m}}{\partial x_{\mu}};$$

moreover:

$$h_{\lambda m} \frac{\partial h^{\alpha m}}{\partial x_{\mu}} = - \Delta^{\alpha}_{\lambda\mu}, \qquad \nabla_{\mu} A_{\lambda} = \frac{\partial A_{\lambda}}{\partial x_{\mu}} - \Delta^{\alpha}_{\lambda\mu} A_{\alpha}. \tag{8}$$

The quantities $\Delta^{\nu}_{\lambda\mu}$ obey the transformation law:

$$\Delta^{\nu}_{\lambda\mu} = \frac{\partial^2 \bar{x}^{\alpha}}{\partial x_{\lambda} \partial x_{\mu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\alpha}} + \frac{\partial \bar{x}^{\alpha}}{\partial x_{\lambda}} \frac{\partial \bar{x}^{\beta}}{\partial x_{\mu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{\gamma}} \bar{\Delta}^{\gamma}_{\alpha\beta}. \tag{9}$$

For a parallel infinitesimal displacement of the vector (A), one has:

$$dA_m = 0, \qquad \nabla_{\mu} A^{\alpha} = 0, \qquad \nabla_{\mu} A_{\alpha} = 0, \tag{10}$$

and, in particular:

$$\nabla_{\mu} g_{\alpha\beta} = 0, \qquad \nabla_{\mu} h_{\alpha m} = 0, \qquad \nabla_{\mu} h^{\alpha m} = 0. \tag{11}$$

The vierbeins in the neighborhood of a point are then arranged so as to be parallel to each other. We set:

$$\Delta^{\nu}_{\sigma\rho} - \Delta^{\nu}_{\rho\sigma} = h^{\nu m} \left(\frac{\partial h_{\sigma m}}{\partial x_{\rho}} - \frac{\partial h_{\rho m}}{\partial x_{\sigma}} \right) = \Lambda^{\cdot\cdot\nu}_{\sigma\rho}. \tag{12}$$

After some calculation, one gets:

$$\Delta^{\nu}_{\sigma\rho} = \Gamma^{\nu}_{\sigma\rho} + \Pi^{\cdot\cdot\nu}_{\sigma\rho}, \tag{13}$$

with

(*) All vierbeins $\overset{*}{h}_{\alpha m} = \vartheta_{mr}\, h_{\alpha r}$ that go to each other under proper, orthogonal substitutions with constant coefficients ϑ_{mr} are equivalent, so the metric connection ($g_{\alpha\beta}$) will not be influenced by them.

$$\Gamma_{\sigma\rho}^{\nu} = \tfrac{1}{2} g^{\nu\alpha} \left\{ \frac{\partial g_{\rho\alpha}}{\partial x_\sigma} + \frac{\partial g_{\sigma\alpha}}{\partial x_\rho} - \frac{\partial g_{\sigma\rho}}{\partial x_\alpha} \right\}$$

and

$$\Pi_{\sigma\rho}^{\cdot\cdot\nu} = \tfrac{1}{2} \{ \Lambda_{\sigma\rho}^{\cdot\cdot\nu} + \Lambda_{\sigma\cdot\rho}^{\nu} + \Lambda_{\rho\cdot\sigma}^{\nu} \} .$$

If we denote the Riemannian derivative of a quantity $A_{\alpha\dots}^{\dots\beta\dots}$ with respect to x^μ by $\delta_\mu A_{\alpha\dots}^{\dots\beta\dots}$ then it follows, in particular, that:

$$\delta_\mu h_{\alpha m} = \tfrac{1}{2} h^{\nu m} (\Lambda_{\alpha\mu\nu} + \Lambda_{\alpha\nu\mu} + \Lambda_{\mu\nu\alpha}). \tag{14}$$

We further define:

$$(d_1 d_2 - d_2 d_1) s_m = \Lambda_{\sigma\rho}^{\cdot\cdot\nu} \; h_{\nu m} \, d_1 x^\rho \, d_2 x^\sigma. \tag{15}$$

This formula defines the "torsion." If we assume that we have:

$$(d_1 d_2 - d_2 d_1) s_m = 0 \tag{16}$$

everywhere then one must have:

$$\Lambda_{\sigma\rho}^{\cdot\cdot\nu} = 0. \tag{17}$$

Conversely, when (17) is true, (16) would also follow, and we would have:

$$\frac{\partial h_{\sigma m}}{\partial x_\rho} - \frac{\partial h_{\rho m}}{\partial x_\sigma} = 0. \tag{18}$$

In that case, we could define four functions:

$$\xi_m = \xi_m (x^1, x^2, x^3, x^4) \tag{19}$$

in such a way that:

$$h_{\alpha m} = \frac{\partial \xi_m}{\partial x^\alpha} . \tag{20}$$

We make the coordinate transformation (19), so that $\bar{h}_{\alpha m} = \dfrac{\partial x^\nu}{\partial \xi_\alpha} h_{\nu m}$. It then follows from (7), (9), and (20) that:

$$\bar{h}_{\alpha m} = \varepsilon_{\alpha m} , \qquad \bar{g}_{\alpha\beta} = \varepsilon_{\alpha\beta} , \qquad \bar{\Delta}_{\sigma\rho}^{\nu} = 0. \tag{21}$$

In a metrically-integrable world, one must then have $(d_1 d_2 - d_2 d_1) s_m \neq 0$. By means of the non-integrable equations:

$$dx_0^\alpha = \kappa_\nu^{\cdot\cdot\alpha} dx^\nu , \qquad \kappa_\nu^{\cdot\cdot\alpha} (x^1, x^2, x^3, x^4) , \tag{22}$$

we can extend the transformation law:

$$h_{\alpha m} = \kappa_{\alpha}^{..\nu} \overset{\circ}{h}_{\nu m}, \qquad A^{\alpha} = \kappa_{\nu}^{..\alpha} \overset{\circ}{A}{}^{\nu}, \qquad \kappa_{\sigma}^{..\nu}\kappa_{\nu}^{..\rho} = \varepsilon_{\sigma}^{..\rho}, \tag{23}$$

where $\overset{\circ}{h}_{\nu m}$, $\overset{\circ}{A}{}^{\nu}$ represent functions of the x^{α}, but not of the x_0^{α}! If we choose (22) in such a way that $\overset{\circ}{h}_{\nu m} = \varepsilon_{\nu m}$ then it follows that $h_{\alpha m} = \kappa_{\alpha}^{..m}$, $dx_0^m = ds_m$. If there then exists torsion as in (15) then the metric of the integrable world can be made to vanish only by a transformation of the type (22). However, if (16) is true then one arrives at the usual transformation (19).

§ 3. Since:

$$(d_1\, d_2 - d_2\, d_1)\, A_m = A^{\nu} \left\{ \frac{\partial}{\partial x_{\rho}} \left(\frac{\partial h_{\nu m}}{\partial x_{\sigma}} \right) - \frac{\partial}{\partial x_{\sigma}} \left(\frac{\partial h_{\nu m}}{\partial x_{\rho}} \right) \right\} d_1 x^{\rho}\, d_2 x^{\rho} \equiv 0, \tag{24}$$

it then follows by an application of Stokes's theorem that:

$$\iint (d_1 d_2 - d_2 d_1) A_m = \oint dA_m \equiv 0. \tag{25}$$

The distant comparison of direction is also possible; one further has:

$$P_{\nu\rho\sigma}^{...\alpha} = - h^{\alpha m} \left\{ \frac{\partial}{\partial x_{\rho}} \left(\frac{\partial h_{\nu m}}{\partial x_{\sigma}} \right) - \frac{\partial}{\partial x_{\sigma}} \left(\frac{\partial h_{\nu m}}{\partial x_{\rho}} \right) \right\} = R_{\nu\rho\sigma}^{...\alpha} - \delta_{\rho} \Pi_{\nu\alpha}^{...\alpha} \tag{26}$$
$$+ \delta_{\sigma} \Pi_{\nu\rho}^{...\alpha} + \Pi_{\nu\rho}^{...\kappa} \Pi_{\kappa\sigma}^{...\alpha} - \Pi_{\nu\sigma}^{...\kappa} \Pi_{\kappa\rho}^{...\alpha} \equiv 0,$$

with

$$R_{\nu\rho\sigma}^{...\alpha} = - \frac{\partial \Gamma_{\nu\sigma}^{\alpha}}{\partial x_{\rho}} + \frac{\partial \Gamma_{\nu\rho}^{\alpha}}{\partial x_{\sigma}} + \Gamma_{\nu\rho}^{...\kappa} \Gamma_{\kappa\sigma}^{...\alpha} - \Gamma_{\nu\sigma}^{...\kappa} \Gamma_{\kappa\rho}^{...\alpha}.$$

The identities (26), which express the idea that the curvature ratios that are described by $P_{\nu\rho\sigma}^{...\alpha}$ vanish identically, are invariant under not only the transformations (19), but also the transformations (22). In non-integrable geometry, the $dh_{\nu m}$ are not complete differentials, and therefore one will have $P_{\nu\rho\sigma}^{...\alpha} \neq 0$. We set:

$$\Lambda_{\mu\kappa}^{...\kappa} = \Lambda_{\mu}, \tag{27}$$

from which it also follows that:

$$\Pi_{\mu\kappa}^{...\kappa} = \Lambda_{\mu}.$$

Furthermore:

$$R_{\nu\rho\sigma}^{...\alpha} = R_{\alpha\beta}, \quad R_{\alpha\beta}\, g^{\alpha\beta} = R.$$

One then has:

$$\tfrac{1}{2}\{P^{\cdots\kappa}_{\alpha\beta\kappa}+P^{\cdots\kappa}_{\beta\alpha\kappa}\} = R_{\alpha\beta} - \tfrac{1}{2}\{\delta_\alpha A_\beta + \delta_\beta A_\alpha\} + \tfrac{1}{2}\delta_\mu\{\Lambda^{\ \mu}_{\alpha\cdot\beta}+\Lambda^{\ \mu}_{\beta\cdot\alpha}\}$$
$$+\tfrac{1}{4}\{\Lambda_{\mu\kappa\alpha}\Lambda^{\mu\kappa}_{\ldots\beta}-\Lambda_{\alpha\mu\kappa}\Lambda^{\mu\kappa}_{\ldots\beta}-\Lambda_{\beta\mu\kappa}\Lambda^{\mu\kappa}_{\ldots\alpha}\}+\tfrac{1}{2}\{\Lambda^{\ \mu}_{\alpha\cdot\beta}+\Lambda^{\ \mu}_{\beta\cdot\alpha}\}\Lambda_\mu \equiv 0, \tag{28}$$

$$\tfrac{1}{2}\{P^{\cdots\kappa}_{\alpha\beta\kappa}+P^{\cdots\kappa}_{\beta\alpha\kappa}\} = \tfrac{1}{2}\{\delta_\alpha A_\beta + \delta_\beta A_\alpha\} + \tfrac{1}{2}\delta_\mu\Lambda^{\cdot\cdot\mu}_{\alpha\beta}$$
$$+\tfrac{1}{4}\{\Lambda_{\alpha\mu\kappa}\Lambda^{\mu\kappa}_{\ldots\beta}-\Lambda_{\beta\mu\kappa}\Lambda^{\mu\kappa}_{\ldots\alpha}\}+\tfrac{1}{2}\Lambda^{\cdot\cdot\mu}_{\alpha\beta}\Lambda_\mu \equiv 0, \tag{29}$$

$$P = R - \delta_\mu A^\mu + \tfrac{1}{2}\Lambda_{\mu\alpha\beta}\Lambda^{\mu\beta\alpha} + \tfrac{1}{4}\Lambda_{\alpha\beta\gamma}\Lambda^{\alpha\beta\gamma} - \Lambda_\mu\Lambda^\mu \equiv 0. \tag{30}$$

We set:

$$\frac{c^2}{2}g_{\alpha\beta} = \psi_{\alpha\beta} = \text{gravitational potentials,}$$

$$\frac{\varepsilon}{2}\Lambda_\mu = \Phi_\mu = \text{electromagnetic potentials.}$$

c = vacuum speed of light, ε = constant with the dimensions of electric charge. The fundamental equations from which the $h_{\mu m}$ will be determined must include the $h_{\mu m}$ and their derivatives of at most second order.

We can demand the derivability of the fundamental equations from a variational principle:

$$\delta\int H\, dx^1\, dx^2\, dx^3\, dx^4 = 0. \tag{31}$$

$\delta h_{\mu m}$, $\delta\left(\dfrac{\partial h_{\mu m}}{\partial x_\rho}\right)$ vanish on the boundary of the domain of integration.

Part two. Fundamental equations.

§ 1. The function H can either include only the $h_{\mu m}$ and its first-order derivatives of or also the second-order derivatives, but these must be linear with coefficients that relate to only the $h_{\mu m}$.

Examples of the first kind:

$$H_1 = \Lambda_{\alpha\beta\gamma}\Lambda^{\alpha\beta\gamma}\sqrt{g}\,, \quad H_2 = \Lambda_{\mu\alpha\beta}\Lambda^{\mu\beta\alpha}\sqrt{g}\,, \quad H_3 = \Lambda_\mu\Lambda^\mu\sqrt{g}\,.$$

Examples of the second kind:

$$H_5 = \delta_\mu\Lambda^\mu\sqrt{g}\,, \qquad H_6 = R\sqrt{g}\,.$$

From (30), all five of these Ansätze are connected by the identity:

$$H_1 + 2H_2 - 4H_3 - 8H_4 + 4H_5 \equiv 0. \tag{32}$$

One can also examine a linear combination:

$$H = \sum_{m=1}^{5} C_m H_m \,, \qquad\qquad C_m = \text{constants.} \tag{33}$$

A. Einstein discussed only the Ansatz H_2 in the first approximation, and added that H_1 would lead to similar results.

We would now like to assume that the $h_{\alpha m}$ deviate from the Euclidian values $\varepsilon_{\alpha m}$ only slightly. We can then solve the fundamental equations by successive approximations if we first solve them in the first approximation, and then in the second approximation, etc.

By restricting to the first and second approximation, one gets:

$$h_{\alpha m} = \varepsilon_{\alpha m} + \bar{K}_{\alpha m} + \bar{\bar{K}}_{\alpha m}, \qquad h^{\alpha m} = \varepsilon_{\alpha m} - \bar{K}_{\alpha m} - \bar{\bar{K}}_{m\alpha} + \bar{K}_{mr}\bar{K}_{r\alpha}, \tag{34}$$

and if we set:

$$\bar{K}_{\alpha\beta} + \bar{K}_{\beta\alpha} = \bar{g}_{\alpha\beta}, \qquad \bar{\bar{K}}_{\alpha\beta} + \bar{\bar{K}}_{\beta\alpha} + \bar{K}_{\alpha\kappa}\bar{K}_{\beta\kappa} = \bar{\bar{g}}_{\alpha\beta}, \tag{35}$$

then it follows that:

$$\left.\begin{aligned} g_{\alpha\beta} = \varepsilon_{\alpha\beta} + \bar{g}_{\alpha\beta} + \bar{\bar{g}}_{\alpha\beta},\ g^{\alpha\beta} = \varepsilon_{\alpha\beta} - \bar{g}_{\alpha\beta} - \bar{\bar{g}}_{\alpha\beta} + \bar{g}_{\alpha\kappa}\bar{g}_{\beta\kappa}, \\ \sqrt{g} = 1 + \tfrac{1}{2}\bar{g}_{\mu\mu} + \tfrac{1}{2}\bar{\bar{g}}_{\mu\mu} + \tfrac{1}{2}\bar{g}_{\mu\mu}\bar{g}_{\kappa\kappa} - \tfrac{1}{4}\bar{g}_{\mu\kappa}\bar{g}_{\mu\kappa}. \end{aligned}\right\} \tag{36}$$

One further has:

$$\left.\begin{aligned} \Lambda_{\mu\alpha}^{\cdot\cdot\beta} &= \frac{\partial\bar{K}_{\mu\beta}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\beta}}{\partial x_\mu} + \frac{\partial\bar{\bar{K}}_{\mu\beta}}{\partial x_\alpha} - \frac{\partial\bar{\bar{K}}_{\alpha\beta}}{\partial x_\mu} - \bar{K}_{\kappa\beta}\left(\frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\kappa}}{\partial x_\mu}\right), \\ \Lambda_\alpha &= \frac{\partial\bar{K}_{\alpha\mu}}{\partial x_\mu} - \frac{\partial\bar{K}_{\mu\mu}}{\partial x_\alpha} + \frac{\partial\bar{\bar{K}}_{\alpha\mu}}{\partial x_\mu} - \frac{\partial\bar{\bar{K}}_{\mu\mu}}{\partial x_\alpha} - \bar{K}_{\kappa\mu}\left(\frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\kappa}}{\partial x_\mu}\right). \end{aligned}\right\} \tag{37}$$

A calculation also produces:

$$\left.\begin{aligned} R_{\alpha\beta} &= \bar{R}_{\alpha\beta} + \bar{\bar{R}}_{\alpha\beta}, \\ \bar{R}_{\alpha\beta} &= \frac{1}{2}\left\{-\frac{\partial^2\bar{g}_{\alpha\beta}}{\partial x_\mu^2} + \frac{\partial^2\bar{g}_{\alpha\beta}}{\partial x_\mu\,\partial x_\kappa} + \frac{\partial^2\bar{g}_{\mu\beta}}{\partial x_\mu\,\partial x_\alpha} - \frac{\partial^2\bar{g}_{\mu\mu}}{\partial x_\alpha\,\partial x_\beta}\right\} \\ \bar{R} &= -\frac{\partial^2\bar{g}_{\mu\mu}}{\partial x_\rho^2} + \frac{\partial^2\bar{g}_{\mu\kappa}}{\partial x_\mu\,\partial x_\kappa}, \\ \bar{\bar{R}}_{\alpha\beta} &= \frac{1}{2}\left\{-\frac{\partial^2\bar{\bar{g}}_{\alpha\beta}}{\partial x_\mu^2} + \frac{\partial^2\bar{\bar{g}}_{\alpha\beta}}{\partial x_\mu\,\partial x_\kappa} + \frac{\partial^2\bar{\bar{g}}_{\mu\beta}}{\partial x_\mu\,\partial x_\alpha} - \frac{\partial^2\bar{\bar{g}}_{\mu\mu}}{\partial x_\alpha\,\partial x_\beta}\right\} \\ &\quad -\frac{1}{4}\bar{g}_{\kappa\sigma}\left\{\frac{\partial}{\partial x_\kappa}\left(\frac{\partial\bar{g}_{\beta\sigma}}{\partial x_\alpha} + \frac{\partial\bar{g}_{\alpha\sigma}}{\partial x_\beta}\right) - \frac{\partial^2\bar{g}_{\alpha\beta}}{\partial x_\kappa\,\partial x_\sigma} - \frac{\partial^2\bar{g}_{\kappa\alpha}}{\partial x_\alpha\,\partial x_\beta}\right\} \\ &\quad -\frac{1}{4}\left(\frac{\partial\bar{g}_{\alpha\kappa}}{\partial x_\beta} + \frac{\partial\bar{g}_{\beta\kappa}}{\partial x_\alpha} - \frac{\partial\bar{g}_{\alpha\beta}}{\partial x_\kappa}\right)\cdot\left(\frac{\partial\bar{g}_{\mu\kappa}}{\partial x_\mu} + \frac{\partial\bar{g}_{\mu\mu}}{\partial x_\kappa}\right) \\ &\quad -\frac{1}{4}\left(\frac{\partial\bar{g}_{\alpha\kappa}}{\partial x_\sigma} + \frac{\partial\bar{g}_{\kappa\sigma}}{\partial x_\alpha} - \frac{\partial\bar{g}_{\alpha\sigma}}{\partial x_\kappa}\right)\cdot\left(\frac{\partial\bar{g}_{\beta\sigma}}{\partial x_\kappa} + \frac{\partial\bar{g}_{\sigma\kappa}}{\partial x_\beta} - \frac{\partial\bar{g}_{\beta\kappa}}{\partial x_\sigma}\right). \end{aligned}\right\} \tag{38}$$

The Bianchi identities $\delta_\mu\,[R_{\alpha}^{\cdot\cdot\mu} - \tfrac{1}{2}\varepsilon_{\alpha}^{\cdot\cdot\mu}R] \equiv 0$ deliver, in the first approximation:

$$\frac{\partial}{\partial x_\beta}\left(\bar{R}_{\alpha\beta} - \tfrac{1}{2}\varepsilon_{\alpha\beta}\bar{R}\right) \equiv 0. \tag{39}$$

If we set:

$$\bar{X}_{\alpha\beta} = \frac{\partial\bar{\Lambda}_\beta}{\partial x_\alpha} - \frac{\partial\bar{\Lambda}_\alpha}{\partial x_\beta},$$

then (29) yields:

$$\bar{X}_{\alpha\beta} = -\frac{\partial}{\partial x_\mu}\bar{\Lambda}_{\alpha\beta\mu}\,. \tag{40}$$

If follows from (30) that:

$$\bar{R} = 2\frac{\partial}{\partial x_\mu}\bar{\Lambda}_\mu\,. \tag{41}$$

Finally, we get from (28):

$$\bar{R}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\bar{\Lambda}_\beta}{\partial x_\alpha} + \frac{\partial\bar{\Lambda}_\alpha}{\partial x_\beta}\right) - \frac{1}{2}(\bar{\Lambda}_{\alpha\mu\beta} + \bar{\Lambda}_{\beta\mu\alpha})\,. \tag{42}$$

We consider the first approximation. One has:

$$\left.\begin{aligned}
\bar{H}_1 &= \left(\frac{\partial\bar{K}_{\mu\beta}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\beta}}{\partial x_\mu}\right)\cdot\left(\frac{\partial\bar{K}_{\mu\beta}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\beta}}{\partial x_\mu}\right),\\
\bar{H}_2 &= \left(\frac{\partial\bar{K}_{\mu\alpha}}{\partial x_\beta} - \frac{\partial\bar{K}_{\beta\alpha}}{\partial x_\mu}\right)\cdot\left(\frac{\partial\bar{K}_{\mu\beta}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\beta}}{\partial x_\mu}\right),\\
\bar{H}_3 &= \left(\frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\kappa} - \frac{\partial\bar{K}_{\kappa\kappa}}{\partial x_\mu}\right)\cdot\left(\frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\kappa} - \frac{\partial\bar{K}_{\kappa\kappa}}{\partial x_\mu}\right),\\
\bar{H}_4 &= \left(\frac{\partial\bar{K}_{\alpha\kappa}}{\partial x_\mu} - \frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\alpha}\right)\frac{\partial}{\partial x_\mu}\bar{K}_{\kappa\alpha} + \left\{\frac{\partial}{\partial x_\mu}(\bar{K}_{\mu\kappa} + \bar{K}_{\kappa\mu})\right.\\
&\quad \left. - \frac{\partial\bar{K}_{\rho\rho}}{\partial x_\kappa}\right\}\left(\frac{\partial\bar{K}_{\alpha\alpha}}{\partial x_\kappa} - \frac{\partial\bar{K}_{\kappa\alpha}}{\partial x_\alpha}\right) + (1 + \bar{K}_{\rho\rho})\frac{\partial}{\partial x_\mu}\left(\frac{\partial\bar{K}_{\mu\alpha}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\alpha}}{\partial x_\mu}\right)\\
&\quad - (\bar{K}_{\mu\kappa} + \bar{K}_{\kappa\mu})\frac{\partial}{\partial x_\mu}\left(\frac{\partial\bar{K}_{\kappa\alpha}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\alpha}}{\partial x_\kappa}\right) - \bar{K}_{\kappa\alpha}\frac{\partial}{\partial x_\mu}\left(\frac{\partial\bar{K}_{\mu\kappa}}{\partial x_\alpha} - \frac{\partial\bar{K}_{\alpha\kappa}}{\partial x_\mu}\right),\\
\bar{H}_5 &= -\frac{1}{4}\bar{H}_1 - \frac{1}{2}\bar{H}_2 + \bar{H}_3 + 2\bar{H}_4.
\end{aligned}\right\} \tag{43}$$

One recognizes that the foregoing fundamental equations represent linear functions of the $\frac{\partial^2 \bar{K}_{\alpha m}}{\partial x_\rho \, \partial x_\sigma}$ with constant coefficients. Any linear combination (33) leads to similar equations.

§ 2. The fundamental equations for $\bar{H}_1$ read:

$$\bar{\Phi}^{(1)}_{\alpha\beta} = \frac{\partial^2 \bar{K}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2 \bar{K}_{\mu\beta}}{\partial x_\mu \, \partial x_\alpha} = \frac{\partial}{\partial x_\mu} \bar{\Lambda}_{\alpha\mu\beta} = 0. \tag{44}$$

From this, it follows that:

$$\frac{\partial \bar{\Phi}^{(1)}_{\alpha\beta}}{\partial x_\alpha} \equiv 0 \tag{45}$$

and

$$\frac{\partial \bar{\Lambda}_\mu}{\partial x_\mu} = -\,\varepsilon_{\alpha\beta} \frac{\partial}{\partial x_\mu} \bar{\Lambda}_{\alpha\mu\beta} = 0. \tag{46}$$

From (41) and (46), one then has:

$$\bar{R} = 0; \tag{47}$$

finally, it follows from (42) and (44) that:

$$\bar{R}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \bar{\Lambda}_\beta}{\partial x_\alpha} + \frac{\partial \bar{\Lambda}_\alpha}{\partial x_\beta} \right). \tag{48}$$

It follows from (39), (46), (48) that:

$$\frac{\partial^2 \bar{\Lambda}_\alpha}{\partial x_\mu^2} = 0. \tag{49}$$

For $\bar{H}_2$, we have:

$$\bar{\Phi}^{(2)}_{\alpha\beta} = \frac{\partial^2 \bar{K}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2 \bar{K}_{\mu\beta}}{\partial x_\mu \, \partial x_\alpha} + \frac{\partial^2 \bar{K}_{\beta\mu}}{\partial x_\mu \, \partial x_\alpha} - \frac{\partial^2 \bar{K}_{\alpha\mu}}{\partial x_\mu \, \partial x_\beta}$$

$$= \frac{\partial}{\partial x_\mu} (\bar{\Lambda}_{\alpha\mu\beta} + \bar{\Lambda}_{\beta\alpha\mu}) = 0, \tag{50}$$

from which:

$$\frac{\partial \bar{\Phi}^{(2)}_{\alpha\beta}}{\partial x_\beta} \equiv 0. \tag{51}$$

From (40) and (50), one has:

$$\frac{\partial}{\partial x_\mu}\bar{\Lambda}_{\beta\mu\alpha} = \bar{X}_{\alpha\beta}, \tag{52}$$

from which, it follows that:

$$\frac{\partial}{\partial x_\mu}(\bar{\Lambda}_{\alpha\mu\beta} + \bar{\Lambda}_{\beta\mu\alpha}) = \bar{X}_{\beta\alpha} + \bar{X}_{\alpha\beta} = 0. \tag{53}$$

From (42), (48) then follows from (52) that:

$$\frac{\partial\bar{\Lambda}_\mu}{\partial x_\mu} = -\varepsilon_{\alpha\beta}\frac{\partial}{\partial x_\mu}\bar{\Lambda}_{\alpha\mu\beta} = -\varepsilon_{\alpha\beta}\bar{X}_{\beta\alpha} = 0,$$

and from this, (41) also yields (47). Moreover, (49) follows from (39), (46), and (48). In fact, $\bar{H}_2$ gives the same results as $\bar{H}_1$.

For the Ansatz $\bar{H}_3$, one has:

$$\begin{aligned}\bar{\Phi}^{(3)}_{\alpha\beta} &= \frac{\partial^2\bar{K}_{\alpha\mu}}{\partial x_\mu\,\partial x_\beta} - \frac{\partial^2\bar{K}_{\mu\mu}}{\partial x_\alpha\,\partial x_\beta} + \varepsilon_{\alpha\beta}\left(\frac{\partial^2\bar{K}_{\mu\mu}}{\partial x_\rho^2} - \frac{\partial^2\bar{K}_{\mu\rho}}{\partial x_\mu\,\partial x_\rho}\right)\\ &= \frac{\partial}{\partial x_\mu}\bar{\Lambda}_\alpha - \varepsilon_{\alpha\beta}\frac{\partial\bar{\Lambda}_\mu}{\partial x_\mu} = 0.\end{aligned} \tag{54}$$

From this, it first follows that:

$$\frac{\partial\bar{\Phi}^{(3)}_{\alpha\beta}}{\partial x_\alpha} \equiv 0, \tag{55}$$

and then, from (54):

$$\bar{X}_{\alpha\beta} = \frac{\partial}{\partial x_\alpha}\bar{\Lambda}_\beta - \frac{\partial}{\partial x_\beta}\bar{\Lambda}_\alpha = 0. \tag{56}$$

$\bar{H}_3$ then leads to paradoxical results.

We consider, in turn, the Ansätze $\bar{H}_1$ and $\bar{H}_2$. From (44) or (50), it follows that:

$$\frac{\partial^2\bar{g}_{\alpha\beta}}{\partial x_\mu^2} = \frac{\partial}{\partial x_\mu}\left(\frac{\partial\bar{K}_{\mu\alpha}}{\partial x_\beta} + \frac{\partial\bar{K}_{\mu\beta}}{\partial x_\alpha}\right); \tag{57}$$

from (40), (57), one gets:

$$\frac{1}{2}\left(\frac{\partial^2\bar{g}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2\bar{g}_{\alpha\mu}}{\partial x_\mu\,\partial x_\beta} + \frac{\partial^2\bar{g}_{\beta\mu}}{\partial x_\mu\,\partial x_\alpha} - \bar{X}_{\alpha\beta}\right) = \frac{\partial^2\bar{K}_{\mu\beta}}{\partial x_\mu\,\partial x_\alpha}. \tag{58}$$

For the Ansatz H_1, it then follows from (44), (58) that:

$$\frac{\partial^2 \bar{K}_{\alpha\beta}}{\partial x_\mu^2} = \frac{1}{2}\left(\frac{\partial^2 \bar{g}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2 \bar{g}_{\alpha\mu}}{\partial x_\mu\,\partial x_\beta} + \frac{\partial^2 \bar{g}_{\beta\mu}}{\partial x_\mu\,\partial x_\alpha} - \bar{X}_{\alpha\beta}\right); \tag{59}$$

for the Ansatz H_2, it follows from (50), (58) that:

$$\frac{\partial^2 \bar{K}_{\alpha\beta}}{\partial x_\mu^2} = \frac{1}{2}\left(\frac{\partial^2 \bar{g}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2 \bar{g}_{\alpha\mu}}{\partial x_\mu\,\partial x_\beta} + \frac{\partial^2 \bar{g}_{\beta\mu}}{\partial x_\mu\,\partial x_\alpha} - 3\bar{X}_{\alpha\beta}\right). \tag{60}$$

Now, formulas (59) and (60) show that the distribution of $\bar{K}_{\alpha\beta}$ can be calculated completely from the distribution of $\bar{g}_{\alpha\beta}$, $\bar{\Lambda}_\alpha$.

We would also like to carry out the second approximation for H_1. It is:

$$H_1 = \bar{H}_1 + \bar{\bar{H}}_1, \tag{61}$$

with

$$\begin{aligned}\bar{\bar{H}}_1 = &-2(\bar{K}_{\sigma\alpha} + \bar{K}_{\alpha\sigma})\cdot\left(\frac{\partial \bar{K}_{\alpha m}}{\partial x_\beta} - \frac{\partial \bar{K}_{\beta m}}{\partial x_\alpha}\right)\cdot\left(\frac{\partial \bar{K}_{\sigma m}}{\partial x_\beta} - \frac{\partial \bar{K}_{\beta m}}{\partial x_\sigma}\right)\\ &+ \bar{K}_{\mu\mu}\left(\frac{\partial \bar{K}_{\alpha m}}{\partial x_\beta} - \frac{\partial \bar{K}_{\beta m}}{\partial x_\alpha}\right)\cdot\left(\frac{\partial \bar{K}_{\alpha m}}{\partial x_\beta} - \frac{\partial \bar{K}_{\beta m}}{\partial x_\alpha}\right)\\ &+ 2\left(\frac{\partial \bar{K}_{\alpha m}}{\partial x_\beta} - \frac{\partial \bar{K}_{\beta m}}{\partial x_\alpha}\right)\cdot\left(\frac{\partial \bar{\bar{K}}_{\alpha m}}{\partial x_\beta} - \frac{\partial \bar{\bar{K}}_{\beta m}}{\partial x_\alpha}\right).\end{aligned}$$

On the basis of the first approximation (44), we obtain:

$$\begin{aligned}\bar{\bar{\Phi}}_\alpha^{(1)} = &\frac{\partial^2 \bar{\bar{K}}_{\alpha\beta}}{\partial x_\mu^2} - \frac{\partial^2 \bar{\bar{K}}_{\mu\beta}}{\partial x_\mu\,\partial x_\alpha} + \frac{\partial}{\partial x_\mu}(\bar{g}_{\mu\kappa}\bar{\Lambda}_{\kappa\alpha\beta}) + \frac{\partial \bar{g}_{\kappa\alpha}}{\partial x_\mu}\bar{\Lambda}_{\mu\kappa\beta}\\ &+ \frac{1}{2}\frac{\partial \bar{g}_{\kappa\kappa}}{\partial x_\mu}\bar{\Lambda}_{\alpha\mu\beta} + \bar{\Lambda}_{\alpha\kappa\mu}\bar{\Lambda}_{\beta\kappa\mu} - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{\Lambda}_{\kappa\rho\mu}\bar{\Lambda}_{\kappa\rho\mu} = 0,\end{aligned} \tag{62}$$

from which:

$$\frac{\partial \bar{\bar{\Phi}}_\alpha^{(1)}}{\partial x_\alpha} \equiv 0. \tag{63}$$

§ 3. We assume that the $h_{\alpha m}$ are arbitrarily large. With the Ansatz:

$$H_1 = h^{\sigma k}\, h^{\alpha k}\, h^{\rho r}\, h^{\beta r}\,|\,h_{\tau\alpha}\,|\cdot\left(\frac{\partial h_{\alpha m}}{\partial x_\beta} - \frac{\partial h_{\beta m}}{\partial x_\alpha}\right)\cdot\left(\frac{\partial h_{\sigma m}}{\partial x_\rho} - \frac{\partial h_{\rho m}}{\partial x_\sigma}\right)$$

the fundamental equations read, upon considering formulas (11), (14):

$$\delta_\mu \Lambda^{\mu}_{\alpha\cdot\beta} + \tfrac{1}{2}\Lambda^{\mu\cdot\kappa}_{\alpha}(\Lambda_{\mu\beta\kappa} - \Lambda_{\kappa\beta\mu} - \Lambda_{\kappa\mu\beta}) - \tfrac{1}{4} g_{\alpha\beta}\Lambda_{\mu\kappa\rho}\Lambda^{\mu\kappa\rho} = 0. \tag{64}$$

One obtains from (64):

$$\delta_\mu \Lambda^\mu = -\tfrac{1}{2}\Lambda_{\alpha\beta\gamma}\Lambda^{\alpha\beta\gamma}. \tag{65}$$

According to (30), it follows from (65) that:

$$R = -\tfrac{5}{4}\Lambda_{\alpha\beta\gamma}\Lambda^{\alpha\beta\gamma} - \tfrac{1}{2}\Lambda_{\mu\alpha\beta}\Lambda^{\mu\beta\alpha} + \Lambda_\mu \Lambda^\mu; \tag{66}$$

according to (28), it follows from (64) that:

$$\begin{aligned} R_{\alpha\beta} = {} & \tfrac{1}{2}(\delta_\alpha \Lambda_\beta + \delta_\beta \Lambda_\alpha) + \tfrac{1}{4}\{(\Lambda_{\mu\alpha\kappa} - \Lambda_{\kappa\mu\alpha}) \cdot (\Lambda^{\mu\cdot\kappa}_{\beta} - \Lambda^{\kappa\cdot\mu}_{\beta}) \\ & - \Lambda_{\kappa\mu\alpha}\Lambda^{\mu\kappa}_{\cdot\cdot\beta} - g_{\alpha\beta}\Lambda_{\mu\kappa\rho}\Lambda^{\mu\kappa\rho} - 2(\Lambda^{\mu}_{\alpha\cdot\beta} + \Lambda^{\mu}_{\beta\cdot\alpha})\Lambda_\mu\}. \end{aligned} \tag{67}$$

Finally, (29) gives:

$$X_{\alpha\beta} = \frac{\partial}{\partial x_\alpha}\Lambda_\beta - \frac{\partial}{\partial x_\beta}\Lambda_\alpha = -\{\delta_\mu \Lambda^{\cdot\cdot\mu}_{\alpha\beta} + \tfrac{1}{2}(\Lambda_{\alpha\mu\kappa}\Lambda^{\mu\kappa}_{\cdot\cdot\beta} - \Lambda_{\beta\mu\kappa}\Lambda^{\mu\kappa}_{\cdot\cdot\alpha}) + \Lambda^{\cdot\cdot\mu}_{\alpha\beta}\Lambda_\mu\}. \tag{68}$$

All of these exact equations can be employed in the formulation of more precise equations. The Bianchi identities and the calculation of the $\delta_\mu X^{\cdot\cdot\mu}_{\alpha}$, $\frac{\partial}{\partial x_\alpha}X_{\beta\gamma} + \frac{\partial}{\partial x_\beta}X_{\gamma\alpha} + \frac{\partial}{\partial x_\gamma}X_{\alpha\beta}$ yield new equations.

Finally, I would like to remark that A. Einstein has already advanced so far in his development of the field theory that the arguments that were outlined here, which are closely linked with his article in the Sitzungsberichten der Preussischen Akademie, can only be of mathematical interest.

Sofia. Physikalisches Institut der Universität, 20 December 1928.

On the classification of the new Einstein Ansatz on gravitation and electricity

By **Hans Reichenbach** in Berlin

(Received on 22 January 1929)

§ 1. The Einstein spatial type with teleparallelism can be regarded as a specialization of the Weyl-Eddington spaces, which is based upon the commutability of specializations; it is not a specialization of the Riemannian spaces, but is logically complementary to it.

§ 2. The epistemological significance of a unified field theory is examined.

§ 1. The geometric foundations. The new attempt by Einstein (*) to change Riemannian geometry in such a way as to arrive at a spatial type that could encompass both gravitation and electromagnetism can arouse the objection that one is then dealing with a concept that that is not included in the geometric theory that has been developed up to now; in particular, one could object to the paradoxical title that combines Riemannian geometry and teleparallelism, since one is now treating a hitherto-unknown concept that is intermediate to Riemannian and Euclidian geometry. In what follows, it will be shown that this is not the case, and that furthermore the new Einstein space already occupies a logical place in the context of Weyl-Eddington geometry that can be understood precisely.

To that end, I would like to draw attention to a presentation (**) in which I developed an extended conception of space using Eddington's approach as its logical structure. The difference in the basic notions of Weyl in his extension seems to me to be that he recognized the independence of the displacement operation that is given by $\Gamma^{\tau}_{\mu\nu}$ from the one that is given by the metric $g_{\mu\nu}$; the general treatment of space problems can be constructed upon these ideas. Once the topological assignment of all space points has been established by a coordinate system, one imagines two systems of functions $g_{\mu\nu}$ and $\Gamma^{\tau}_{\mu\nu}$ as being given arbitrarily; the former shall define the metric by way of:

$$ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}, \tag{1}$$

while the latter shall define the displacement of a vector A^{τ} by way of:

(*) A. Einstein, "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus," Berl. Ber., Phys.-Math. Kl. **17** (1928); cited as E. I. A. Einstein, "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität," Berl. Ber., Phys.-Math. Kl. **18** (1928); cited as E. II.

(**) H. Reichenbach, *Philosophie der Raum-Zeit-Lehre*, Berlin, de Gruyter, 1928. Appendix; cited as Ph.

$$dA^{\tau} = \Gamma^{\tau}_{\mu\nu} A^{\mu}\, dx^{\nu}. \tag{2}$$

The two operations generally involve different situations, since, e.g., the metric is not defined by the comparison of directions, while the displacement hardly gives a measure of the lengths of vectors. However, they can agree when the lengths of two different vectors A^{τ} and $A^{*\tau}$ are compared at different locations. In fact, the metric gives a comparison of length by way of the relation:

$$l^2 = g_{\mu\nu} A^{\mu} A^{\nu}, \tag{3}$$

$$l^* - l = \sqrt{g_{\mu\nu} A^{\mu} A^{\nu}} - \sqrt{g^*_{\mu\nu} A^{*\mu} A^{*\nu}}\,, \tag{4}$$

while the displacement of such vectors is compared by means of the relation:

$$A^{*\tau} - A^{\tau} = \int_s \Gamma^{\tau}_{\mu\nu} A^{\mu} dx^{\nu} \tag{5}$$

(in which the latter integral depends upon the path s). In general, both operations will contradict each other here; e.g., from (5), one can have $A^{*\tau} - A^{\tau} = 0$, even though, from (4), $l^* - l \neq 0$. If one would wish to obtain a "balanced space" (*ausgeglichenen Raum*) in which such contradictions are excluded then one has two paths to choose from: Either one makes the displacement the fundamental principle by taking the metric to be a function of the length comparison at different locations and employs only the ratios of the $g_{\mu\nu}$ or one makes the metric the fundamental principle so one prescribes the displacement by the condition that it leaves the lengths of vectors unchanged, independently of the path. I call the former type of space a *displacement space* and the latter one a *metric space* (†).

Whereas in the previous Ansätze of Weyl, Eddington, and Einstein, the displacement space was employed, or indeed given along with a suitable unbalanced type of space, the new Einstein Ansatz employs the metric space, and this shall be shown in what follows.

The metric space is characterized by the condition:

$$d(l^2) = 0, \tag{6}$$

which, according to Eddington, leads to the relation:

$$K_{\mu\nu,\,\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} + \Gamma_{\mu\sigma,\,\varpi} + \Gamma_{\mu\nu,\,\mu} = 0. \tag{7}$$

It is quite significant that this condition still does not lead to the Riemannian space; this first comes about when one adds the far-reaching condition:

$$\Gamma^{\tau}_{\mu\nu} = \Gamma^{\tau}_{\nu\mu}\,. \tag{8}$$

(†) Ph., § 47.

When that is true, in fact, from (7), one gets the well-known result:

$$\Gamma^{\tau}_{\mu\nu} = -\begin{Bmatrix} \mu\,\nu \\ \tau \end{Bmatrix}. \tag{9}$$

The general metric space is, however, different from a Riemannian space. The Riemannian space is the specialization of a metric space that is given by (8) (*).

Einstein's idea in E. I now consists of the fact that another specialization of the general metric space can be used besides (8). Namely, he demanded that, along with the relation (6) [(7), resp.], one should have integrability of the transfer of direction that is given by (2). Ordinarily, one first employs this far-reaching requirement when one goes from a Riemannian space to a Euclidian space, and thus seeks a specialization of the $\Gamma^{\tau}_{\mu\nu}$ that starts from (8) [(9), resp.]. *By contrast, Einstein idea can be expressed by saying that one can already pose this demand along with* (6) [(7), resp.] *without having to pose the symmetry requirement* (8).

The mathematical formulation of this requirement can be given with no further discussion using the familiar tools. Should the transfer of length and direction be integrable then a vector that is at *P* determines one and only one "congruent" vector at any other location without referring to a connecting path. The partial derivative of this vector field with respect to the coordinates is, from (2), given by:

$$\frac{\partial A^{\tau}}{\partial x^{\nu}} = \Gamma^{\tau}_{\mu\nu} A^{\mu}. \tag{10}$$

The condition that the function $\Gamma^{\tau}_{\mu\nu}$ can establish the partial derivative of a vector field in this way is equivalent to the integrability condition of (10); it is known that this leads to the condition (**):

$$R^{\tau}_{\mu\nu\sigma}(\Gamma) = 0, \tag{11}$$

$$R^{\tau}_{\mu\nu\sigma}(\Gamma) = \frac{\partial \Gamma^{\tau}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\tau}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\tau}_{\alpha\nu}\Gamma^{\alpha}_{\mu\sigma} - \Gamma^{\tau}_{\alpha\sigma}\Gamma^{\alpha}_{\mu\nu}. \tag{12}$$

It is essential to understand that this, as the well-known condition for the vanishing of the Riemann tensor, can be formulated as the sole condition on the $\Gamma^{\tau}_{\mu\nu}$, without anything else being assumed about the connection between the $\Gamma^{\tau}_{\mu\nu}$ and the $g_{\mu\nu}$. It is also important that the symmetry of the $\Gamma^{\tau}_{\mu\nu}$ in (8) was not assumed for (11).

The Einsteinian space is therefore characterized by conditions (7) and (11); the latter is a condition on the $\Gamma^{\tau}_{\mu\nu}$ alone, while (7) represents a prescription for the connection

(*) Ph., pp. 346-351. The notation that I am using differs from Eddington's notation by the sign and sequence of lower indices in (2), as well as by the omission of the numerical factor 2 in (7).

(**) Cf., say, Weyl, *Raum, Zeit, Materie*, 1st ed, Berlin, 1918; pp. 108.

between the $\Gamma^{\tau}_{\mu\nu}$ and the $g_{\mu\nu}$. It is a metric space with teleparallelism, which is, however, different from Euclidian space by the asymmetry of the $\Gamma^{\tau}_{\mu\nu}$; it is the addition of (8) to (7) and (11) that first leads to Euclidian space. The logical classification of Einsteinian spaces can be illustrated by the following diagram:

Metric space

$d(l^2) = 0$

Einsteinian space	Riemannian space
$R^{\tau}_{\mu\nu\sigma}(\Gamma) = 0$	$\Gamma^{\tau}_{\mu\nu} = \Gamma^{\tau}_{\nu\mu}$
Euclidian space	Euclidian space
$\Gamma^{\tau}_{\mu\nu} = \Gamma^{\tau}_{\nu\mu}$	$R^{\tau}_{\mu\nu\sigma}(\Gamma) = 0$

The Einsteinian space is therefore not a special case of the Riemannian space, but should be placed next to it; its possibility rests upon the commutability of the specializations that lead from the metric spaces to the Euclidian spaces.

A two-dimensional illustration might be given by, perhaps, a sphere, upon which the meridians and latitude circles are defined as two families of parallel lines. Two vectors at different locations are called parallel when they make equal angles with the crosses through their locations that are defined by the lines of the family. (One observes that the angle measure is established by the metric $g_{\mu\nu}$, since it can be reduced to the length measure from the angle measure.) For that reason, the functions $\Gamma^{\tau}_{\mu\nu}$ are simply equal to zero for this coordinate system. This is not the Riemannian parallelism of the sphere, since that would be characterized by the $\left\{ {\mu\nu \atop \tau} \right\}$, which naturally do not vanish for the sphere. If, say, a line element that lies in the sphere is perpendicular to a meridian is displaced in its proper length direction then it describes a latitude circle under Einsteinian parallelism and a great circle under Riemannian parallelism that is tangent to the latitude circle at the starting point. In Einsteinian space, as in a general metric space, the straightest lines and the shortest lines coincide.

(11) formulates the condition for the displacement operation to be integrable in length and direction, while (7) implies the requirement that the length that is transported by the displacement is identical with the distant comparison of the metric. Here, one can imagine a generalization in which these two lengths do not coincide; it would then be less desirable to construct an unbalanced space of that kind in which the two types of distant comparison were obtained from each other in some contrived way. By contrast, another generalization of the Einsteinian way of thinking can be of interest that Einstein himself has already thought of, as I learned from him. The displacement operation can be integrable in regard to direction,

while it is not integrable in regard to length. In place of (11), one would then pose a less restrictive condition. The balanced spatial type that this belongs to, which one can call a *direction space*, would be a displacement space in which the $g_{\mu\nu}$ are therefore established only up to their ratios. As a result of this, one can regard the Einsteinian space as a specialization of the metric spaces, as well as the direction spaces. Once again, this is based upon a commutability of the specializations, namely, the integrability of the length transfer and the integrability of the direction transfer.

§ 2. Some applications of the space type constructed. Einstein then gave a certain Ansatz for the $\Gamma^{\tau}_{\mu\nu}$, in which he represented both of the functions $\Gamma^{\tau}_{\mu\nu}$ and $g_{\mu\nu}$ as functions of a parameter (*):

$$g_{\mu\nu} = h_{\mu a}\, h_{\nu a}\,, \tag{13a}$$

$$\Gamma^{\tau}_{\mu\nu} = -\, h^{\tau}_{a}\,\frac{\partial h_{\mu a}}{\partial x^{\nu}}\,. \tag{13b}$$

One easily confirms by calculation that this Ansatz satisfies our equations (7) and (11). It would be interesting to know whether this Ansatz represents the only solution to (7) and (11); as far as that is concerned, one should note that generally (13*b*) is not a covariant equation.

On the physical interpretation of the Einsteinian Ansatz, let us make the following remark, which is true for the more recently published Einstein Ansatz (**) just the same. Indeed, the goal is to combine the basis laws of gravitation and electricity into one law. Now, there are two ways of unifying separate physical theories. The first way is to combine the two theories into a new one in such a way that the new theory says nothing more than the two theories combined; for that reason, such a unification has only a formal significance. It corresponds to the replacement of a system of axioms *A* with another system *B* that contains less theorems in such a way that *A* follows from *B* just as *B* follows from *A*. The second way means embedding the older theory into the new one, in the sense of a special case; this corresponds to the replacement of a system of axioms *A* with a system *B* in such a way that *A* can be derived from *B* when a "specializing" axiom *b* is added to *B*, while, conversely, *B* cannot be derived from *A* (***). This latter way implies the proper process of inductive, physical reasoning; it then replaces existing knowledge with a new, more assertive, knowledge. For that reason, the second way makes the judgment "true or false" in the sense of empirical proof, while the first one makes it only in the sense of the logical consistency of the derivation of *B* from *A*, and conversely. An example of the first way is, perhaps, the replacement of the Lagrangian equations of motions with a variational principle, while an example of the second way is the replacement of the Keplerian laws with Newton's law of gravitation.

(*) E. I., pp. 5: Equation (7*a*) that was given there includes a printing error, and likewise the previous and following equations. Moreover, Einstein wrote $-\Delta^{\tau}_{\mu\nu}$ for our $\Gamma^{\tau}_{\mu\nu}$.

(**) A. Einstein, "Zur einheitlichen Feldtheorie," Berl. Ber., Phys-Math. Kl. **1** (1929). Since this new Ansatz does not differ from the other two in relation to geometry, everything that was done in § 1 is equally true for it.

(***) On this, cf., H. Reichenbach, "Ziele und Wege der physikalischen Erkenntnis," Hand. d. Phys. IV, Berlin, Springer, 1929, pp. 38.

The fact that the first way is practicable, in the sense of a combination of gravitation and electricity into one field that determines geometry in an extended Riemannian space, was shown by the author (*); it is remarkable that one can thus find an immediate geometric interpretation for the displacement operation, namely, in the law of motion for electrically-charged mass points. There, the straightest line is identified with the path of the electrically-charged mass point, while the shortest line remains that of the uncharged mass point. In this, one achieves a certain parallel with Einstein's equivalence principle (**). Moreover, a space that is related to the Einsteinian space will be defined there, namely, a metric space with asymmetric $\Gamma^{\tau}_{\mu\nu}$. The fact that the first way was used in this comes about from an epistemological motive: Namely, with the intention of showing that the geometric interpretation of electricity in itself implies no physical epistemological significance. By contrast, the Einstein Ansatz naturally employs the latter way, since he is indeed concerned with an expansion of physical knowledge; it is the goal of Einstein's new theory to find a concatenation of gravitation and electricity that in the first approximation it would lead to a decomposition into the separate equations of the previous theory, while in a higher approximation it would lead to a reciprocal interaction of the two fields, that might possibly lead to an understanding of previously-unsolved question, such as the riddle of the quantum. However, this goal seems to me to be achievable only at the expense of its immediate physical interpretation, if not, in turn, that of the actual field quantities. For that reason, from the geometric standpoint, such a path appears to be quite unsatisfying; its sole justification is given by the fact that it encompasses more physical facts in the aforementioned concatenation than were put into its definition.

(*) Ph., § 49.
(**) Ph., pp. 367.

On unified field theory

By A. EINSTEIN

In two recently-appearing papers ([1]), I sought to show that one could succeed in obtaining a unified theory of gravitation and electromagnetism by attributing the property of "teleparallelism" to a four-dimensional continuum, in addition to a RIEMANN metric. In fact, one also succeeds in giving a unified meaning to the gravitational field and the electromagnetic field. On the other hand, the derivation of the field equation from HAMILTON's principle does not proceed in a simple and completely unique way. This difficulty grows stronger under more detailed considerations. However, I have since then succeeded in finding a satisfying way of deriving the field equations that I shall communicate in what follows.

1. Formal preparations.

I shall use the notation that WEITZENBÖCK recently proposed in his paper on the subject ([2]). The n-component of the s leg of an n-bein will thus be denoted by ${}_sh^\nu$, and the corresponding normalized sub-determinant by ${}^sh_\nu$. Local n-beins are assumed to be "parallel." Vectors are parallel and equal when they have equal coordinates relative to their respective local n-beins. The parallel translation of a vector is given by the formula:

$$\delta A^\mu = - \Delta^\mu_{\alpha\beta} A^\alpha \delta x^\beta = - {}_sh^\mu \, {}^sh_{\alpha,\beta} A^\alpha \delta x^\beta,$$

where the comma in the ${}^sh_{\alpha,\beta}$ shall suggest differentiation with respect to x^β in the usual sense. The "RIEMANN curvature tensor" that is constructed from the $\Delta^\mu_{\alpha\beta}$ (which is asymmetric in α and β) vanishes identically.

As for the "covariant derivative," we shall use only the one that is constructed by means of the Δ. Let it be denoted by a semi-colon, in the style of the Italian mathematicians, so:

$$A_{\nu;\,\sigma} \equiv A_{\nu,\,\sigma} - A_\alpha \, \Delta^\alpha_{\mu\sigma} ,$$
$$A^\mu{}_{;\,\sigma} \equiv A_{\nu,\,\sigma} + A^\alpha \, \Delta^\mu_{\alpha\sigma} .$$

([1]) These Berichte, VIII.28 and XVII.28.
([2]) These Berichte XXVI.28.

Since the ${}^{s}h_{\nu}$, as well as the $g_{\mu\nu}$ ($\equiv {}^{s}h_{\mu}\, {}^{s}h_{\nu}$) and the $g^{\mu\nu}$, have vanishing covariant derivatives, these quantities can be exchanged arbitrarily as factors under the differentiation sign.

I will now deviate from my previous notation, in that I will now define the tensor Λ (now omitting the factor 1/2) by the equation:

$$\Lambda_{\mu\nu}^{\alpha} \equiv \Delta_{\mu\nu}^{\alpha} - \Delta_{\nu\mu}^{\alpha} .$$

The principal difference between this and the familiar formulas of the absolute differential calculus lies in the construction of the divergence, which comes from the introduction of an asymmetric translation law. Let $T_{\dots}^{\dots\sigma}$ be an arbitrary tensor with an upper index σ. Its covariant derivative reads, when we include only those additional terms that relate to the index σ:

$$T_{\dots;\tau}^{\dots\sigma} \equiv \frac{\partial T_{\dots}^{\dots\sigma}}{\partial x^{\tau}} + \dots + T_{\dots}^{\dots\alpha} \Delta_{\alpha\tau}^{\sigma} .$$

If one multiplies this equation by the determinant h, in which one has contracted σ and τ, then by introducing the tensor density $\mathfrak{T}$ on the right-hand side one obtains:

$$h T_{\dots;\sigma}^{\dots\sigma} \equiv \frac{\partial \mathfrak{T}_{\dots}^{\dots\sigma}}{\partial x^{\sigma}} + \dots + \mathfrak{T}_{\dots}^{\dots\alpha} \Delta_{\alpha\sigma}^{\sigma} .$$

The last term on the right-hand side is missing when the translation law is symmetric. It is itself a tensor density, as well as the remaining terms on the right-hand side collectively, which we will refer to as the divergence of the tensor density $\mathfrak{T}$, in agreement with the usual notation, and write:

$$\mathfrak{T}_{\dots/\sigma}^{\dots\sigma} .$$

One then gets:

$$h T_{\dots;\sigma}^{\dots\sigma} \equiv \mathfrak{T}_{\dots,\sigma}^{\dots\sigma} + \dots + \mathfrak{T}_{\dots}^{\dots\alpha} \Delta_{\alpha\sigma}^{\sigma} . \tag{1}$$

Finally, we would like to introduce that a notation that – it seems to me – improves the clarity of the presentation. I will often suggest the raising (lowering, resp.) of an index in such a way that the index in question is underlined. Therefore, I will – e.g. – denote the purely contravariant tensor that is associated with $(\Lambda_{\mu\nu}^{\sigma})$ by $(\Lambda_{\underline{\mu\nu}}^{\sigma})$, and the purely covariant tensor that is associated with $(\Lambda_{\mu\nu}^{\sigma})$ by $(\Lambda_{\mu\nu}^{\underline{\sigma}})$.

2. The derivation of some identities.

The vanishing of "curvature is expressed by the identity:

$$0 \equiv -\Delta^{i}_{kl,m} + \Delta^{i}_{km,l} + \Delta^{i}_{\sigma l}\Delta^{\sigma}_{km} - \Delta^{i}_{\sigma m}\Delta^{\sigma}_{kl}\,. \tag{2}$$

We will use this identity in order to derive another one that is true for the tensor Λ. One constructs both of the equations that are obtained from (2) by the cyclic permutation of the indices *klm* and then adds the three equations. By an appropriate summation, one then immediately gets the identity:

$$0 \equiv (\Lambda^{i}_{kl,m} + \Lambda^{i}_{lm,k} + \Lambda^{i}_{mk,l}) + (\Lambda^{i}_{\sigma k}\Lambda^{\sigma}_{lm} + \Lambda^{i}_{\sigma l}\Lambda^{\sigma}_{mk} + \Lambda^{i}_{\sigma m}\Lambda^{\sigma}_{kl})\,.$$

We form these equations in such a way that we introduce the covariant derivatives of Λ, instead of the usual ones. By a suitable summation, one then effortlessly obtains the identity:

$$0 \equiv (\Lambda^{i}_{kl;m} + \Lambda^{i}_{lm;k} + \Lambda^{i}_{mk;l}) + (\Lambda^{i}_{k\sigma}\Lambda^{\sigma}_{lm} + \Lambda^{i}_{l\sigma}\Lambda^{\sigma}_{mk} + \Lambda^{i}_{m\sigma}\Lambda^{\sigma}_{kl})\,. \tag{3}$$

In fact, this is the requirement for the Λ to be expressed in the stated way in terms of the *h*.

By contracting (3) once, in which one replaces $\Lambda^{\alpha}_{\mu\alpha}$ with the abbreviated symbol ϕ_μ, one gets the identity:

$$0 \equiv \Lambda^{\alpha}_{kl;\alpha} + \phi_{l;\,k} - \phi_{k;\,l} + \phi_\alpha \Lambda^{\alpha}_{kl}\,, \tag{3a}$$

which will important in what follows. We transform this by introducing the tensor density (which is anti-symmetric in *k* and *l*):

$$\mathfrak{B}^{\alpha}_{kl} = h(\Lambda^{\alpha}_{kl} + \phi_l\,\delta^{\alpha}_{k} - \phi_k\,\delta^{\alpha}_{l})\,. \tag{4}$$

Equation (3*a*) is then converted into the simple form:

$$(\mathfrak{B}^{\alpha}_{kl})_{/\alpha} \equiv 0. \tag{3b}$$

The tensor density $\mathfrak{B}^{\alpha}_{kl}$ satisfies a second identity that will be significant in what follows. For its derivation, we lean upon the following commutation law for the construction of the divergence of a tensor density of arbitrary rank:

$$\mathfrak{A}^{\dots ik}_{\dots/i/k} - \mathfrak{A}^{\dots ik}_{\dots/k/i} \equiv -(\mathfrak{A}^{\dots ik}_{\dots}\Lambda^{\sigma}_{ik})_{/\sigma}\,. \tag{5}$$

The ellipses with $\mathfrak{A}$ mean any arbitrary indices that are the same for all three terms of the equation, namely, the ones that were not related to taking the divergence.

The proof of (5) relies upon the defining formula (*):

$$\mathfrak{A}^{\sigma\dots i}_{\tau\dots/i} = \mathfrak{A}^{\sigma\dots i}_{\tau\dots,i} + \mathfrak{A}^{\alpha\dots i}_{\tau\dots}\Delta^{\sigma}_{\alpha i} - \mathfrak{A}^{\sigma\dots i}_{\alpha\dots}\Delta^{\alpha}_{\tau i}\,, \tag{6}$$

(*) Translator's note: The last term on the right-hand side was misprinted in the original.

and particularly upon the identity (2). Equation (5) is closely connected with the commutation laws for covariant differentiation, which I will likewise state, for the sake of completeness. Let T be an arbitrary tensor whose indices I will disregard for the sake of brevity. We then have:

$$T_{;i\,;\,k} - T_{;k\,;\,i} \equiv T_{;\,\sigma}\,\Delta^{\sigma}_{ik}\,. \tag{7}$$

From the identity (5), we now make use of the tensor density $\mathfrak{B}^{\alpha}_{\underline{kl}}$, whose lower indices we presume to be raised. We thus find for the single non-trivial identity:

$$\mathfrak{B}^{\alpha}_{\underline{kl}/l/\alpha} - \mathfrak{B}^{\alpha}_{\underline{kl}/\alpha/l} \equiv - (\mathfrak{B}^{\alpha}_{\underline{kl}}\Lambda^{\sigma}_{l\alpha})_{/\alpha}\,,$$

which, in light of (3*b*), one can put into the form:

$$(\mathfrak{B}^{\alpha}_{\underline{kl}/l} - \mathfrak{B}^{\sigma}_{\underline{k\tau}}\Lambda^{\alpha}_{\sigma\tau})_{/\alpha} \equiv 0. \tag{8}$$

3. The field equations.

When I discovered the identity (3*b*), it was clear to me that the tensor density $\mathfrak{B}^{\alpha}_{kl}$ must play an important role for a naturally-restricted characterization of a manifold of the type under scrutiny. Since its divergence $\mathfrak{B}^{\alpha}_{kl/\alpha}$ vanishes identically, the next thing that came to mind was that in order to express the requirements (i.e., field equations), the other divergence $\mathfrak{B}^{\alpha}_{\underline{kl}/l}$ should also vanish. In fact, one arrives at equations that produce the vacuum field equations in the first approximation that are well-known from the earlier general theory of relativity.

On the other hand, one obtains no vector relation for the ϕ_α in such a way that all ϕ_α that have vanishing would be compatible with those field equations. This is based upon the fact that, in the first approximation (due to the commutability of ordinary differentiation), there is the identity:

$$\mathfrak{B}^{\alpha}_{\underline{kl}/l/\alpha} \equiv \mathfrak{B}^{\alpha}_{\underline{kl}/\alpha/l}\,.$$

However, due to (3*b*), the quantity on the right-hand side vanishes identically. In this way, in fact, four equations drop out of the system $\mathfrak{B}^{\alpha}_{kl/l} = 0$.

However, I recognize that this deficiency can be easily remedied by postulating the equation:

$$\bar{\mathfrak{B}}^{\alpha}_{kl/l} = 0,$$

instead of the vanishing of $\mathfrak{B}^{\alpha}_{kl/l}$, in which $\bar{\mathfrak{B}}^{\alpha}_{kl}$ refers to the tensor that differs from $\mathfrak{B}^{\alpha}_{kl}$ by an arbitrarily small amount ([1]):

([1]) This is certainly the method that was always used in order to remove the degeneracies that arose in the singular case.

$$\bar{\mathfrak{B}}_{kl}^{\alpha} = \mathfrak{B}_{kl}^{\alpha} - \varepsilon\, h(\phi_l\, \delta_k^{\alpha} - \phi_k\, \delta_l^{\alpha})\,. \tag{9}$$

One then indeed obtains the MAXWELL equations (all in the first approximation) when one takes the divergence of the field equations (for the index α). Moreover, when one passes to the limit $\varepsilon = 0$, one obtains the equations $\mathfrak{B}_{kl/l}^{\alpha} = 0$, as before, which likewise give the correct gravitational laws in the first approximation.

The field equations of electricity and gravitation will then be produced correctly in the first approximation by the Ansatz:

$$\bar{\mathfrak{B}}_{kl/l}^{\alpha} = 0,$$

with the associated restriction that one must pass to the limit $\varepsilon = 0$. This brings with it, the identity (which is valid in the first approximation):

$$\mathfrak{B}_{kl/l/\alpha}^{\alpha} = 0, \tag{8a}$$

which brings about a division of the field equations in the first approximation into the laws of gravitation, on the one hand, and electricity, on the other, a separation that certainly represents a characteristic feature of nature.

We must now make those considerations that were obtained in the first approximation just as useful in the more rigorous case. It is clear that we also have to arrive at an identity here that corresponds to (8*a*). This is obviously the identity (8), especially since both identities, except for (3*b*), are based upon a permutation of the differential operations.

We must therefore propose:

$$\bar{\mathfrak{B}}_{\underline{k}l/l}^{\alpha} - \bar{\mathfrak{B}}_{\underline{k\tau}}^{\sigma}\Lambda_{\sigma\tau}^{\alpha} = 0 \tag{10}$$

as the field equations, with the prescription that we subsequently (i.e., after performing the operation "$/\alpha$") pass to $\varepsilon = 0$. When one denotes the left-hand side of (10) by $\mathfrak{G}^{k\sigma}$, one then obtain the field equations:

$$\mathfrak{G}^{k\sigma} = 0, \tag{10a}$$

$$\tfrac{1}{\varepsilon}\bar{\mathfrak{G}}^{kl}{}_{/\alpha} = 0. \tag{10b}$$

Considering (8) and (9), (10*b*) next gives:

$$\{[h(\phi_{\underline{k}}\delta_{\underline{l}}^{\alpha} - \phi_{\underline{l}}\delta_{k}^{\alpha})]_{/l} - h(\phi_{\underline{k}}\delta_{\underline{\tau}}^{\sigma} - \phi_{\underline{\tau}}\delta_{\underline{k}}^{\sigma})\Lambda_{\sigma\tau}^{\alpha}\}_{/\alpha} = 0.$$

For the sake of brevity, we now introduce the tensor density:

$$\mathfrak{W}_{kl}^{\alpha} = h(\phi_k\delta_l^{\alpha} - \phi_l\delta_k^{\alpha})\,.$$

According to (5), we have:

$$\mathfrak{W}_{\underline{kl}/l/\alpha}^{\alpha} = \mathfrak{W}_{\underline{kl}/\alpha/l}^{\alpha} - (\mathfrak{W}_{\underline{kl}}^{\alpha}\Lambda_{l\alpha}^{\sigma})_{/\sigma}\,,$$

such that the equation that we are deriving can also be written in form:

$$(\mathfrak{W}^{\alpha}_{\underline{kl}/l} - \mathfrak{W}^{\alpha}_{\underline{kl}}\Lambda^{\sigma}_{l\alpha} - \mathfrak{W}^{\sigma}_{\underline{k\tau}}\Lambda^{\alpha}_{\sigma\tau})_{/\sigma} = 0,$$

in which the last two terms have been raised. By straightforward computations, we get:

$$\mathfrak{W}^{l}_{\underline{k\alpha}/l} \equiv h(\phi_{\underline{k};\underline{\alpha}} - \phi_{\underline{\alpha};\underline{k}}) .$$

The transformed equations (10*b*) then read:

$$[h(\phi_{\underline{k};\underline{\alpha}} - \phi_{\underline{\alpha};\underline{k}})]_{/\alpha} = 0, \tag{11}$$

an equation that, together with:

$$\mathfrak{B}^{\alpha}_{\underline{kl}/l} - \mathfrak{B}^{\sigma}_{\underline{k\tau}}\Lambda^{\alpha}_{\sigma\tau} = 0, \tag{10a}$$

defines the complete system of field equations.

Had we started from (10*a*), instead of (10), then we would have obtained the "electromagnetic" equation (11). We would also have no clue that systems (11) and (10*a*) are compatible with each other. Thus, it seems certain that these equations are consistent with each other, since the original equations (10) are sixteen relations between the sixteen quantities ${}^{s}h_{\mu}$. There necessarily exist four identities between these sixteen equations (10) due to the covariance of these equations. Therefore, there exist a total of eight relations between the twenty field equations (11), (10*b*), of which only four of them are stated explicitly in the text.

The facts that equations (10*a*) include the gravitational equations in the first approximation and equations (11) (in conjunction with the existence of a vector potential) include the MAXWELL equations for the vacuum has already been asserted. I can also show that, conversely, for every solution of these equations there exists an *h*-field that satisfies equations (10*a*). By contracting equations (10*a*), one obtains a divergence relation for the electromagnetic potential:

$$\left.\begin{aligned} &\mathfrak{f}^{l}_{/l} - \tfrac{1}{2}\mathfrak{B}^{\sigma}_{k\underline{\tau}}\Lambda^{k}_{\sigma\tau} = 0, \\ &(2\mathfrak{f}^{l} = \mathfrak{B}^{\sigma}_{\alpha\underline{l}} = 2h\phi^{l}). \end{aligned}\right\} \tag{12}$$

A deeper examination of the consequences of the field equations (11), (10*a*) will have to show whether the RIEMANN metric, in conjunction with teleparallelism, actually delivers an adequate description of the character of space. Prior to such an examination, this is not improbable.

It is a pleasant task for me to thank Dr. H. MUNTZ for the difficult rigorous computation of the centrally-symmetric problem on the basis of HAMILTON's principle; it was by the results of that research that I was brought closer to the findings of the path that was taken here. Likewise, at this time, I would like to thank the "physical fund," which made it possible for me to employ a research assistant, in the person of Dr. GROMMER, during the last year.

Supplementary correction: The field equations that were proposed in this paper are formally the opposite of the ones that are usually conceivable. By leaning upon the identity (8), we found that the sixteen quantities ${}^{s}h_{\nu}$ could be subject to, not merely sixteen, but twenty autonomous differential equations. One should understand term "autonomous" to mean that none of these equations can follow from the remaining ones when there also exist eight (differential) identity relations between them.

Received on 30 January.

On the foundations of a new field theory of A. Einstein

(Second part)

By **Raschco Zaycoff** in Sofia.

(Received on 4 March 1929)

In a recently-appearing article (*), A. Einstein abandoned the variational method of deriving the fundamental equations, since it led to no unique results (**). Einstein then subjected the metric of his world to natural restrictions that would imply the correct fundamental equations. In the present publication, in conjunction with my first publication (***), I will show that the new method gives the natural laws quite well. From it, one gets a summary of Einstein's investigations.

§ 1. The identities (28), (29), I, can also be written in the form:

$$R_{\alpha\beta} - \tfrac{1}{2}(\nabla_\alpha \Lambda_\beta + \nabla_\beta \Lambda_\alpha) + \tfrac{1}{2}\nabla_\mu(\Lambda^{\mu}_{\alpha\cdot\beta} + \Lambda^{\mu}_{\beta\cdot\alpha})$$
$$+ \tfrac{1}{4}\{(\Lambda_{\alpha\mu\kappa} + \Lambda_{\alpha\kappa\mu}) \cdot (\Lambda_{\beta}^{\cdot\mu\kappa} + \Lambda_{\beta}^{\cdot\kappa\mu}) - \Lambda_{\mu\kappa\alpha}\Lambda^{\mu\kappa}_{\dots\beta}\} - \tfrac{1}{2}(\Lambda^{\mu}_{\alpha\cdot\beta} + \Lambda^{\mu}_{\beta\cdot\alpha})\Lambda_\mu \equiv 0, \tag{1}$$

$$\nabla_\alpha \Lambda_\beta - \nabla_\beta \Lambda_\alpha + \nabla_\mu \Lambda^{\cdot\cdot\mu}_{\alpha\beta} + \Lambda^{\mu}_{\alpha\beta}\Lambda_\mu \equiv 0. \tag{2}$$

We would now like to give these identities a useful form. We then set:

$$V^{\cdot\cdot\gamma}_{\alpha\beta} = \Lambda^{\cdot\cdot\gamma}_{\alpha\beta} - \Lambda_\alpha \varepsilon^{\cdot\gamma}_{\beta} + \Lambda_\beta \varepsilon^{\cdot\gamma}_{\alpha}, \tag{3}$$

from which, it follows that:

$$V^{\cdot\cdot\gamma}_{\alpha\beta} = -2\,\Lambda_\alpha. \tag{4}$$

Moreover, one has:

$$D_\mu \Lambda^{\cdot\cdot\mu}_{\dots} = (\nabla_\mu - \Lambda_\mu)\;\Lambda^{\cdot\cdot\mu}_{\dots}, \tag{5}$$

where $\Lambda^{\cdot\cdot\mu}_{\dots}$ represents any tensor, and:

$$W^{\cdot\cdot\gamma}_{\alpha\beta} = \Lambda_\alpha \varepsilon^{\cdot\gamma}_{\beta} - \Lambda_\beta \varepsilon^{\cdot\gamma}_{\alpha}, \tag{6}$$

(*) A. Einstein, Sitzungsber. d. Preuss. Akad. (1929), no. 1.
(**) The same, *ibid.* (1928), no. 17/18; R. Weitzenböck, *ibid.*, no. 26.
(***) R. Zaycoff, Zeit. Phys. **53** (1929), 719; referred to as I in what follows.

from which, if follows from (3) that:

$$\Lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma} = V_{\alpha\beta}^{\cdot\cdot\cdot\gamma} + W_{\alpha\beta}^{\cdot\cdot\cdot\gamma}\,. \tag{7}$$

With this notation, from (1) and (2), one gets:

$$\begin{aligned} R_{\alpha\beta} + \tfrac{1}{2}\{D_\mu V_{\alpha\cdot\beta}^{\mu} + V_{\alpha}^{\cdot\mu\kappa}\Lambda_{\mu\kappa\beta} + D_\mu V_{\beta\cdot\alpha}^{\mu} + V_{\beta}^{\cdot\mu\kappa}\Lambda_{\mu\kappa\alpha}\} - g_{\alpha\beta}\{D_\mu \Lambda^\mu - V^{\mu\kappa\sigma}\Lambda_{\mu\sigma\kappa}\} \\ + \tfrac{1}{2}\{V_{\alpha}^{\cdot\mu\kappa}(V_{\beta\mu\kappa} - V_{\mu\kappa\beta}) + V_{\beta}^{\cdot\mu\kappa}(V_{\alpha\mu\kappa} - V_{\mu\kappa\alpha})\} - \tfrac{1}{4}\{V_{\mu\kappa\alpha}V_{\dots\beta}^{\mu\kappa} - 5W_{\mu\kappa\alpha}W_{\dots\beta}^{\mu\kappa}\} \\ - \tfrac{1}{2}g_{\alpha\beta}\{V_{\mu\sigma\kappa}V^{\mu\kappa\sigma} + \tfrac{2}{3}W_{\mu\sigma\kappa}W^{\mu\kappa\sigma}\} \equiv 0, \end{aligned} \tag{8}$$

$$D_\mu V_{\alpha\beta}^{\cdot\cdot\cdot\mu} \equiv 0. \tag{9}$$

§ 2. For any sort of tensor $A_{\dots}^{\cdots}$, one has the identity:

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\ A_{\dots}^{\cdots} \equiv \Lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma}\nabla_\mu\ A_{\dots}^{\cdots}\ . \tag{10}$$

From (26), I, (2) and (10), one gets the identity:

$$(D_\alpha D_\beta - D_\beta D_\alpha)\ A_{\dots}^{\cdots} \equiv D_\mu(\Lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma}\ A_{\dots}^{\cdots}). \tag{11}$$

It follows from (9) and (11) that:

$$D_\mu\{D_\rho V^{\alpha\rho\mu} + V^{\alpha\kappa\rho}\ \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu}\} \equiv 0. \tag{12}$$

In order to avoid the degeneracy in the D_μ operation, we would like to replace $V_{\alpha\beta}^{\cdot\cdot\cdot\gamma}$ with:

$$*V_{\alpha\beta}^{\cdot\cdot\cdot\gamma} = V_{\alpha\beta}^{\cdot\cdot\cdot\gamma} + \varepsilon W_{\alpha\beta}^{\cdot\cdot\cdot\gamma}, \qquad \lim \varepsilon = 0. \tag{13}$$

We now choose the restriction in question in such a way that:

$$\lim_{\varepsilon\to 0}\{D_\rho *V^{\alpha\rho\mu} + *V^{\alpha\kappa\rho}\ \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu}\} = 0, \tag{14}$$

or, due to (13):

$$D_\rho V^{\alpha\rho\mu} + V^{\alpha\kappa\rho}\ \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} = 0, \tag{15}$$

which represent the 16 equations of the first group of fundamental equations.

It follows from (4) and (15) that:

$$D_\mu \Lambda^\mu - \tfrac{1}{2}V^{\alpha\rho\mu}\Lambda_{\mu\sigma\kappa} = 0. \tag{16}$$

Now, the equation:

$$\frac{1}{\varepsilon}\, D_\mu \{D_\rho {}^*V^{\alpha\rho\mu} + W^{\alpha\kappa\rho}\, \Lambda_{\kappa\rho}^{\cdot\cdot\mu}\} = 0 \tag{17}$$

next gives, when one considers (12) and (13):

$$D_\mu \{D_\rho\, W^{\alpha\rho\mu} + W^{\alpha\kappa\rho}\, \Lambda_{\kappa\rho}^{\cdot\cdot\mu}\} = 0, \tag{18}$$

and then, from (11) and (18):

$$D_\rho\, D_\mu\, W^{\alpha\rho\mu} = 0. \tag{19}$$

We set:

$$F_{\alpha\beta} \equiv \nabla_\alpha \Lambda_\beta - \nabla_\beta \Lambda_\alpha\,. \tag{20}$$

It follows from (2) that:

$$F^{\alpha\beta} \equiv - D_\mu\, \Lambda^{\alpha\rho\mu}\,, \tag{21}$$

so, from ((7), (9), and (21):

$$F^{\alpha\beta} \equiv - D_\mu\, W^{\alpha\rho\mu}\,. \tag{22}$$

(19) then assumes the form:

$$D_\mu\, F^{\alpha\beta} = 0. \tag{23}$$

Equations (23) define four more fundamental equations (viz., the second group).

§ 3. It follows from (8), (15), and (16) that:

$$\begin{aligned} R_{\alpha\beta} &+ \tfrac{1}{2}\{V_\alpha^{\cdot\mu\kappa}(V_{\beta\mu\kappa} - V_{\mu\kappa\beta}) + V_\beta^{\cdot\mu\kappa}(V_{\alpha\mu\kappa} - V_{\mu\kappa\alpha})\} \\ &- \tfrac{1}{4}\{V_{\mu\kappa\alpha} V^{\mu\kappa}_{\dots\beta} - 5W_{\mu\kappa\alpha} W^{\mu\kappa}_{\dots\beta}\} - \tfrac{1}{2} g_{\alpha\beta}\{V_{\mu\sigma\kappa} V^{\mu\kappa\sigma} + \tfrac{2}{3} W_{\mu\sigma\kappa} W^{\mu\kappa\sigma}\} = 0. \end{aligned} \tag{24}$$

These are the equations of gravity.

Now, one also has:

$$F_{\alpha\beta} = D_\alpha\, \Lambda_\beta - D_\beta\, \Lambda_\alpha\,; \tag{25}$$

from this and (23), it follows that:

$$D_\rho\, D^\rho\, \Lambda_\alpha = D_\rho\, D_\alpha\, \Lambda^\rho. \tag{26}$$

From (7), (9), (11), and (12), one has:

$$D_\rho\, D^\rho\, \Lambda_\alpha = D_\alpha\, D_\rho\, \Lambda^\rho + F_{\alpha\rho}\, \Lambda^\rho + \Lambda_{\rho\alpha\mu}\, D_\mu\, \Lambda^\rho + \Lambda_{\rho\alpha\mu}\, \Lambda^\mu\, \Lambda^\rho, \tag{27}$$

or, upon considering (15) and (16):

$$D_\rho\, D_\alpha\, \Lambda^\rho = F_{\alpha\rho}\, \Lambda^\rho + S_\alpha\,, \tag{28}$$

in which:

$$\begin{aligned} S_\alpha = \tfrac{1}{2} D_\alpha (V^{\mu\kappa\sigma}\, \Lambda_{\mu\sigma\kappa}) &+ \Lambda_{\rho\alpha\mu}\, D_\sigma\, \Lambda^{\rho\sigma\mu} - \tfrac{1}{2}\, \Lambda_\alpha\, V^{\mu\kappa\sigma}\, \Lambda_{\mu\sigma\kappa} \\ &+ V^{\rho\kappa\sigma}\, \Lambda_{\kappa\sigma}^{\cdot\cdot\mu}\, \Lambda_{\rho\alpha\mu} + \Lambda_{\rho\alpha\mu}\, \Lambda^\mu\, \Lambda^\rho. \end{aligned} \tag{29}$$

It then follows from (26) and (28) that:

$$D_\rho D^\rho \Lambda_\alpha - F_{\alpha\rho} \Lambda^\rho - S_\alpha = 0. \tag{30}$$

These equations exhibit a strong analogy with the wave equations of quantum mechanics (*).

§ 4. We consider the case of infinitely weak fields. One then arrives at the first approximation (**). From the identity (9), one will have:

$$\frac{\partial}{\partial x_\mu} \bar{V}_{\alpha\beta\mu} = 0, \tag{31}$$

which agrees with (40), since the following identity exists:

$$\bar{V}_{\alpha\beta\gamma} = \bar{\Lambda}_{\alpha\beta\gamma} - \bar{\Lambda}_\alpha \varepsilon_{\beta\gamma} + \bar{\Lambda}_\beta \varepsilon_{\alpha\gamma}. \tag{32}$$

Equation (16) reads:

$$\frac{\partial \bar{\Lambda}_\mu}{\partial x_\mu} = 0. \tag{33}$$

Equations (15) become:

$$\frac{\partial}{\partial x_\rho} \bar{V}_{\alpha\rho\mu} = 0, \tag{34}$$

or, upon consideration of (33):

$$\frac{\partial}{\partial x_\rho} \bar{\Lambda}_{\alpha\rho\mu} - \frac{\partial}{\partial x_\mu} \bar{\Lambda}_\alpha = 0. \tag{35}$$

Equations (23) assume the form:

$$\frac{\partial}{\partial x_\mu} \bar{F}_{\alpha\mu} = 0, \tag{36}$$

and equations (24) assume the form:

$$\bar{R}_{\alpha\beta} = 0. \tag{37}$$

Finally, equation (30) becomes:

$$\frac{\partial^2 \bar{\Lambda}_\alpha}{\partial x_\rho^2} = 0. \tag{38}$$

(*) Cf., R. Zaycoff, "Zur neuen Quantentheorie," Zeit. Phys. **54** (1929), 588-589.
(**) Cf., I.

One can also calculate equations (15), (16), (24), (30) in the second approximation with the use of formulas (34), (35), (36), (37), (38) in I and the equations (33), (35), (37), (38) for the first approximation.

Sofia, Physical Institute of the University, 25 February 1929.

On the foundations of a new field theory of A. Einstein

(Third part)

By **Raschco Zaycoff** in Sofia.

(Received on 11 March 1929)

The question that was posed by A. Einstein on the compatibility of the 20 fundamental equations with the fundamental identities is resolved.

In my second part (*), I followed the methods of A. Einstein precisely. In his most recent paper (**), Einstein still did not confirm that the chosen restrictions on the world metric were permissible. A little investigation shall resolve this question.

§ 1. We have used the identities:

$$(D_\alpha D_\beta - D_\beta D_\alpha) A^{\cdots}_{\cdots} \equiv D_\mu (\Lambda^{\cdot\cdot\mu}_{\alpha\beta} A^{\cdots}_{\cdots}) , \tag{1}$$

where $A^{\cdots}_{\cdots}$ is an arbitrary tensor, and:

$$D_\mu V^{\cdot\cdot\mu}_{\alpha\beta} \equiv 0, \tag{2}$$

$$D_\mu (D_\rho V^{\alpha\rho\mu} + V^{\alpha\kappa\rho} \Lambda^{\cdot\cdot\mu}_{\kappa\rho}) \equiv 0, \tag{3}$$

$$F_{\alpha\beta} \equiv - D_\mu W^{\cdot\cdot\mu}_{\alpha\beta} , \tag{4}$$

in which we have used:

(*) R. Zaycoff, "Zur Begründung einer neuen Feldtheorie von A. Einstein (Zweite Mitteilung.)," Zeit. Phys. **54** (1929), 590.

(**) A. Einstein, "Zur einheitlichen Feldtheorie," Ber. d. Preuss. Akad. (1929), no. 1.

(the same) "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus," Ber. d. Preuss. Akad. (1928), no. 17.

(the same) "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität," *ibid.*. (1928), no. 18.

R. Weitzenböck, "Differentialinvarianten in der Einsteinschen Theorie des Fernparallelismus," *ibid.* (1928), no. 26.

R. Zaycoff, "Zur Begründung einer neuen Feldtheorie von A. Einstein," Zeit. Phys. **53** (1929), 719.

$$\left.\begin{aligned} V_{\alpha\beta}^{\cdot\cdot\cdot\gamma} &= \Lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma} - W_{\alpha\beta}^{\cdot\cdot\cdot\gamma}, \\ W_{\alpha\beta}^{\cdot\cdot\cdot\gamma} &= \Lambda_\alpha \varepsilon_\beta^{\cdot\gamma} - \Lambda_\beta \varepsilon_\alpha^{\cdot\gamma}, \\ F_{\alpha\beta} &= \nabla_\alpha \Lambda_\beta - \nabla_\beta \Lambda_\alpha. \end{aligned}\right\} \tag{5}$$

Instead of setting the expression:

$$D_\rho V^{\alpha\rho\mu} + V^{\alpha\kappa\rho} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu}$$

equal to zero, we would like to assume that it is equal to $S^{\alpha\mu}$ (by definition), from which, due to (3), it emerges in any event that:

$$D_\mu S^{\alpha\mu} \equiv 0. \tag{6}$$

It then follows naturally from (6) that:

$$D_\alpha D_\mu S^{\alpha\mu} \equiv 0. \tag{7}$$

The identities (1) and (7) give, moreover:

$$D_\alpha (D_\mu S^{\alpha\mu} + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} S^{\kappa\rho}) \equiv 0. \tag{8}$$

Let $X^{\alpha\beta\dots}$ be any tensor that is anti-symmetric in α and β. From (1), one then has:

$$(D_\alpha D_\beta - D_\beta D_\alpha) X^{\alpha\beta\dots} \equiv D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} X^{\alpha\beta\dots}), \tag{9}$$

or, since:

$$- D_\beta D_\alpha X^{\alpha\beta\dots} \equiv D_\alpha D_\beta X^{\alpha\beta\dots}, \tag{10}$$

one also has:

$$D_\beta D_\alpha X^{\alpha\beta\dots} = \tfrac{1}{2} D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} X^{\kappa\rho\dots}). \tag{11}$$

When (11) is applied to the identity:

$$D_\rho V^{\alpha\rho\mu} + V^{\alpha\kappa\rho} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} - S^{\alpha\mu} \equiv 0, \tag{12}$$

that yields the relation (*):

$$D_\alpha (\tfrac{1}{2} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\alpha} V^{\kappa\rho\mu} + V^{\alpha\kappa\rho} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} - S^{\alpha\mu}) \equiv 0. \tag{13}$$

After manipulating this formula, with the use of (5), it follows that:

$$D_\alpha [\Lambda_\kappa (\Lambda^{\kappa\alpha\mu} - \Lambda^{\kappa\mu\alpha}) + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} (\tfrac{1}{2} \Lambda^{\kappa\rho\alpha} - \Lambda^{\kappa\alpha\rho})] - D_\rho S^{\alpha\mu} \equiv 0. \tag{14}$$

Thus, should the restriction:

(*) The tensor $V^{\alpha\rho\mu}$ is indeed anti-symmetric in α and ρ!

$$S^{\alpha\mu} = 0 \tag{15}$$

be in force, the identity (8) would be fulfilled, but the identity (14) gives the further restrictions (*):

$$D_\alpha [\Lambda_\kappa (\Lambda^{\kappa\alpha\mu} - \Lambda^{\kappa\mu\alpha}) + \Lambda^{\cdot\cdot\cdot\mu}_{\kappa\rho} (\tfrac{1}{2}\Lambda^{\kappa\rho\alpha} - \Lambda^{\kappa\alpha\rho})] = 0, \tag{16}$$

which are indeed of second order in magnitude, in such a way the first approximation would not be influenced by them.

However, it is conceivable that one might avoid this difficulty in Einstein's theory by postulating some restrictions of the form:

$$S^{\alpha\mu} \neq 0 \tag{17}$$

instead of the restrictions (15), such that the identities (13) or (14) would then give no further conditions on the $\Lambda^{\cdot\cdot\gamma}_{\alpha\beta}$ quantities, and only the identities (6) and (8) accomplish this.

However, a choice of the form (15) is free of contradictions, in itself, since (16) must then be fulfilled on the basis of (14).

§ 2. If we apply the identity (11) to the quantities $F^{\alpha\beta}$ then it follows that:

$$D_\alpha (D_\mu F^{\alpha\mu} - \tfrac{1}{2} \Lambda^{\cdot\cdot\cdot\alpha}_{\kappa\rho} F^{\kappa\rho}) \equiv 0. \tag{18}$$

If – as is, in fact, the case in the theory of A. Einstein – the further restrictions must be true:

$$D_\mu F^{\alpha\mu} = 0 \tag{19}$$

then, from (18), the additional restriction must be fulfilled that:

$$D_\alpha (\Lambda^{\cdot\cdot\cdot\alpha}_{\kappa\rho} F^{\kappa\rho}) = 0. \tag{20}$$

This is also of second order in magnitude. Should we desire to postulate the restrictions:

$$D_\mu F^{\alpha\mu} - \tfrac{1}{2} \Lambda^{\cdot\cdot\cdot\alpha}_{\kappa\rho} F^{\kappa\rho} \equiv \delta_\mu F^{\alpha\mu} = 0, \tag{21}$$

in place of (19) then (18) would indeed be fulfilled, but the entire theory would then lose its elegance and potential for success. However, in itself, the choice (19) is free of contradictions, since (20) must then necessarily exist, on the basis of (18).

Sofia, Physical Institute of the University, 6 March 1929.

(*) Which are four in number.

On the most recent formulation of Einstein's unified field theory

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(Received on 6 June 1929)

A. Einstein (*) succeeded in basing his theory on Hamilton's principle. The following report extends and completes Einstein's investigations relating to that, and is closely linked with them. Many of the objections to the geometric foundations of Einstein's theory will be briefly refuted. My previous three papers (**) will serve as a starting point.

§ 1. The difficulties regarding the admissibility of the twenty unified field equations to which I referred in III [Formulas (16), (20)] become more concrete upon more precise consideration. It does not allow one to exhibit the four identity relations for these equations that follow from the demand of covariance.

In I, pp. 723, we have given all possible Ansätze for the Hamilton function. They are, in fact, of the form (***):

$$H = \sum_{m=1}^{5} C_m H_m , \tag{1}$$

where

$$H_1 = \Lambda_{\alpha\beta\gamma} \Lambda^{\alpha\beta\gamma} \sqrt{g} , \quad H_2 = \Lambda_{\mu\alpha\beta} \Lambda^{\mu\beta\alpha} \sqrt{g} , \quad H_3 = \Lambda_\mu \Lambda^\mu \sqrt{g} ,$$
$$H_4 = \delta_\mu \Lambda^\mu \sqrt{g} = D_\mu \Lambda^\mu \sqrt{g} .$$

$H_5 = R \sqrt{g}$, and C_m are arbitrary constants.

Now, A. Einstein considered solely homogeneous quadratic functions of the $\Lambda_{\alpha\beta}^{\cdot\cdot\gamma}$. These are all of the form (****):

(*) A. Einstein, Sitzungsber. d. Preuss. Akad., supplement to volume 17/18 (1928), 1, and (1929), 10.

(**) R. Zaycoff, Zeit. Phys. **53** (1929), 719; **54** (1929), 590, 738. In the sequel, referred to as I, II, III. See also *R. Weitzenböck*, Sitzungsber. d. Preuss. Akad. (1929), supplement to volume 26 (1928), as well as the papers of *T. Levi-Civita*, *ibid.* (1929) and *H. Reichenbach*, Zeit. Phys. **53** (1929), 683. The papers of *Müntz* and *C. Lanczos* are still unknown to me.

(***) Cf., I, pp. 723.

(****) From I, pp. 723, the equations that determine the $h_{\alpha s}$ must include only those quantities and their derivatives up to second order, which is possible only when the Hamilton function has the form (1). On the other hand, from I (32), one has:

$$H_5 \equiv 2\, H_4 + H_3 - \frac{1}{2} H_2 - \frac{1}{4} H_1 .$$

Now, H_4, when used as the chosen Hamilton function, gives equations that are fulfilled identically. One convinces oneself of that from I (43) in the first approximation. It follows that we can employ only the linear combination (2) for the Hamilton function, in which the choice of constants C_1, C_2, C_3 remains free.

$$H = \sum_{m=1}^{3} C_m H_m \,. \tag{2}$$

One must have:

$$\delta \int H\, d\omega = 0 \qquad\qquad (d\omega = dx_1\, dx_2\, dx_3\, dx_4) \tag{3}$$

for all variations of $[h_{\alpha s}]$ that vanish on the boundary.

These sixteen equations follow from (3) after multiplying by $h^{\beta s}$:

$$\left\{ \frac{\partial H}{\partial h_{\alpha s}} - \frac{\partial}{\partial x_\rho}\left(\frac{\partial H}{\partial \frac{\partial h_{\alpha s}}{\partial x_\rho}} \right) \right\} h^{\beta s} = 0, \tag{4}$$

If we set:

$$\frac{\partial H}{\partial g_{\alpha\beta}} = H^{\alpha\beta}, \qquad\qquad \frac{\partial H}{\partial \Lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma}} = H^{\alpha\beta}_{\cdot\cdot\cdot\gamma}, \tag{5}$$

which are quantities that represent tensor densities, then it follows from I (2), (4), (12) that:

$$\left.\begin{aligned} \frac{\partial H}{\partial h_{\alpha s}} &= H^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial h_{\alpha s}} + H^{\mu\nu}_{\cdot\cdot\cdot\gamma} \frac{\partial \Lambda_{\mu\nu}^{\cdot\cdot\cdot\gamma}}{\partial h_{\alpha s}} = (2H^{\alpha\sigma} - H^{\mu\nu\rho}\Lambda_{\mu\nu}^{\cdot\cdot\cdot\alpha}) h_{\rho s}, \\ \frac{\partial H}{\partial \frac{\partial h_{\alpha s}}{\partial x_\rho}} &= H^{\mu\nu}_{\cdot\cdot\cdot\sigma} \frac{\partial \Lambda_{\mu\nu}^{\cdot\cdot\cdot\sigma}}{\partial \frac{\partial h_{\alpha s}}{\partial x_\rho}} = 2H^{\alpha\rho\kappa} h_{\kappa s}. \end{aligned}\right\} \tag{6}$$

Substituting (6) in (4) produces equations that, from I (7), (8), (12), II (5), can also be written:

$$G^{\alpha\beta} = H^{\alpha\beta} - D_\mu\, H^{\alpha\mu\beta} = 0. \tag{7}$$

After performing an infinitesimal coordinate transformation:

$$\delta h_{\alpha s} = \frac{\partial h_{\alpha s}}{\partial x_\mu} \xi^\mu + h_{\mu s} \frac{\partial \xi^\mu}{\partial x_\alpha}, \tag{8}$$

it follows from:

$$\int G^{\rho\kappa}\, h_{\kappa s}\, \delta h^{\rho s}\, d\omega = 0, \tag{9}$$

in which the ξ^μ vanish on the boundary, that:

$$\int {}_\rho \big| G^{\rho}_{\cdot\cdot\cdot\mu} \xi^\mu d\omega \equiv 0. \tag{10}$$

From the cited formulas in I and II, this yields:

$$ {}_{\rho}| \, G^{\rho\alpha} = D_\rho \, G^{\rho\alpha} + G^{\rho\kappa} \, \Lambda^{\alpha}_{\cdot\rho\kappa} \equiv 0, \tag{11} $$

which are four identities that can be applied to any Hamilton function of the form (2).

A. Einstein built his new theory upon equations (7) and the identities (11).

It follows from (7), (11), II (21), and III (11) that:

$$ {}_{\rho}| \, G^{\rho\alpha} = {}_{\rho}| \, H^{\rho\alpha} + \tfrac{1}{2} F_{\kappa\rho} \, H^{\kappa\rho\alpha} - \tfrac{1}{2} \Lambda^{\cdots\mu}_{\kappa\rho} \, \nabla_\mu H^{\kappa\rho\alpha} - D_\mu H^{\rho\mu\kappa} \Lambda^{\alpha}_{\cdot\rho\kappa} \equiv 0, \tag{12} $$

where, from II (20), (25):

$$ F_{\alpha\beta} = \nabla_\alpha \Lambda_\beta - \nabla_\beta \Lambda_\alpha = D_\alpha \Lambda_\beta - D_\beta \Lambda_\alpha \, . \tag{13} $$

We present the condition for $G^{\alpha\beta}$ to be symmetric in α, β; i.e.:

$$ G^{\alpha\beta} - G^{\beta\alpha} \equiv 0. \tag{14} $$

It follows from (7) and (14) that, from (5), one has:

$$ H^{\alpha\beta} - H^{\beta\alpha} \equiv 0, \tag{15} $$

$$ D_\mu \, (H^{\alpha\mu\beta} - H^{\beta\mu\alpha}) \equiv 0. \tag{16} $$

A calculation that uses (1) and (2) now yields (*):

$$ H^{\alpha\mu\beta} = \{2C_1 \, \Lambda^{\alpha\mu\beta} + C_2 \, (\Lambda^{\alpha\beta\mu} + \Lambda^{\beta\mu\alpha}) + C_3 \, (\Lambda^{\alpha} g^{\mu\beta} - \Lambda^{\mu} g^{\alpha\beta})\} \sqrt{g} \, . \tag{17} $$

From formulas II (6), (7), and identities II (9), it follows from (16) and (17) that:

$$ D_\mu \, \{(2C_1 - C_2) \, (\Lambda^{\alpha\mu\beta} - \Lambda^{\beta\mu\alpha}) + (2C_2 + C_3) \, W^{\alpha\beta\mu}\} \equiv 0. \tag{18} $$

This identity is possible only when one also has:

$$ 2C_1 - C_2 = 0, \qquad \text{as well as} \qquad 2C_2 + C_3 = 0. \tag{19} $$

Since one of the constants can be chosen freely, we would like to set $C_3 = 1$, from which, (19) yields:

$$ C_1 = -\tfrac{1}{4}, \qquad C_2 = -\tfrac{1}{2}. \tag{20} $$

It is thus proved that the special Hamilton function:

$$ H = H_3 - \tfrac{1}{2} H_2 - \tfrac{1}{4} H_1 \tag{21} $$

is the only one that produces equations (*):

(*) One has: $H^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} H + \{C_1 (\Lambda^{\cdots\alpha}_{\mu\kappa} \Lambda^{\mu\kappa\beta} - 2\Lambda^{\alpha}_{\mu\cdot\kappa} \Lambda^{\mu\beta\kappa}) - C_2 \Lambda^{\alpha}_{\mu\cdot\kappa} \Lambda^{\beta\kappa\mu} - C_3 \Lambda^{\alpha} \Lambda^{\beta}\} \sqrt{g}$.

$$G^{\alpha\beta} = 0 \tag{22}$$

that are symmetric in α, β. These are the gravitational equations.

From I (32), one has:

$$H \equiv H_5 - 2\,H_4\,. \tag{23}$$

In the earlier theory of relativity, it was postulated that: $H = H_5$, and due to the assumption that $\delta_\mu\,\Lambda^\mu = 0$, also that $H_4 = 0$.

We then see that in (23) we have arrived at a connection with the earlier theory.

§ 2. In order to obtain the missing six equations, we would like to replace the constants C_1, C_2, C_3 in (18), using (20), with constants that differ from them infinitely little:

$$\bar{C}_1 = -\tfrac{1}{4}(1-\varepsilon_1), \qquad \bar{C}_2 = -\tfrac{1}{2}(1+\varepsilon_1), \qquad \bar{C}_3 = (1-\varepsilon_2), \quad \begin{cases} \lim \varepsilon_1 = 0, \\ \lim \varepsilon_2 = 0. \end{cases} \tag{24}$$

We thus have, in place of (18):

$$D_\mu\{(2\bar{C}_1 - \bar{C}_2)(\Lambda^{\alpha\mu\beta} - \Lambda^{\beta\mu\alpha}) + (2\bar{C}_2 + \bar{C}_3)W^{\alpha\beta\mu}\} = 0. \tag{25}$$

If we set:

$$S_{\alpha\beta\gamma} = \Lambda_{\alpha\beta\gamma} + \Lambda_{\beta\gamma\alpha} + \Lambda_{\gamma\alpha\beta} \tag{26}$$

then it follows from (25) and II (7), (9), (22) that:

$$D_\mu\,S^{\alpha\mu\beta} + \sigma F^{\alpha\beta} = 0, \tag{27}$$

in which:

$$\sigma = \frac{2\bar{C}_1 + \bar{C}_2 + \bar{C}_3}{\bar{C}_2 - 2\bar{C}_1} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{28}$$

If we would like to replace the constants $\bar{C}_m$ in σ, not with the values (24), but with the values (20), then we would get $\sigma = 0 / 0$, which is meaningless.

Einstein obtained the six equations (27) when he replaced the function H with the function:

$$\left.\begin{aligned} \bar{H} &= H + \varepsilon_1 H^* + \varepsilon_2 H^{**}, \\ H^* &= \tfrac{1}{4}H_1 - \tfrac{1}{2}H_4, \quad H^{**} = -H_3, \end{aligned}\right\} \tag{29}$$

which differs from it infinitely little.

Equations (22) and (27) are the sixteen fundamental equations for the determination of the sixteen quantities $h_{\alpha s}$.

(*) Which are ten in number.

One also has:

$$H^* = \tfrac{1}{12} S_{\alpha\beta\gamma} S^{\alpha\beta\gamma} \sqrt{g} \,. \tag{30}$$

If we set:

$$*\Lambda^{\alpha\beta\gamma} = \pm \frac{1}{\sqrt{g}} \Lambda_\delta \,,$$

in which $*\Lambda^{\alpha\beta\gamma}$ means the tensor that is dual to Λ_δ and is anti-symmetric in all indices, and one chooses ± according to whether $(\alpha\beta\gamma\delta)$ represents an even or odd permutation of the numbers 1, 2, 3, 4 (†), then it follows that:

$$*F^{\alpha\beta} = D^*_\mu \Lambda^{\alpha\beta\mu} \,. \tag{31}$$

From (27), one has, moreover:

$$F^{\alpha\beta} = D_\mu \left(\frac{1}{\sigma} S^{\alpha\beta\mu} \right). \tag{32}$$

§ 3. We would like to consider the case in which:

$$\varepsilon_1 \gg \varepsilon_2 \,, \tag{33}$$

and thus lim $\sigma = 0$.

It follows from (27) that:

$$- D_\kappa D_\mu S^{\alpha\kappa\mu} + \sigma D_\mu F^{\alpha\mu} = 0. \tag{34}$$

From III (11):

$$-\tfrac{1}{2} D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} S^{\alpha\kappa\rho}) + \sigma D_\mu F^{\alpha\mu} = 0, \tag{35}$$

and again, from III (11):

$$-\tfrac{1}{2} D_\mu D_\lambda (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} S^{\mu\kappa\rho}) + \frac{\sigma}{2} D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} F^{\kappa\rho}) = 0, \tag{36}$$

or

$$D_\mu \{ D_\lambda (\Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} S^{\mu\kappa\rho}) + \sigma \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} F^{\kappa\rho} \} = 0. \tag{37}$$

From II (21), one has:

$$D_\mu \{ -F^{\kappa\rho} S^{\mu\kappa\rho} + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} \nabla_\lambda S^{\mu\kappa\rho} + \sigma \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} F^{\kappa\rho} \} = 0, \tag{38}$$

or

$$D_\mu \{ -F^{\kappa\rho} (S_{\kappa\rho}^{\cdot\cdot\cdot\mu} - \sigma \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu}) + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} \nabla_\lambda S^{\mu\kappa\rho} \} = 0, \tag{39}$$

From (27) and II (21), one gets:

$$\begin{aligned} &-\nabla_\mu F^{\kappa\rho} (S^{\mu\kappa\rho} - \sigma \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu}) - F^{\kappa\rho} (\sigma F_{\kappa\rho} + \sigma F_{\kappa\rho}) \\ &+ \nabla_\mu \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} \nabla_\lambda S^{\mu\kappa\rho} + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} D_\mu (D_\lambda S^{\mu\kappa\rho} + S^{\mu\kappa\rho} \Lambda_\lambda) = 0. \end{aligned} \tag{40}$$

(†) Cf., the papers of the author on Whittaker's theory, Zeit. Phys. **54** (1929), 588, formula (6); **55** (1929), 278, formula (6).

From (27), II (11), (21), one gets:

$$D_\mu (D_\lambda S^{\mu\kappa\rho}) = D_\mu D_\lambda S^{\mu\kappa\rho} + D_\sigma (\Lambda_{\mu\lambda}^{\cdot\cdot\cdot\sigma} S^{\mu\kappa\rho}),$$
$$= \sigma D_\lambda F^{\kappa\rho} + F_{\cdot\lambda}^{\mu} S_{\mu}^{\cdot\kappa\rho} + \Lambda_{\mu\lambda}^{\cdot\cdot\cdot\sigma} \nabla_\sigma S^{\mu\kappa\rho}, \tag{41}$$

and from (27):

$$D_\mu (S^{\mu\kappa\rho} \Lambda_\lambda) = \sigma F^{\kappa\rho} \Lambda_\lambda + S^{\mu\kappa\rho} \nabla_\mu \Lambda_\lambda . \tag{42}$$

From this, it follows that we can express (40) as:

$$A + \sigma B = 0, \tag{43}$$

where:

$$\left.\begin{aligned} A &= - S_{\kappa\rho}^{\cdot\cdot\cdot\mu} \nabla_\mu F^{\kappa\rho} + \nabla_\mu \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} \nabla_\lambda S^{\mu\kappa\rho} \\ &\quad - \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} (F_{\cdot\lambda}^{\mu} S_{\mu}^{\cdot\kappa\rho} - \Lambda_{\mu\lambda}^{\cdot\cdot\cdot\sigma} \nabla_\sigma S^{\mu\kappa\rho} - \nabla_\mu \Lambda_\lambda S^{\mu\kappa\rho}), \\ B &= \nabla_\mu F^{\kappa\rho} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} - 2 F_{\kappa\rho} F^{\kappa\rho} + \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\lambda} (D_\lambda F^{\kappa\rho} + F^{\kappa\rho} \Lambda_\lambda). \end{aligned}\right\} \tag{44}$$

The quantities $S_{\alpha\beta\gamma}$ and their first derivatives enter into A linearly and homogeneously. The quantities $S_{\alpha\beta\gamma}$ and their derivatives are missing from B. Since $F_{\alpha\beta}$, $\Lambda_{\alpha\beta}^{\cdot\cdot\cdot\nu}$, Λ_α, and their derivatives are finite, it then follows that we can set:

$$\lim_{\sigma\to 0} S_{\alpha\beta\gamma} = 0. \tag{45}$$

Moreover, it indeed emerges from (43) and (44) that the quantities $S_{\alpha\beta\gamma}$ tend to zero like the constant σ. This is the theorem that was conjectured by A. Einstein.

From (7), one has:

$$G^{*\alpha\beta} = H^{*\alpha\beta} - D_\mu H^{*\alpha\mu\beta}, \tag{46}$$

and from (29):

$$H^{*\alpha\mu\beta} = \tfrac{1}{2} S^{\alpha\mu\beta}, \tag{47}$$

such that:

$$G^{*\alpha\beta} = H^{*\alpha\beta} - \tfrac{1}{2} D_\mu S^{\alpha\mu\beta}. \tag{48}$$

Therefore, the $H^{*\alpha\mu\beta}$ cannot be homogeneous, quadratic functions of the $S_{\alpha\beta\gamma}$. Now, from (11), one must have:

$$_\rho| \, G^{*\rho\alpha} \equiv 0. \tag{49}$$

After converting this, using (12) and (47), and with consideration to (27), it follows that:

$$_\rho|H^{*\rho\alpha} + \tfrac{1}{4} F_{\kappa\rho} S^{\kappa\rho\alpha} - \tfrac{1}{4} \Lambda_{\kappa\rho}^{\cdot\cdot\cdot\mu} \nabla_\mu S^{\kappa\rho\alpha} + \tfrac{1}{2} \sigma F^{\rho\kappa} \Lambda_{\cdot\rho\kappa}^{\alpha} \equiv 0. \tag{50}$$

It follows from (48), using (27), that:

$$G^{*\alpha\beta} - H^{*\alpha\beta} = \sigma F^{\alpha\beta}, \tag{51}$$

and from (49) that:

$$2_\rho|H^{*\rho\alpha} + \sigma D_\rho F^{\rho\alpha} + \sigma F^{\rho\kappa}\,\Lambda^{\alpha}_{\cdot\rho\kappa} = 0. \tag{52}$$

By subtracting this from (50), one gets:

$$\sigma D_\rho F^{\rho\alpha} + \tfrac{1}{4}\Lambda_{\kappa\rho}^{\cdot\cdot\mu}\nabla_\mu S^{\kappa\rho\alpha} - \tfrac{1}{2}\sigma F_{\rho\kappa}S^{\kappa\rho\alpha} = 0. \tag{53}$$

It follows from (27) that:

$$\lim_{\sigma\to 0} D_\mu S^{\alpha\mu\beta} = 0, \tag{54}$$

which then consists of six equations, from the theorem that was proved above, that are true only when one likewise has the validity of the four equations:

$$\lim_{\sigma\to 0} S^{\alpha\mu\beta} = 0. \tag{55}$$

Equations (53) can also be derived from (35) and II (21). Equations (52) can also be represented as follows:

$$_\rho\left|\left\{\frac{2H^{*\rho\alpha}}{\sigma} + F^{\rho\alpha}\right\}\right. = 0. \tag{56}$$

We set:

$$\lim_{\sigma\to 0}\, _\rho\left|\frac{2H^{*\rho\alpha}}{\sigma}\right. = J^\alpha, \tag{57}$$

from which, it follows that:

$$\lim_{\sigma\to 0} \{_\rho|\, F^{\rho\alpha}\} + J^\alpha = 0. \tag{58}$$

The eight equations for the passage to the limit are then (55) and (58).

Due to (30), it follows from (55) that:

$$\lim_{\sigma\to 0} H^* = 0, \tag{59}$$

or

$$H_1 = 2H_2\,, \tag{60}$$

from (29). From (21), one then has:

$$\lim_{\sigma\to 0} H = H_3 - H_2\,. \tag{61}$$

§ 4. As a result of (14), the gravitational equations:

$$G^{\alpha\beta} = H^{\alpha\beta} - D_\mu H^{\alpha\mu\beta} = 0 \tag{62}$$

can also be written (†):

$$\tfrac{1}{2}(G^{\alpha\beta} + G^{\beta\alpha}) = H^{\alpha\beta} - \tfrac{1}{2}\, D_\mu (H^{\alpha\mu\beta} + H^{\beta\mu\alpha}) + g_{\alpha\beta} D_\mu \Lambda^\mu = 0, \tag{63}$$

or, after some calculation:

$$\frac{1}{\sqrt{g}} H_{\alpha\beta} - \tfrac{1}{2}(D_\alpha \Lambda_\beta + D_\beta \Lambda_\alpha) + \tfrac{1}{2} D_\mu (\Lambda^{\mu}_{\alpha\cdot\beta} + \Lambda^{\mu}_{\beta\cdot\alpha}) = 0. \tag{64}$$

Now, the identities II (1) can also be written:

$$\begin{aligned} R_{\alpha\beta} - \tfrac{1}{2}(D_\alpha \Lambda_\beta + D_\beta \Lambda_\alpha) - \Lambda_\alpha \Lambda_\beta + \tfrac{1}{2} D_\mu (\Lambda^{\mu}_{\alpha\cdot\beta} + \Lambda^{\mu}_{\beta\cdot\alpha}) \\ + \{(\Lambda_{\alpha\mu\kappa} + \Lambda_{\alpha\mu\kappa})(\Lambda_{\beta}^{\cdot\mu\kappa} + \Lambda_{\beta}^{\cdot\kappa\mu}) - \Lambda_{\mu\kappa\alpha} \Lambda^{\mu\kappa}_{\dots\beta}\} \equiv 0, \end{aligned} \tag{65}$$

or, from (64):

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} \left(\frac{H}{\sqrt{g}} + 2 D_\mu \Lambda^\mu \right) = 0. \tag{66}$$

One then has:

$$\frac{H}{\sqrt{g}} + 2 D_\mu \Lambda^\mu \equiv R. \tag{67}$$

From (66) and (67), one then has:

$$R_{\alpha\beta} = 0. \tag{68}$$

We have thus brought the gravitational equations into the usual form, and indeed they are true for any arbitrarily large value of the constant σ (††).

§ 5. From I (4), (12), and (26), one has:

$$S_{\alpha\beta\gamma} = h_{\alpha m} \left(\frac{\partial h_{\beta m}}{\partial x_\gamma} - \frac{\partial h_{\gamma m}}{\partial x_\beta} \right) + h_{\beta m} \left(\frac{\partial h_{\gamma m}}{\partial x_\alpha} - \frac{\partial h_{\alpha m}}{\partial x_\gamma} \right) + h_{\gamma m} \left(\frac{\partial h_{\alpha m}}{\partial x_\beta} - \frac{\partial h_{\beta m}}{\partial x_\alpha} \right). \tag{69}$$

(†) One has: $H_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} H + \{\frac{1}{2}(\Lambda_{\alpha\kappa\rho} \Lambda_{\beta}^{\cdot\rho\kappa} - \frac{1}{2} \Lambda_{\kappa\rho\alpha} \Lambda_{\beta}^{\cdot\rho\kappa} + \Lambda_{\kappa\rho\alpha} \Lambda^{\rho\kappa}_{\dots\beta}) - \Lambda_\alpha \Lambda_\beta\} \sqrt{g}$.

(††) One also has: $R = 0$ and $D_\mu \Lambda^\mu = -\dfrac{1}{2\sqrt{g}} H$.

In the first approximation, it follows from I (34) that:

$$\left.\begin{aligned} h_{\alpha m} &= \varepsilon_{\alpha m} + \bar{K}_{\alpha m}, \\ \bar{g}_{\alpha\beta} &= \bar{K}_{\alpha\beta} + \bar{K}_{\beta\alpha}. \end{aligned}\right\} \tag{70}$$

We also set:

$$\bar{K}_{\alpha\beta} - \bar{K}_{\alpha\beta} = \bar{d}_{\alpha\beta}. \tag{71}$$

Now, from (73), (74), and (75), in the first approximation:

$$\lim_{\sigma\to 0} \bar{S}_{\alpha\beta\gamma} = -\lim_{\sigma\to 0}\left\{\frac{\partial}{\partial x_\alpha}\bar{d}_{\beta\gamma} + \frac{\partial}{\partial x_\beta}\bar{d}_{\gamma\alpha} + \frac{\partial}{\partial x_\gamma}\bar{d}_{\alpha\beta}\right\} = 0. \tag{72}$$

It follows from this that:

$$\lim_{\sigma\to 0} d_{\alpha\beta} = \frac{\partial \bar{d}_\beta}{\partial x_\alpha} - \frac{\partial \bar{d}_\alpha}{\partial x_\beta}, \tag{73}$$

where the $\bar{d}_\alpha$ define the components of a four-vector.

§ 6. We have seen that the unified theory is free of logical objections and actually includes the two groups of phenomena – viz., gravitation and electromagnetism. The geometric foundations of the unified theory are simple, since the integrable connection arises from an entirely elementary group (*).

I cannot presently understand what A. S. Eddington (**) meant when he said that the new theory offered no advantage over the theories of H. Weyl, A. S. Eddington, et al. Indeed, one already knows that these latter theories have complicated the problem of the unity of the field laws more than they have led to any reasonable result.

The objection of J. A. Schouten (***) is also unfounded.

In fact, the $h_{\alpha m}$ define an orthogonal, anholonomic net; i.e., as H. Reichenbächer eloquently remarked: There are parallels, but no parallelograms. The Einstein world is flat, because the curvature ratios vanish identically, but it is also not Euclidian in the usual sense, but so-to-speak "anholonomic Euclidian," due to its non-vanishing torsion (****).

The Riemannian curvature will be compensated for by the torsion curvature everywhere. One can perhaps say that A. Einstein has constructed a flat world that is no longer barren, like the Euclidian space-time of H. Minkowski, but, on the contrary, includes everything that we care to call physical reality.

The $h_{\alpha m}$ define an orthogonal system of vectors at every point, but they do not define an orthogonal system in the large. For that reason, the arguments of Schouten do not

(*) Namely, orthogonal substitutions.
(**) A. S. Eddington, Nature **123** (1929), no. 3095.
(***) J. A. Schouten, C. R. **188** (1929), no. 14.
(****) Cf., I, pp. 722.

relate to the Einsteinian world. If we demand that they also define an orthogonal system in the large then either the torsion must vanish, in which case, the world would also be Euclidian in the usual sense, or the connection would become semi-symmetric. From the theorems of Cartan and Schouten, the latter possibility is excluded for a world with more than two dimensions that is based upon an integrable connection that comes from a simple group. As one says, the "$h_{\alpha m}$" do not then define an orthogonal system in the large. Schouten forgot that A. Einstein allowed only the uniform (i.e., equal everywhere) rotations of the vierbeins (*). It is clear that a non-uniform rotation would have the annihilation of torsion as a consequence.

The single objection that the can be made to the unified field theory is the following one:

It neglects the existence of wave-mechanical phenomena. Wave mechanics is entering into a self-sufficient phase by the work of Dirac, and the single successful attempt to connect this new group of phenomena with the remaining ones is the theory of J. M. Whittaker (**).

Indeed, he based it upon the old gravitation theory of A. Einstein and the Maxwell-Lorentz electrodynamics, and then extended these latter theories by wave fields and included the Dirac wave equations, as well as the entire theory of spin transformations, but this sort of unification is very contrived. Many difficulties that I referred to in my papers (***) can be lifted by means of the ideas of W. Anderson (****). On the basis of the Fermi statistics that are based in quantum theory, one can show that both types of electrical quanta and light quanta define different "phases" of a ground state. The difference becomes smaller and smaller with increasing pressure and temperature.

Müntz has already given solutions to the field equations in the first approximation for the spherically-symmetric case on the basis of the original formulation of the unified field theory. Solutions of the new formulation are completely lacking, and should perhaps be expected in the future. If one no longer separates gravitation from electromagnetism then one certainly gets more, but it can be very questionable whether the Dirac wave equation can be replaced with solutions of the new theory. Perhaps the current theory will admit yet another complete re-formulation, but ultimately completely different conceptions of the world are also imaginable.

Sofia, Physical Institute of the University, 1 June 1929.

Addendum added in proofreading: It follows from (70) and (71) in the first approximation that:

$$\lim_{\sigma\to 0} h_{\alpha\beta} = \varepsilon_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta} + \left(\frac{\partial \bar{d}_\beta}{\partial \kappa_\alpha} - \frac{\partial \bar{d}_\alpha}{\partial \kappa_\beta}\right). \tag{74}$$

A generalization of the theory can be made in the direction where we do not carry out the passage to the limit of lim $\sigma = 0$, and thus regard "σ" as a finite quantity, from which,

(*) Cf., I, footnote, pp. 721.
(**) J. M. Whittaker, Proc. Roy. Soc. (A) **121** (1928), 543.
(***) Zeit. Phys. **55** (1929), 273.
(****) *Ibidem* **54** (1929), 433.

it follows that $S_{\alpha\beta}$ can also be remain finite. In fact, the passage to the limit of lim $\sigma = 0$ is employed in order to be able to obtain the equations $\lim_{\sigma\to 0} \{_\rho \mid F^{\rho\alpha}\} = 0$, when one would like to conclude from (30) that $H^{*\alpha\beta}$ is constructed from $S^{\alpha\beta}$ homogeneously and quadratically. However, a calculation refutes this objection. We have obtained equations (58) with non-zero J^α as the electromagnetic equations.

We can replace (74) with:

$$h_{\alpha\beta} = \varepsilon_{\alpha\beta} + \tfrac{1}{2} g_{\alpha\beta} + \tfrac{1}{2} \bar{d}_{\alpha\beta}, \tag{75}$$

in which $\bar{d}_{\alpha\beta}$ is not determined by (73) now.

In a following communication: "Herleitung der Dirac-Whittakerschen Wellengleichungen aus der Einsteinschen einheitlichen Feldtheorie," I have set $\lim_{\sigma\to 0} \dfrac{S^{\alpha\beta\mu}}{\sigma} = {}^*K^{\alpha\beta\mu}$, but one can also, in the sense of the statements above, simply write $\dfrac{S^{\alpha\beta\mu}}{\sigma} = {}^*K^{\alpha\beta\mu}$ for a finite value of "σ" such that all of the reasoning in the cited communication remains in force. However, it seems to me that the choice (29) of Hamilton function with infinitely small constants ε_1 and ε_2 – a choice that leads to the gravitational equations *in vacuo* $R_{\alpha\beta} = 0$ – is not the appropriate one, since these equations do not have the form that K. Lanczos considered that is consistent with Mach's principle.

Furthermore, equations (58) are not the symmetrically-constructed electromagnetic field equations. I hope to come back to all of the questions that I have raised in a later publication that takes the most general viewpoint possible.

Sofia, Physical Institute of the University, 15 July 1929.

Derivation of the Dirac-Whittaker wave equations from Einstein's unified field theory

By **Raschko Zaycoff** in Sofia.

(Received on 21 June 1929)

It is asserted, in connection with the author's treatise (†) "Zu der neuen Formulierung der Einsteinschen einheitlichen Feldtheorie" and with the use of all the notations in that treatise, that the Dirac-Whittaker form of quantum theory (††) does not contradict the unified theory.

If we set:

$$\frac{\partial h_{\alpha m}}{\partial x_\beta} - \frac{\partial h_{\beta m}}{\partial x_\alpha} = h_{\alpha\beta m} \tag{1}$$

then it follows from IV and I (19) that:

$$S_{\alpha\beta\gamma} = h_{\alpha m}\, h_{\beta\gamma m} + h_{\beta m}\, h_{\gamma\alpha m} + h_{\gamma m}\, h_{\alpha\beta m}\,, \tag{2}$$

or, from dual relations that were mentioned in * (6):

$$-\, h^{\kappa m}\, {}^*h_{\alpha\kappa m} = {}^*S_\alpha\,. \tag{3}$$

We have an analogy with the formula that follows from I (12), (27), and (1) (†††):

$$h^{\kappa m}\, {}^*h_{\alpha\kappa m} = {}^*\Lambda_\alpha\,.$$

Since the quantities $S_{\alpha\beta\gamma}$ have the same order as σ, $\lim_{\sigma\to 0}\frac{S^{\alpha\beta\gamma}}{\sigma}$ has a well-defined limiting value, which we might denote by $^*K^{\alpha\beta\gamma}$, from which, IV (27) yields:

$$F^{\alpha\beta} = D_\mu\, {}^*K^{\alpha\beta\mu}, \tag{4}$$

(†) R. Zaycoff, Zeit. Phys. **56** (1929), 717. Referred to as IV in what follows.
(††) See the same author, *ibid.* **55** (1929), 273. Referred to by * in what follows.
(†††) One also has: $-\,(h_{\alpha m}\, {}^*h_{\beta\gamma m} + h_{\beta m}\, {}^*h_{\gamma\alpha m} + h_{\gamma m}\, {}^*h_{\alpha\beta m}) = {}^*\Lambda_{\alpha\beta\gamma}$. The four-vectors $^*S_\alpha$ and Λ_α are then dually similar.

On the other hand, from IV:

$$*F^{\alpha\beta} = D_\mu *\Lambda^{\alpha\beta\mu}. \tag{5}$$

The dual components of formulas (4) and (5) are:

$$\left.\begin{aligned} *F_{\alpha\beta} &= D_\alpha K_\beta - D_\beta K_\alpha, \\ F_{\alpha\beta} &= D_\alpha \Lambda_\beta - D_\beta \Lambda_\alpha. \end{aligned}\right\} \tag{6}$$

If we set:

$$\left.\begin{aligned} \Lambda^\alpha + K^\alpha &= \Phi^\alpha, \\ \Lambda^\alpha - K^\alpha &= \Psi^\alpha \end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} 2(F^{\alpha\beta} + *F^{\alpha\beta}) &= X^{\alpha\beta}, \\ 2(F^{\alpha\beta} - *F^{\alpha\beta}) &= Y^{\alpha\beta} \end{aligned}\right\} \tag{8}$$

then it follows from (4), (5), (6), (7), (8) that:

$$\left.\begin{aligned} X^{\alpha\beta} &= D^\alpha \Phi^\beta - D^\beta \Phi^\alpha + D_\mu *\Phi^{\alpha\beta\mu}, \\ Y^{\alpha\beta} &= D^\alpha \Psi^\beta - D^\beta \Psi^\alpha - D_\mu *\Psi^{\alpha\beta\mu} \end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} X^{\alpha\beta} &= \;\; 2D_\mu *\Phi^{\alpha\beta\mu} = 2(D^\alpha \Phi^\beta - D^\beta \Phi^\alpha), \\ Y^{\alpha\beta} &= -2D_\mu *\Psi^{\alpha\beta\mu} = 2(D^\alpha \Psi^\beta - D^\beta \Psi^\alpha). \end{aligned}\right\} \tag{10}$$

It now follows from the identity III (11) and from (10) that:

$$\left.\begin{aligned} D_\kappa D^\kappa \Phi^\alpha - (D_\kappa D^\alpha \Phi^\kappa - D^\alpha D_\kappa \Phi^\kappa) - D^\alpha D_\kappa \Phi^\kappa + \tfrac{1}{2} D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\mu} *\Phi^{\alpha\kappa\rho}) = 0, \\ D_\kappa D^\kappa \Psi^\alpha - (D_\kappa D^\alpha \Psi^\kappa - D^\alpha D_\kappa \Psi^\kappa) - D^\alpha D_\kappa \Psi^\kappa - \tfrac{1}{2} D_\mu (\Lambda_{\kappa\rho}^{\cdot\cdot\mu} *\Psi^{\alpha\kappa\rho}) = 0. \end{aligned}\right\} \tag{11}$$

We set:

$$\left.\begin{aligned} D_\kappa \Phi^\kappa &= \Phi, \\ D_\kappa \Psi^\kappa &= \Psi. \end{aligned}\right\} \tag{12}$$

Furthermore, from IV and (7), one has (†):

$$\left.\begin{aligned} \Phi &= \delta_\kappa \Phi^\kappa = \delta_\kappa \Lambda^\kappa + \delta_\kappa K^\kappa = -\tfrac{1}{2} H + \delta_\kappa K^\kappa, \\ \Psi &= \delta_\kappa \Psi^\kappa = \delta_\kappa \Lambda^\kappa - \delta_\kappa K^\kappa = -\tfrac{1}{2} H - \delta_\kappa K^\kappa. \end{aligned}\right\} \tag{13}$$

Then, from the identity II (11), one has:

(†) Here, one has: $H = \Lambda_\mu \Lambda^\mu \;\; -\frac{1}{2} \;\; \Lambda_{\mu\alpha\beta} \Lambda^{\mu\alpha\beta} \;\; -\frac{1}{4} \;\; \Lambda_{\alpha\beta\gamma} \Lambda^{\alpha\beta\gamma}$. [Translator's note: This is the way it appeared in the original.]

$$\left.\begin{aligned} D_\kappa D^\alpha \Phi^\kappa - D^\alpha D_\kappa \Phi^\kappa &\equiv D_\mu (\Lambda_\kappa^{\cdot\,\alpha\mu} \Phi^\kappa), \\ D_\kappa D^\alpha \Psi^\kappa - D^\alpha D_\kappa \Psi^\kappa &\equiv D_\mu (\Lambda_\kappa^{\cdot\,\alpha\mu} \Psi^\kappa). \end{aligned}\right\} \tag{14}$$

It follows from II (21) and * (20) that:

$$\left.\begin{aligned} -D_\mu \Lambda^{\alpha\beta\mu} &= F^{\alpha\beta}, \\ \tfrac{1}{2} F_{\kappa\rho} * \Phi^{\alpha\kappa\rho} &= *F^{\alpha\kappa} \Phi_\kappa, \\ \tfrac{1}{2} F_{\kappa\rho} * \Psi^{\alpha\kappa\rho} &= *F^{\alpha\kappa} \Psi_\kappa, \end{aligned}\right\} \tag{15}$$

such that from (12), (14), (15), equations (11) become:

$$\left.\begin{aligned} D_\kappa D^\kappa \Phi^\alpha - (F^{\alpha\kappa} + *F^{\alpha\kappa}) \Phi_\kappa - D^\alpha \Phi - \Lambda_\kappa^{\cdot\,\alpha\mu} \nabla_\mu \Phi^\kappa + \tfrac{1}{2} \Lambda_{\kappa\rho}^{\cdot\cdot\,\mu} \nabla_\mu * \Phi^\kappa = 0, \\ D_\kappa D^\kappa \Psi^\alpha - (F^{\alpha\kappa} - *F^{\alpha\kappa}) \Psi_\kappa - D^\alpha \Psi - \Lambda_\kappa^{\cdot\,\alpha\mu} \nabla_\mu \Psi^\kappa - \tfrac{1}{2} \Lambda_{\kappa\rho}^{\cdot\cdot\,\mu} \nabla_\mu * \Psi^\kappa = 0. \end{aligned}\right\} \tag{16}$$

Now, from I (12) and (1), with consideration given to I (11) and * (20):

$$\left.\begin{aligned} -\Lambda_\kappa^{\cdot\,\alpha\mu} \nabla_\mu \Phi^\kappa &= h_m^{\alpha\kappa} \nabla_\mu (h^{\mu m} \Phi_\kappa), \\ -\Lambda_\kappa^{\cdot\,\alpha\mu} \nabla_\mu \Psi^\kappa &= h_m^{\alpha\kappa} \nabla_\mu (h^{\mu m} \Psi_\kappa) \end{aligned}\right\} \tag{17}$$

and

$$\left.\begin{aligned} -\Lambda_{\kappa\rho}^{\cdot\cdot\,\mu} \nabla_\mu * \Phi^{\alpha\kappa\rho} &= *h_m^{\alpha\kappa} \nabla_\mu (h^{\mu m} \Phi_\kappa), \\ -\Lambda_\kappa^{\cdot\cdot\,\mu} \nabla_\mu * \Psi^{\alpha\kappa\rho} &= *h_m^{\alpha\kappa} \nabla_\mu (h^{\mu m} \Psi_\kappa). \end{aligned}\right\} \tag{18}$$

From (16), (17), and (18), it then follows that:

$$\left.\begin{aligned} D_\kappa D^\kappa \Phi^\alpha - (F^{\alpha\kappa} + *F^{\alpha\kappa}) \Phi_\kappa - D^\alpha \Phi + (h_m^{\alpha\kappa} + *h_m^{\alpha\kappa}) \nabla_\mu (h^{\mu m} \Phi_\kappa) = 0, \\ D_\kappa D^\kappa \Psi^\alpha - (F^{\alpha\kappa} - *F^{\alpha\kappa}) \Psi_\kappa - D^\alpha \Psi + (h_m^{\alpha\kappa} + *h_m^{\alpha\kappa}) \nabla_\mu (h^{\mu m} \Psi_\kappa) = 0. \end{aligned}\right\} \tag{19}$$

We now consider equations * (26).

If we set:

$$\left.\begin{aligned} p_\alpha - q_\alpha &= \Phi_\alpha, \\ p_\alpha + q_\alpha &= \Psi_\alpha \end{aligned}\right\} \tag{20}$$

then it follows from those equations that:

$$\left.\begin{aligned} D\Phi^\alpha - R^{\alpha\kappa} \Phi_\kappa + \alpha (F^{\alpha\kappa} + *F^{\alpha\kappa}) \Phi_\kappa = 0, \\ D\Psi^\alpha - R^{\alpha\kappa} \Psi_\kappa + \alpha (F^{\alpha\kappa} - *F^{\alpha\kappa}) \Psi_\kappa = 0, \end{aligned}\right\} \tag{21}$$

from which, using * (21), one has:

$$D = (\delta_\kappa + \alpha\,\varphi_\kappa) \cdot (\delta^\kappa + \alpha\,\varphi^\kappa) + \beta^2. \tag{22}$$

If we set:

$$\alpha\,\varphi_\rho = -\Lambda_\rho \tag{23}$$

then it follows that the quantities $F_{\alpha\beta}$, $*F_{\alpha\beta}$ in (21) can be replaced by the quantities:

$$\left.\begin{aligned} f_{\alpha\beta} &= -\alpha\ \ F_{\alpha\beta}, \\ *f_{\alpha\beta} &= -\alpha * F_{\alpha\beta}, \end{aligned}\right\} \tag{24}$$

and if we again write $F_{\alpha\beta}$, $*F_{\alpha\beta}$, in place of $f_{\alpha\beta}$, $*f_{\alpha\beta}$, then it follows from (21) that:

$$\left.\begin{aligned} D_\kappa D^\kappa \Phi^\alpha - R^{\alpha\kappa}\Phi_\kappa - (F^{\alpha\kappa} + *F^{\alpha\kappa})\Phi_\kappa + \beta^2\Phi^\alpha = 0, \\ D_\kappa D^\kappa \Psi^\alpha - R^{\alpha\kappa}\Psi_\kappa - (F^{\alpha\kappa} - *F^{\alpha\kappa})\Psi_\kappa + \beta^2\Psi^\alpha = 0, \end{aligned}\right\} \tag{25}$$

whereby Einstein's "∇_ρ" will now be used in place of Riemann's "δ_ρ".

We now compare equations (19) with (25).

Obviously the terms:

$$\left.\begin{aligned} (h_m^{\alpha\kappa} + *h_m^{\alpha\kappa})\nabla_\mu(h^{\mu m}\Phi_\kappa), \\ (h_m^{\alpha\kappa} - *h_m^{\alpha\kappa})\nabla_\mu(h^{\mu m}\Psi_\kappa) \end{aligned}\right\} \tag{26}$$

in equations (19) play the same role as the terms:

$$\left.\begin{aligned} -R^{\alpha\kappa}\Phi_\kappa, \\ -R^{\alpha\kappa}\Psi_\kappa \end{aligned}\right\} \tag{27}$$

do in equations (25).

Likewise, the terms:

$$\left.\begin{aligned} -D^\alpha\Phi, \\ -D^\alpha\Psi \end{aligned}\right\} \tag{28}$$

in (19) correspond to the terms:

$$\left.\begin{aligned} \beta^2\Phi^\alpha, \\ \beta^2\Psi^\alpha \end{aligned}\right\} \tag{29}$$

in (25).

The analogy is therefore far-reaching.

As is known, equations (25) can be linked with Dirac's theory.

Sofia, Physical Institute of the University, 15 June 1929.

Unified field theory and HAMILTON's principle

By A. EINSTEIN

In a recently-appearing paper (these Berichte, 1929, I), I set down field equations for a unified field without establishing a variational principle. The justification for these equations rested upon the assumption that the 16 field equations (10) of *loc. cit.* were compatible. Since it was not possible to produce four identity relations between these equations LANCZOS and MUNTZ expressed doubts concerning the admissibility of the field equations that were given there, without presenting a clear decision on what had been done. Meanwhile, I found that it is possible to solve the problem in a completely satisfactory way on the basis of a HAMILTON principle, in which the compatibility of the equations is certain from the outset. The identities that were derived in the earlier work, as well as the notation that was used there, will be used (assumed, resp.) here.

§ 1. Generalities on HAMILTON's principle, applied to a continuum with a RIEMANN metric and teleparallelism.

Let $\mathfrak{H}$ be a scalar density that can be expressed algebraically in terms of the $g_{\mu\nu}$ and the $\Lambda^{\alpha}_{\mu\nu}$. HAMILTON's principle:

$$\delta\left\{\int \mathfrak{H}\, d\tau\right\} = 0, \tag{1}$$

in which the ${}^{s}h_{\nu}$ are varied, is associated with the field equations:

$$\mathfrak{G}^{\mu\alpha} = \mathfrak{H}^{\mu\alpha} - (\mathfrak{H}^{\mu\nu}_{\underline{\alpha}})_{/\nu} = 0, \tag{2}$$

in which the quantities $\mathfrak{H}^{\mu\alpha}$ and $\mathfrak{H}^{\mu\nu}_{\underline{\alpha}}$ are defined by the equations:

$$\left.\begin{aligned} \mathfrak{H}^{\mu\nu} &= \frac{\partial \mathfrak{H}}{\partial g_{\mu\nu}}, \\ \mathfrak{H}^{\mu\nu}_{\alpha} &= \frac{\partial \mathfrak{H}}{\partial \Lambda^{\alpha}_{\mu\nu}}. \end{aligned}\right\} \tag{3}$$

This follows immediately from (1), if one considers the defining equation:

$$\Lambda^{\alpha}_{\mu\nu} = {}_s h^{\alpha} \, ({}^s h_{\mu,\nu} - {}^s h_{\nu,\mu}), \tag{4}$$

in which the comma means ordinary differentiation.

The fact that (1) is, in itself, satisfied for any variations (that vanish on the boundary) of the ${}^s h_\nu$ that can be generated by mere infinitesimal coordinate transformations leads to a four-identity:

$$D_\mu(\mathfrak{G}^{\mu\alpha}) = \mathfrak{H}^{\mu\alpha}{}_{/\mu} + \mathfrak{H}^{\mu\beta}\Lambda^{\beta}_{\underline{\alpha\mu}} \equiv 0. \tag{5}$$

as it does in the previous theory of relativity. In this, D_μ is the divergence-like differential operator that is given by (5). An identity of type (5) is always true for a tensor density $\mathfrak{G}^{\mu\alpha}$ that is the HAMILTON derivative of a scalar density $\mathfrak{H}$ that depends upon only ${}^s h_\nu$ and their derivatives.

§ 2. A special choice of HAMILTON function.

The simplest choice of HAMILTON function is characterized by the property that $\mathfrak{H}$ is of second degree in the $\Lambda^{\alpha}_{\mu\nu}$. From this, we derive the fact that $\mathfrak{H}$ is a linear combination of the quantities:

$$\left.\begin{aligned} \mathfrak{J}_1 &= h\,\Lambda^{\alpha}_{\mu\beta}\Lambda^{\beta}_{\underline{\mu\alpha}}, \\ \mathfrak{J}_2 &= h\,\Lambda^{\alpha}_{\mu\beta}\Lambda^{\alpha}_{\underline{\mu\beta}}, \\ \mathfrak{J}_3 &= h\,\Lambda^{\alpha}_{\mu\alpha}\Lambda^{\beta}_{\underline{\mu\beta}}. \end{aligned}\right\} \tag{6}$$

Amongst all possible linear combinations, only one of them is distinguished by the fact that the associated $\mathfrak{G}^{\mu\alpha}$ is symmetric, namely:

$$\mathfrak{H} = \tfrac{1}{2}\mathfrak{J}_1 + \tfrac{1}{4}\mathfrak{J}_2 - \mathfrak{J}_3 . \tag{7}$$

The proof of this is based upon the symmetry of $\mathfrak{H}^{\mu\alpha}$, as well as on the identity that was derived in the earlier paper:

$$\mathfrak{B}^{\alpha}_{\mu\nu/\alpha} = [h(\Lambda^{\alpha}_{\mu\nu} + \phi_\nu\delta^{\alpha}_{\mu} - \phi_\mu\delta^{\alpha}_{\nu})]_{/\alpha} \equiv 0. \tag{8}$$

By variation, this yields ten equations:

$$\mathfrak{G}^{\mu\alpha} = 0, \tag{9}$$

which agree with the gravitational field equations that are based in RIEMANNIAN geometry in the first approximation.

One obtains the desired field equations when one chooses a linear combination $\overline{\mathfrak{H}}$ of the $\mathfrak{J}$ that differs from it by infinitely small quantities, instead of the HAMILTON function in (7). For the sake of clarity, we choose it in such a way that:

$$\overline{\mathfrak{H}} = \mathfrak{H} + \varepsilon_1 \mathfrak{H}^* + \varepsilon_2 \mathfrak{H}^{**}, \tag{10}$$

in which:

$$\mathfrak{H}^* = \tfrac{1}{2}\mathfrak{J}_1 - \tfrac{1}{4}\mathfrak{J}_2, \tag{11}$$

$$\mathfrak{H}^{**} = \mathfrak{J}_3 . \tag{12}$$

By computation, we get:

$$\mathfrak{H}^* = -\frac{1}{12} h S^{\underline{\alpha}}_{\mu\nu} S^{\alpha}_{\underline{\mu\nu}}, \tag{11a}$$

in which we have set:

$$S^{\alpha}_{\mu\nu} = \Lambda^{\underline{\alpha}}_{\mu\nu} + \Lambda^{\underline{\nu}}_{\mu\alpha} + \Lambda^{\underline{\mu}}_{\nu\alpha}, \tag{13}$$

which is a quantity that is anti-symmetric in all three indices. By performing the variation of $\mathfrak{H}$ and splitting the tensor equation that is thus obtained into symmetric and anti-symmetric components, one gets, besides (9), the equations:

$$(\mathfrak{G}^{*\mu\alpha} - \mathfrak{G}^{*\alpha\mu}) + \sigma(\mathfrak{G}^{**\mu\alpha} - \mathfrak{G}^{**\alpha\mu}) = 0, \tag{14}$$

in which σ means the ratio of the infinitely small quantities ε_1 and ε_2. These equations can also be written in the form:

$$(\mathfrak{H}^{*\mu\nu}{}_{\underline{\alpha}} - \mathfrak{H}^{*\alpha\nu}{}_{\underline{\mu}}) + \sigma(\mathfrak{H}^{**\mu\nu}{}_{\underline{\alpha}} - \mathfrak{H}^{**\alpha\nu}{}_{\underline{\mu}}) = 0. \tag{14a}$$

By computation, one gets:

$$\mathfrak{H}^{*\mu\nu}{}_{\underline{\alpha}} - \mathfrak{H}^{*\alpha\nu}{}_{\underline{\mu}} = -h\, S^{\alpha}_{\underline{\mu\nu}} = -\mathfrak{S}^{\alpha}_{\underline{\mu\nu}}, \tag{15}$$

$$\mathfrak{H}^{**\mu\nu}{}_{\underline{\alpha}} - \mathfrak{H}^{**\alpha\nu}{}_{\underline{\mu}} = h\,(\phi^{\mu} g^{\alpha\nu} - \phi^{\alpha} g^{\mu\nu}), \tag{16}$$

and by performing the operation $/\nu$ on (14a), one gets:

$$\mathfrak{S}^{\alpha}_{\underline{\mu\nu}/\nu} - s[h(\phi_{\underline{\mu};\underline{\alpha}} - \phi_{\underline{\alpha};\underline{\mu}})]_{/\nu} = 0, \tag{17}$$

or, after introducing the contravariant tensor density $\mathfrak{f}^{\mu\alpha}$:

$$\mathfrak{S}^{\alpha}_{\underline{\mu\nu}/\nu} - \sigma \mathfrak{f}^{\mu\alpha} = 0. \tag{17a}$$

One sees immediately that these equations include the MAXWELL theory in the first approximation. We then first find that the dependence of the “field strengths” $\mathfrak{f}^{\mu\alpha}$ on the “potentials” ϕ_μ is the same as it is for MAXWELL in the first approximation. Secondly –

in the first approximation – the symbol $/\nu$ means ordinary differentiation so differentiation by α yields the vanishing of $\mathfrak{f}^{\mu\alpha}{}_{/\alpha}$, due to the anti-symmetry of $\mathfrak{S}$.

However, in order for the existence of electrical charge to be justified, it is necessary to pass to the limiting case of $\sigma = 0$.

§ 3. The limiting case of $\sigma = 0$.

In order to carry out the passage to the limit in question, we first require a lemma: $\mathfrak{G}^{*\mu\alpha}$ can be written in the form:

$$\mathfrak{G}^{*\mu\alpha} = \tfrac{1}{2}\mathfrak{S}^{\alpha}_{\underline{\mu\nu}/\nu} + \mathfrak{H}^{*\mu\alpha}. \tag{18}$$

It is apparent from (3) and (11*a*) that $\mathfrak{H}^{*\mu\alpha}$ depends upon the $S^{\alpha}_{\underline{\mu\nu}}$ in a quadratic and homogeneous fashion. Furthermore, $\mathfrak{G}^{*\mu\alpha}$ satisfies the identity:

$$D_\mu(\mathfrak{G}^{*\mu\alpha}) \equiv 0. \tag{5a}$$

Now, according to (17*a*), passing to the limit $\sigma = 0$ immediately gives the relations:

$$\mathfrak{S}^{\alpha}_{\underline{\mu\nu}/\nu} = 0. \tag{19}$$

These six equations have – aside from some special cases – the vanishing of the four quantities $\mathfrak{S}^{\alpha}_{\underline{\mu\nu}}$ as a result. I shall assume in what follows that upon passing to $\sigma = 0$, the quantities $\mathfrak{S}^{\alpha}_{\mu\nu}$, which are proportional to σ, go to zero, a statement for which I cannot produce a proof up to now.

When one eliminates $\mathfrak{S}^{\alpha}_{\underline{\mu\nu}/\nu}$ from (18) and (17*a*), one then obtains the equation:

$$2\,[\mathfrak{G}^{*\mu\alpha} - \mathfrak{G}^{*\mu\alpha}] - \sigma\,\mathfrak{f}^{\mu\alpha} = 0,$$

or, after performing the operation D_μ, due to (5*a*):

$$D_\mu\left(\mathfrak{f}^{\mu\alpha} + 2\frac{\mathfrak{H}^{*\mu\alpha}}{\sigma}\right) = 0. \tag{20}$$

The second term vanishes upon passing to the limit $\sigma = 0$, since its numerator, like $(\mathfrak{S}^{\alpha}_{\mu\nu})^2$ – i.e., from our assumption above – goes to zero like σ^2, such that one gets:

$$D_\mu(\mathfrak{f}^{\mu\alpha}) = 0, \tag{21}$$

an equation that, along with:

$$S_{\underline{\mu\nu}}^{\alpha} = 0, \tag{22}$$

defines the result of the passage to the limit.

The combined system of equations (9), (21), and (22) is then to be regarded as the final result of this investigation, in which the derivation of of (21) is not completely rigorous.

It must be remarked that equation (22) brings with it the fact that the HAMILTON function:

$$\mathfrak{H} = \mathfrak{J}_1 - \mathfrak{J}_3 \tag{7a}$$

can be used just as well in equations (9), in place of the HAMILTON function (7).

On the unified field theory that is based upon the Riemann metric and absolute parallelism

By

A. Einstein in Berlin

In the following paper, the theory that I have been developing for some years now will be presented in such a way that anyone who is knowledgeable in general theory of relativity will understand it comfortably. The following presentation is necessary, because the readers of the previous papers were required to waste their time unnecessarily when further connections and improvements were found since that time. Here, the situation is presented in a way that seems most preferable to me if one is to advance into it comfortably. In particular, I learned from Weitzenböck and Cartan that the treatment of continua of the type in question is not new in itself. Cartan was friendly enough to compose a treatise on the history of the mathematical situation that comes under consideration, by which he extended the scope of my presentation; it is published in this journal immediately after my own. At this point, I would like to extend my deepest thanks to Cartan for his worthwhile contribution. The most important, or at least new, aspect of the present treatise is the discovery of the simplest field laws that a Riemannian manifold with teleparallelism can be subject to. I will go into the question of the physical interpretation of the theory only briefly.

§ 1. The structure of the continuum.

Since dimension plays no role in the following arguments, we shall base it upon an n-dimensional continuum. In order for the facts about the metric and gravitation to be valid, we assume the existence of a Riemann metric. In nature, however, electromagnetic fields also exist, which cannot be represented by the Riemann metric. The question arises: How can we ascribe yet another structure to our Riemannian space in a logically natural way so that the totality has a unified character?

The continuum is (pseudo-) Euclidian in the neighborhood of a point P. A local Cartesian coordinate system (orthogonal n-bein, resp.) exists at every point, relative to which the Pythagorean Theorem is valid. The orientation of this n-bein plays no role in a Riemannian manifold. We would now like to assume that a direction relationship also prevails between these elementary Euclidian spaces. We would like to assume that it is meaningful to speak of a parallel orientation of all local n-beins (which is meaningless in a space with *only* a metric structure) by means of spatial structure, as it is in Euclidian geometry. In the sequel, we shall imagine that orthogonal n-beins are always oriented parallel to each other. The, in itself, arbitrary, orientation of the local n-beins at *one point P* then determines the orientation of the local n-beins at all points of the continuum uniquely. Our first problem now consists in describing such a continuum mathematically, and then to present the simplest restricting laws in it that such a continuum can be subject to. We do this in the hope of deriving the general laws of nature, as the earlier general theory of relativity sought to do for gravitation when one starts with only a metric space structure as a foundation.

§ 2. Mathematical description of the structure of the space.

The local n-bein consists of n mutually perpendicular unit vectors, whose components with respect to an arbitrary Gaussian coordinate system are $h_s{}^\nu$. Here, as always, a lower Latin index shall imply the association with a certain bein of the n-bein, while a Greek index, whether in the upper or lower position, shall express the contravariant (covariant, resp.) transformation character of the quantity in question relative to a change of Gaussian coordinate system.

The general transformation property of the $h_s{}^\nu$ is the following one: If one rotates all local systems (n-beins, resp.) in the same way (which is permitted), and one then likewise introduces a new Gaussian coordinate system then the transformation law:

(1) $$h_s{}^{\nu'} = \alpha_{st}\,\frac{\partial x^{\nu'}}{\partial x^\alpha}\,h_t{}^\alpha,$$

exist between the new and the old $h_s{}^\nu$, where the constant coefficients α_{st} define an orthogonal system:

(2) $$\alpha_{sa}\,\alpha_{sb} = \alpha_{as}\,\alpha_{bs} = \delta_{ab} = \begin{cases} 1, & \text{when } a = b, \\ 0, & \text{when } a \neq b. \end{cases}$$

The transformation law (1) can be generalized, with no further assumptions, to structures whose components have arbitrarily many local and coordinate indices. We shall call such structures *tensors.* The algebraic laws of tensors (e.g., addition, multiplication, contraction of Latin and Greek indices) follow from that immediately.

We call the $h_s{}^\nu$ the components of the *fundamental tensor.* If a vector has the components A_s in a local system relative to the Gaussian system of coordinates A_ν then, from the meaning of the $h_s{}^\nu$, one has:

(3) $$A^\nu = h_s{}^\nu A_s,$$

or – when this is solved for the A_s :

(4) $$A_s = h_{s\nu} A^\nu.$$

The tensor character of the normalized sub-determinant $h_{s\nu}$ of the $h_s{}^\nu$ is clear from (4). $h_{s\nu}$ are the covariant components of the fundamental tensor. The following relations exist between the $h_{s\nu}$ and the $h_s{}^\nu$:

$$h_{s\mu}\, h_s{}^\nu = \delta_\mu{}^\nu = \begin{cases} 1, & \text{when } \mu = \nu, \\ 0, & \text{when } \mu \neq \nu, \end{cases} \tag{5}$$

$$h_{s\mu}\, h_t^\mu = \delta_{st}\,. \tag{6}$$

Due to the orthogonality of the local system, one has:

$$A^2 = A_s{}^2 = h_{s\mu}\, h_{s\nu} A^\mu A^\nu = g_{\mu\nu} A^\mu A^\nu, \tag{6}$$

for the magnitude of the vector, so:

$$g_{\mu\nu} = h_{s\mu}\, h_{s\nu} \tag{7}$$

are the coefficients of the metric.

The fundamental tensor [cf. (3) and (4)] allows local indices to be converted into coordinate indices, and conversely (by multiplication and contraction), such that the question of which index character one will operate on a tensor with implies only a question of form.

It is clear that one also has the relations:

$$A_\nu = h_{s\nu}\, A_s\,, \tag{3a}$$

$$A_s = h_s{}^\nu A_\nu\,. \tag{4a}$$

Moreover, one has the determinant relation:

$$g = |\, g_{\sigma\tau}\,| = |\, h_{\alpha\sigma}\,|^2 = h^2, \tag{8}$$

such that the invariant volume element $\sqrt{g}\; d\tau$ assumes the form $h\, d\tau$.

In our 4-dimensional continuum of space and time, the special character of time will appear most conveniently by arranging that the x^4-coordinate (local, as well as general) is taken to be pure imaginary, and likewise all tensor components with an odd number of 4 indices.

§ 3. Differential relations.

We let δ denote the increase that the components of a vector or tensor experience under a "parallel displacement," in the Levi-Civita sense, under the transition to an infinitely close point of the continuum, so, from the above, one has:

$$0 = \delta A_s = \delta(h_{s\alpha} A^\alpha) = d(h_s{}^\alpha A_\alpha). \tag{9}$$

Performing the operation on the parentheses yields:

$$h_{s\alpha}\, \delta A^\alpha + A^\alpha\, h_{s\alpha,\,\beta}\, \delta x^\beta = 0,$$

$$h_s^{\ \alpha}\,\delta A_\alpha + A_\alpha\,h_{s\ \ ,\beta}^{\ \alpha}\,\delta x^\beta = 0,$$

where the comma in the second term means ordinary differentiation with respect to x^β.

By solving these equations, one gets:

$$\delta A^\sigma = -\,A^\alpha\,\Delta_{\alpha\ \beta}^{\ \sigma}\,\delta x^\beta, \tag{10}$$

$$\delta A_\sigma = \quad A_\alpha\,\Delta_{\sigma\ \beta}^{\ \alpha}\,\delta x^\beta, \tag{11}$$

in which we have set:

$$\Delta_{\alpha\ \beta}^{\ \sigma} = h_s^{\ \alpha}\,h_{s\alpha,\,\beta} = -\,h_{s\alpha}\,h_{s\ \ ,\beta}^{\ \alpha}\,. \tag{12}$$

[The last conversion is based upon (5).]

In contrast to Riemann geometry, this law of parallel displacement is generally not symmetric. If it is symmetric then one has Euclidian geometry; one then has:

$$\Delta_{\alpha\ \beta}^{\ \sigma} - \Delta_{\beta\ \alpha}^{\ \sigma} = 0,$$

or

$$h_{s\alpha,\,\beta} - h_{s\beta,\,\sigma} = 0.$$

However, one then has:

$$h_{s\alpha} = \frac{\partial\psi_s}{\partial x_\alpha}\,.$$

If one chooses the ψ_s to be the new variables x'_s then one has:

$$h_{s\alpha} = \delta_{s\alpha}\,, \tag{13}$$

which proves the assertion.

Covariant differentiation. The local components A_s of a vector are invariant under an arbitrary coordinate transformation. The tensor character of the differential quotients:

$$A_{s,\,\alpha} \tag{14}$$

follows from this. If one replaces this with:

$$(h_s^{\ \sigma}\,A_\sigma)_{,\,\alpha}\,,$$

based upon (4*a*), then this yields the tensor character of:

$$h_s^{\ \sigma}\,A_{\sigma,\,\alpha} + A_\sigma\,h_{s\ \ ,\,\alpha}^{\ \sigma},$$

and therefore (after multiplying by $h_{s\tau}$) also that of:

$$A_{\tau,\,\alpha} + A_\sigma\,h_{s\ \ ,\,\alpha}^{\ \sigma}h_{s\tau}$$

and

$$A_{\tau,\,\alpha} - A_\sigma\,h_s^{\ \sigma}\,h_{s\tau,\,\alpha}\,,$$

or, according to (16), of:

$$A_{\tau,\,\alpha} - A_\sigma \Delta_\tau{}^\sigma{}_\alpha\,.$$

We refer to this as the *covariant derivative* ($A_{\tau;\,\alpha}$) of A_τ.

We have thus arrived at:

$$A_{\sigma;\,\tau} = A_{\sigma,\,\tau} - A_\alpha \Delta_\sigma{}^\alpha{}_\tau \tag{15}$$

as the law of covariant differentiation. Analogously, the formula:

$$A^\sigma{}_{;\,\tau} = A^\sigma{}_{,\,\tau} + A^\alpha \Delta_\alpha{}^\sigma{}_\tau \tag{16}$$

also follows from (3).

We now have the analogous law for the covariant differentiation of arbitrary tensors. We describe it by the example:

$$A_a{}^\sigma{}_{\tau;\,\rho} = A_a{}^\sigma{}_{\tau,\,\rho} + A_a{}^\alpha{}_\tau \Delta_\alpha{}^\sigma{}_\rho - A_a{}^\sigma{}_\alpha \Delta_\tau{}^\alpha{}_\rho\,. \tag{17}$$

Since local (i.e., Latin) indices can be converted into coordinate (i.e., Greek) indices by means of the fundamental tensor $h_s{}^\alpha$, one can freely choose whether one prefers either type of tensor index in the formulation of any tensor relations. The former approach would be preferred by the Italian school (e.g., Levi-Civita, Palatini), while I prefer to use the coordinate indices.

Divergence. One gets the divergence by contracting the differential quotients, as one does in the absolute differential calculus that is based upon the metric alone. For example, by contracting the indices σ and ρ in (17), one gets the tensor:

$$A_{a\tau} = A_a{}^\sigma{}_{\tau;\,\sigma}\,.$$

In previous papers, I have introduced other divergence operations, but I have deviated from them here in order to ascribe a special meaning to the operations.

Covariant differential quotients of the fundamental tensor. One easily finds that the covariant derivatives and divergences of the fundamental tensor vanish from the formulas that we derived. For instance, one has:

$$\begin{aligned} h_s{}^\nu{}_{;\,\tau} &\equiv h_s{}^\nu{}_{,\tau} + h_s{}^\nu \Delta_\alpha{}^\nu{}_\tau \equiv \delta_{st}\,(h_t{}^\nu{}_{,\tau} + h_t{}^\nu \Delta_\alpha{}^\nu{}_\tau) \\ &\equiv h_s{}^\alpha\,(h_{t\alpha}\, h_t{}^\nu{}_{,\tau} + \Delta_\alpha{}^\nu{}_\tau) \equiv h_s{}^\alpha\,(-\Delta_\alpha{}^\nu{}_\tau + \Delta_\alpha{}^\nu{}_\tau) \equiv 0. \end{aligned} \tag{18}$$

One also proves analogously that:

$$h_s{}^\nu{}_{;\,\tau} \equiv g^{\mu\nu}{}_{;\,\tau} \equiv g_{\mu\nu;\,\tau} \equiv 0. \tag{18a}$$

Differentiation of tensor products. As in the familiar differential calculus, the covariant differential quotient of a tensor product can be expressed in terms of the differential quotients of the factors. If $S^{\cdots}_{\cdots}$ and $T^{\cdots}_{\cdots}$ are tensors of arbitrary index character then one has:

(19) $$(S^{\cdots}_{\cdots}\, T^{\cdots}_{\cdots})_{;\,\alpha} = S^{\cdots}_{\cdots;\alpha}\, T^{\cdots}_{\cdots} + S^{\cdots}_{\cdots}\, T^{\cdots}_{\cdots;\alpha} .$$

It follows from this and the vanishing of the covariant differential quotients of the fundamental tensor that one can exchange them with the differentiation sign (;) at will.

"*Curvature*". The hypothesis of "teleparallelism" [equation (9), resp.] yields the integrability of the displacement law (10) [(11), resp.]. From this, it follows that:

(20) $$0 \equiv -\Delta_{\kappa}{}^{\iota}{}_{\lambda;\,\mu} \equiv -\Delta_{\kappa}{}^{\iota}{}_{\lambda,\,\mu} + \Delta_{\kappa}{}^{\iota}{}_{\mu,\,\lambda} + \Delta_{\sigma}{}^{\iota}{}_{\lambda}\,\Delta_{\kappa}{}^{\sigma}{}_{\mu} - \Delta_{\sigma}{}^{\iota}{}_{\mu}\,\Delta_{\kappa}{}^{\sigma}{}_{\lambda} .$$

The Δ must satisfy this condition in order for them to be expressed in terms of the h quantities according to (12). One sees from (20) that the mandated characterization of a manifold of the type that is considered here must be very different from the one that the previous theory obeyed. Indeed, according to the new theory, all tensors of the previous theory still exist, and in particular, the Riemannian curvature tensor that is defined by the Christoffel symbols. However, according to the new theory, simpler and more intrinsic tensorial constructions also exist that can be used in the formulation of the field laws.

The tensor Λ. If we covariant differentiate a scalar φ twice then, from (15), we get the tensor:

$$\varphi_{,\,\sigma,\,\tau} - \varphi_{,\,\alpha}\,\Delta_{\sigma}{}^{\alpha}{}_{\lambda} .$$

A new tensor arises by switching σ and τ, and by subtracting both expressions, one gets the tensor:

$$\frac{\partial\varphi}{\partial x_{\alpha}}(\Delta^{\ \alpha}_{\sigma\ \tau} - \Delta^{\ \alpha}_{\tau\ \sigma}) .$$

The tensor character of:

(21) $$\Lambda^{\ \alpha}_{\sigma\ \tau} = \Delta^{\ \alpha}_{\sigma\ \tau} - \Delta^{\ \alpha}_{\tau\ \sigma}$$

follows from that immediately. Therefore, a tensor exists in this theory that includes only the components $h_{s\alpha}$ of the fundamental tensor and its first differential quotients. The fact that its vanishing has the validity of Euclidian geometry as a consequence was already proved earlier [cf., (13)]. A natural way of determining such a continuum will then exist in the form of conditions on this tensor.

By contracting the tensor Λ, one gets the vector:

(22) $$\varphi_{\sigma} = \Lambda^{\ \alpha}_{\sigma\ \alpha} ,$$

which I had previously assumed would play the role of the electromagnetic potential in this theory. However, I have recently deviated from this assumption.

Commutation rule for differentiation. If one covariant differentiates an arbitrary tensor $T^{\cdots}_{\cdots}$ twice then one gets the important commutation rule:

$$T^{\cdots}_{\dots;\sigma;\tau} - T^{\cdots}_{\dots;\tau;\sigma} \equiv - T^{\cdots}_{\dots;\alpha}\Lambda^{\alpha}_{\sigma\tau}. \tag{23}$$

Proof. If T is a scalar (i.e., a tensor with no Greek indices) then the theorem follows effortlessly from (15). We would like to base the general proof of the theorem upon this special case.

We first remark about that statement that according to the theory that is treated here there are parallel vector fields. They are vector fields that have the same components in all local systems. If (a^{α}) [(a_{α}), resp.] is such a vector field then it fulfills the condition:

$$a^{\alpha}{}_{;\,\sigma} = 0 \qquad (a_{\alpha;\,\sigma} = 0, \text{ resp.}),$$

as is easily proved.

The commutation rule effortlessly leads back to the rule for a scalar with the use of such parallel vector fields. For ease of notation, we carry out the proof for a tensor T^{λ} with only one index. If φ is a scalar then it follows from the defining equations (16) and (21) that:

$$\varphi_{;\,\sigma;\,\tau} - \varphi_{;\,\tau;\,\sigma} \equiv - \varphi_{;\,\alpha}\Delta_{\sigma}{}^{\alpha}{}_{\tau}.$$

If we replace the scalar φ with $a_{\lambda}\,T^{\lambda}$ in this equation, where a_{λ} is a parallel vector field then a_{λ} can be commuted with differentiation sign under any covariant differentiation, such that a_{λ} appears as a factor in all terms. One then obtains:

$$[T^{\lambda}{}_{;\,\sigma;\,\tau} - T^{\lambda}{}_{;\,\tau;\,\sigma} + T^{\lambda}{}_{;\,\alpha}\Delta_{\sigma}{}^{\alpha}{}_{\tau}]\, a_{\lambda} \equiv 0.$$

Since this identity must exist for any choice of a_{λ} in the position that we consider, the vanishing of the square brackets must follow, with which the proof is complete. The generalization to tensors with arbitrarily many Greek indices is immediate.

Identities for the tensor Λ. If one adds the three identities that emerge from (20) by cyclic permutation of κ, λ, μ then, by a suitable way of regarding the terms, and recalling (21), it follows that:

$$0 \equiv (\Lambda_{\kappa}{}^{\iota}{}_{\lambda,\,\mu} + \Lambda_{\lambda}{}^{\iota}{}_{\mu,\,\kappa} + \Lambda_{\mu}{}^{\iota}{}_{\kappa,\,\lambda}) + (\Delta_{\sigma}{}^{\iota}{}_{\kappa}\Lambda_{\lambda}{}^{\sigma}{}_{\mu} + \Delta_{\sigma}{}^{\iota}{}_{\lambda}\Lambda_{\mu}{}^{\sigma}{}_{\kappa} + \Delta_{\sigma}{}^{\iota}{}_{\mu}\Lambda_{\kappa}{}^{\sigma}{}_{\lambda}).$$

We convert this identity in such a way that we introduce the covariant derivatives of the tensor Λ (according to (17)), instead of ordinary differentiation; we then obtain the identity:

$$0 \equiv (\Lambda_{\kappa}{}^{\iota}{}_{\lambda;\,\mu} + \Lambda_{\lambda}{}^{\iota}{}_{\mu;\,\kappa} + \Lambda_{\mu}{}^{\iota}{}_{\kappa;\,\lambda}) + (\Delta_{\sigma}{}^{\iota}{}_{\kappa}\Lambda_{\lambda}{}^{\sigma}{}_{\mu} + \Delta_{\sigma}{}^{\iota}{}_{\lambda}\Lambda_{\mu}{}^{\sigma}{}_{\kappa} + \Delta_{\sigma}{}^{\iota}{}_{\mu}\Lambda_{\kappa}{}^{\sigma}{}_{\lambda}). \tag{24}$$

This is the condition for the Λ to be expressible in terms of the h in the given way.

By contracting this equation over the indices ι and μ, one further obtains the identity:

$$0 \equiv \Lambda_{\kappa}{}^{\alpha}{}_{\lambda\,;\,\alpha} + \varphi_{\lambda\,;\,\kappa} - \varphi_{\kappa\,;\,\lambda} - \varphi_{\alpha}\,\Lambda_{\kappa}{}^{\alpha}{}_{\lambda}\,,$$

or

(25) $$\Lambda_{\kappa}{}^{\alpha}{}_{\lambda\,;\,\alpha} \equiv \varphi_{\kappa,\,\lambda} - \varphi_{\lambda\,,\,\kappa},$$

where the φ_λ is the abbreviation for $\Lambda_{\lambda}{}^{\alpha}{}_{\alpha}$ [cf., (22)].

§ 4. The field equations.

The simplest field equations that we seek will be conditions that the tensor $\Lambda_{\mu}{}^{\alpha}{}_{\nu}$ is subject to. Since the number of h-components is n^2 and n of them must remain undetermined due to general covariance, the number of mutually-independent field equations must be $n^2 - n$. On the other hand, it is clear that a theory becomes all the more satisfying the more that it restricts the possibilities (without coming into contradiction with experiment). The number Z of field equations shall be as large as possible. If $\bar{Z}$ is the number of identities that exist between them then $Z - \bar{Z}$ must equal $n^2 - n$.

According to the commutation rule for differentiation, one has:

(26) $$\Lambda^{\alpha}_{\underline{\mu}\,\underline{\nu};\nu;\alpha} - \Lambda^{\alpha}_{\underline{\mu}\,\underline{\nu};\alpha;\nu} - \Lambda^{\sigma}_{\underline{\mu}\,\underline{\tau};\alpha}\Lambda^{\alpha}_{\sigma\,\tau} \equiv 0.$$

Here, the underline below an index means the "raising" ("lowering," resp.) of an index, so, e.g.:

$$\Lambda^{\alpha}_{\underline{\mu}\,\underline{\nu}} \equiv \Lambda^{\alpha}_{\beta\,\gamma} g^{\mu\beta} g^{\nu\gamma},$$

$$\Lambda^{\underline{\alpha}}_{\mu\,\nu} \equiv \Lambda^{\beta}_{\mu\,\nu} g_{\alpha\beta}.$$

We now write the identity (26) in the form:

(26*a*) $$G^{\mu\alpha}{}_{;\,\alpha} - F^{\mu\nu}{}_{;\,\nu} + \Lambda^{\sigma}_{\underline{\mu}\,\underline{\tau}}\,F_{\sigma\tau} \equiv 0,$$

in which we have set:

(27) $$G^{\mu\alpha} \equiv \Lambda^{\alpha}_{\underline{\mu}\,\underline{\nu};\,\nu} - \Lambda^{\sigma}_{\underline{\mu}\,\underline{\tau}}\Lambda^{\alpha}_{\sigma\,\tau}\,,$$

(28) $$F^{\mu\alpha} \equiv \Lambda^{\alpha}_{\underline{\mu}\,\underline{\nu};\,\alpha}\,.$$

We now propose that the *field equations* should be:

(29) $$G^{\mu\alpha} = 0,$$

(29) $$F^{\mu\alpha} = 0.$$

These equations seem to contain an inadmissible indeterminacy. The number of them is $n^2 + \frac{n(n-1)}{2}$, while, for the time being, all that is known about them is that they must satisfy n identities, namely, (26*a*).

However, from the identity (25), in conjunction with (30), it emerges that the φ_κ must be derivable from a potential. We accordingly set:

$$F_\kappa \equiv \varphi_\kappa - \frac{\partial \ln \psi}{\partial x^\kappa} = 0. \tag{31}$$

(31) is completely equivalent to (30). Equations (29), (31) are collectively $n^2 + n$ equations for the $n^2 + 1$ functions $h_{s\nu}$ and ψ. However, in addition to (26*a*), yet another system of identities exists between these equations, which we would now like to derive.

If one denotes the anti-symmetric part of $G^{\mu\alpha}$ by $\underline{G}^{\mu\alpha}$ then one obtains from straightforward calculations that start with (27):

$$2\underline{G}^{\mu\alpha} \equiv -S_{\mu}{}^{\nu}{}_{\alpha;\nu} + \tfrac{1}{2} S_{\underline{\sigma}\,\underline{\tau}}^{\;\mu} \Lambda_{\sigma\tau}^{\;\alpha} - \tfrac{1}{2} S_{\underline{\sigma}\,\underline{\tau}}^{\;\alpha} \Lambda_{\sigma\tau}^{\;\mu} + F^{\mu\alpha}, \tag{32}$$

where, to abbreviate, the tensor:

$$S_{\underline{\mu}\,\underline{\nu}}^{\;\alpha} = \Lambda_{\underline{\mu}\,\underline{\nu}}^{\;\alpha} + \Lambda_{\underline{\alpha}\,\underline{\mu}}^{\;\nu} + \Lambda_{\underline{\nu}\,\underline{\alpha}}^{\;\mu}, \tag{33}$$

which is anti-symmetric in all indices, has been introduced. By calculating the first term of (32), this yields:

$$2\underline{G}^{\mu\alpha} \equiv -S_{\mu}{}^{\nu}{}_{\alpha;\nu} + S_{\underline{\mu}\,\underset{\sim}{\alpha}}^{\;\nu} - S_{\underline{\mu}\,\underset{\sim}{\alpha}}^{\;\sigma} \Lambda_{\sigma\nu}^{\;\nu} + F^{\mu\alpha}. \tag{34}$$

However, if one recalls the definition of F_κ – viz., (31) – then one now has:

$$\Delta_{\sigma}{}^{\nu}{}_{\nu} - \Delta_{\nu}{}^{\nu}{}_{\sigma} \equiv \Lambda_{\sigma}{}^{\nu}{}_{\nu} \equiv \varphi_\sigma \equiv F_\sigma + \frac{\partial \ln \psi}{\partial x^\sigma}$$

or

$$\Delta_{\sigma}{}^{\nu}{}_{\nu} = \frac{\partial \ln \psi h}{\partial x^\sigma} + F_\sigma. \tag{35}$$

(34) then assumes the form:

$$h\,\psi(2\underline{G}^{\mu\alpha} - F^{\mu\alpha} + S_{\underline{\mu}\,\underset{\sim}{\alpha}}^{\;\sigma} F_\sigma) \equiv -\frac{\partial}{\partial x^\sigma}(h\psi S_{\underline{\mu}\;\underline{\alpha}}^{\;\sigma}). \tag{34b}$$

The desired system of identity equations follows from this due to anti-symmetry:

$$\frac{\partial}{\partial x^\sigma}[h\,\psi(2\underline{G}^{\mu\alpha} - F^{\mu\alpha} + S_{\underline{\mu}\,\underset{\sim}{\alpha}}^{\;\sigma} F_\sigma)] \equiv 0. \tag{36}$$

These are n identities, of which, however, only $n - 1$ are mutually independent, and in which, due to anti-symmetry, one has $[\ ]_{,\alpha,\mu} \equiv 0$, independently of whatever one introduces for $G^{\mu\alpha}$ and F_μ .

In the identities (26*a*) and (36), $F^{\mu\alpha}$ is to be regarded as being expressed in terms of F_μ according to the following relation, which follows from (31):

$$F_{\mu\alpha} \equiv F_{\mu,\alpha} - F_{\alpha,\mu} . \tag{31a}$$

We are now in a position to prove the compatibility of the field equations (29), (30) [(29), (31), resp.].

We first show that the number of field equations, minus the number of (independent) identities is n less than the number of field variables. We have:

Number of equations (29), (30):	$n^2 + n$,
Number of (independent) identities:	$n + n - 1$,
Number of field variables:	$n^2 + 1$,

and

$$(n^2 + n) - (n + n - 1) = (n^2 + 1) - n.$$

The number of identities is thus precisely the correct one. However, we do not content ourselves with this, but prove the following:

Theorem. *If all differential equations are fulfilled in a section x^n = const. and, in addition, $(n^2 + 1) - n$ of them (suitably chosen) are fulfilled everywhere then all $n^2 + n$ equations are satisfied everywhere.*

Proof. Let all of the equations be fulfilled in the section x^n = const., and in addition, the equations that correspond to setting:

$$\begin{matrix} F_1 & \ldots F_{n-1} & F_n \\ G^{11} & \ldots G^{1\,n-1} & \\ \ldots\ldots & \ldots\ldots & \\ G^{n-1\,1} & \ldots G^{n-1\,n-1} & \end{matrix}$$

equal to zero. It next follows from (31*a*) that the $F^{\mu\alpha}$ then vanish everywhere. Now, from (36), it follows that the anti-symmetric $\underline{G}^{\mu\alpha}$ must also vanish for $\alpha = n$ in the neighboring section $x^n = a + da$ [1]). Furthermore, it then follows analogously from (26*a*) that, in addition, the symmetric $\underline{\underline{G}}^{\mu\alpha}$ must vanish for $\alpha = n$ for the neighboring section $x^n = a + da$. The assertion follows by repeating this argument.

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[1]) The $\partial \underline{G}^{\mu n} / \partial x^n$ vanish for $x^n = a$.

§ 5. First approximation.

We now consider a field that differs only infinitely little from a Euclidian one with the usual teleparallelism. We can then set:

$$h_{s\nu} = \delta_{s\nu} + \bar{h}_{s\nu} \,, \tag{37}$$

in which the $\bar{h}_{s\nu}$ are infinitely small of first order and small quantities of higher order have been neglected. According to (5) [(6), resp.], one then sets:

$$h_s{}^{\nu} = \delta_{s\nu} - \bar{h}_{s\nu} \,. \tag{38}$$

In the first approximation, the field equations (29), (30) read:

$$\bar{h}_{a\mu,\nu,\nu} - \bar{h}_{a\nu,\nu,\mu} = 0, \tag{39}$$

$$\bar{h}_{a\mu,a,\nu} - \bar{h}_{a\nu,a,\mu} = 0. \tag{40}$$

We replace equation (40) with:

$$\bar{h}_{a\nu,a} = \chi_{,\,\nu} \,. \tag{40a}$$

We now assert that there is an infinitesimal coordinate transformation $x^{\nu'} = x^{\nu} - \xi^{\nu}$ that makes all of the quantities $\bar{h}_{a\nu,\nu}$ and $\bar{h}_{a\nu,a}$ vanish.

Proof. One first proves that:

$$\bar{h}'_{\mu\nu} = \bar{h}_{s\nu} + \xi^{\mu}{}_{,\,\nu} \,. \tag{41}$$

From this, one has:

$$\bar{h}'_{a\nu,\nu} = \bar{h}_{a\nu,\nu} + \xi^{a}{}_{,\,\nu,\,\nu} \,,$$
$$\bar{h}'_{a\nu,a} = \bar{h}_{a\nu,a} + \xi^{a}{}_{,\,a,\,\nu} \,.$$

The right-hand sides vanish because of (40*a*) when the following equations are fulfilled:

$$\begin{aligned} \xi^{a}{}_{,\,\nu,\,\nu} &= -\,\bar{h}'_{a\nu,\nu} \,, \\ \xi^{a}{}_{,\,a} &= -\,\chi. \end{aligned} \tag{42}$$

These $n + 1$ equations for the n quantities ξ^a are, however, compatible, because, according to (40*a*):

$$(-\,\bar{h}_{a\nu,\nu})_{,\,a} - (-\,\chi)_{,\,\nu,\,\nu} = 0.$$

With the new choice of coordinates, the field equations read:

$$\bar{h}_{a\mu,\nu,\nu} = 0,$$
$$\bar{h}_{a\mu,a} = 0,$$
$$\bar{h}_{a\mu,\mu} = 0.$$

We now split the $\bar{h}_{a\mu}$ according to the equations:

$$\bar{h}_{a\mu} + \bar{h}_{\mu a} = \bar{g}_{a\mu},$$
$$\bar{h}_{a\mu} - \bar{h}_{\mu a} = a_{a\mu},$$

in which $\delta_{a\mu} + \bar{g}_{a\mu}$ ($= g_{\mu\nu}$) determines the metric in the first approximation, and the field equations assume the neat form:

(44) $$\bar{g}_{a\mu,\sigma,\sigma} = 0,$$

(45) $$\bar{g}_{a\mu,\mu} = 0,$$

(46) $$a_{a\mu,\sigma,\sigma} = 0,$$

(47) $$a_{a\mu,\mu} = 0.$$

This suggests that the $\bar{g}_{a\mu}$ represent the gravitational field and the $a_{a\mu}$ represent the electromagnetic field, in the first approximation. (44), (45) correspond to Poisson's equation, while (46), (47) correspond to Maxwell's equations in empty space. It is interesting that the field laws of gravitation seem to separate from those of the electromagnetic field, which corresponds to the independence of the two fields in experiments. However, in full rigor, neither of these fields takes on a separate existence in this theory.

As far as the covariance of equations (44) to (47) is concerned, we have the following: The transformation law:

$$h'_{s\mu} = \alpha_{st} \frac{\partial x^{\sigma}}{\partial x^{\mu'}} h_{t\sigma}$$

is generally true for the $h_{s\mu}$. If one chooses the coordinate transformation to be linear and orthogonal, as well as conforming to the rotation of the local system, so:

(48) $$x^{\mu'} = \alpha_{\mu\sigma} x^{\sigma},$$

then this yields the transformation law:

(49) $$h'_{s\mu} = \alpha_{st}\, \alpha_{\mu\sigma}\, h_{t\sigma},$$

which is thus precisely the same as for tensors in the special theory of relativity. Since the same transformation law is true for the $\delta_{s\mu}$, due to (48), it is also true for the quantities $\bar{h}_{a\mu}$, $\bar{g}_{a\mu}$, and $a_{a\mu}$. Equations (44) to (47) are covariant under such transformations.

Concluding remarks.

For me, the great allure of the theory that was set down here stems from its unified character and the fact that is has the highest (allowable) indeterminacy in the field variables. I have also been able to show that the field equations, in the first approximation, lead to equations that correspond to the Newton-Poisson theory of gravitation, as well as Maxwell's theory of the electromagnetic field. Nevertheless, I am still far from being able to assert the physical validity of the equations thus derived. The basis for that lies in the fact that I have still not arrived at the derivation of the laws of motion for corpuscles from them.

(Received on 19.8.1929)

Historical survey of the notion of absolute parallelism

By

E. Cartan in Paris

Einstein, whom I have apprised of some of my papers that contain the notion of a Riemannian manifold with absolute parallelism, has kindly requested that I write a historical survey of that notion, as described from the geometric viewpoint. I do this all the more willingly because, aside from some questions of priority (which are, after all, of interest to only a small number of people), there exist several problems that I will thus have occasion to point out, and whose solution is likely to be of interest to physicists. As a matter of preference, I will address the geometrical aspect of the problem, while leaving the corresponding analytical developments in the background.

I.

1. The notion of absolute parallelism (or *Fernparallelismus*) in a Riemannian manifold can be defined independently of any metric ideas. Suppose that the manifold is n-dimensional. Two infinitely small vectors with different origins will be called *parallel* (or rather, *equipollent*) if n linearly-independent Pfaff forms:

$$^{i}L = {}^{i}h_k \, dx_k \qquad (i = 1, 2, \ldots, n)$$

are numerically equal to each other for these two vectors. One naturally gets the same absolute parallelism if one substitutes linear combinations with *constant* coefficients of the n forms ^{i}L for those forms.

In 1923 ([**4**] ([1]), pp. 320), Weitzenböck, and myself in 1921 ([**1**], pp. 51) defined a certain covariant derivative with respect to a system of n linearly-independent Pfaff forms. However, one cannot see the first appearance of the notion of absolute parallelism in that purely formal operation. Ricci, in his method for the calculus of n-tuples of orthogonal congruences that was published in 1895, utilized a system of n Pfaff expressions as the basis for his study of Riemannian manifolds; this is also what one does in differential geometry whenever one appeals to *local* systems of moving reference

([1]) The numbers in boldface refer to the articles that are cited in the bibliography that is placed at the end of this survey.

frames. There is a general method in it that is completely foreign to the notion of absolute parallelism ([1]).

2. On the contrary, that notion was introduced explicitly in 1923 in a paper [**5***a*] that was dedicated to the development of a general theory that I had sketched out in the previous year in two notes to *Comptes rendus* ([**2**] and [**3**]), and which I then presented in its various geometrical aspects in several articles and conferences ([**6**], [**9**], [**19**]).

That theory makes every space with a fundamental group, in the sense of F. Klein (e.g., Euclidian space, affine space, projective space, etc.), correspond to a *non-holonomic space* that likewise has a fundamental group (e.g., a space with Euclidian, affine, projective connection, etc.). The Riemann spaces that one envisions in the classical theory belong to the most general class of spaces with Euclidian connection whose fundamental group is the group of Euclidian displacements.

A general space with Euclidian connection can be imagined as composed of an infinitude of infinitely small pieces of a Euclidian space, with a law of agreement that permits one to integrate two contiguous pieces into one and the same Euclidian space. In a more precise manner, here is the nature of that law of agreement: Consider two infinitely close points A and A', as well as two local rectangular reference systems (R_A) and $(R_{A'})$ that are attached to those points. An observer that is placed at A can be imagined to be in a Euclidian space, and once the law of agreement is known, he will have localized the point A' and the frame $(R_{A'})$ in that space. In other words, he will know the rectangular coordinates of A' with respect to (R_A), which amounts to knowing the ds^2 of the space and the angles that the axes of $(R_{A'})$ make with those of (R_A), which amounts to knowing the law of parallel transport. He will, in turn, know the angle that an arbitrary vector that issues from A' makes with an arbitrary vector that issues from A. If one imagines that a continuous series of observers is arranged along an arc AB of a curve then the observer that is placed at A will thus be capable of localizing, step-by-step, the various points of AB and the various vectors that issue from these points in that same Euclidian space (viz., the Euclidian space that is tangent to A). One can say that he has *developed* the line AB and the portion of space that is immediately neighboring that line onto his Euclidian space.

The observer A will be aware that it is not in a true Euclidian space that he is experimenting by following two different paths ACB and $AC'B$ for localizing the point B and the vectors that issue from B into his Euclidian space. Depending upon the path that is followed, he will not attribute the same position to the point B *in his Euclidian space*, any more than he will attribute the same orientation to the vectors that issue from B. The rotation that he perceives the vectors to be subjected when one passes from one path to the other constitutes the *curvature* that is associated with the cycle $BC'ACB$. The translation that brings the two different positions that are attributed to the point B into coincidence constitutes the *torsion* that is associated with the same cycle; the vectors that represents that translation is the *torsion vector* of the cycle. If the cycle is infinitely small then the curvature translates analytically into the well-known tensor with four indices,

([1]) This is not to say that the research of Weitzenböck has no geometric importance, because it is immediately utilizable in the analytic theory of absolute parallelism, once that geometric notion is introduced. See the papers [**4**], [**10**], [**11**], [**12**], [**15**], [**21**], [**23**], which contain analytical developments of Weitzenböck's theory. Cf., note [5]) of [**7**].

and the torsion into the tensor Λ_{ij}^{k} with three indices that was used by Einstein. The necessary and sufficient condition for the torsion tensor to be zero is that the parallel transport be the one that was defined by Levi-Civita in 1917 (i.e., the *geodätische Übertragung* of Schouten).

All of the preceding extends, *mutatis mutandis*, to spaces with affine connection.

3. We now return to absolute parallelism. I have proved ([**5*a***], pp. 368) – and this is not entirely obvious – that if the curvature that is associated with any *infinitely small* cycle is zero (i.e., a space without curvature) then the space is endowed with absolute parallelism; in other words, a vector that issues from a point *A*, when transported parallel to itself step-by-step from *A* to *B*, will always give the same final vector (provided, however, that the intermediate paths followed are reducible to each other by continuous deformations). If one chooses a reference system at a point *A* that is defined by *n* independent vectors and one takes the reference system at an arbitrary point that is defined by *n* vectors that are parallel to the first ones then the affine connection of the space is defined completely ([**5*a***], pp. 368; [**5*c***], pp. 20) by the *n* Pfaff forms ω^j that represent the projections of an infinitely small vector onto the local coordinate axes that are attached to the origin of the vector ([1]).

The proof, which is given in the general case of a space with affine connection, is naturally valid in the particular case of a space with Euclidian connection. One then obtains Einstein's Riemannian spaces with absolute parallelism. I have, moreover, pointed out, still in the same paper ([**5*a***], pp. 404-409; cf., [**6**], pp. 301-302), the simplest example of such a space: For $n = 2$, it is that of the terrestrial surface, which is assumed to be spherical, when one regards two directions as parallel when the form the same angle with the compass needle; there, the torsion vector is tangent to the meridian circles.

It is interesting to remark that Einstein's first theory of relativity rests upon the notion of a Riemannian space without torsion, while the present theory rests upon that of Riemannian space without curvature.

4. Here, we make the obvious remark that one can pass from a space with affine connection without curvature to a Riemannian space with absolute parallelism by taking the fundamental quadratic differential form to be the sum of the squares of the *n* Pfaff expressions ω^j – so the *n* coordinate vectors become unitary and rectangular – or furthermore, a quadratic form with arbitrary *constant* coefficients that is constructed from the ω^j, so that the *n* coordinate vectors form a figure that is invariant in size and form. Conversely, one can arrive at the most general absolute parallelism in a given Riemannian space by decomposing its ds^2 into a sum of *n* squares.

([1]) The torsion vector that is associated with a cycle can be defined with precision only if one chooses the origin of the cycle, unless the cycle is infinitely small. This is not the case when there is absolute parallelism ([**16**], pp. 37). With the notations of [**1**], the torsion vector that is associated with a finite cycle is the one that has components with respect to the chosen reference that are the *n* integrals $\int {}^i h_k \, dx^k$ that are taken over the cycle. Here, the general theorem of the conservation of curvature and torsion ([**5*a***], pp. 373-375), which comprises the Bianchi identities, in particular, amounts to a classical theorem of H. Poincaré [Acta Math. **9** (1887), 321]; geometrically, it implies that the geometric sum of the torsion vectors that are associated with a closed surface is zero.

5. In the case of a space with *affine* connection, the torsion tensor Λ_{ij}^{k} decomposes ([**5c**], pp. 30-33) into two *irreducible* tensors. One of them is Einstein's vector $\Lambda_{ik}^{k} = \varphi_i$, which has a purely affine significance. The other one can be interpreted geometrically: It is zero in the case – and only in that case – where the torsion vector that is associated with an elementary cycle is situated in the planar element of that cycle; the corresponding spaces are J. A. Schouten's spaces with *semi-symmetric* connection ([1]).

If the space has a *Euclidian* connection then the second torsion tensor ceases to be irreducible ([**5c**], pp. 50-52); in particular, in the case where n is equal to 4, which is important to relativity, one of the two irreducible tensors into which it is decomposed is a *vector ψ_i that therefore has an essentially metric significance* ([**5c**], pp. 69-71). With the usual notations, one has:

$$\psi_i = \frac{1}{\sqrt{-g}}(g_{j\alpha}\Lambda_{kh}^{\alpha} + g_{k\alpha}\Lambda_{hj}^{\alpha} + g_{h\alpha}\Lambda_{jh}^{\alpha}),$$

in which the indices i, j, k, h define an *even* permutation of the indices 1, 2, 3, 4.

6. According to Weitzenböck, the covariant derivative with respect to a system of n Pfaff expressions was discovered recently by G. Vitali ([**7**] and [**8**]) in 1924. However, that author attached a geometrical significance to it and recognized the possibility of deducing an affine connection that proved to be without curvature. The converse theorem that I proved in 1923 was proved more recently by E. Bortolotti in 1927 in the case of a Euclidian connection [**17**]. Since then, absolute parallelism has been considered by various authors, and one will find a (probably incomplete) list of them in the bibliography.

7. I would now like to give a rapid outline of the principal problems that one poses in relation to absolute parallelism.

We first take the strictly *affine* point of view. In 1926, Schouten and I [**13**] showed that there exist two remarkable absolute parallelisms in the representative space of the transformations of a finite, continuous group. If one lets T_x denote the general transformation of the group whose parameters are $x_1, x_2, \ldots, x_n$ then the expressions ω^j that define the first absolute parallelism are the parameters of the infinitesimal transformation $T_x^{-1}T_{x+dx}$; the ones that define the second parallelism are the parameters of the infinitesimal transformation $T_{x+dx}T_x^{-1}$ ([2]). The torsion vectors that correspond to these

([1]) The affine connections that I have introduced go back to even more general connections that are due to Schouten [Math. Zeit. **13** (1922), 56-81]; however, Schouten's viewpoint is different from mine. For him, parallel transport (*lineare Übertragung*) is the essential geometric notion; for me, it is only a means of grasping some particular properties of affine space, and which can no longer be utilized in order to establish the notion of a space with projective (or conformal, etc.) connection.

([2]) The Pfaff expressions ω^j play an important role in my theory of the structure of continuous groups, a theory that goes back to a general method of differential geometry by the utilization of a system of moving reference frames. On the other hand, I have [Ann. Ec. Norm. **25** (1908), 60-88] converted the search for differential invariants of an arbitrary differential system under a group of continuous transformations – *whether finite or infinite* – into the search for invariants of a system of n independent Pfaff expressions in n variables under the general group of these n variables. The only analytic operations that the solution two parallelisms are equal and opposite, and the quantities are just the *structure constants* c_{ijk} of S. Lie, up to sign.

8. These group spaces are physically interesting. Indeed, with Einstein's new theory, it is natural to call a universe *homogeneous* when the torsion vectors that are associated with two *parallel* surface elements are themselves *parallel*; i.e., when parallel transport preserves torsion. Now ([**13**], pp. 813; [**16**], pp. 50-51), the only spaces with absolute parallelism that enjoy that property are the representative spaces of groups.

One can further characterize them in another way [**16**]. Call a point-like transformation in a space with absolute parallelism for which the various points of the spaces describe infinitely small, equipollent vectors an *infinitesimal translation*. One can associate the affine connection without curvature that is defined by the given absolute parallelism with a second affine connection that carries curvature and torsion, in general; it suffices ([**16**], pp. 52-53) to agree that two vectors whose infinitely close origins are parallel (in the second sense) can be deduced from each other by the infinitesimal translation that brings their two origins into coincidence. The torsion of that new connection is always equal and opposite to that of the first. In order for the new connection to be without curvature, as well, it is necessary and sufficient that the given space be a group space ([**16**], pp. 53). The two absolute parallelisms on that space are then deduced from each other by the process that we just indicated.

9. No matter what the ds^2 that one attributes to a group space in order to make it a *homogeneous* Riemannian space with absolute parallelism, the vector φ_i is always the same, *and one finds that its rotation is always zero, which therefore excludes electromagnetism from any homogeneous universe.* This conclusion will break down if one can define the electromagnetic potential by means of a vector ψ_i (no. 5); however, we would then leave the domain of geometry. We simply remark that, in principle, mechanical phenomena are of a purely affine nature, while electromagnetic phenomena are of an essentially metric nature. It can thus seem quite natural to seek to represent the electromagnetic potential by a vector that is not purely affine.

10. Another problem that Schouten and I were likewise occupied in 1926 [**14**] relates exclusively to Riemannian spaces with absolute parallelism. Is it possible to define an absolute parallelism in a Riemannian space that is given by its ds^2 in such a way that the geodesics of that parallelism coincide with the Riemannian geodesics? One can formulate this problem in many other ways. For example, one can, by appealing to a general theorem that I proved in 1923 ([***5a***], pp. 408), demand that it is possible to find an absolute parallelism such that the torsion vector that is associated with an arbitrary surface element is normal to that element. One can further specify in which cases the affine connection that is associated with the absolute parallelism by following the procedure that was defined in no. 8 will preserve the length of vectors. Finally, one can attach the question to a problem of classical mechanics: Being given a material system with n degrees of freedom, is it possible to choose the *velocity characteristics* p_i such that

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demands are the covariant derivative of a scalar with respect to the given system of Pfaff expressions and the formation of the *bilinear covariant* (i.e., rotation) of a Pfaff expression.

the spontaneous motions of the system are given by the equations $dp_i / dt = 0$ ([1])? This problem presents itself, for example, when one considers a solid body that moves around a fixed point O, where the ellipsoid of inertia that relates to O is a sphere, and one takes the velocity characteristics to be the components p, q, r of the instantaneous rotation around O.

11. We have succeeded in resolving the problem completely, at least in the case where the given ds^2 is *definite*. If one limits oneself to *irreducible* solutions, since all of the other ones are easily deduced from them, then one finds:

1. The representative spaces of closed, simple groups that are endowed with a ds^2 that is intrinsically linked with the structure of the group, where the absolute parallelism is either of the two absolute parallelisms that are attached to the group.

2. The 7-dimensional elliptic space, which admits two *continuous* families of absolute parallelism that satisfy the desired conditions; a study of these parallelism was made by Vaney [**26**].

In particular, the three-dimensional elliptic space (or spherical space) belongs in the first category: The two absolute parallelisms that were in question above were pointed out a long time ago by Clifford. That space is the representative space of the group of rotations of ordinary space. From the mechanical standpoint, its various points represent the various positions of a solid body that moves around a fixed point. The two parallelisms then admit a remarkable kinematical interpretation ([**6**], pp. 305-308). One sees that the Clifford parallelisms, which define a completely isolated chapter in geometry, are now attached to a very general theory that, despite the apparent conflict between the two notions, subsumes both the parallelism of Levi-Civita and the parallelism of Clifford.

12. The Riemannian spaces that were just now in question belong to a more general category, that of spaces in which the parallel transport preserves curvature and torsion; they then admit a transitive group of rigid displacements that likewise leaves curvature and torsion invariant. Conversely, if a Riemannian space, when envisioned from the classical viewpoint, admits a transitive group of rigid displacements – i.e., it leaves the ds^2 invariant – then one can always (at least, if the ds^2 is definite) define a Euclidian connection in that space such that the corresponding parallel transport preserves curvature and torsion. Once again, the vector φ_i always has a zero rotation. It is true that the ds^2 is indefinite in the possible applications to the theory of relativity; however, for $n = 4$, the conclusion persists, even in this case. The spaces without torsion in which the parallel transport preserves the curvature play an important role in geometry, but that would leave the scope of this survey completely ([2]).

([1]) This problem of mechanics has been the object of research for Georg Hamel [Zeit. f. Math. u. Phys. **50** (1904), 1-53], who has found a subset of the solutions that were described above (no. 11).

([2]) For some other problems of geometry that one can attach to absolute parallelism, one can consult a quite recent note of E. Bortolotti [**25**].

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(Received on 19-8-1929.)

Teleparallelism and wave mechanics. I

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(Received on 21 October 1929)

After a brief discussion of an attempt to modify Einstein's unified field theory (*), a cylindrical five-dimensional geometry will be constructed on the basis of teleparallelism that can be regarded as the foundation for a unified field theory. Then, a guiding potential for matter will be introduced in a suitable way that differs notably from the previous attempts in this direction. The notations are taken from my earlier publications (**).

The essential information in Einstein's geometry is the assumption of a rigid coupling of the vierbeins at the various space-time points. Indeed, the metric, which is described by means of the quantities:

$$g_{\alpha\beta} = h_{\alpha m}\, h_{\beta m}$$

and their ordinary derivatives, is bein-invariant, but the torsion is expressed by the quantities:

$$\Lambda_{\alpha\beta\gamma} = \left(\frac{\partial h_{\alpha m}}{\partial x^{\beta}} - \frac{\partial h_{\beta m}}{\partial x^{\alpha}}\right) h_{\gamma m}\,,$$

which is preserved only under everywhere equal rotations of the vierbeins. Only the components of the curvature tensor of the torsion, which are equal and opposite (***) to the components of the metric curvature tensor (****), define the unique bein-invariant functions of the quantities Λ and their first derivatives. Therefore, the vierbeins define a rigid, but globally rotatable, framework in space-time. In this, and only in this, case are the bein-components "$h_{\alpha m}$" well-defined coordinate functions. Should, as H. Weyl intended, the vierbeins at each space-time point be freely rotatable, so an arbitrariness would exist in their orientations, then the "$dh_{\alpha m}$" would obviously not represent complete differentials, and then teleparallelism would be impossible (†). It also seems to me that the interpretation of Einstein's geometry in relation to that of H. Mandel as the geometry

(*) Which is treated in the papers: Levi-Civita, Sitz. d. Preuss. Akad., supplement to volume 9, 1929; H. Weyl, Zeit. Phys. **56** (1929), 330; H. Mandel, *ibid.*, **56** (1929), 838; V. Fock, D. Iwanenko, C. R. **188**, pp. 1470, 3 June 1929.

(**) R. Zaycoff, Zeit. Phys. **53**, 719; **54**, 588, 590, 738; **55**, 273; **56**, 717, 862; **58**, 143, 280, (1929).

(***) The total curvature tensor vanishes identically.

(****) In the case considered (viz., Riemannian curvature).

(†) As, e.g., in the geometry of Cartan.

of a pseudo-projection of a cylindrical Riemannian universe with five dimensions would be unjustified since the choice of components for the fünfbeins is very unnatural. A phenomenological unity, in the sense of Levi-Civita, and thus a direct replacement for the classical field laws, is just as improbable, on the following grounds: a) One cannot present the four identities (viz., the covariance requirement), simply because Hamilton's principle is lacking. b) The ten gravitational field equations are of second order in the "$h_{\alpha m}$", while the remaining six (here, there are eight Maxwell-Lorentz equations, between which two identities exist) are of third order in the "$h_{\alpha m}$", and additionally: c) Quantities of the form:

$$a\,\nabla_\mu\,(\tfrac{1}{2}S^{\alpha\beta\mu} - \Lambda^{\alpha\beta\mu}) + b\,\Lambda^{\alpha\beta\mu}\,\Lambda_\mu$$

appear as the components of the electromagnetic field tensor, which have no direct relationship with experience. However, the unity of Einstein's attempts ([*]) with components of electromagnetic potentials that were identified with the "Λ" have a certain drawback, since the aforementioned electromagnetic equations exhibit only an apparent similarity with the Maxwell-Lorentz theory ([**]). In addition to the $h_{\alpha m}$ H. Weyl introduced four more components f_α of a quantity that he set equal to the electromagnetic potential. Since he affected a rotational freedom of the vierbeins ([***]), he believed he could manage with the quantities "$g_{\alpha\beta}$, f_α," but when it came to explaining spin phenomena, he also appealed to the sixteen components "$h_{\alpha m}$." I believe that, in fact, the restriction to only the metric quantities "$g_{\alpha\beta}$" and the quantities "f_α" cannot lead to an explanation for wave-mechanical phenomena, and in particular, spin. We cannot assert, *a priori*, that only the aforementioned quantities are required for the unique solution to the field problem! However, it is, on the other hand, clear that the quantities "$h_{\alpha m}$" alone will not produce the electromagnetic laws. Another difficulty in the unified field theory was the impossibility of deriving quantum theory from its structure. Here, I must openly state that, up to now, the attempts on my own part to reconcile the Dirac-Whittaker theory with unified field theory, especially the attempt to follow through on Whittaker's idea along a different path, have not taken me very far. A return to the older, four-dimensional, theory of relativity (A. Einstein, H. Weyl, A. S. Eddington, L. Infeld, K. Hattori, et al.), as well as five-dimensional ones (Th. Kaluza, O. Klein, H. Mandel, E. Reichenbächer, the author, et al.), thus seems to be excluded from the outset. With the concepts of teleparallelism, one has truly taken a step in the direction of understanding! In addition to the $h_{\alpha m}$, f_α , V. Fock and D. Iwanenko have introduced the Dirac ψ-functions in a suitable way, while H. Weyl introduced only the two Pauli functions $\psi_{(1)}$, $\psi_{(2)}$. Both directions define a law of covariant differentiation for the ψ. Although Dirac's theory yields more that is perhaps necessary in experiment, nonetheless, Pauli's theory seems to be superior on several grounds. However, the use of several wave quantities (J. M. Whittaker, E. Madelung, J. Frenkel, et al.) complicates the problem more than it

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([*]) A. Einstein, Sitz. d. Preuss. Akad., supplement to volume 17 and 18, 1928, volume 1 and 10, 1929; also R. Weitzenböck, *ibid.*, supplement to volume 26, 1928; H. Reichenbach, Zeit. Phys. **53** (1929), 683, and the cited papers of the author.

([**]) Where is the gauge invariance of the electromagnetic field tensor and the expression for the Lorentz force? Furthermore, what are the equations of motion for charged matter?

([***]) As one says, this rotational freedom is incompatible with the well-defined functions "$h_{\alpha m}$!"

simplifies it. It then remains hard to understand why the universal constants e, m_0 are still present in the in the Dirac interpretation, as they already were in Schrödinger's. H. Weyl embarked upon another path, along which he first introduced these constants by integrating the equations of continuity (e.g., energy-stress, four-current) over a space-like section t = const., and then considered the wave equations to be macroscopic laws. However, what kind of sense do the microscopic equations that emerge from the variation of the ψ-functions in the Hamilton integral then have? The attempts of G. Mie and K. Bollert to derive the wave equations from Mie's electrodynamics seem compelling, but they adhered to known serious difficulties that are still not resolved, and secondly, the problem of unity would then have to be relinquished in favor of Weyl's theory of relativity. The extension of Maxwell's theory that was made by Thomson would also prove to be not sufficiently suitable for the quantum problem. The path that I would like to follow here is the following one: a) Extension of the theory of teleparallelism by a new dimension. b) Operate with the ψ-functions directly.

§ 1. We set:

$$H_{\alpha m} = h_{\alpha m}\,, \qquad H_{\alpha 0} = -f_\alpha\,, \qquad H_{0m} = 0, \qquad H_{00} = 1, \tag{1}$$

where the quantities $h_{\alpha m}$, f_α do not include the fifth coordinate x^0 (*) and "τ" is a constant with the dimensions of length (**).

It follows that:

$$H^{\alpha m} = h^{\alpha m}, \qquad H^{\alpha 0} = 0, \qquad H^{0m} = f_\rho\, h^{\rho m}, \quad H^{00} = 1. \tag{2}$$

Furthermore, one has:

$$\left.\begin{aligned} ds_m &= h_{\rho m} dx^\rho, \\ ds_0 &= dx^0 - f_\rho dx^\rho \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} \frac{\partial\Lambda}{\partial s_m} &= h^{\rho m}\frac{\partial\Lambda}{\partial x^\rho} + f_\rho h^{\rho m}\frac{\partial\Lambda}{\partial x^0}, \\ \frac{\partial\Lambda}{\partial s_0} &= \frac{\partial\Lambda}{\partial x^0}. \end{aligned}\right\} \tag{4}$$

A calculation yields:

$$\left.\begin{aligned} \delta^\gamma_{\alpha\beta} &= \Delta^\gamma_{\alpha\beta}, \\ \delta^0_{\alpha\beta} &= -\nabla_\beta f_\alpha, \end{aligned}\right\} \tag{5}$$

while the remaining δ vanish.

From this, it follows that:

(*) It has a spatial, and in what follows, absolute character. In contrast to Th. Kaluza, et al., we do not couple the transformation $'f_\alpha = f_\alpha - \partial\lambda / \partial x^\alpha$ with a transformation of the type $\bar{x}^0 = x^0 - \lambda(x^1, x^2, x^3, x^4)$, despite the fact that this seems closely-related, but to a transformation of the functions ψ of the type $'\psi = e^{i\lambda/\tau} \cdot \psi$. We thus define the "$x^0$" as an absolute dimension.

(**) The f_α are proportional to the electromagnetic potentials and are dimensionless.

$$\left.\begin{aligned} \lambda_{\alpha\beta}^{\cdot\cdot\cdot\gamma} &= \Delta_{\alpha\beta}^{\cdot\cdot\cdot\gamma}, \\ \lambda_{\alpha\beta}^{\cdot\cdot\cdot 0} &= \nabla_\alpha f_\beta - \nabla_\beta f_\alpha, \end{aligned}\right\} \tag{6}$$

while the remaining λ vanish.

If we set:

$$\left.\begin{aligned} f_m &= f_\rho h^{\rho m}, \\ f_{mn} &= f_{\rho\kappa} h^{\rho m} h^{\kappa n}, \end{aligned}\right\} \tag{7}$$

in which:

$$f_{\alpha\beta} = \frac{\partial f_\beta}{\partial x^\alpha} - \frac{\partial f_\alpha}{\partial x^\beta}, \tag{8}$$

then one has:

$$\left.\begin{aligned} \lambda_{klm} &= \Lambda_{klm}, \\ \lambda_{kl0} &= f_{kl}, \end{aligned}\right\} \tag{9}$$

while the remaining λ vanish.

For the torsion, we then have:

$$\left.\begin{aligned} (d_1 d_2 - d_2 d_1)\, s_m &= \Lambda_{klm} d_1 s_l d_2 s_k, \\ (d_1 d_2 - d_2 d_1)\, s_0 &= f_{kl} d_1 s_l d_2 s_k. \end{aligned}\right\} \tag{10}$$

Now, one also has:

$$\left.\begin{aligned} \lambda_m &= \Lambda_m, & \sigma_{klm} &= s_{klm}, \\ \lambda_0 &= 0, & \sigma_{klm} &= f_{kl}. \end{aligned}\right\} \tag{11}$$

Furthermore:

$$\left.\begin{aligned} \pi_{klm} &= \Pi_{klm}, & \pi_{k0m} &= \tfrac{1}{2} f_{km}, \\ \pi_{0lm} &= \tfrac{1}{2} f_{lm}, & \pi_{00m} &= 0. \end{aligned}\right\} \tag{12}$$

For the metric, we have:

$$\left.\begin{aligned} \gamma_{\alpha\beta} &= g_{\alpha\beta} + f_\alpha f_\beta, & \gamma_{\alpha 0} &= -f_\alpha, & \gamma_{00} &= 1, \\ \gamma^{\alpha\beta} &= g^{\alpha\beta}, & \gamma^{\alpha 0} &= f^\alpha, & \gamma^{00} &= 1 + f_\rho f^\rho, \\ \gamma &= g, \end{aligned}\right\} \tag{13}$$

and for the Riemannian curvature:

$$\rho = R - f_{\pi\rho} f^{\rho\pi}. \tag{14}$$

Moreover, if the k', l', m', …, α', β', … vary from 0 to 4, one has (†):

$$\mathfrak{G} = \{c_1\, \lambda_{k'l'm'} \lambda_{k'l'm'} + c_2\, \lambda_{k'l'm'} \lambda_{k'\,m'l'} + c_3\, \lambda_{k'} \lambda_{k'}\} \sqrt{\gamma}$$

(†) Translator’s note: This equation was undoubtedly misprinted in the original.

$$= \{c_1\,\Lambda_{klm}\Lambda_{klm} + c_2\,\Lambda_{klm}\Lambda_{kml} + c_3\,\Lambda_k\Lambda_k + c_1\,f_{mn}\,f_{mn}\}\sqrt{g}\,. \tag{15}$$

In particular, if we set:

$$c_1 = -\tfrac{1}{4}, \qquad c_2 = -\tfrac{1}{2}, \qquad c_3 = 1 \tag{16}$$

then if $\mathfrak{G}$ is chosen to be the Hamilton function it will produce the classical field equations *in vacuo*. However, the restriction (16) is not absolutely mandatory (*).

§ 2. Now that we have presented the geometry of our five-dimensional cylindrical universe, we go on to the interpretation of its relationship with wave-mechanics. Let:

$$\omega = \psi\, e^{ix^0/\tau} \tag{17}$$

be any four complex scalar functions, where the ψ does not depend upon x^0, τ is a small constant with the dimensions of a length, and:

$$\tilde{\omega} = \tilde{\psi}\, e^{-ix^0/\tau} \tag{18}$$

are their conjugate complements.

Moreover, we would like to introduce the constant four-rowed Hermitian matrices γ_m and subject them to the conditions:

$$\left.\begin{array}{c} \tfrac{1}{2}\{\gamma_m\gamma_n + \gamma_n\gamma_m\} = \varepsilon_{mn}\cdot\varepsilon, \qquad (m, n = 1, 2, 3, 4), \\ \varepsilon = \text{identity matrix.} \end{array}\right\} \tag{19}$$

One also has the general conditions (†):

(*) Naturally, the degenerate forms of $\mathfrak{G}$ are excluded.

(†) Let:

$$\omega(0) = a\,e^{ix^0/\tau}, \qquad \tilde{\omega}(0) = a\,e^{-ix^0/\tau},$$

in which "a" is a real constant. According to (17), (18), we have:

$$\omega(s) = \psi(s)\,e^{ix^0/\tau}, \qquad \tilde{\omega}(s) = \tilde{\psi}(s)\,e^{-ix^0/\tau}.$$

We would like to choose the matrix components $\gamma_{m'}(s', t')$ as follows:

$$\begin{gathered} \tfrac{1}{2}\{\gamma_m(s, t)\,\gamma_n(r, t) + \gamma_n(s, t)\,\gamma_m(r, t)\} = \varepsilon_{mn}\cdot\varepsilon_{st}\,, \\ \gamma_m(0, 0) = \gamma_m(s, 0) = \gamma_m(0, s) = \gamma_0(s, 0) = \gamma_0(0, s) = 0, \\ \gamma_0(0, 0) = 1. \end{gathered}$$

Relations (20) are then fulfilled with no further assumptions. This choice of $\gamma_{m'}(s', t')$ is justified by the absolute character of the fifth dimension.

$$\tfrac{1}{2}\{\gamma_{m'}\,\gamma_{n'} + \gamma_{n'}\,\gamma_{m'}\} = \varepsilon_{m'n'}\cdot\varepsilon \quad (m', n' = 1, 2, 3, 4, 0). \tag{20}$$

In addition, let:

$$l_0 = l_1 = l_2 = l_3 = 1, \qquad l_4 = i, \tag{21}$$

and if the sign in front of any quantity $A_{m'}$ is set to $l_{m'}$ then the sum is not taken over the products $l_1\,A_1$, $l_2\,A_2$, $l_3\,A_3$, $l_4\,A_4$, $l_0\,A_0$.

We now define the quantities:

$$\left.\begin{aligned} J_m &= \tilde{\omega} l_m\,\gamma_m\,\omega, \\ J_0 &= \tilde{\omega} l_0\,\gamma_0\,\omega. \end{aligned}\right\} \tag{22}$$

Under the rotation of the bein-framework:

$$\overset{*}{H}_{\alpha'm'} = \vartheta_{m'r'}\,H_{\alpha'r'}\,, \qquad \left\{\begin{aligned} \vartheta_{0m} = \vartheta_{m0} = 0, \quad \vartheta_{00} = 1, \\ \vartheta_{mr}\vartheta_{ms} = \vartheta_{rm}\vartheta_{sm} = \varepsilon_{rs}\,, \end{aligned}\right\} \tag{23}$$

one gets:

$$\left.\begin{aligned} \overset{*}{J}_m &= \vartheta_{mr}\,J_r\,, \\ \overset{*}{J}_0 &= J_0\,. \end{aligned}\right\} \tag{24}$$

Likewise, the ψ, $\bar{\psi}$ will be transformed into each other in a certain way (*). One can prove that the quantities:

$$ds_{l'}\,[\omega]_{l'} = ds_{l'}\left\{\frac{\partial\Lambda}{\partial s_{l'}} \pm \frac{1}{4}\pi_{k'l'm'}l_{k'}l_{m'}\gamma_{k'}\gamma_{m'}\right\}\omega \tag{25}$$

are subjected to the same “spin transformation” as the quantity ω itself. In this, one takes the + or – sign according to whether the index 4 appears once or twice $\pi_{k'l'm'}$, respectively. From § 1, one has (**):

$$\left.\begin{aligned} [\omega]_l &= h^{\rho l}\left(\frac{\partial}{\partial x^\rho} + \frac{i}{\tau}f_\rho\right)\omega \pm \Pi_{klm}l_k l_m\gamma_k\gamma_m\omega, \\ [\omega]_0 &= \frac{i}{\tau}\omega + \frac{1}{8}f_{km}l_k l_m\gamma_k\gamma_m\omega. \end{aligned}\right\} \tag{26}$$

We define the quantity:

$$\mathfrak{D} = \frac{1}{i}\tilde{\omega} l_{m'}\,\gamma_{m'}[\omega]_{m'}\cdot\sqrt{g}\;. \tag{27}$$

If we set:

(*) Namely: a spin transformation.

(**) One also has: $[\omega(0)]_l = \frac{i}{\tau}f_l\cdot\omega(0)$, $\qquad [\omega(0)]_0 = \frac{i}{\tau}\,\omega(0)$.

$$\left.\begin{array}{ll}\left\{\begin{array}{l} S_{123} = \;\; 2iM_4, \\ S_{421} = -2iM_3, \\ S_{413} = -2iM_2, \\ S_{432} = -2iM_1, \end{array}\right. & \left\{\begin{array}{l} i\gamma_1\gamma_2\gamma_3 = \bar{\gamma}_4, \\ i\gamma_4\gamma_2\gamma_1 = \bar{\gamma}_3, \\ i\gamma_4\gamma_1\gamma_3 = \bar{\gamma}_2, \\ i\gamma_4\gamma_3\gamma_2 = \bar{\gamma}_1, \end{array}\right. \\ \quad \bar{J}_m = \tilde{\psi}\, l_m\, \bar{\gamma}_m\, \psi, & \end{array}\right\} \tag{28}$$

and from (22):

$$J_m = \tilde{\psi}\; l_m\; \gamma_m\; \psi\,, \qquad J_0 = a^2$$

then it follows that:

$$\frac{1}{\sqrt{g}}\mathfrak{D} = -\, i\tilde{\psi}\;\, l_m\; \gamma_m\; h^{\rho m}\, \frac{\partial \psi}{\partial x^\rho} + \frac{i}{2}\Lambda_m\, J_m + \frac{1}{2} M_m \bar{J}_m + \frac{1}{\tau} f_m\, J_m + \frac{1}{\tau} J_0\,. \tag{29}$$

The matrices $\bar{\gamma}_m$ are also Hermitian.

§ 3. We now choose the quantity:

$$\mathfrak{Z} = \mathfrak{G} + k\,\mathfrak{D}, \tag{30}$$

to be the Hamilton function, where k is a constant, and $h_{\alpha m}$, f_α, but not say $H_{\alpha' m'}$, must be varied (*)! If the ψ are normalized in such a way that they have the dimension [$\mathrm{cm}^{-3/2}$] then the constant k will have the dimension [cm^2]. The variation of the $h_{\alpha m}$, f_α, ψ, $\bar{\psi}$ in (30) thus yields the 28 field equations that contain $h_{\alpha m}$, f_α, ψ, $\bar{\psi}$, and their derivatives. Some identities now exist between these equations (**).

If we set $h_{\alpha m} = \varepsilon_{\alpha m}$ then certainly the electromagnetic field exists. The Maxwell-Lorentz equations in this case read simply:

$$\frac{\partial f_{mr}}{\partial x^r} = p_m\,, \tag{31}$$

with the current components (***):

$$p_m = \frac{k}{\tau} J_m\,. \tag{32}$$

Sofia, Physical Institute of the University, 1 October 1929.

(*) Then, from (1), the conditions of sharpened cylindricity $H_{0m} = 0$, $H_{00} = 1$ exist between the 32 quantities $H_{\alpha' m'}$, and only the 20 quantities $h_{\alpha m}$, f_α remain. In addition, ψ, $\tilde{\psi}$ (but not, say, $\tilde{\omega}$, ω) will be varied.

(**) Which are five in number.

(***) The matrices γ_m, $\bar{\gamma}_m$ are the analogues of the classical quantities: four-velocity, mechanical spin, resp. In a second publication, I will concern myself with the presentation of the field equations.

Teleparallelism and wave mechanics. II

By **Raschko Zaycoff** in Sofia

(Received on 4 November 1929)

The 28 field equations will be presented, in connection with the first publication on this situation (*). An explanation of the character of the matrices employed then emerges from this.

§ 1. We set:

$$\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 = \gamma_0\,. \tag{1}$$

$\bar{\gamma}_m$ and γ_m are then the matrices of Eddington's theory (**), which are Hermitian and satisfy the conditions I (20):

$$\tfrac{1}{2}\{\bar{\gamma}_{m'}\bar{\gamma}_{n'} + \bar{\gamma}_{n'}\bar{\gamma}_{m'}\} = \varepsilon_{m'n'}\cdot\varepsilon \qquad (m', n' = 0, 1, 2, 3, 4). \tag{2}$$

The relations then follow:

$$\left.\begin{array}{c}\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_4\bar{\gamma}_0 = \varepsilon, \qquad \gamma_m = -i\bar{\gamma}_0\bar{\gamma}_m \qquad (m = 0, 1, 2, 3, 4),\\ \tfrac{1}{2}\{\gamma_m\bar{\gamma}_n - \bar{\gamma}_n\gamma_m\} = i\bar{\gamma}_0\varepsilon_{mn}.\end{array}\right\} \tag{3}$$

We set (***):

$$\delta\int \mathfrak{Z}\; dx = 0 \qquad\qquad (dx = dx^1\, dx^2\, dx^3\, dx^4) \tag{4}$$

for all variations $[h_{\alpha m}]$, $[f_\alpha]$, $[\psi]$, $[\bar{\psi}]$ that vanish on the boundary. The 20 field equations then follow:

(*) R. Zaycoff, "Fernparallelismus und Wellenmechanik I," Zeit. Phys. **58** (1929), 833. Cited as I in what follows.

(**) A. S. Eddington, Proc. Roy. Soc. (A) **121** (1928), 524; **122** (1929), 358.

(***) $\mathfrak{Z}$ has the value that was given in I (30).

$$\left.\begin{aligned}A^{\alpha\beta} = {} & 2\{c_1(\Lambda_{\mu\kappa}^{\cdot\cdot\alpha}\Lambda^{\mu\kappa\beta} - 2\Lambda_{\mu\cdot\kappa}^{\alpha}\Lambda^{\mu\beta\kappa}) - c_2\Lambda_{\cdot\mu\kappa}^{\alpha}\Lambda^{\beta\kappa\mu} - c_3\Lambda^{\alpha}\Lambda^{\beta} \\ & + \tfrac{1}{2}g^{\alpha\beta}(c_1\Lambda_{\mu\kappa\rho}\Lambda^{\mu\kappa\rho} + c_2\Lambda_{\mu\kappa\rho}\Lambda^{\mu\rho\kappa} + c_3\Lambda_{\mu}\Lambda^{\mu}) \\ & - D_{\mu}[2c_1\Lambda^{\alpha\mu\beta} + c_2(\Lambda^{\alpha\beta\mu} + \Lambda^{\beta\mu\alpha}) + c_3(\Lambda^{\alpha}g^{\mu\beta} - \Lambda^{\mu}g^{\alpha\beta})]\} \\ & + c_1\{-4f_{\cdot\kappa}^{\alpha}f^{\beta\kappa} + f_{\kappa\rho}f^{\kappa\rho}\cdot g^{\alpha\beta}\} \\ & - \frac{k}{2i}\left\{\tilde{\psi}\,l_k\gamma_k\left(\frac{\partial}{\partial x_\rho} + \frac{i}{\tau}f_\rho\right)\psi - \left(\frac{\partial}{\partial x_\rho} - \frac{i}{\tau}f_\rho\right)\tilde{\psi}\,l_k\gamma_k\psi\right\}\cdot(h^{\alpha k}g^{\rho\beta} - h^{\rho k}g^{\alpha\beta}) \\ & + \frac{k}{4i}\{D_{\mu}\Theta^{\alpha\rho\beta} - \tfrac{1}{2}\Theta^{\beta\kappa\rho}\Lambda_{\kappa\rho}^{\cdot\cdot\alpha} + \Theta^{\alpha\kappa\rho}\Lambda_{\cdot\kappa\rho}^{\beta} - \tfrac{1}{2}\Theta^{\kappa\rho\alpha}\Lambda_{\kappa\rho\mu}g^{\alpha\beta}\} \\ & + \frac{k}{\tau}a^2\cdot g^{\alpha\beta} = 0,\end{aligned}\right\} \tag{5}$$

$$A^{\alpha} \equiv 4c_1\ \delta_\rho f^{\alpha\rho} + \frac{k}{\tau}J^{\alpha} = 0. \tag{6}$$

Here, one has: δ_ρ is the Riemannian derivative with respect to x^ρ, $D_\rho = \nabla_\rho - \Lambda_\rho$, where ∇_ρ means the Einsteinian derivative with respect to x^ρ, and:

$$\Theta_{klm} = \tilde{\psi}\,l_k\,l_l\,l_m\,\gamma_k\,\gamma_l\,\gamma_m\,\psi \qquad (k \neq l \neq m).$$

Equations (5) describe the gravitational and spin phenomena, while equations (6) describe the electromagnetic phenomena. The choice of constants c_1, c_2, c_3, a, k, τ is then arbitrary in them (*). Moreover, equations (5) are of second order in the $h_{\alpha m}$ and of first order in the f_α, ψ, $\tilde{\psi}$, and equations (6) are of second order in the f_α and of first order in the $h_{\alpha m}$. However, with that, we have, in fact, proved that they represent causality equations.

§ 2. The eight remaining equations, which arise by varying the ψ, $\tilde{\psi}$, do not have the character of causality equations, since they are of first order in the $h_{\alpha m}$, ψ, $\tilde{\psi}$. They can thus be regarded as eight auxiliary conditions to equations (5), (6). They now read:

$$\frac{1}{k}A \equiv -\frac{1}{i}\left(\frac{\partial}{\partial x_\rho} - \frac{i}{\tau}f_\rho\right)\tilde{\psi}\cdot l_m\gamma_m h^{\rho m} + \frac{1}{2i}\Lambda_m\tilde{\psi}\,l_m\,\gamma_m + \frac{1}{2}M_m\tilde{\psi}\,l_m\,\overline{\gamma}_m = 0, \tag{7}$$

$$\frac{1}{k}\tilde{A} \equiv \frac{1}{i}l_m\gamma_m h^{\rho m}\left(\frac{\partial}{\partial x_\rho} - \frac{i}{\tau}f_\rho\right)\psi - \frac{1}{2i}\Lambda_m\,l_m\,\gamma_m\psi + \frac{1}{2}M_m\,l_m\,\overline{\gamma}_m\psi = 0. \tag{8}$$

Here, the quantities M_m are determined from formula I (28), and let it be further mentioned that the following relations are valid:

(*) The constants a, τ, and k have very small values.

$$\Lambda_m \equiv - \delta_\rho h^{\rho m}, \quad \delta_\alpha \psi \equiv \frac{\partial \psi}{\partial x^\alpha}, \quad \delta_\alpha \tilde{\psi} \equiv \frac{\partial \tilde{\psi}}{\partial x^\alpha}, \quad \delta_\alpha h_{\beta m} \equiv \Pi_{\beta\alpha\nu} h^{\nu m}. \tag{9}$$

Under an infinitesimal coordinate transformation, the variations of the quantities $h_{\alpha m}$, f_α , ψ, $\tilde{\psi}$ read:

$$\left.\begin{aligned} [h_{\alpha m}] &= \frac{\partial h_{\alpha m}}{\partial x_\rho} \xi^\rho + h_{\rho m} \frac{\partial \xi^\rho}{\partial x^\alpha}, \quad [f_\alpha] = \frac{\partial f_\alpha}{\partial x^\rho} \xi^\rho + f_\rho \frac{\partial \xi^\rho}{\partial x^\alpha}, \\ [\psi] &= \frac{\partial \psi}{\partial x^\rho} \xi^\rho, \\ [\tilde{\psi}] &= \frac{\partial \tilde{\psi}}{\partial x^\rho} \xi^\rho. \end{aligned}\right\} \tag{10}$$

Under an infinitesimal change in the normalization of the guiding potentials ψ, $\tilde{\psi}$, the variations read:

$$'[h_{\alpha m}] = 0, \qquad '[f_\alpha] = \frac{\partial \lambda}{\partial x^\alpha}, \quad '[\psi] = - i \frac{\lambda}{\tau} \psi, \qquad '[\tilde{\psi}] = i \frac{\lambda}{\tau} \tilde{\psi}. \tag{11}$$

§ 3. It now follows that:

$$\left.\begin{aligned} &\int \{A^{\alpha\beta} h_{\beta m} [h_{\alpha m}] + A^\alpha [f_\alpha] + A[\psi] + [\tilde{\psi}]\tilde{A}\} \sqrt{g}\, dx \equiv 0, \\ &\int \{A^{\alpha\prime} [f_\alpha] + A'[\psi] + '[\tilde{\psi}]\tilde{A}\} \sqrt{g}\, dx \equiv 0. \end{aligned}\right\} \tag{12}$$

The following five identities emerge from this:

$$- \{D_\rho A^{\rho\alpha} + A^{\rho\kappa} \Lambda^{\alpha}_{\cdot\rho\kappa}\} + f^{\alpha}_{\cdot\kappa} A^\kappa - \delta_\kappa A^\kappa \cdot f^\alpha + A\, g^{\rho\alpha} \frac{\partial}{\partial x^\rho} \psi + g^{\rho\alpha} \frac{\partial}{\partial x^\rho} \tilde{\psi} A\psi \equiv 0, \tag{13}$$

$$- \delta_\rho A^\rho - \frac{1}{i} A\, \psi + \frac{1}{i} \tilde{\psi} \tilde{A} \equiv 0. \tag{14}$$

We now split $A^{\alpha\beta}$ into two parts:

$$A^{\alpha\beta} = 2\, G^{\alpha\beta} + 2 \cdot k\, T^{\alpha\beta}, \tag{15}$$

where 2 $G^{\alpha\beta}$ is the part of $A^{\alpha\beta}$ that includes only the quantities $h_{\alpha m}$ and their derivatives, and $\kappa = \frac{8\pi g}{c_4}$ (g = Newtonian gravitational constant). Since one has (*):

$$D_\rho G^{\rho\alpha} + G^{\rho\kappa}\, \Lambda^{\alpha}_{\cdot\rho\kappa} \equiv 0 \tag{16}$$

identically, the identities (13) are also true when one replaces the $A^{\alpha\beta}$ with the quantities $2\kappa T^{\alpha\beta}$. It follows from (6) that:

$$\delta_\rho J^\rho = 0, \tag{17}$$

and from (5), (15), (16) that:

$$D_\rho T^{\rho\alpha} + T^{\rho\kappa}\, \Lambda^{\alpha}_{\cdot\rho\kappa} = 0. \tag{18}$$

Now, some consideration of (17) yields:

$$\frac{1}{2k}(\tilde{\psi}\tilde{A} + A\psi) = \frac{1}{\sqrt{g}}\mathfrak{D}. \tag{19}$$

It then follows from (7), (8) that:

$$\mathfrak{D} = 0. \tag{20}$$

The action density $\mathfrak{D}$ then vanishes on the basis of the field equations (**).

Concluding remark. One obtains the macroscopic quantum laws upon integrating the continuity equations (17), (18), and suitably combining the formulas that are thus found.

Sofia, Physical Institute of the University, 20 October 1929.

Added during proof (23 November 1929): We now choose the coefficients c_1, c_2, c_3 that enter into $G_{\alpha\beta}$ as follows:

$$c_1 = -\tfrac{1}{4}(1+\tfrac{1}{2}\alpha), \qquad c_2 = -\tfrac{1}{4}(1-\tfrac{1}{2}\alpha), \qquad c_3 = 1 - \alpha, \tag{21}$$

where α is a dimensionless constant. The bein-structure can be made pseudo-Cartesian iff ψ, $\tilde{\psi}$ vanish, and in this case, one will have:

(*) $G^{\alpha\beta} \equiv G^{\beta\alpha}$ iff the coefficients c_1, c_2, c_3 fulfill the condition I (16). In this, and only in this, case will the identities (16) assume the classical form: $\delta_\rho G^{\rho\alpha} \equiv 0$. In general, however, the $G^{\alpha\beta}$ are not symmetric in α, β, since the $T^{\alpha\beta}$ is also not symmetric in α, β.

(**) We also have the same state of affairs in Whittaker's theory.

$$\left.\begin{aligned} G_{\alpha\beta} \equiv\ & -(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R) + \alpha\{\tfrac{1}{2} g_{\alpha\beta}(-\tfrac{1}{8}\Lambda_{\mu\kappa\rho}\Lambda^{\mu\kappa\rho} + \tfrac{1}{4}\Lambda_{\mu\kappa\rho}\Lambda^{\mu\rho\kappa} - \Lambda_{\mu}\Lambda^{\mu}) \\ & -\tfrac{1}{2}(\Lambda_{\mu\kappa\alpha}\Lambda^{\mu\kappa}_{\cdot\cdot\cdot\beta} - 2\Lambda_{\alpha\mu\kappa}\Lambda_{\beta}^{\cdot\mu\kappa}) - \tfrac{1}{4}\Lambda_{\alpha\mu\kappa}\Lambda_{\beta}^{\cdot\kappa\mu} + \Lambda_{\alpha}\Lambda_{\beta} \\ & + \tfrac{1}{2}(D_{\alpha}\Lambda_{\beta} + D_{\beta}\Lambda_{\alpha}) - g_{\alpha\beta} D_{\mu}\Lambda^{\mu}\} - \frac{\alpha}{2} D_{\mu}(\tfrac{1}{2} S_{\alpha\beta}^{\cdot\cdot\cdot\mu} - \Lambda_{\alpha\beta}^{\cdot\cdot\cdot\mu}) \\ & + \frac{ka^2}{2\tau} g_{\alpha\beta}. \end{aligned}\right\} \tag{22}$$

We further set:

$$f_{\alpha} = \rho\, \varphi_{\alpha}\,, \tag{23}$$

where the φ_{α} represent the electromagnetic potentials, multiplied by $1/\sqrt{4\pi}$. Empirically, one has:

$$\left.\begin{aligned} \rho &= \frac{4}{c^2}\sqrt{\frac{\pi g}{1+\frac{1}{2}\alpha}}, & \tau &= \frac{h\sqrt{g}}{\pi e c\sqrt{1+\frac{1}{2}\alpha}}, \\ k &= \frac{8gh}{c^3}, & a^2 &= \frac{\lambda_0 \cdot c^2(1-\frac{3}{2}\alpha)}{4\pi e\sqrt{1+\frac{1}{2}\alpha}}. \end{aligned}\right\} \tag{24}$$

α then remains undetermined.

Unitary theory of the physical field

By

A. EINSTEIN

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1. – The "unitary theory of the physical field" proposes to restate the general theory of relativity and to unite the theories of the electromagnetic field and the gravitational field into a unique discipline.

At present, this new theory is only a mathematical edifice that is only weakly coupled by some very loose links to physical reality. It was discovered by exclusively formal considerations, and its mathematical consequences have not been developed sufficiently enough to permit its comparison with experiment. Nevertheless, this attempt seems very interesting to me in its own right; above all, it offers magnificent possibilities for development and it is in the hope that the mathematicians will find it interesting that I shall present and analyze it here.

2. – From the formal viewpoint, the fundamental idea of the general theory of relativity is the following one: The four-dimensional space in which the phenomena take place is not amorphous, but possesses a structure; its existence translates into the existence of a Riemannian metric in that space.

Physically, this signifies that there exists a fundamental quadratic form:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

that is characteristic of that space and which expresses its metric, and which, when equated to zero:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = 0,$$

defines the law of propagation of light in that space. That quadratic form is therefore intimately linked with physical reality. Its introduction is not simply a mind game, and its use is justified by the correspondence that one can establish between its coefficients $g_{\mu\nu}$ and a class of known phenomena – viz., gravitational phenomena.

Since the structure of space is defined by the fundamental quadratic form, the problem that is posed is then the following one: What is the *simplest* law that one can impose upon the coefficients $g_{\mu\nu}$? The answer is given by RIEMANN's tensorial theory.

One can form a tensor $R^i_{k.lm}$ by starting with the $g_{\mu\nu}$ and their derivatives, which is called the RIEMANN curvature tensor. Upon contracting it with respect to the indices i and m, one deduces another second-rank tensor R_{kl} . The simplest law to which one can subject the $g_{\mu\nu}$ is expressed simply by the equation:

$$R_{kl} = 0.$$

This theory will be the ideal physical theory if it can completely describe the field of forces that actually exists in nature; i.e., the set that is composed of the gravitational field and the electromagnetic field. However, the equations $R_{kl} = 0$, which seem to describe gravitational phenomena, do not account for electromagnetic phenomena. The metric alone does not suffice to describe that set.

In order to completely characterize space, one seeks to give, in addition to the fundamental form $g_{\mu\nu}\, dx^\mu\, dx^\nu$, a linear form $\varphi_\mu\, dx^\mu$, whose coefficients φ_μ will be the components of the electromagnetic vector potential. The complete equations of the field will then be of the form $R_{kl} + T_{kl} = 0$, where T_{kl} is a term that depends upon the potentials – for example, the MAXWELL electromagnetic tensor – or some analogous thing. Meanwhile, this manner of proceeding is not satisfactory. Indeed, the equation $R_{kl} + T_{kl} = 0$ involves two *independent* terms; one can logically change one without affecting the other one. In this way, one introduces two independent elements into the theory that correspond to *two* "states" of the space. Nature then presents a lack of unity that our mind absolutely refuses to believe. On the contrary, it seems more satisfactory to attribute this flaw to an imperfection of the theory, and to seek to complete and enrich it in order to realize the unity to which our spirit aspires so ineluctably.

The unitary theory of the physical field thus begins with the affirmation that the metric alone does not suffice to describe phenomena. Meanwhile, it provides at least one part of truth: It certainly occupies a physical substratum. The problem that one then poses consists in finding what will complete the metric and what will permit us to describe the structure of space without leaving anything out.

3. – To that end, we seek to find what sense one might attribute to the notion of a Riemannian metric and what sort of representation one can give to it.

Consider an n-dimensional continuum that presents a Riemannian structure. Such a continuum is characterized by the fact that Euclidian geometry is valid in an infinitely small domain around a given point. Moreover, if one is given two points A and B at a finite distance apart then one can compare the *lengths* of the two linear elements that are situated at A and B, but one cannot say the same thing of their *directions*; there exists no distant parallelism in Riemannian geometry.

4. – Imagine a Cartesian system of coordinates at a given point in such a space; i.e., a system of n rectangular axes, each of which is given a unit vector. We call such a system of axes an *n-pode* (n-Bein).

The infinitesimal Euclidian domain that surrounds a point is characterized completely when one is given an n-pode at that point. The metric of space is known if one has fixed an n-pode *at each point* of that space. Indeed, the metric of space remains the same if one subjects all of the n-podes to arbitrary rotations. The orientation of the n-podes is not fixed when one is given only the metric; there then remains a certain arbitrariness in the determination of the structure of space. In this manner, one thus sees that the description of space by n-podes is, in some way, richer than the description with the aid of the fundamental quadratic form. One imagines that one can find the cause of electromagnetic phenomena in this arbitrariness that is attached to the structure of space, and these are phenomena that have not found their place in the theory.

This is not the first time that such spaces have been envisioned. They had already been studied previously from the purely mathematical viewpoint. CARTAN was kind enough to produce a note for the *Mathematischen Annalen* that summarizes the various phases of the formal development of the concept.

Suppose that one is given an n-pode at A; the structure of space will be defined if we give an arbitrary n-pode at every other point that we regard as parallel to the first one, by definition. One can thus establish a relation of direction between two points of space, in addition to a relation of length. The notion of distance parallelism now possesses a precise sense that it cannot have in RIEMANN's theory. Two vectors that have their origins at finitely-separated points will be parallel if they have the same components in their local systems. When one characterizes the structure of space by a field of n-podes, one *simultaneously* expresses the existence of a Riemannian metric and that of a distant parallelism; between two infinitesimal elements of that space there then exists not only a relation of length that is expressed by the metric, but also a relation of direction that is expressed by the orientation of the n-podes.

In summary, the only new hypothesis that one introduces in order to arrive at a more complete geometry than that of RIEMANN concerns the existence of "directions" in space and the relations between these directions. This notion of "direction" is not contained in either the notion of a continuum or that of space. One must then make a supplementary hypothesis in order to assume that there exist something like direction relations in the space that are expressed by the existence of parallelism at a finite distance.

Meanwhile, it is easy to see that, likewise with the hypothesis of parallelism at a distance combined with that of a Riemannian metric, the field of n-podes is defined only up to a rotation (that is common to all n-podes).

5. – Introduce a general system of GAUSSian coordinates and consider the n-pode that is attached to the point P. Let h_s^{ν} be the components of the unit vectors of the n-pode in the GAUSSian coordinate system. In what follows, any Greek index will relate to the coordinates and any Latin index to the n-pode. h_s^{ν} will thus represent the ν^{th} component of the unit vector that corresponds to the s axis of the n-pode. In a quadri-dimensional space – i.e., $n = 4$ – we thus have 16 quantities h_s^{ν} that describe the structure of that space perfectly.

If these quantities are given then one can calculate the components of an arbitrary vector A in a local system as functions of its components in the GAUSSian system. One has:

$$A^{\nu} = h_s^{\nu} A_s , \tag{1}$$

and conversely:

(2) $$A_s^* = h_{s\nu} A^\nu,$$

where the $h_{s\nu}$ are the minors of the determinant $h = | h_s{}^\nu |$, divided by h. By convention, one can perform the summation over indices that appear twice.

In order to get the metric of that space, one calculates the magnitude of a vector A. In a local system, since Euclidian geometry is valid, one has:

(3) $$A^2 = \sum A_s^2 = \sum h_{s\mu}\, h_{s\nu}\, A^\mu A^\nu.$$

The coefficients of the fundamental metric form $g_{\mu\nu}\, dx^\mu\, dx^\nu$ will thus be given by:

(4) $$g_{\mu\nu} = h_{s\mu}\, h_{s\nu}\,.$$

One then sees that a field of n-podes ($h_s{}^\nu$) determines the metric ($g_{\mu\nu}$) completely, but the converse is not true.

The quantities $h_s{}^\nu$ form the fundamental tensor that is analogous to the tensor $g_{\mu\nu}$ of the old theory; for the case of $n = 4$, there are sixteen quantities $h_s{}^\nu$ and only ten $g_{\mu\nu}$.

The concept of tensor is found to be broader in this theory. Indeed, here we can consider not only transformations that change the system of coordinates, but also ones that modify the orientations of the n-podes. The n-podes are determined up to a rotation; the only admissible relations must then be invariant with respect to such a rotation. For example, change the coordinate system and the orientation of the local system simultaneously. Since the rotation is characterized by the constant coefficients α_{st} , independently of the coordinates and such that:

(5) $$\alpha_{s\mu} \cdot \alpha_{s\nu} = \alpha_{\nu s} \cdot \alpha_{\mu s} = \delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu, \end{cases}$$

one will have:

(6) $$h_s{}^{\nu'} = \alpha_{st} \frac{\partial x^{\nu'}}{\partial x^\rho} h_t{}^\rho\,.$$

Each local index corresponds to a transformation α and each Greek index to an ordinary transformation.

8. – The algebraic laws to which these tensors are subject are almost the same as the ones for the tensors of the absolute differential calculus. One can define the sum and the difference of two tensors T and S that have the same indices. The product of two tensors has the same law of transformation as a tensor of higher rank.

The contraction operation is applicable for both the Greek indices and the Latin indices. For the former, one must always equate an upper index and a lower one. The permutation of these indices is possible; in particular, one can replace a Latin index by a

Greek one by means of the fundamental tensor $h_s{}^\nu$. For example, take the tensor $T^{\dots\lambda}_{\dots s}$. One has:

(7) $$h_s{}^\tau \, T^{\dots\lambda}_{\dots s} = T^{\dots\tau\lambda}_{\dots}.$$

One can then pass from the local components to the components of the same tensor in the GAUSSian system, and conversely.

Finally, calculate the volume element in this new theory. That important quantity has the following expression in the general theory of relativity:

$$d\Omega = \sqrt{g} \cdot T^{\dots\lambda}_{\dots s} \; d\tau,$$

where

$$g = |\, g_{\mu\nu} \,| \qquad \text{and} \qquad d\tau = dx^1 \cdot dx^2 \dots$$

Now, one has:

$$g_{\mu\nu} = h_{\mu s} \cdot h_{\nu s} \qquad \text{and} \qquad g = h^2;$$

thus:

(8) $$d\Omega = h\, d\tau.$$

The fact that the radical has disappeared is then another advantage of the new theory.

7. – Now consider the parallel displacement of a vector A^μ. In a Riemannian multiplicity, this displacement is given by the formula:

$$dA^\mu = -\, \Gamma^\mu_{\alpha\beta} A^\alpha \, dx^\beta.$$

The $\Gamma^\mu_{\alpha\beta}$ are the CHRISTOFFEL brackets, and must satisfy two conditions:

1. The translation that they define must preserve the metric; i.e., it must leave the lengths of the vectors in question invariant, and
2. The $\Gamma^\mu_{\alpha\beta}$ must be symmetric in α and β:

$$\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}.$$

Parallel displacement is not integrable in this geometry. If one performs it along a closed curve then the initial vector does not coincide with the final vector, and the difference is measured by the RIEMANN tensor $R^i_{k,lm}$.

Things present themselves differently in the new theory. The parallel displacement of a vector A is given by an analogous formula:

(6) $$\delta A^\mu = -\, \Delta^\mu_{\alpha\beta} A^\alpha \, \delta x^\beta.$$

However, the displacement is integrable here: If one displaces a vector along a closed curve then the initial vector will always coincide with the final vector. Consequently, the RIEMANN tensor that is formed by starting with the $\Delta^{\mu}_{\alpha\beta}$ will be zero. Moreover, the $\Delta^{\mu}_{\alpha\beta}$ are no longer symmetric in α and β. One easily verifies these results by calculating the expression for the $\Delta^{\mu}_{\alpha\beta}$ as functions of the h.

Let x^{β} and $x^{\beta} + dx^{\beta}$ be two neighboring points, such that their n-podes are "parallel" to each other. The vectors A_s and $A_s + \delta A_s$ will be parallel if they have the same components in the two n-podes. The condition of parallel displacement of x^{β} to $x^{\beta} + dx^{\beta}$ is therefore $\delta A_s = 0$. By expressing the A_s as functions of the components of A in the GAUSSian system:

$$A_s = h_{s\mu}\, A^{\mu},$$

one has:

(10) $$\delta(h_{s\mu}\, A^{\mu}) = 0.$$

Upon multiplying by $h_s^{\ \sigma}$, one deduces that:

(11) $$0 = h_s^{\sigma}\left\{h_{s\mu}\delta A^{\mu} + A^{\alpha}\cdot\frac{\partial h_{s\alpha}}{\partial x^{\beta}}\delta x^{\beta}\right\},$$

or, upon denoting the ordinary derivative by a comma (,):

(12) $$\Delta^{\mu}_{\alpha\beta} = h_s^{\ \mu}\, h_{s\alpha,\,\beta}\,,$$

and also:

(13) $$\Delta^{\mu}_{\alpha\beta} = -\, h_{s\alpha}\, h_s^{\ \mu}{}_{,\,\beta}\,.$$

By the same mechanism as the one that is used in the absolute differential calculus, one can form the covariant derivative operator by starting with the $\Delta^{\mu}_{\alpha\beta}$. Upon denoting it by the semi-colon (;), one has for a contravariant, first-rank tensor:

(14) $$A^{\mu}{}_{;\,\sigma} = A^{\mu}{}_{,\,\sigma} + A^{\alpha}\ \Delta^{\mu}_{\alpha\sigma}\,,$$

and for a covariant, first-rank tensor:

(15) $$A_{\mu;\,\sigma} = A_{\mu,\,\sigma} - A_{\alpha}\ \Delta^{\alpha}_{\mu\sigma}\,.$$

One finds analogous formulas for the tensors of higher rank. They are parallel to the formulas of absolute differential calculus that are based upon the metric exclusively, and are deduced in the same fashion.

One easily conforms that the covariant derivative of the fundamental tensor is identically zero:

(16) $$h_s^{\ \nu}{}_{;\,\tau} = h_{s\nu\,;\,\tau} = g_{\sigma\tau;\,\rho} = g^{\sigma\tau}{}_{;\,\rho} \equiv 0.$$

Indeed, one has:

$$h_s{}^{\nu}{}_{;\tau} = h_s{}^{\nu}{}_{,\tau} + h_s^{\alpha}\Delta^{\nu}_{\alpha\tau} = (\delta_{s\iota}\, h_\iota{}^{\nu}{}_{,\tau} + h_s^{\alpha}\Delta^{\nu}_{\alpha\tau}) = h_s^{\alpha}(h_{\iota\alpha}h^{\nu}_{\iota,\tau} + \Delta^{\nu}_{\alpha\tau}) \equiv 0.$$

The covariant derivative of a product of two tensors is obtained by the usual rule of differential calculus. For example, if $T^{\cdots}_{\cdots}$ and $S^{\cdots}_{\cdots}$ are two tensors of arbitrary rank then one has:

(17) $$(T^{\cdots}_{\cdots}\, S^{\cdots}_{\cdots})_{;\tau} = T^{\cdots}_{\cdots;\tau}\, S^{\cdots}_{\cdots} + T^{\cdots}_{\cdots}\, S^{\cdots}_{\cdots;\tau} \;.$$

Two covariant differentiations do not commute – i.e., the order of differentiation is not immaterial. Let $T^{\cdots}_{\cdots}$ be an arbitrary tensor. Take successive covariant derivatives – first, in the order σ, τ and then in the order τ, σ – and then take the difference between them. We then have the fundamental formula:

(18) $$T^{\cdots}_{\cdots;\sigma;\tau} - T^{\cdots}_{\cdots;\tau;\sigma} \equiv - T^{\cdots}_{\cdots;\alpha}\Lambda^{\alpha}_{\sigma\tau} ,$$

where:

$$\Lambda^{\alpha}_{\sigma\tau} = \Delta^{\alpha}_{\sigma\tau} - \Delta^{\alpha}_{\tau\sigma} .$$

It is easy to prove this formula in some simple cases. First, suppose that $T^{\cdots}_{\cdots}$ reduces to a scalar ψ. In this case, the covariant derivative coincides with the ordinary derivative:

$$\psi_{;\sigma} = \psi_{,\sigma} ,$$

and we have:

$$\begin{aligned} \psi_{;\sigma;\tau} &= \psi_{,\sigma,\tau} - \psi_{,\alpha}\, \Delta^{\alpha}_{\sigma\tau} , \\ \psi_{;\tau;\sigma} &= \psi_{,\tau,\sigma} - \psi_{,\alpha}\, \Delta^{\alpha}_{\tau\sigma} , \\ \psi_{;\sigma;\tau} - \psi_{;\tau;\sigma} &= -\psi_{,\alpha}\,(\Delta^{\alpha}_{\sigma\tau} - \Delta^{\alpha}_{\tau\sigma}) = -\psi_{,\alpha}\,\Lambda^{\alpha}_{\sigma\tau} . \end{aligned}$$

The difference indeed has the stated form. One recalls that $\Lambda^{\alpha}_{\sigma\tau}$ is a tensor.

The case of a vector $T^{\cdots}_{\cdots} = A^{\mu}$ reduces to the preceding case if we take into account the fact that distant parallelism exists in this theory. In effect, the existence of that parallelism entails the possibility of the existence of a uniform vector field (i.e., a parallel field); it is possible to imagine that there is a vector that is equipollent to the given vector at each point of space.

This being the case, consider an *arbitrary* uniform vector field a_μ ; one easily shows that $a_{\mu;\sigma} = a^{\mu}{}_{;\sigma} = 0$. With the given vector A^{μ}, for the scalar:

$$\psi = A^{\mu} \cdot a_{\mu} .$$

We may apply the formula for the difference D that was established above to this scalar. Then, upon taking into account the rule for the differentiation of a product, one has $\left(A^{\mu}_{\alpha\mu}\right)_{;\sigma} = a_\mu\, A^{\mu}{}_{;\sigma}$. The arbitrary quantities a_μ turn into factors and disappear, and finally, one has a relation of the same form:

$$A^{\mu}{}_{;\,\sigma;\,\tau} - A^{\mu}{}_{;\,\sigma;\,\tau} \equiv - A^{\mu}{}_{;\,\alpha}\,\Lambda^{\alpha}_{\sigma\tau}\,,$$

which is easy to generalize to a tensor of arbitrary rank.

8. – An important difference between the theory that is presented here and RIEMANN's theory deserves our attention. In RIEMANN's theory, there is no tensor that can be expressed solely by means of the first derivatives of the fundamental tensor. In ours, the difference:

(19) $$\Lambda^{\alpha}_{\mu\nu} = \Delta^{\alpha}_{\mu\nu} - \Delta^{\alpha}_{\nu\mu}$$

is a tensor that contains only first derivatives. In addition, this tensor is remarkable because, in a certain sense, it is the analogue of the RIEMANN tensor: *If* Λ *is zero then the continuum is Euclidian.*

This result is easy to establish. From the formula that was given for $\Delta^{\alpha}_{\mu\nu}$, one has:

$$\Lambda^{\alpha}_{\mu\nu} = h_s{}^{\alpha}\,(h_{s\mu,\,\nu} - h_{s\nu,\,\mu}) = 0.$$

Upon multiplying by $h_t{}^{\alpha}$, one deduces, since $h_s{}^{\alpha}\,h_{t\alpha} = \delta_{st}$, that:

$$h_{t\mu,\,\nu} - h_{t\nu,\,\mu} = 0,$$

so $h_{t\mu}$ is of the form:

$$h_{t\mu} = \frac{\partial \psi_t}{\partial x^{\mu}}\,.$$

If we justifiably take the ψ_t to be GAUSSian coordinates, which is possible – so $\psi^t = x^t$ – then the:

$$h_{t\mu} = \delta_{t\mu} = \begin{cases} 1 & t = \mu \\ 0 & t \neq \mu \end{cases}$$

are constants; they define a matrix in which only the diagonal terms are equal to 1, while the others are zero. Since the $h_{s\mu}$ and the $g_{\mu\nu}$ are constant, the continuum is Euclidian.

9. – Consider the quantity Λ, which plays a fundamental role in the new theory. There are $6 \times 4 = 24$ quantities Λ, in all; meanwhile, the h are 16 in number. Therefore, there are some relations between the various Λ that must be satisfied. In order to find them, start with the expression for Λ as a function of the Δ. Since parallel displacement is integrable, the "curvature" tensor that is analogous to the RIEMANN tensor will thus be identically zero. Consequently, we have:

(20) $$\Delta^{\iota}_{\kappa\lambda,\mu} - \Delta^{\iota}_{\kappa\mu,\lambda} - \Delta^{\iota}_{\sigma\lambda}\cdot\Delta^{\sigma}_{\kappa\mu} + \Delta^{\iota}_{\sigma\mu}\Delta^{\sigma}_{\kappa\lambda} \equiv 0.$$

Make a cyclic permutation of the indices κ; λ, μ, and take the sum; then introduce the covariant derivative in place of the ordinary derivative. One thus arrives at the following identity for the Λ:

$$(\Lambda^{\iota}_{\kappa\lambda;\mu} + \Lambda^{\iota}_{\lambda\mu;\kappa} + \Lambda^{\iota}_{\mu\kappa;\lambda}) + (\Lambda^{\iota}_{\kappa\alpha}\Lambda^{\alpha}_{\lambda\mu} + \Lambda^{\iota}_{\lambda\alpha}\Lambda^{\alpha}_{\mu\kappa} + \Lambda^{\iota}_{\mu\alpha}\Lambda^{\alpha}_{\kappa\lambda}) \equiv 0. \tag{21}$$

By contracting this once with respect to i and μ and setting $\Lambda^{\alpha}_{\mu\alpha} = \varphi_\mu$, one finds another important identity:

$$\Lambda^{\alpha}_{\mu\nu;\alpha} - \left(\frac{\partial\varphi_\mu}{\partial x^\nu} - \frac{\partial\varphi_\nu}{\partial x^\mu}\right) \equiv 0. \tag{22}$$

In order to deduce another one, one must appeal to the rule for the permutation of covariant derivatives, which is expressed by:

$$T^{\cdots}_{\ldots;\sigma;\tau} - T^{\cdots}_{\ldots;\tau;\sigma} = - T^{\cdots}_{\ldots;\alpha}\Lambda^{\alpha}_{\sigma\tau}.$$

We introduce a new notation: We agree that an underlined index signifies that an index has changed position – i.e., it has been raised or lowered. For example, if we write $\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}$ then that signifies that we take the contravariant components of the $\Lambda^{\alpha}_{\mu\nu}$:

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}} = \Lambda^{\alpha}_{\mu\nu}\, g^{\mu\sigma} g^{\nu\tau}.$$

With that definition, we apply the preceding rule to the $\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}$ upon differentiating it with respect to ν and α. One has:

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\nu;\alpha} - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha;\nu} \equiv - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\sigma}\Lambda^{\sigma}_{\nu\alpha}.$$

The right-hand side can be written:

$$- \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\sigma}\Lambda^{\sigma}_{\nu\alpha} \equiv - (\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}\Lambda^{\sigma}_{\nu\alpha})_{;\sigma} + \Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}\Lambda^{\sigma}_{\nu\alpha;\sigma}.$$

In the first term of the right-hand side, we change the names of the dummy indices σ, α, and ν into α, σ, and τ; that term becomes:

$$- (\Lambda^{\sigma}_{\underline{\mu}\underline{\nu}}\Lambda^{\alpha}_{\tau\sigma})_{;\alpha} \equiv + (\Lambda^{\sigma}_{\underline{\mu}\underline{\nu}}\Lambda^{\alpha}_{\sigma\tau})_{;\alpha}.$$

One thus has:

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\nu;\alpha} - (\Lambda^{\sigma}_{\underline{\mu}\underline{\nu}}\Lambda^{\alpha}_{\sigma\tau})_{;\alpha} - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha;\nu} - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}\Lambda^{\sigma}_{\nu\alpha;\sigma} \equiv 0,$$

or

$$(\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\nu} - \Lambda^{\sigma}_{\underline{\mu}\underline{\nu}}\Lambda^{\alpha}_{\sigma\tau})_{;\alpha} - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha;\nu} - \Lambda^{\alpha}_{\underline{\mu}\underline{\nu}}\Lambda^{\sigma}_{\nu\alpha;\sigma} \equiv 0, \tag{23}$$

which constitutes the desired identity. Introduce the notations:

$$G^{\mu\alpha} \equiv \Lambda^{\alpha}_{\underline{\mu\nu};\nu} - \Lambda^{\sigma}_{\underline{\mu\tau}}\Lambda^{\alpha}_{\sigma\tau},$$
$$F^{\mu\nu} \equiv \Lambda^{\alpha}_{\underline{\mu\nu};\alpha}.$$

The identity (23) is then written:

(24) $$G^{\mu\alpha}{}_{;\,\alpha} - F^{\mu\alpha}{}_{;\,\alpha} - \Lambda^{\alpha}_{\underline{\mu\nu}}F_{\nu\alpha} \equiv 0.$$

10. – having defined the manner by which we write the structure of the space mathematically, we now examine the fundamental problem of the theory, which is to establish the field equations. As in the general theory of relativity, this problem consists of finding the simplest conditions that one can impose upon the elements that define the structure of space – i.e., the quantities $h_s{}^{\nu}$. It thus amounts to making a choice amongst the possibilities; the difficulty in making that choice then resides in the absence of benchmarks that could guide us. Before writing the defining equations of the field, it seems interesting to me to point out the path that I followed in order to discover them.

My point of departure consisted of the identities that the quantities $\Lambda^{\alpha}_{\mu\nu}$ satisfied. In a more general manner, the search for certain identities can be a great help for the choice of field equations by suggesting some possible forms for the desired relations. The study of these identities must therefore logically precede the choice of a system of equations. However, one cannot know, *a priori*, what the quantities are between which one can establish these identities.

A primary benchmark that appears here seems to be the following one: The desired relations must most likely contain $\Lambda^{\alpha}_{\mu\nu}$ and its derivatives, since that tensor is the only one that can be expressed solely as a function of the first derivatives of the fundamental tensor.

The simplest condition for one to impose would be:

$$\Lambda^{\alpha}_{\mu\nu} = 0.$$

It is obvious that this condition is too restrictive: viz., The space would be Euclidian. Moreover, *it contains only first derivatives* and it is likely that the equations that regulate natural phenomena are of second order; for example, the POISSON equation.

We then attempt to set:

$$\Lambda^{\alpha}_{\mu\nu;\sigma} = 0.$$

This relation is not acceptable either, because it is almost equivalent to the first one; however, it is useful because it immediately suggests that we try to annul the divergences that one can form by starting with the $\Lambda^{\alpha}_{\mu\nu;\sigma}$. We thus start with that covariant derivative and contract it in all possible manners (which is equivalent to taking the divergence). We have two possibilities:

Either:

(25) $$\Lambda^{\alpha}_{\mu\nu;\alpha} = 0$$

or

(26) $$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\nu} = 0.$$

One immediately sees that the set of these systems is not appropriate, since the number of equations cannot be chosen arbitrarily: One cannot guarantee the compatibility of these equations without a special study. Now, it is indispensible that the chosen system should be such that the equations are compatible.

11. – In general, for a space of n dimension, there are n^2 variables $h_s{}^\nu$. However, in a general covariant theory, since the choice of coordinate system is arbitrary, among the n^2 variables, n of them can be taken arbitrarily. Consequently, the number of independent equations will be $n^2 - n$. Similarly, the number of equations can be larger than $n^2 - n$, provided that they are related by a convenient number of identities that render the system compatible. In any case, the system must satisfy the rule that *the excess of the number of equations over the number of identities is equal to the number of variables minus n.*

For example, consider the equations of general relativity. We have ten unknown functions $g_{\mu\nu}$; since the coordinate system is arbitrary, we can choose it in such a fashion that four of the functions $g_{\mu\nu}$ are arbitrary. The six unknowns will thus satisfy ten equations. However, as one knows, one has, at the same time, the four identities:

$$(R^{ik} - \tfrac{1}{2} g^{ik} R)_{;\,\kappa} \equiv 0,$$

which re-establish the compatibility ([1]).

One can cite another case for which the number of equations exceeds the number of unknowns without the equations being incompatible. For example, the MAXWELL equations:

$$\text{rot}\,\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0, \qquad \text{rot}\,\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0,$$

$$\text{div}\,\mathbf{E} = 0, \qquad \text{div}\,\mathbf{H} = 0$$

are eight in number with six unknowns; the system is nonetheless compatible, since the equations are related by two known identities.

What intrinsically signifies the presence of a greater number of equations than unknowns?

In the example chosen, the two vectorial MAXWELL equations determine the problem canonically. If the fields **E** and **H** are given at the instant t then they are determined at all remaining times. However, the other scalar relations imply that the initial conditions are not arbitrary. Therefore, a stronger determination of the problem – viz., a number of equations that is larger than the number of unknowns (with identities

([1]) The symbol ";" is employed here with a well-known significance that is different from the one that is defined in the rest of this article.

that render them compatible, moreover) – partially eliminates the arbitrariness that exists in this case for the initial conditions. It is, moreover, clear that a theory that is compatible with experiment is all the more satisfying if it limits that arbitrariness in a more complete fashion. Having said that, we return to our problem.

12. – For a four-dimensional space – i.e., $n = 4$ – we have 16 unknowns $h_s{}^n$, four of which are arbitrary, so only 12 of them can be determined by the field equations. On first glance, the number of equations that form a convenient system is 22 – namely, 6 equations (25) and 16 equations (26). There must then be 10 identities, which do not exist, in this case. In that way, one understands how the compatibility condition permits us to limit the arbitrariness in the choice of field equations in an efficacious manner.

We then examine the identity (24). It suggests that we take the field equations to be the system:

$$G^{\mu\alpha} = 0, \tag{27}$$

$$F^{\mu\alpha} = 0, \tag{28}$$

or, explicitly:

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\nu} - \Lambda^{\sigma}_{\underline{\mu}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau} = 0, \tag{27a}$$

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha} = 0. \tag{28a}$$

This system, which is a little different from the system (25), (26), is always comprised of 22 equations, but ones that are chosen in such a manner as to satisfy the 4 identities (24).

Nevertheless, the excess 22 – 4 = 18 is always greater than the difference 16 – 4 = 12. In order for the new system of equations to be compatible, it is necessary that there further exist 6 supplementary identities between its equations. We prove that these necessary identities exist. In order to show this, we first give equations (28) another form that is equivalent to the first one, and which is guided by the identity (22):

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha} - (\varphi_{\mu,\,\nu} - \varphi_{\nu,\,\mu}) \equiv 0. \tag{22}$$

We have set $\Lambda^{\alpha}_{\mu\nu;\alpha} = 0$; from (22), it results that one also has:

$$\frac{\partial\varphi_\mu}{\partial x^\nu} - \frac{\partial\varphi_\nu}{\partial x^\mu} = 0.$$

Therefore, φ_μ is the derivative of a scalar, which is conveniently denoted by $\log \psi$, here:

$$\varphi_\mu = \frac{\partial \log\psi}{\partial x^\mu}.$$

Therefore, set:

$$F_\mu = \varphi_\mu - \frac{\partial \log\psi}{\partial x^\mu};$$

one has $F_\mu = 0$. We can then replace the equations:

$$\Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha} = 0$$

with the equations $F_\mu = 0$, and write our system of equations as follows:

$$(29) \qquad G^{\mu\alpha} = 0$$

$$(30) \qquad F_\mu = 0,$$

or

$$(29a) \qquad \Lambda^{\alpha}_{\underline{\mu}\underline{\nu};\alpha} - \Lambda^{\sigma}_{\underline{\mu}\underline{\nu}}\Lambda^{\alpha}_{\sigma\nu} = 0,$$

$$(30a) \qquad \varphi_\mu - \frac{\partial \log \psi}{\partial x^\mu} = 0.$$

We now have 16 equations (29) and 4 equations (30), and therefore, 20 equations, in all. We have introduced a new variable – viz., the scalar, ψ; there are thus 16 + 1 = 17 unknowns, four of which are arbitrary. In order for the system to be compatible, it is necessary that there be:

$$20 - (17 - 4) = 7$$

identities between the $G^{\mu\alpha}$ and the $F^{\mu\alpha}$. We have found only 4 of them, namely, the identities (24). Now, there further exist some identities between quantities envisioned and – miraculously, one can say – there are just three. I cannot say what the profound reason for their existence is. It essentially comes down to the nature of the space in question. Moreover, this type of space was imagined before me by some mathematicians, notably, by WEITZENBÖCK, EISENHART and CARTAN; it is my hope that they can assist us in discovering the hidden origin of these new identities.

Be that as it may, they exist; I would like to point out how one can arrive at them.

Decompose the tensor $G^{\mu\alpha}$ into its symmetric part $\underline{\underline{G}}^{\mu\alpha}$ and its anti-symmetric part $\underline{G}^{\mu\alpha}$. One has:

$$\begin{cases} 2\underline{G}^{\mu\alpha} = (\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}} - \Lambda^{\mu}_{\underline{\alpha}\underline{\nu}})_{;\nu} - \Lambda^{\sigma}_{\underline{\mu}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau} + \Lambda^{\sigma}_{\underset{\sim}{\alpha}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} \\ \qquad = (\Lambda^{\alpha}_{\underline{\mu}\underline{\nu}} + \Lambda^{\mu}_{\underline{\nu}\underline{\alpha}})_{;\nu} - \Lambda^{\sigma}_{\underline{\mu}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau} + \Lambda^{\sigma}_{\underset{\sim}{\alpha}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau}, \end{cases}$$

since the $\Lambda^{\mu}_{\underline{\alpha}\underline{\nu}}$ are anti-symmetric in α, ν.

One can express $2\underline{G}^{\mu\alpha}$ as functions of $F^{\mu\alpha} = \Lambda^{\nu}_{\mu\alpha;\nu}$ and a tensor that is anti-symmetric with respect to an arbitrary pair of the indices α, μ, ν, namely:

$$(31) \qquad S^{\mu}_{\underline{\alpha}\underline{\nu}} = \Lambda^{\alpha}_{\underline{\mu}\underline{\nu}} + \Lambda^{\nu}_{\underline{\alpha}\underline{\mu}} + \Lambda^{\mu}_{\underline{\nu}\underline{\alpha}}.$$

One obviously has (†):

$$2\underline{G}^{\mu\alpha} = S^{\mu}_{\underline{\alpha}\underline{\nu};\nu} + F^{\mu\alpha} + C^{\mu\alpha},$$

(†) Translator's note: In the original, the C term was given without indices.

where the complementary $C^{\mu\alpha}$ is given by:

$$C^{\mu\alpha} = \Lambda^{\sigma}_{\underline{\alpha}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} - \Lambda^{\sigma}_{\underline{\mu}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau}.$$

In order to calculate it, observe that upon changing the dummy indices σ and τ, one has:

$$\Lambda^{\sigma}_{\underline{\alpha}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} = \Lambda^{\tau}_{\underline{\alpha}\underline{\sigma}}\Lambda^{\mu}_{\tau\sigma} = -\Lambda^{\tau}_{\underline{\alpha}\underline{\sigma}}\Lambda^{\mu}_{\sigma\tau}$$

and

$$\Lambda^{\sigma}_{\underline{\alpha}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} = \Lambda^{\tau}_{\underline{\mu}\underline{\sigma}}\Lambda^{\alpha}_{\tau\sigma} = -\Lambda^{\tau}_{\underline{\mu}\underline{\sigma}}\Lambda^{\alpha}_{\sigma\tau}.$$

On the other hand, we have the equality:

$$\Lambda^{\alpha}_{\underline{\tau}\underline{\sigma}}\Lambda^{\mu}_{\sigma\tau} = \Lambda^{\alpha}_{\tau\sigma}\Lambda^{\mu}_{\underline{\sigma}\underline{\tau}},$$

due to the fact that:

$$\Lambda^{\alpha}_{\underline{\tau}\underline{\sigma}}\Lambda^{\mu}_{\sigma\tau} = \Lambda^{\alpha}_{\beta\gamma}g^{\beta\tau}g^{\gamma\sigma}\Lambda^{\mu}_{\sigma\tau} = \Lambda^{\alpha}_{\beta\gamma}\Lambda^{\mu}_{\underline{\gamma}\underline{\beta}} = \Lambda^{\alpha}_{\tau\sigma}\Lambda^{\mu}_{\underline{\sigma}\underline{\tau}}.$$

Therefore:

$$-C^{\mu\alpha} = \Lambda^{\sigma}_{\underline{\tau}\underline{\alpha}}\Lambda^{\mu}_{\sigma\tau} - \Lambda^{\sigma}_{\underline{\tau}\underline{\mu}}\Lambda^{\alpha}_{\sigma\tau} = \tfrac{1}{2}(\Lambda^{\sigma}_{\underline{\tau}\underline{\alpha}} + \Lambda^{\tau}_{\underline{\alpha}\underline{\sigma}} + \Lambda^{\alpha}_{\underline{\sigma}\underline{\tau}})\Lambda^{\mu}_{\sigma\tau} - \tfrac{1}{2}(\Lambda^{\sigma}_{\underline{\tau}\underline{\mu}} + \Lambda^{\tau}_{\underline{\mu}\underline{\sigma}} + \Lambda^{\mu}_{\underline{\sigma}\underline{\tau}})\Lambda^{\alpha}_{\sigma\tau}$$

or

$$-C^{\mu\alpha} = \tfrac{1}{2}S^{\alpha}_{\underline{\sigma}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} - \tfrac{1}{2}S^{\mu}_{\underline{\sigma}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau}.$$

Finally, one then has:

(32) $$2\underline{G}^{\mu\alpha} = -S^{\nu}_{\underline{\mu}\underline{\alpha};\nu} + \tfrac{1}{2}S^{\mu}_{\underline{\sigma}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau} - \tfrac{1}{2}S^{\alpha}_{\underline{\sigma}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} + F^{\mu\alpha}.$$

We develop the covariant derivative (the underlined indices are contravariant). One has:

$$-S^{\nu}_{\underline{\mu}\underline{\alpha};\nu} = S^{\tau}_{\underline{\alpha}\underline{\mu};\tau} = S^{\tau}_{\underline{\alpha}\underline{\mu},\tau} + S^{\sigma}_{\underline{\alpha}\underline{\mu}}\Delta^{\tau}_{\sigma\tau} + S^{\tau}_{\underline{\sigma}\underline{\mu}}\Delta^{\alpha}_{\sigma\tau} + S^{\tau}_{\underline{\alpha}\underline{\sigma}}\Delta^{\mu}_{\sigma\tau}.$$

Now, upon switching σ and τ, we get:

$$\begin{aligned} S^{\tau}_{\underline{\sigma}\underline{\mu}}\Delta^{\alpha}_{\sigma\tau} &= S^{\sigma}_{\underline{\tau}\underline{\mu}}\Delta^{\alpha}_{\tau\sigma} = \tfrac{1}{2}(S^{\tau}_{\underline{\sigma}\underline{\mu}}\Lambda^{\alpha}_{\sigma\tau} - S^{\sigma}_{\underline{\tau}\underline{\mu}}\Lambda^{\alpha}_{\tau\sigma}) \\ &= \tfrac{1}{2}S^{\mu}_{\underline{\sigma}\underline{\tau}}(\Lambda^{\alpha}_{\tau\sigma} - \Lambda^{\alpha}_{\sigma\tau}) = -\tfrac{1}{2}S^{\mu}_{\underline{\sigma}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau}, \end{aligned}$$

because

$$S^{\tau}_{\underline{\sigma}\underline{\mu}} = S^{\mu}_{\underline{\tau}\underline{\sigma}} = S^{\sigma}_{\underline{\mu}\underline{\tau}}$$

and also

$$S^{\tau}_{\underline{\alpha}\underline{\sigma}}\Delta^{\mu}_{\sigma\tau} = S^{\sigma}_{\underline{\alpha}\underline{\tau}}\Delta^{\mu}_{\tau\sigma} = \tfrac{1}{2}S^{\alpha}_{\underline{\sigma}\underline{\tau}}(\Lambda^{\mu}_{\sigma\tau} - \Lambda^{\mu}_{\tau\sigma}) = -\tfrac{1}{2}S^{\alpha}_{\underline{\sigma}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau}.$$

Therefore:

$$-S^{\nu}_{\underline{\mu}\underline{\alpha};\nu} = -S^{\nu}_{\underline{\mu}\underline{\alpha},\nu} + \tfrac{1}{2}S^{\alpha}_{\underline{\sigma}\underline{\tau}}\Lambda^{\mu}_{\sigma\tau} - \tfrac{1}{2}S^{\mu}_{\underline{\sigma}\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau} - S^{\sigma}_{\underline{\mu}\underline{\alpha}}\Lambda^{\nu}_{\sigma\nu},$$

so consequently:

(33) $$2\underline{G}^{\mu\alpha} = -S^{\nu}_{\underline{\mu}\underline{\alpha};\nu} - S^{\sigma}_{\underline{\mu}\underline{\alpha}}\Delta^{\nu}_{\sigma\nu} + F^{\mu\alpha}.$$

We calculate the term $\Delta^{\nu}_{\sigma\nu}$ from its definition. One has:

$$\Lambda^{\tau}_{\sigma\tau} = \Delta^{\tau}_{\sigma\tau} - \Delta^{\tau}_{\tau\sigma}.$$

Now, in general, by definition:

$$\Delta^{\mu}_{\alpha\beta} = h_s{}^{\mu} h_{s\alpha,\beta}, \qquad \Lambda^{\tau}_{\tau\sigma} = \frac{1}{h}\frac{\partial h}{\partial x^{\sigma}} = \frac{\partial \log h}{\partial x^{\sigma}}.$$

On the other hand, we have set:

$$\Lambda^{\tau}_{\sigma\tau} = \varphi_{\sigma}$$

and

$$F_{\sigma} = \varphi_{\sigma} - \frac{\partial \log \psi}{\partial x^{\sigma}}.$$

Therefore:

$$\Delta^{\sigma}_{\sigma\tau} = \varphi_{\sigma} + \frac{\partial \log h}{\partial x^{\sigma}} = F_{\sigma} + \frac{\partial \log(\psi h)}{\partial x^{\sigma}}.$$

We substitute this in the previous equation, after multiplying by ψh:

$$\psi h\,(2\underline{G}^{\mu\alpha} - F^{\mu\alpha}) = \psi h\, S^{\sigma}_{\underline{\alpha}\underline{\mu};\sigma} - \psi h\, F_{\sigma} S^{\sigma}_{\underline{\mu}\underline{\alpha}} - \psi h \frac{\partial \log(\psi h)}{\partial x^{\sigma}} S^{\sigma}_{\underline{\mu}\underline{\alpha}}.$$

Upon moving the second term to the left-hand side, one has:

$$\psi h\,(2\underline{G}^{\mu\alpha} - F^{\mu\alpha} + S^{\sigma}_{\underline{\mu}\underline{\alpha}} F_{\sigma}) = \frac{\partial}{\partial x^{\sigma}}(\psi h S^{\sigma}_{\underline{\mu}\underline{\alpha}}).$$

Now, if one differentiates the right-hand side with respect to x^{α} then it vanishes, and we have the identities:

(34) $$\frac{\partial}{\partial x^{\alpha}}[\psi h\,(2\underline{G}^{\mu\alpha} - F^{\mu\alpha} + S^{\sigma}_{\underline{\mu}\underline{\alpha}} F_{\sigma})] \equiv 0.$$

Indeed, the right-hand side is written, upon changing the names of the dummy indices:

$$(\psi h S^{\sigma}_{\underline{\mu}\underline{\alpha}})_{,\sigma,\alpha} = (\psi h S^{\alpha}_{\underline{\mu}\underline{\sigma}})_{,\alpha,\sigma} = -(\psi h S^{\sigma}_{\underline{\mu}\underline{\alpha}})_{,\alpha,\sigma},$$

since:

$$S^{\alpha}_{\underline{\mu}\underline{\sigma}} = -S^{\sigma}_{\underline{\mu}\underline{\alpha}}.$$

There are three independent identities (34). If $A^{\mu\alpha}$ is an anti-symmetric tensor then:

$$A^{\mu\alpha} = -A^{\alpha\mu},$$

such that:

$$(A^{\mu\alpha})_{,\,\alpha} \equiv 0,$$

then one has:

$$(A^{\mu\alpha})_{,\,\alpha,\,\mu} = (A^{\alpha\mu})_{,\,\mu,\,\alpha} = -(A^{\mu\alpha})_{,\,\mu,\,\alpha} \equiv 0.$$

This is true for any $A^{\mu\alpha}$ provided that it is anti-symmetric. If we take $A^{\mu\alpha}$ to be the left-hand side of (34) then we have a relation that is independent of the values that the $G^{\mu\alpha}$ and F_σ take, which diminishes the number of independent identities by one. Finally, the number of these identities is 4 + 3 = 7, the number of equations is 20, and the number of unknowns is 17. One has:

$$20 - 7 = 17 - 4,$$

so the system is compatible.

13. – One can, moreover, seek to prove the compatibility of the proposed system of equations directly. In order to do this, suppose that *all* of the equations:

$$G^{\mu\alpha} = 0, \qquad F_\sigma = 0$$

are satisfied for an x^4 = constant = a section. Separate them into two groups. The first one contains 13 equations ([1]):

$$\begin{array}{llll} F_1 = 0, & F_2 = 0, & F_3 = 0, & F_4 = 0, \\ G^{11} = 0, & G^{12} = 0, & G^{13} = 0, & \\ G^{21} = 0, & G^{22} = 0, & G^{23} = 0, & \\ G^{31} = 0, & G^{32} = 0, & G^{33} = 0, & \end{array}$$

and the second group contains the other seven. One can easily prove the following proposition: *If all of the equations are satisfied in an $x^4 = a$ section then if the* 13 *equations of the first group are satisfied in all of four-dimensional space then the equations of the second group are also all satisfied automatically*.

Indeed, one has:

$$F_{\mu\alpha} = F_{\mu,\,\alpha} - F_{\alpha,\,\mu}\ .$$

Since F_μ is everywhere zero, the $F_{\mu\alpha}$ will also be so.

In the section $x^4 = a$, one has:

$$\frac{\partial G^{\mu 4}}{\partial x^4} = 0,$$

as the following identity shows:

$$\frac{\partial}{\partial x^\alpha}\left[h\psi(2\underline{G}^{\mu\alpha} - \mathcal{F}^{\mu\alpha} + S^{\sigma}_{\underline{\mu}\,\underline{\alpha}}F_\sigma)\right] \equiv 0. \tag{34}$$

([1]) The compatibility of these 13 equations is not in doubt.

Consider an infinitely close section $x^4 = a + da$. Since the $F_{\mu\alpha}$ and F_σ are everywhere zero, one deduces from the preceding identity that for $\alpha = 4$, the $G_{\underline{\mu\alpha}}$ will be likewise zero in that section. An analogous argument that uses the identity:

(24) $$G^{\mu\alpha}{}_{;\,\alpha} - F^{\mu\alpha}{}_{;\,\alpha} - \Lambda^{\sigma}_{\underline{\mu\tau}} F_{\sigma\tau} \equiv 0$$

shows us that the symmetric part of $G^{\mu\alpha}$ – viz., $\underline{\underline{G}}^{\mu\alpha}$ – is also annulled for $\alpha = 4$ in the infinitely close section $x^4 = a + da$. The conclusion is therefore valid for:

$$G^{\mu\alpha} = \underline{G}^{\mu\alpha} + \underline{\underline{G}}^{\mu\alpha}$$

in a section $x^4 = a + da$, and can be extended step-wise to all of space.

14. – We now examine the physical aspect of the theory – to the extent that it is possible. It is difficult to give a physical interpretation for the equations in full generality; one must limit oneself to a first approximation.

In order to do this, consider a space that differs from a Euclidian space infinitely little. Since the latter is characterized by having the $h_{s\nu}$ equal to $\delta_{s\nu} = \begin{cases} 1 \\ 0 \end{cases}$, ($x^4$ imaginary), this amounts to setting:

(35*a*) $$h_{s\nu} = \delta_{s\nu} + \bar{h}_{s\nu}\,.$$

One deduces that one must set:

(35*b*) $$h_s{}^{\nu} = \delta_{s\nu} - \bar{h}_{\nu s}\,.$$

We thus replace the $h_{s\nu}$ with that expression in the given equations and retain only the first approximation. One will have:

$$\begin{aligned} \Delta^{\mu}_{\alpha\beta} &= h_s{}^{\mu} h_{s\alpha,\beta} = \bar{h}_{\mu\alpha,\beta}\,, \\ \Lambda^{\mu}_{\alpha\beta} &= \bar{h}_{\mu\alpha,\beta} - \bar{h}_{\mu\beta,\alpha}\,. \end{aligned}$$

The field equations will then be:

(36) $$\bar{h}_{\alpha\mu,\nu,\nu} - \bar{h}_{\alpha\nu,\mu,\nu} = 0$$

(37) $$\bar{h}_{\alpha\mu,\nu,\alpha} - \bar{h}_{\alpha\nu,\mu,\alpha} = 0$$

or

$$\begin{cases} \bar{h}_{\alpha\mu,\nu,\nu} - \bar{h}_{\alpha\nu,\nu,\mu} = 0, \\ \bar{h}_{\alpha\mu,\alpha,\nu} - \bar{h}_{\alpha\nu,\alpha,\mu} = 0. \end{cases}$$

The second equation signifies simply that one can set:

(38) $$\bar{h}_{\alpha\mu,\alpha} = \frac{\partial\chi}{\partial x^{\mu}},$$

in such a way that the system reduces to:

(39) $$\bar{h}_{\alpha\mu,\nu,\nu} - \bar{h}_{\alpha\nu,\nu,\mu} = 0,$$

(40) $$\bar{h}_{\alpha\mu,\alpha} - \chi_{,\mu} = 0.$$

This form is not, moreover, very satisfactory, because, on first glance, it does not give sufficiently clear information about the field envisioned. In order to arrive at something more easily interpretable, recall that the coordinate system is arbitrary, up to a certain point, and subject it to an infinitesimal transformation:

(41) $$x^{\mu'} = x^{\mu} - \xi^{\mu},$$

where the ξ^{μ} are infinitely small or first order, which we choose conveniently in order to give the system a simple form.

Applying the infinitesimal transformation amounts to replacing the $h_{\mu\nu}$ with:

(42) $$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \xi^{\mu}_{,\nu}$$

(an equation that obeys the transformation rule for tensors).

One will then have:

$$\bar{h}'_{\alpha\nu,\nu} = \bar{h}_{\alpha\nu,\nu} + \xi^{\alpha}_{,\nu,\nu},$$
$$\bar{h}'_{\alpha\nu,\alpha} = \bar{h}_{\alpha\nu,\alpha} + \xi^{\alpha}_{,\nu,\alpha}.$$

Choose the ξ^{μ} in such a fashion that these two quantities are annulled in the new coordinate system. I say that it suffices to take:

(43) $$\xi^{\mu}_{,\nu,\nu} = -\bar{h}_{\alpha\nu,\nu},$$

(44) $$\xi^{\alpha}_{,\alpha} = -\chi.$$

Indeed, one first has:

$$\xi^{\alpha}_{,\nu,\alpha} = \xi^{\alpha}_{,\alpha,\nu} = -\chi_{,\nu} = -\bar{h}_{\alpha\nu,\alpha}.$$

The system (43), (44) is then compatible, even though it constitutes five equations for four unknowns; indeed, one has the identity relation:

$$(-\chi)_{,\nu,\nu} - (-\bar{h}_{\alpha\nu,\nu})_{,\alpha} \equiv 0.$$

Therefore the solution of this system gives us quantities ξ^{μ} such that one has:

(45) $$\bar{h}'_{\alpha\nu,\nu} = 0,$$

(46) $$\bar{h}'_{\alpha\nu,\alpha} = 0.$$

Now make a change of coordinates. Our equations become (upon suppressing the primes):

(47) $$\begin{cases} \bar{h}_{\alpha\mu,\nu,\nu} = 0, \\ \bar{h}_{\alpha\mu,\alpha} = 0, \\ \bar{h}_{\alpha\mu,\mu} = 0. \end{cases}$$

If we decompose the $\bar{h}_{\alpha\mu}$ into a symmetric part $S_{\alpha\mu}$ and an anti-symmetric part $A_{\alpha\mu}$ then the system decomposes into two other ones that contain only symmetric or anti-symmetric terms, respectively:

(48) $$\begin{cases} S_{\alpha\mu,\nu,\nu} = 0, \qquad A_{\alpha\mu,\mu,\nu} = 0, \\ \qquad\qquad \text{and} \\ S_{\alpha\mu,\mu} = 0, \qquad A_{\alpha\mu,\mu} = 0. \end{cases}$$

We have thus arrived at two groups of equations. *The symmetric group gives the laws of the gravitational field that are compatible with the* NEWTON-POISSON *law;* however, the result is not completely identical to the one that is given by the theory that is based upon RIEMANNian geometry. *The anti-symmetric group gives the* MAXWELL *equation* in a more general form. I basically believe that the anti-symmetric system must be interpreted as giving the general equations of the electromagnetic field (in the first approximation).

In this case, there thus exists a very neat separation between the laws of electromagnetism, on the one hand, and those of gravitation, on the other. However, this separation is valid *only in the first approximation;* it does not exist in the general case: The theory is ruled by a single law.

In the present state of the theory, one cannot meanwhile judge whether *the interpretation* of the quantities that represent the field is correct or not. In effect, a field is defined, in the first place, by the motivating actions that it exerts on particles, and one does not presently known the law of these actions; the discovery of this law demands the integration of the field equations, which has not yet been realized.

15. – To conclude, we can say, upon condensing the results that have presented up to now:

The particular structure of space that we have taken as the fundamental hypothesis led us to certain general field equations that reduced in the first approximation to the well-known equations of gravitation and electromagnetism. Despite this, the results obtained up to the present do not give us the possibility of verifying the theoretical predictions experimentally. Indeed, one has not, moreover, been able to deduce the laws of the structure of particles and their motions in the field by starting with the given equations and integrating them. The hurdle that the theory must overcome will then be the discovery of integrals – devoid of singularities – that satisfy the differential equations of the field and are capable of providing a correct solution to the problem of particles and their motion. It is only after this has been done that the comparison with experiment will become possible.

(Conference talk that was given at l'Institut H. POINCARÉ in November 1929 and edited by AL. PROCA.)

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The compatibility of the field equations in unified field theory

By A. EINSTEIN

(Received on 12 December 1929 [cf. Jahrg. 1929, pp. 683].)

Several months ago, I presented the mathematical foundations of a unified field theory in a survey article that appeared in the Mathematischen Annalen. In this report, I would like to briefly summarize its essentials and simultaneously show the points at which my previously-appearing papers (these Berichte, "Zur einheitlichen Feldtheorie," 1929, I and "Einheitliche Feldtheorie und HAMILTONsches Prinzip," 1929, X) can be improved. The proof of compatibility is based upon a brief communication by CARTAN, for which I am grateful [cf., § **3**, (16)], and which is somewhat simpler than the one that was given in the Mathematischen Annalen.

§ 1. Critique of my earlier papers.

The divergence operation on a tensor density that was introduced in § 1 of the first-mentioned paper is not preferable. It is better to remain with the divergence operator that is defined as the contraction of the expansion of a tensor. The divergence of the fundamental tensor then vanishes identically by the latter definition.

The identity (3*a*) [(3*b*), resp.] of *loc. cit.* then assumes the form:

$$\Lambda^{\alpha}_{\kappa l;\alpha} - (\phi_{\kappa,\,l} - \phi_{l,\,\kappa}) \equiv 0, \tag{1}$$

in which we have set:

$$\phi_{\kappa} = \Lambda^{\sigma}_{\kappa\sigma}\,. \tag{1a}$$

As we have already explained, the proof of compatibility for the field equations that was given in that paper rests upon the incorrect assumption that four identities exist between equations (10) that were given in it.

The second of the papers mentioned contains a fatal error. It is likewise incorrect that the $G^{*\mu\alpha}$ depend upon the $\underline{S^{\alpha}_{\mu\nu}}$ homogeneously and quadratically. Hence, the derivation that was given in that article of equation (21), which is interpreted as the electromagnetic field equation, fails.

§ Overview of the mathematical apparatus of the theory.

The structure of space (the field, resp.) is described by the GAUSSIAN components $h_s{}^{\nu}$ of local orthogonal 4-beins (viz., the ν^{th} component of the s^{th} bein). The transformation law for a change of GAUSSIAN coordinate system, with a simultaneous rotation of all local 4-beins, is:

$$h_s{}^{\nu'} = \alpha_{st} \frac{\partial x^{\nu'}}{\partial x^{\sigma}} h_t{}^{\sigma}, \tag{2}$$

in which the constants α_{st} define an orthogonal system.

The normalized sub-determinant $h_{s\nu}$ of the $h_s{}^{\nu}$ obeys the transformation law:

$$h'_{s\nu} = \alpha_{st} \frac{\partial x^{\sigma}}{\partial x^{\nu'}} h_{t\sigma}. \tag{3}$$

Systems of quantities whose transformation properties differ from those of the h only by the number of their indices are called *tensors*. The quantities $(h_{s\nu})$ [$(h_s{}^{\nu})$, resp.] comprise the *fundamental tensor*.

Addition, subtraction, and multiplication are defined as they are in the usual theory of tensors. Contraction relative to two local (Latin) or coordinate (Greek) indices of differing character is possible.

Changing the index character as a tensor by means of the fundamental tensor is always possible by multiplication and contraction; e.g.:

$$A_s = h_{s\nu} A^{\nu}.$$

If $A_s\ A_s$ is to be the magnitude of the vector (As) then it follows that the $g_{\mu\nu}$ coefficients of the RIEMANN metric must be given by the quadratic construction:

$$g_{\mu\nu} = h_{s\mu}\, h_{s\nu}\,. \tag{4}$$

The elementary (i.e., integrable) law of parallel translation:

$$\left.\begin{aligned} \delta A^{\mu} &= -\Delta^{\mu}_{\alpha\beta} A^{\alpha} \delta x^{\beta}, \\ \Delta^{\mu}_{\alpha\beta} &= h_s^{\alpha} h_{s\alpha,\beta}, \end{aligned}\right\} \tag{5}$$

follows from the assumption of the parallelism of the local 4-bein, in which the comma means ordinary differentiation. The laws of (absolute) differentiation follow from this:

$$A^{\mu}{}_{;\,\sigma} = A^{\mu}{}_{,\,\sigma} + A^{\alpha} \Delta^{\mu}_{\alpha\sigma}, \tag{6}$$

$$A_{\mu;\,\sigma} = A_{\mu,\,\sigma} - A_{\alpha} \Delta^{\alpha}_{\mu\sigma}. \tag{7}$$

For tensors with more Greek and Latin indices, a corresponding term for each Greek index appears.

The tensor character of:

$$\Lambda^{\alpha}_{\mu\nu} = \Delta^{\alpha}_{\mu\nu} - \Delta^{\alpha}_{\nu\mu} \tag{8}$$

follows easily from a double differentiation of the tensor $\Phi_{;\,\sigma;\,\tau}$ that is constructed from a scalar Φ [from the tensor character of ($\Phi_{;\,\sigma;\,\tau} - \Phi_{;\,\tau;\,\sigma}$), resp.]. The vanishing of all $\Lambda^{\alpha}_{\mu\nu}$ is the condition for the continuum to be Euclidian.

Due to its expressibility in terms of the h-quantities (due to the integrability of the Δ-parallel translation, resp.), the tensor (Λ) satisfies the identity:

$$(\Lambda^{\iota}_{\kappa\lambda;\mu} + \Lambda^{\iota}_{\lambda\mu;\kappa} + \Lambda^{\iota}_{\mu\kappa;\lambda}) + (\Lambda^{\iota}_{\kappa\alpha}\Lambda^{\alpha}_{\lambda\mu} + \Lambda^{\iota}_{\lambda\alpha}\Lambda^{\alpha}_{\mu\kappa} + \Lambda^{\iota}_{\mu\alpha}\Lambda^{\alpha}_{\kappa\lambda}), \tag{9}$$

from which identity (1) follows by contraction.

The product rule is valid for absolute differentiation. The absolute differential quotients of the h, as well as the $g_{\mu\nu}$ ($g^{\mu\nu}$, resp.) vanish identically. The fundamental tensor also commutes with the differentiation sign (;) as a factor.

As for the second absolute derivative of an arbitrary tensor $T^{\cdots}_{\cdots}$ (the ellipses mean arbitrary indices), we have the following commutation law for differentiation:

$$T^{\cdots}_{\cdots;\sigma;\tau} - T^{\cdots}_{\cdots;\tau;\sigma} \equiv - T^{\cdots}_{\cdots;\alpha}\Lambda^{\alpha}_{\sigma\tau}\,. \tag{10}$$

The proof follows directly when $T^{\cdots}_{\cdots}$ has no Greek indices (i.e., it has a scalar character). The proof for arbitrary tensors is obtained by multiplying them with parallel vectors (i.e., vectors that have absolute derivatives that vanish everywhere) in such a way as to impart a scalar character upon them.

If the tensor $T^{\cdots}_{\cdots}$ in question has two contravariant indices then one can contract relative to them and σ (τ, resp.); one obtains a commutation theorem for the divergence from (10).

The special character of the four-dimensional continuum of physics is established by defining the coordinate x^4 to be pure imaginary (also the fourth local coordinate), while the remaining ones are real. Tensor components are pure imaginary when the have an odd number of indices; otherwise, they are real.

Finally, we make a formal convention: Changing the location of a Greek index (i.e., "raising" or "lowering," resp.) shall also be expressed by underlining the index in question.

§ 3. The field equations and their compatibility.

The field equations must naturally be covariant. One must also assume that they are of second order and linear in the coordinates of the twice-differentiated field variables. Whereas in the previous general theory of relativity these requirements sufficed, at least, for the determination of the field equations, in the present theory this is not the case. Due to the tensor character of Λ, one likewise has a much larger variety of tensors than the ones that one finds in the context of the RIEMANN schema.

General covariance brings with it the fact that four of the field variables must remain arbitrary. Thus, the sixteen quantities h can be subject to only twelve independent conditions. Hence, if the number N of field equations is larger than twelve then at least N – 12 identities must exist between them.

A simple possibility for the statement of a covariant system of only twelve equations does not present itself. We must therefore state equations, between which identity relations must exist. The larger the number of equations (and, as a result, the identities that exist between them), the more definite the statements that come out of the theory will be, beyond the requirement of mere determinism; hence, the more valid the theory will be in the event that it is consistent with the facts of experience ([1]). The requirement of the existence of an "over-determined" system of equations with the required number of identities gives us the means to find the field equations.

As field equations, I propose the two systems of equations:

$$G^{\mu\alpha} = \Lambda^{\alpha}_{\underline{\mu\nu};\nu} - \Lambda^{\sigma}_{\underline{\mu\tau}}\Lambda^{\alpha}_{\sigma\tau} = 0, \tag{11}$$

$$F_{\mu\alpha} = \Lambda^{\alpha}_{\mu\alpha;\sigma} = 0; \tag{12}$$

these are 16+6 equations for the 16 field variables $h_{s\nu}$. I came upon them by using the fact that:

$$\Lambda^{\alpha}_{\underline{\mu\nu};\nu;\alpha} - \Lambda^{\alpha}_{\underline{\mu\nu};\alpha;\nu} \equiv - \Lambda^{\alpha}_{\underline{\mu\nu};\sigma}\Lambda^{\sigma}_{\nu\alpha}.$$

With regard to this, one can bring the identity into the form:

$$G^{\mu\alpha}{}_{;\,\alpha} - F^{\mu\alpha}{}_{;\,\alpha} + \Lambda^{\sigma}_{\underline{\mu\alpha}}F_{\sigma\tau} \equiv 0 \tag{13}$$

by a suitable naming of the summation indices. These are four identity relations between equations (11) and (12), which gave rise to their being written down.

Equations (12), when combined with the identity (1), lead immediately to the identity:

$$F_{\mu\nu;\,\rho} + F_{\nu\rho;\,\mu} + F_{\rho\mu;\,\nu} \equiv 0. \tag{14}$$

We remark that equations (12) can also be replaced with:

$$F_{\mu\nu} = \phi_{\mu,\,\alpha} - \phi_{\alpha,\,\mu} = 0 \tag{12a}$$

or

$$F_{\mu} = \phi_{\mu} - \frac{\partial\psi}{\partial x^{\mu}} = 0, \tag{12b}$$

([1]) In the earlier theory of gravitation, there were – e.g. – ten equations for the ten field variables, with four identities existing between them.

in which ψ is a scalar. We further have that $F_{\mu\nu}$ can be expressed in terms of F_μ by means of the relation:

$$F_{\mu\nu} \equiv F_{\mu,\,\nu} - F_{\nu,\,\mu}\,. \tag{15}$$

We obtain a third system of identities upon forming $G^{\mu\alpha}{}_{;\,\mu}$. We first find that (11) yields:

$$G^{\mu\alpha}{}_{;\,\mu} \equiv \Lambda^{\alpha}_{\underline{\mu\nu};\nu;\mu} - \Lambda^{\tau}_{\underline{\sigma\mu};\mu}\Lambda^{\alpha}_{\sigma\tau} - \Lambda^{\sigma}_{\underline{\mu\tau}}\Lambda^{\alpha}_{\sigma\tau;\mu}\,.$$

If one uses the commutation relations for the divergence of $\Lambda^{\alpha}_{\underline{\mu\nu}}$ with respect to the indices ν and μ then one will obtain:

$$\Lambda^{\alpha}_{\underline{\mu\nu};\nu;\mu} \equiv -\tfrac{1}{2}\Lambda^{\alpha}_{\underline{\mu\nu};\sigma}\Lambda^{\sigma}_{\nu\mu}\,.$$

If one replaces the first term of the right-hand side of the identity above by means of this relation then one can replace the first and third term collectively with:

$$-\Lambda^{\sigma}_{\underline{\mu\tau}}(\Lambda^{\alpha}_{\underline{\sigma\tau};\mu} + \tfrac{1}{2}\Lambda^{\alpha}_{\tau\mu;\sigma})$$

or with

$$-\tfrac{1}{2}\Lambda^{\sigma}_{\underline{\mu\tau}}(\Lambda^{\alpha}_{\sigma\tau;\mu} + \Lambda^{\alpha}_{\tau\mu;\sigma} + \Lambda^{\alpha}_{\mu\sigma;\tau})\,.$$

However, in light of (9), the bracketed term itself can be expressed in terms of the Λ, such that one gets:

$$\tfrac{1}{2}\Lambda^{\sigma}_{\underline{\mu\tau}}(\Lambda^{\alpha}_{\sigma\lambda}\Lambda^{\lambda}_{\tau\mu} + \Lambda^{\alpha}_{\tau\lambda}\Lambda^{\lambda}_{\mu\sigma} + \Lambda^{\alpha}_{\mu\lambda}\Lambda^{\lambda}_{\sigma\tau})\,,$$

or, since the first term in the bracket goes away and the other two can be combined:

$$\Lambda^{\sigma}_{\underline{\mu\tau}}\Lambda^{\lambda}_{\sigma\tau}\Lambda^{\alpha}_{\mu\lambda}\,.$$

We thus get:

$$G^{\mu\alpha}{}_{;\,\mu} \equiv -\Lambda^{\alpha}_{\sigma\tau}(\Lambda^{\tau}_{\underline{\sigma\mu};\mu} - \Lambda^{\rho}_{\underline{\sigma\lambda}}\Lambda^{\tau}_{\rho\lambda})\,,$$

or finally:

$$G^{\mu\alpha}{}_{;\,\mu} + \Lambda^{\alpha}_{\sigma\tau}G^{\sigma\tau} \equiv 0. \tag{16}$$

(13), (14), and (16) are the identities that exist between the field equations (11), (12).

The fact that these identities actually imply the compatibility of equations (11), (12) is clear from the following argument: It might be possible for equations (11), (12) to both be satisfied for a slice $x^4 = a$. Likewise, it might be possible for those twelve equations that are characterized by setting the following quantities to zero:

$$\begin{matrix} G^{11} & G^{12} & G^{13} \\ G^{21} & G^{22} & G^{23} \\ G^{31} & G^{32} & G^{33} \\ F_{14} & F_{24} & F_{34} \end{matrix}$$

to be satisfied in all of space.

Furthermore, one might choose the latter solution in such a way that it is a continuous extension of the solution for the slice $x^4 = a$. We then assert that this solution also everywhere satisfies the equations that are characterized by setting the following quantities to zero:

$$G^{14}, G^{24}, G^{34}, G^{41}, G^{42}, G^{43}, F_{23}, F_{31}, F_{13}\,.$$

It then follows from this that F_{14}, F_{24}, F_{34} must vanish everywhere, and as a result of (14), that $\partial F_{23} / \partial x^4$, $\partial F_{31} / \partial x^4$, $\partial F_{12} / \partial x^4$ must vanish everywhere. However, since F_{23}, F_{31}, F_{12} vanish on the slice $x^4 = a$ they vanish everywhere. Furthermore, it follows from (13) and (16) that the derivatives of G^{14}, G^{41}, …, G^{44} with respect to x^4 must all vanish on the slice $x^4 = a$. These quantities, and therefore all $G^{\mu\alpha}$, then vanish in the infinitesimally-neighboring slice $x^4 = a + da$. By repeating this argument, it finally follows that all of the $G^{\mu\alpha}$ must vanish everywhere. Hence, the proof of the compatibility of the field equations (11), (12) is complete.

First approximation. We shall examine fields that differ from the special case of Euclidian ones only by an infinitely small amount:

$$h_{s\nu} = \delta_{s\nu} + \bar{h}_{s\nu}\,. \tag{17}$$

$\delta_{s\nu}$ equals 1 (0, resp.) whenever $s = \nu$ ($s \neq \nu$, resp.), while the $\bar{h}_{s\nu}$ are infinitely small compared to 1. If one neglects terms that are quadratic in the h (i.e., second-order terms) then one can replace the field equations with:

$$\bar{h}_{\alpha\mu,\nu,\nu} - \bar{h}_{\alpha\nu,\nu,\mu} = 0, \tag{11a}$$

$$\bar{h}_{\alpha\mu,\alpha,\nu} - \bar{h}_{\alpha\nu,\alpha,\mu} = 0. \tag{12a}$$

The Ansatz (17) allows one to make an infinitesimal transformation to GAUSSIAN coordinates. It can now be shown that because of equations (12*a*), a choice of coordinates is possible such that:

$$\bar{h}_{\mu\alpha,\alpha} = \bar{h}_{\alpha\mu,\alpha} = 0 \tag{18}$$

is satisfied, so the only field equation that remains is:

$$\bar{h}_{\alpha\mu,\nu,\nu} = 0. \tag{11b}$$

If one denotes twice the symmetric part of $\bar{h}_{\alpha\nu}$ by $\bar{g}_{\alpha\mu}$ and twice the anti-symmetric part by $a_{\alpha\mu}$ then the field equations in the two systems split into:

$$\left.\begin{aligned} \bar{g}_{\alpha\mu,\nu,\nu} &= 0, \\ \bar{g}_{\alpha\mu,\mu} &= 0, \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} a_{\alpha\mu,\nu,\nu} &= 0, \\ a_{\alpha\mu,\mu} &= 0. \end{aligned}\right\} \tag{20}$$

In my opinion, equations (19) express the laws of the gravitational field, while (20) express those of the electromagnetic field, in which the $a_{\alpha\mu}$ play the role of electromagnetic fields. For more rigorous considerations, a splitting of the field into a gravitational field and an electromagnetic field is not possible. One can find the details of this in my paper in Mathematischen Annalen.

The most important question that is connected with the (rigorous) field equations is that of the existence of solutions that are free of singularities that could represent electrons and photons.

Presented on 6 February

Two rigorous static solutions to the field equations of unified field theory

By A. EINSTEIN and W. MAYER

Two special cases shall be treated in what follows:

a) The spatially centrally-symmetric (i.e., rotationally symmetric) case, in which there is likewise mirror symmetry.

When regarded physically, this will be treated as the external field of an electrically-charged ball of non-vanishing mass.

b) The static solution that corresponds to an arbitrary number of electrically-uncharged mass points.

Remark. The development in § 1 up to equation (27) involves only the rigorous mathematical proof that the $h_s{}^\alpha$ can take the form that is given by (27) for suitable choice of coordinates in the case of central symmetry and spatial mirror symmetry.

§ 1. The spatially centrally-symmetric case.

We shall look for the most general three-dimensional continuum:

$$x_1, x_2, x_3, h_s{}^\alpha(x_1, x_2, x_3) \qquad s, \alpha = 1, 2, 3$$

that has the property of rotational symmetry; i.e., that it possesses invariance under the group:

$$\bar{x}_\alpha = a_{\alpha\beta}\, x_\beta \qquad \alpha, \beta = 1, 2, 3, \tag{1}$$

where $\| a_{\alpha\beta} \|$ is an orthogonal matrix.

The point $P(x_1, x_2, x_3)$ is transformed into the point $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ under (1), and the normalized dreibein $h_s{}^\alpha(x)$ at the point P is transformed into the dreibein:

$$\bar{h}_s{}^\alpha(\bar{x}) = a_{\alpha\beta}\, h_s{}^\alpha(\bar{x}) \qquad s, \alpha, \beta = 1, 2, 3 \tag{2}$$

at the point $\bar{P}$.

In order for there to be rotational symmetry, it is necessary and sufficient that there exist a "local rotation" (i.e., a rotation of the local 3-bein) that is the same for all points of R_3, under which the dreibein $\bar{h}_s^{\ \alpha}(\bar{x})$ comes from the original dreibein $h_s^{\ \alpha}(\bar{x})$ by way of:

(3) $$\bar{h}_s^{\ \alpha}(\bar{x}) = A_{st}\ h_s^{\ \alpha}(\bar{x}) \qquad s, t, \alpha = 1, 2, 3.$$

R_3 is asymptotically Euclidian at infinity; i.e., the $\bar{h}_s^{\ \alpha}(x)$ converge to $\delta_{s\alpha}$ when the x_1, x_2, x_3 (that is, at least one of these coordinates) go to infinity. We write this briefly as: $h_s^{\ \alpha}(\infty) = \delta_{s\alpha}$.

It follows from (2) that $\bar{h}_s^{\ \alpha}(\bar{x}) = a_{s\alpha}$ and from (3) that $a_{s\alpha} = A_{s\alpha}$ at infinity. In place of (3), we would then have:

(3′) $$\bar{h}_s^{\ \alpha}(\bar{x}) = a_{st}\ h_s^{\ \alpha}(\bar{x}) \qquad s, t, \alpha = 1, 2, 3,$$

which compares with (2):

(4) $$a_{\alpha\beta}\, h_{s\alpha}(x_1, x_2, x_3) = a_{st}\, h_t^{\ \alpha}(a_{1j}\, x_j,\ a_{2j}\, x_j,\ a_{3j}\, x_j) \qquad \alpha, \beta, s, t = 1, 2, 3$$

as the functional equation that yields the desired bein-components. The relations (4) are identities in the quantities x_1, x_2, x_3, $a_{\alpha\beta}$, as long as the matrix $\| a_{\alpha\beta} \|$ is orthogonal.

We now direct our attention to the point $P(x_1, x_2, x_3)$ and choose the $a_{\alpha\beta}$ to be the dreibein:

(5) $$a_{\alpha\beta} = {}_{(\alpha)}\xi_\beta \qquad \alpha, \beta = 1, 2, 3,$$

which is normal Euclidian, since $a_{\alpha\beta}\, a_{\alpha\gamma} = {}_{(\alpha)}\xi_\beta\ {}_{(\alpha)}\xi_\gamma$, and in which we have set:

(5′) $${}_{(1)}\xi_\alpha = \frac{x_\alpha}{s} \qquad (s^2 = x_\alpha\, x_\alpha) \qquad \alpha = 1, 2, 3.$$

For this choice of the matrix $\| a_{\alpha\beta} \|$, (4) yields:

(6) $${}_{(\alpha)}\xi_\beta\, h_s^{\ \beta}(x_1, x_2, x_3) = {}_{(t)}\xi_s\, h_t^{\ \alpha}(s, 0, 0) \qquad s, t, \alpha = 1, 2, 3.$$

We move the ${}_{(\alpha)}\xi_\beta$ to the other side of (6) and obtain:

(7) $$\begin{aligned} h_s^{\ \gamma}(x_1, x_2, x_3) &= {}_{(t)}\xi_s\ {}_{(\alpha)}\xi_\gamma\, h_t^{\ \alpha}(s) = \\ &= {}_{(1)}\xi_s\ {}_{(1)}\xi_\gamma\, h_1^{\ 1}(s) + {}_{(1)}\xi_s\ {}_{(\alpha)}\xi_\gamma\, h_1^{\ \alpha}(s) + {}_{(1)}\xi_s\ {}_{(t)}\xi_\gamma\, h_t^{\ 1}(s) + {}_{(t)}\xi_s\ {}_{(\alpha)}\xi_\gamma\, h_t^{\ \alpha}(s). \end{aligned}$$

We now use the indeterminacy in fixing the vectors ${}_{(2)}\xi_\alpha$, ${}_{(3)}\xi_\alpha$, which, together with ${}_{(1)}\xi_\alpha$, must form a normal Euclidian dreibein.

If we introduce the zweibeins ${}_{(2)}\xi_\alpha$, ${}_{(3)}\xi_\alpha$ in (8), after they have been rotated by way of:

$$(9) \qquad \begin{cases} {}_{(2)}\xi_\alpha = \cos\phi\ {}_{(2)}\eta_\alpha + \sin\phi\ {}_{(3)}\eta_\alpha, \\ {}_{(3)}\xi_\alpha = -\sin\phi\ {}_{(2)}\eta_\alpha + \cos\phi\ {}_{(3)}\eta_\alpha, \end{cases}$$

then, in place of the chosen zweibeins ${}_{(2)}\xi_\alpha$, ${}_{(3)}\xi_\alpha$, we obtain a new representation of the dreibein $h_s^{\gamma}(x_1, x_2, x_3)$, into which the arbitrary angle $\angle\phi$ enters. This representation has the form:

$$(10) \qquad h_s^{\gamma}(x_1, x_2, x_3) = P_{(s\gamma)} + Q_{(s\gamma)} \sin\phi + R_{(s\gamma)} \cos\phi + S_{(s\gamma)} \cos^2\phi + T_{(s\gamma)} \sin\phi \cos\phi.$$

Since (10) is valid for an arbitrary ϕ, it then follows that:

$$(11) \qquad h_s^{\gamma}(x_1, x_2, x_3) = P_{(s\gamma)}\,, \quad Q_{(s\gamma)} = R_{(s\gamma)} = S_{(s\gamma)} = T_{(s\gamma)} = 0.$$

If one carries out these simple computations then one gets $Q_{(s\gamma)} = 0$, $R_{(s\gamma)} = 0$:

$$(12) \qquad h_2^{\ 1}(s) = h_3^{\ 1}(s) = h_1^{\ 2}(s) = h_1^{\ 3}(s) = 0.$$

From $S_{(s\gamma)} = 0$, $T_{(s\gamma)} = 0$, it follows, moreover, that:

$$(13) \qquad h_2^{\ 2}(s) = h_3^{\ 3}(s), \qquad h_2^{\ 3}(s) = -h_3^{\ 2}(s).$$

Due to (12) and (13), (8) becomes:

$$(14) \qquad h_s^{\gamma}(x_1, x_2, x_3) = \frac{x_s x_\gamma}{s^2} h_1^1(s) + h_2^2(s)[{}_{(2)}\xi_s\ {}_{(2)}\xi_\gamma + {}_{(3)}\xi_s\ {}_{(3)}\xi_\gamma] + h_2^3(s)[{}_{(2)}\xi_s\ {}_{(3)}\xi_\gamma - {}_{(3)}\xi_s\ {}_{(2)}\xi_\gamma].$$

Now, ${}_{(2)}\xi_s\ {}_{(2)}\xi_\gamma + {}_{(3)}\xi_s\ {}_{(3)}\xi_\gamma = \delta_{s\gamma} - {}_{(1)}\xi_s\ {}_{(1)}\xi_\gamma$ is independent of the special choice of normalized zweibein ${}_{(2)}\xi_s$, ${}_{(3)}\xi_s$. On the other hand, by permuting the vectors ${}_{(2)}\xi_s$, ${}_{(3)}\xi_s$, the quantity ${}_{(2)}\xi_s\ {}_{(3)}\xi_\gamma - {}_{(3)}\xi_s\ {}_{(2)}\xi_\gamma$ changes sign.

However, if we allow transformations (1) for which the matrix $\| a_{ik} \|$ has a determinant of plus one then ${}_{(2)}\xi_s\ {}_{(3)}\xi_\gamma - {}_{(3)}\xi_s\ {}_{(2)}\xi_\gamma$ is also independent of the special choice of zweibein. (We must therefore have that $|\ {}_{(\alpha)}\xi_\gamma\ | = 1$, $\alpha, \beta = 1, 2, 3$, in we are given which of the two vectors ${}_{(2)}\xi_s$, ${}_{(3)}\xi_s$ is to be regarded as the second and third ones.) We call such transformations *proper rotations*.

If we introduce the alternating tensor $\varepsilon_{\alpha\beta\gamma}$ with $\varepsilon_{123} = 1$ then we have ${}_{(2)}\xi_s\ {}_{(3)}\xi_\gamma - {}_{(3)}\xi_s\ {}_{(2)}\xi_\gamma = \varepsilon_{s\beta\gamma}\ {}_{(1)}\xi_\tau$, and instead of (14), we can write:

$$(15) \qquad h_s^{\gamma}(x_1, x_2, x_3) = x_s\, x_\gamma\, \Lambda(s) + \delta_{s\gamma} B(s) + \varepsilon_{s\gamma\tau}\, x_\tau\, C(s),$$

in which:

$$A(s) = -\frac{1}{s^2}(h_1^1(s) - h_2^2(s))\,, \quad B(s) = h_2^2(s)\,, \qquad C(s) = h_2^3(s) \cdot \frac{1}{s} \qquad (15')$$

are arbitrary functions of s that only need to correspond to the condition that $h_s^{\gamma}(\infty) = \delta_{s\alpha}$.

This necessary form (15) for the bein-components is, as simple computation will show, also sufficient for R_3 to be rotationally symmetric.

Indeed, as long as the improper rotations (1) ($| a_{\alpha\beta} | = -1$, i.e., "reflections") are also to be admitted we must set $C(s) = 0$.

In what follows, we shall be occupied with this case alone, which is why (15), with $C(s) = 0$, then represents the most general form of the bein-components.

We extend our continuum x_1, x_2, x_3, $h_s^{\alpha}(x_1, x_2, x_3)$ to a four-dimensional one and associate a vierbein $h_s^{\alpha}(x_1, x_2, x_3, x_4)$, s, $\alpha = 1, \ldots, 4$ to the point x_1, x_2, x_3, x_4, in such a way that:

(16) $$h_s^{\alpha}(x_1, x_2, x_3, x_4) = h_s^{\alpha}(x_1, x_2, x_3) \qquad s,\ \alpha = 1, 2, 3,$$

and the remaining vector components, which depend upon only x_1, x_2, x_3, are determined such that R_4 is invariant under the group:

(17) $$\bar{x}_{\alpha} = a_{\alpha\beta}\, x_{\beta} \qquad \alpha, \beta = 1, 2, 3, \qquad \bar{x}_4 = x_4 .$$

Thus, this R_4 has a pseudo-RIEMANNIAN structure – i.e., the metric tensor $g^{\alpha\beta}$ can be represented in terms of the normalized vierbein $h_s^{\alpha}(x_1, \ldots, x_4)$ ([1]) as:

(18) $$g^{\alpha\beta} = h_1^{\alpha} h_1^{\beta} + h_2^{\alpha} h_2^{\beta} + h_3^{\alpha} h_3^{\beta} - h_4^{\alpha} h_4^{\beta} \qquad \alpha, \beta = 1, \ldots, 4.$$

We again have that $h_s^{\alpha}(\infty) = \delta_{s\alpha}$ at infinity.

The transformation (17) takes the vierbein $h_s^{\alpha}(x)$ at the point $P(x_1, x_2, x_3, x_4)$ to the vierbein:

(19) $$\bar{h}_s^{\alpha}(\bar{x}) = a_{\alpha\beta}\, h_s^{\beta}(x) \qquad \alpha, \beta = 1, 2, 3, \quad \bar{h}_s^{4}(\bar{x}) = h_s^{4}(x) \qquad s = 1, \ldots, 4,$$

and now a local rotation shall be given such that:

(20) $$\bar{h}_s^{\alpha}(\bar{x}) = B_{st}\, \bar{h}_t^{\alpha}(\bar{x}) \qquad s, t, \alpha = 1, 2, 3, 4,$$

in which B_{st} are constant quantities.

From the behavior at infinity, it follows from (19) that $\bar{h}_s^{\alpha}(\infty) = a_{s\alpha}$, α, $s = 1, 2, 3$, $\bar{h}_4^{\alpha}(\infty) = 0$, $\alpha = 1, 2, 3$, and furthermore $\bar{h}_s^{4}(\infty) = h_s^{4}(\infty) = \delta_{s4}$. Inserting this into (20) gives:

$$a_{s\alpha} = B_{s\alpha}, \qquad s, \alpha = 1, 2, 3, \quad B_{4\alpha} = 0, \qquad \alpha = 1, 2, 3 \qquad \delta_{s4} = B_{s4} .$$

Due to the choice of vierbein (16), relations (19) and (20) are satisfied, except for:

(21) $$\bar{h}_s^{4}(\bar{x}) = h_s^{4}(x) = a_{ts}\, h_t^{4}(\bar{x}), \qquad s, t = 1, 2, 3,$$

(21′) $$\bar{h}_4^{4}(\bar{x}) = h_4^{4}(x) = h_4^{4}(\bar{x}),$$

([1]) Here, we shall do without the introduction of imaginaries in order to produce a definite metric tensor.

(21″) $$\bar{h}_s^{\ \alpha}(\bar{x}) = a_{\alpha\beta}\, h_4^{\beta}(x) = h_4^{\alpha}(\bar{x}) \qquad \alpha = 1, 2, 3,$$

which are the functional equations for the remaining bein-components.

One addresses these equations by the method that was employed for (6), and obtains:

(22) $$h_s^{\ 4}(x_1, x_2, x_3) = D(s)\, x_s \qquad s = 1, 2, 3,$$
(22′) $$h_4^{\ \alpha}(x_1, x_2, x_3) = E(s)\, x_\alpha \qquad \alpha = 1, 2, 3,$$
(22″) $$h_4^{\ 4}(x_1, x_2, x_3) = F(s).$$

Since we should have $h_s^{\ \alpha}(\infty) = \delta_{s\alpha}$ at infinity, we have the following developments for the functions that appear in (15) and (22) for $s = \infty$:

(23) $$\begin{cases} A(s) = \dfrac{K}{s^\alpha}(1+(.)), \ \alpha > 2, \quad B = 1+(.), \quad F = 1+(.), \\ \qquad C, D, E = \dfrac{K}{s^b}(1+(.)), \quad b > 1, \end{cases}$$

where the (.) brackets contain the factor $1/s$.

In the coordinate system:

(24) $$x_i = \phi(s)\, x_i \qquad i = 1, 2, 3, \qquad \bar{x}_4 = x_4 ,$$

in which the dreibein x_1, x_2, x_3, $h_s^{\ \alpha}(x_1, x_2, x_3)$, $\alpha = 1, 2, 3$ likewise exhibits rotational symmetry, one justifies the vanishing of the terms in (15) that correspond to the function A by a corresponding choice of the function ϕ:

(25) $$\phi = e^{-\int \frac{As\, ds}{B+As^2}} .$$

Since ϕ tends towards a finite value at infinity, we then have – as simple computation will show – the validity of condition (23) for the new values $\bar{B}(s)$, $\bar{F}(s)$, $\bar{C}(s)$, $\bar{D}(s)$, $\bar{E}(s)$.

One again justifies the vanishing of the function $D(s)$ that appears in (22) by the further coordinate change:

(26) $$\bar{x}_4 = x_4 + \psi(s),$$

whereby (23) are valid in the new coordinate system, as before.

Thus, with no loss of generality, we can assume that $A(s) = D(s) = 0$.

We further assume that our dreibein x_1, x_2, x_3, $h_s^{\ \alpha}(x_1, x_2, x_3)$ is itself invariant under reflection, so the vanishing of the function $C(s)$ is implied by this assumption. The vierbein (15), (22) consequently assumes the most general form:

(27) $$\begin{cases} h_s^{\alpha} = \lambda(s)\delta_{s\alpha}, \ \alpha, s = 1, 2, 3, \quad h_s^{4} = 0, \ s = 1, 2, 3, \\ h_4^{\alpha} = \tau(s) x_\alpha, \ \alpha = 1, 2, 3, \qquad h_s^{4} = \mu(s), \end{cases}$$

in which we have renamed the functions that appear in it.

We shall now look for solutions to the field equations $G^{\mu\alpha} = 0$, $F^{\mu\alpha} = 0$ of unified field theory that have the form (27).

If we now denote the covariant vierbein that is adjoint to h_s^{α} by $k_{s\beta}$, s, $\beta = 1, \ldots, 4$, which is defined by the system:

(28) $$h_s^{\alpha} k_{s\beta} = \delta_{\beta}^{\alpha} \qquad s, \alpha, \beta = 1, \ldots, 4$$

(we have $k_{s\alpha} = h_{s\alpha}$, $s = 1, 2, 3$, $k_{4\alpha} = - h_{4\alpha}$), then from (27) it has the components:

(29) $$\begin{cases} k_{s\alpha} = \dfrac{1}{\lambda}\delta_{s\alpha}, \ \alpha, s = 1, 2, 3, \quad k_{s4} = -\dfrac{\tau}{\mu\lambda} x_s, \ s = 1, 2, 3, \\ k_{4\alpha} = 0, \qquad \alpha = 1, 2, 3, \quad k_{44} = \dfrac{1}{\mu}. \end{cases}$$

We must now write down the formulas that will be used in what follows: One computes the $\Delta_{ik}^{l} = -\sum_{s=1}^{3} \dfrac{\partial h_s^{l}}{\partial x_k} k_{si} - \dfrac{\partial h_4^{l}}{\partial x_k} k_{4i}$ as follows:

(30) $$\begin{cases} \Delta_{i4}^{l} = 0, \quad i, l = 1, \ldots, 4, \\ \Delta_{ik}^{l} = -\dfrac{\partial \ln\lambda}{\partial x_k}\delta_{il}, \quad i, k, l = 1, 2, 3 \qquad \Delta_{ik}^{4} = 0, \quad i, k = 1, 2, 3, \\ \Delta_{4k}^{l} = -\dfrac{\lambda}{\mu}\dfrac{\partial}{\partial x_k}\left(\dfrac{\tau}{\lambda}x_l\right) = -\dfrac{\lambda}{\mu}\dfrac{\partial}{\partial x_l}\left(\dfrac{\tau}{\lambda}x_k\right), \ k, l = 1, 2, 3, \\ \Delta_{4k}^{4} = -\dfrac{\partial \ln\mu}{\partial x_k}, \quad k = 1, 2, 3. \end{cases}$$

The quantities $\Lambda_{ik}^{l} = \Delta_{ik}^{l} - \Delta_{ki}^{l}$, $i, j, k = 1, \ldots, 4$ then follow from this:

(31) $$\begin{cases} \Lambda_{ik}^{l} = \dfrac{\partial \ln\lambda}{\partial x_i}\delta_{kl} - \dfrac{\partial \ln\lambda}{\partial x_k}\delta_{il}, \quad i, k, l = 1, 2, 3, \\ \Lambda_{i4}^{l} = \dfrac{\lambda}{\mu}\dfrac{\partial}{\partial x_i}\left(\dfrac{\tau}{\lambda}x_l\right), \quad i, l = 1, 2, 3, \qquad \Lambda_{ik}^{4} = 0, \quad i, k = 1, 2, 3, \\ \Lambda_{ik}^{4} = 0, \qquad i, k = 1, 2, 3, \qquad k, l = 1, 2, 3, \\ \Lambda_{4k}^{4} = -\dfrac{\partial \ln\mu}{\partial x_k}, \quad k = 1, 2, 3. \end{cases}$$

Furthermore, we require the need the contravariant metric tensor, whose system of components has the form:

(32)
$$\begin{cases} g^{\alpha\beta} = \lambda\delta_{\alpha\beta} - \tau^2 x_\alpha x_\beta, \quad \alpha, \beta = 1, 2, 3, \\ g^{4\alpha} = -\mu\tau\, x_\alpha, \\ g^{44} = -\mu^2. \end{cases}$$

We first address the system of field equations $F^{\mu\nu} \equiv \Lambda^{\alpha}_{\underline{\mu\nu;\alpha}} = 0$ ($\phi_{\mu,\,\nu} - \phi_{\nu,\,\mu} = 0$, where $\phi_\mu = \Lambda^{\alpha}_{\mu\alpha}$, resp.).

Because of (31), we have:

(33)
$$\phi_i = \Lambda^{\alpha}_{i\alpha} = \xi_i \left(\frac{\mu'}{\mu} + 2\frac{\lambda'}{\lambda} \right), \qquad \xi_i = \frac{x_i}{s}, \qquad i = 1, 2, 3,$$

(33′)
$$\phi_4 = \Lambda^{\alpha}_{4\alpha} = -\frac{\lambda}{\mu}\frac{\partial}{\partial x_\alpha}\left(\frac{\tau}{\lambda} x_\alpha \right), \qquad \alpha = 1, 2, 3.$$

Since the system $\phi_{i,k} - \phi_{k,i} = 0$, i, $k = 1, 2, 3$ is satisfied identically, all that remains is $\phi_{4,i} - \phi_{i,4} = 0$, or $\phi_{4,i} = 0$, since $\phi_{i,4} = 0$; i.e.:

(34)
$$\phi_4 = \text{const.}$$

From (33′), this gives the equation:

(35)
$$\frac{\lambda}{\mu}\left[\left(\frac{\tau}{\lambda} \right)' s + 3\frac{\tau}{\lambda} \right] = k, \qquad k = \text{constant},$$

which can also be written in the form:

(36)
$$\left(\frac{\tau}{\lambda} s^3 \right)' = k\frac{\mu}{\lambda} s^2,$$

and integration gives:

(36′)
$$\frac{\tau}{\lambda} s^3 = k\int \frac{\mu}{\lambda} s^2 ds + k_1, \qquad k_1 = \text{constant}.$$

λ, μ, τ have the developments $\lambda = 1 + (.)$, $\mu = 1 + (.)$, $\tau = c / s^b\,(1 + (.))$, $b > 1$ at infinity, from which $k = 0$, due to (36′); i.e., it then follows that:

(37)
$$\tau = e\,\frac{\lambda}{s^3}, \qquad e = \text{constant}.$$

(k_1 was set to e.)

This system $F^{\mu\nu} = 0$ is then exhausted with that.

We now treat the other system of field equations:

(38) $$G^{\mu\alpha} \equiv \Lambda^{\alpha}_{\underline{\mu\nu};\nu} - \Lambda^{\sigma}_{\underline{\mu\tau}}\Lambda^{\alpha}_{\sigma\tau} = 0,$$

which, by a simple conversion, we put into the form:

(39) $$G_{\sigma}{}^{\alpha} \equiv g^{\nu\beta}\left[\frac{\partial\Lambda^{\alpha}_{\sigma\rho}}{\partial x_{\nu}} - \Delta^{j}_{\sigma\nu}\Lambda^{\alpha}_{j\rho} - \Delta^{j}_{\rho\nu}\Lambda^{\alpha}_{\sigma j} + \Delta^{\alpha}_{\nu j}\Lambda^{j}_{\sigma\rho}\right], \qquad \alpha, \mu = 1, .., 4.$$

We first treat the system of components $\alpha = 4$, $\sigma \neq 4$.
From (39), we get for them:

(40) $$0 = g^{4\rho}\left[\frac{\partial\Lambda^{4}_{\sigma 4}}{\partial x_{\rho}} - \Delta^{j}_{\sigma\rho}\Lambda^{4}_{j4} - \Delta^{4}_{\sigma 4}\Lambda^{4}_{\rho 4} + \Delta^{4}_{4j}\Lambda^{j}_{\sigma\rho}\right] + g^{44}\Delta^{4}_{4j}\Lambda^{j}_{\sigma 4}.$$

From (31), (32), and by performing the computations, we get ξ_{σ} times the factor:

(41) $$\tau\left(\frac{\mu'}{\mu}\right)' s + \frac{\lambda'\mu'}{\lambda\mu}\tau - \frac{\mu'\lambda}{\mu}\left[\left(\frac{\tau}{\lambda}\right)' s + \frac{\tau}{\lambda}\right] + \left(\frac{\mu'}{\mu}\right)^{2} s\tau = 0.$$

Due to (35), we have (k = 0!): $\left(\frac{\tau}{\lambda}\right)' s + \frac{\tau}{\lambda} = -\frac{2\tau}{\lambda}$.

(41) is satisfied for $\tau = 0$, so we assume that $\tau \neq 0$. Hence, we can divide (41) by τ and get:

(41′) $$\left(\frac{\mu'}{\mu}\right)' s + \frac{\lambda'\mu'}{\lambda\mu} + 2\frac{\mu'}{\mu} + s\left(\frac{\mu'}{\mu}\right)^{2} = 0,$$

which leads immediately to:

(42) $$\mu'\,\lambda\, s^2 = \text{constant},$$

and then to:

(43) $$\mu = k\int\frac{ds}{\lambda s^2} + k_1 .$$

Since λ and μ tend towards unity at infinity, we have $k_1 = 1$, so:

(44) $$\mu = 1 + m\int\frac{ds}{\lambda s^2}, \qquad m = \text{constant}.$$

We now address the system of components in (39) where $\alpha \neq 4$, $\sigma = 4$.
For it, we have:

(45) $$g^{\nu\rho}\left[\frac{\partial\Lambda^{\alpha}_{4\rho}}{\partial x_\nu}-\Delta^{j}_{4\nu}\Lambda^{\alpha}_{j\rho}-\Delta^{j}_{\rho\nu}\Lambda^{\alpha}_{4j}+\Delta^{\alpha}_{\nu j}\Lambda^{j}_{4\rho}\right]+g^{4\rho}[\Delta^{\alpha}_{4j}\Lambda^{j}_{4\rho}]=0.$$

By performing the computations, this time we get ξ_α times:

(46) $$\left(1-\frac{e^2}{s^4}\right)\left[2\lambda^2 e\left(\frac{\lambda}{\mu s^3}\right)'+2\frac{\mu'\lambda^3 e}{\mu^2 s^3}\right]+\frac{6\lambda^3 e}{\mu s^4}-\frac{4e^3\lambda^3}{\mu s^8}=0.$$

Since:

$$\frac{6\lambda^3 e}{\mu s^4}-\frac{4e^3\lambda^3}{\mu s^8}=\frac{2\lambda^3 e}{\mu s^4}+\frac{4\lambda^3 e}{\mu s^4}\left(1-\frac{e^2}{s^4}\right),$$

it follows from (46) that:

(47) $$\left(1-\frac{e^2}{s^4}\right)\left[\lambda^2\left(\frac{\lambda}{\mu s^3}\right)'+\frac{\mu'\lambda^3}{\mu^2 s^3}+\frac{2\lambda^3}{\mu s^4}\right]+\frac{\lambda^3}{\mu s^4}=0.$$

Here, we divided through by $2e$; $e = 0$ already satisfies (46).
An elementary conversion of (47) gives:

(48) $$\left(1-\frac{e^2}{s^4}\right)\left(\ln\frac{\lambda}{s}\right)'+\frac{1}{s}=0,$$

so

$$\ln\frac{\lambda}{\mu}=-\int\frac{ds}{s\left(1-\frac{e^2}{s^4}\right)}+k=-\ln\sqrt[4]{s^4-e^2}+k,$$

or finally:

(49) $$\lambda=c\,\frac{s}{\sqrt[4]{s^4-e^2}}.$$

Since λ is unity at infinity, we must set $c = 1$, and we ultimately get:

(50) $$\lambda=\frac{1}{\sqrt[4]{1-\frac{e^2}{s^4}}}.$$

We already know the functions λ, μ, and τ that are characteristic of the rotationally-symmetric case from (37), (44), and (50).

The still-unused relations (39), which are the ones for which $\alpha = \sigma = 4$ and $\alpha, \sigma \neq 4$, must be satisfied identically for the functions (37), (44), and (50).

For $\alpha = \sigma = 4$, (39) becomes:

$$g^{\nu\rho}\left[\frac{\partial\Lambda^4_{4\rho}}{\partial x_\nu}-\Delta^4_{4\nu}\Lambda^4_{4\rho}-\Delta^j_{\rho\nu}\Lambda^4_{4j}\right]+g^{4\rho}\Delta^4_{4j}\Lambda^j_{4\rho}=0, \tag{51}$$

or

$$(\lambda^2\,\delta_{\nu\rho}-\tau^2\,x_\nu\,x_\rho)\left[-\frac{\partial}{\partial x_\nu}\left(\frac{\mu'}{\mu}\xi_\rho\right)-\left(\frac{\mu'}{\mu}\right)'\xi_\nu\xi_\rho-\frac{\lambda'\mu'}{\lambda\mu}\xi_\nu\xi_\rho\right] \tag{51′}$$

$$-\,\tau\,\mu\,x_\rho\,\frac{\mu'\lambda}{\mu^2}\xi_j\,\frac{\partial}{\partial x_\rho}\left(\frac{\tau}{\lambda}x_j\right)=0,$$

resp.

This equation will, in fact, be satisfied due to (37), (44), and (50). For α, $\sigma\neq 4$, (39) reads:

$$\begin{cases}(\lambda^2\delta_{\nu\rho}-\tau^2x_\nu x_\rho)\left[\frac{\partial}{\partial x_\nu}\left(\frac{\lambda'}{\lambda}(\xi_\sigma\delta_{\rho\alpha}-\xi_\rho\delta_{\sigma\alpha})\right)+\left(\frac{\lambda'}{\lambda}\right)^2\xi_\nu\delta_{j\sigma}(\xi_j\delta_{\rho\alpha}-\xi_\rho\delta_{j\alpha})\right.\\ \qquad\left.+\left(\frac{\lambda'}{\lambda}\right)^2\xi_\nu\delta_{j\sigma}(\xi_\sigma\delta_{j\alpha}-\xi_j\delta_{\sigma\alpha})-\left(\frac{\lambda'}{\lambda}\right)^2\xi_j\delta_{\nu\sigma}(\xi_\sigma\delta_{j\rho}-\xi_\rho\delta_{j\sigma})\right]\\ +\lambda'\tau x_i\frac{\partial}{\partial x_j}\left(\frac{\tau}{\lambda}x_\alpha\right)(\xi_\sigma\delta_{j\rho}-\xi_\rho\delta_{j\sigma})\\ -\mu\tau x_\nu\left[\frac{\partial}{\partial x_\nu}\left(\frac{\lambda}{\mu}\cdot\frac{\partial}{\partial x_\sigma}\left(\frac{\tau}{\lambda}x_\alpha\right)\right)+\frac{\lambda'}{\mu}\xi_\nu\delta_{j\sigma}\frac{\partial}{\partial x_j}\left(\frac{\tau}{\lambda}x_\alpha\right)+\frac{\lambda'}{\mu}\frac{\partial}{\partial x_\nu}\left(\frac{\tau}{\lambda}x_j\right)(\xi_\sigma\delta_{j\alpha}-\xi_j\delta_{\sigma\alpha})\right.\\ \qquad\left.+\frac{\mu'}{\mu}\xi_\nu\frac{\lambda}{\mu}\frac{\partial}{\partial x_\sigma}\left(\frac{\tau}{\lambda}x_\alpha\right)-\frac{\lambda'}{\mu}\xi_j\delta_{\nu\sigma}\frac{\partial}{\partial x_\sigma}\left(\frac{\tau}{\lambda}x_j\right)\right]\\ +\lambda^2\frac{\partial}{\partial x_j}\left(\frac{\tau}{\lambda}x_\alpha\right)\frac{\partial}{\partial x_j}\left(\frac{\tau}{\lambda}x_\sigma\right)=0.\end{cases} \tag{52}$$

This system will also be satisfied for the functions (37), (44), and (50). We shall let the reader carry out these computations, of which, only one requires any attention.

We note the result: The vierbein:

$$\begin{cases}h_s^\alpha=\dfrac{\delta_{s\alpha}}{\sqrt[4]{1-\dfrac{e^2}{s^4}}},\quad \alpha,s=1,2,3,\qquad h_s^4=0,\\[2ex] h_4^\alpha=\dfrac{e}{\sqrt[4]{1-\dfrac{e^2}{s^4}}}\dfrac{x_\alpha}{s^3},\quad\alpha=1,2,3\qquad h_4^4=1+m\displaystyle\int\sqrt[4]{1-\frac{e^2}{s^4}}\frac{ds}{s^2}\end{cases} \tag{53}$$

is the most general solution to the centrally-symmetric (and mirror-symmetric) case. As far as the physical interpretation is concerned, e is to be regarded as the electrical charge, and m, as the ponderomotive mass. This interpretation is, in itself, arbitrary, if one disregards the fact that it conforms to the meaning of the field that is given by considering the field equations in the first approximation. The appearance of two – and only two – constants is noteworthy, since, in retrospect, this is required by experiment.

§ 2. Static, pure gravitational field.

From equation (53), we derive the fact that for vanishing charge e, all of the $h_s{}^\alpha$, except for $h_4{}^4$, are constants, while $h_4{}^4 = 1 - m / s$. This finding leads us to conjecture that there are static solutions of a general sort for which only $h_4{}^4$ is variable.

To that end, we set:

$$h_s{}^\alpha = \delta_{s\alpha}, \qquad s = 1, 2, 3, \qquad h_4{}^\alpha = \delta_{4\alpha}\ \sigma(x_1, x_2, x_3), \tag{1}$$

such that all of the $\Delta^\gamma_{\alpha\beta}$ are zero, except for:

$$\Delta^4_{4\beta} = \Lambda^4_{4\beta} = -k_{44}\ h^4_{4,\beta} = -\frac{\partial \ln \sigma}{\partial x_\beta}. \tag{2}$$

All of the field equations are satisfied identically, except for:

$$G_4{}^4 = g^{\varpi\rho}\left[\frac{\partial \Lambda^4_{4\rho}}{\partial x_\nu} - \Delta^4_{4\nu}\Lambda^4_{4\rho}\right] = 0, \tag{3}$$

or

$$0 = \sum_\rho \frac{\partial \Lambda^4_{4\rho}}{\partial x_\rho} - \Delta^4_{4\rho}\Lambda^4_{4\rho} = \sum_\rho \frac{\partial^2 \ln \sigma}{\partial x_\rho^2} + \frac{\partial \ln \sigma}{\partial x_\rho}\frac{\partial \ln \sigma}{\partial x_\rho}, \tag{3′}$$

resp. This says that σ satisfies:

$$\sum_\rho \frac{\partial^2 \sigma}{\partial x_\rho^2} = 0; \tag{4}$$

i.e., σ is a potential.

Since σ converges to unity at infinity, the solution (in the case of finitely-many mass points) reads:

$$\sigma = 1 + \sum_j \frac{m_j}{r_j}, \qquad m_j = \text{constant}. \tag{5}$$

This rigorous result is, in retrospect, important for the physical interpretation of the theory on the following grounds: Formula (5) shows us that there is a rigorous solution that corresponds to the case in which two or more unbound electrically-neutral masses at arbitrary distances from each other are at rest. There is no such case in nature. One is then inclined to judge this to be a failure of the theory when it is compared to experiment. This was also precisely the case when one attempted to derive the law of motion that followed from the field equations for such singularities in the original statement of the theory. However, this does not seem to be the case in the present theory ([1]).

Hence, no argument for the utility of the theory can be derived from the existence of the static solution that is considered here. However, one knows full well that in the new theory one must demand freedom from singularities for any solutions that could represent the elementary particles of matter.

Prior to the discovery of such solutions, it would not seem possible for us to deduce the law of motion for the particles from the field equations.

Received on 11 March

([1]) The derivability of the law of motion in the earlier statement of the theory rested upon the fact that we had a field equation in the form of a symmetric tensor equation whose divergence vanished identically.

However, this condition is not satisfied in the present theory.

Solution with Axial Symmetry of Einstein's Equations of Teleparallelism

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(Received 9th June 1930. Read 7th November 1930.)

§ 1. Introduction. Einstein ([1]) has recently adopted a new set of field-equations in his Unified Field Theory of Gravitation and Electricity – the so-called theory of parallelism at a distance or Teleparallelism – and has given ([2]) a solution of these equations with spherical symmetry, corresponding to the field of a charged mass-particle. In the present paper, we discuss the solution of these equations with axial symmetry, which corresponds to a statical field whose field variables depend upon a single coordinate only – viz., the coordinate which is measured along the axis of symmetry. We begin by finding this solution and showing that it is the only one of this type possible on the theory of teleparallelism. This result contrasts with that of the hitherto-accepted relativity theory of 1916, in which a number of solutions of this type are known, corresponding to different values, assigned *a priori*, of the energy tensor. In particular, the gravitational field of a uniform electric force ([3]) has, on the 1916 theory, the axial type of symmetry defined above. Bearing this in mind, we then show that the single solution with axial symmetry yielded by the theory of teleparallelism has the following three properties: Firstly, it contains no electromagnetic force, according to the definition of this force in the theory of teleparallelism. Secondly, it is not one of the fields of electromagnetic force already found on the 1916 theory. Thirdly, it corresponds, on this latter theory, to a distribution of matter which, although possible in theory, cannot be said to have any physical counterpart.

§ 2. The field equations. The field variables in a four-dimensional manifold are, according to the theory of teleparallelism, sixteen quantities ${}_sh^{\alpha}$. The x_r are Gaussian cooordinates, and the manifold is taken to be Riemannian, so that its metric is:

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu \, .$$

The geometrical interpretation of the ${}_sh^{\alpha}$ is this: Consider a point whose coordinates are (x_1, x_2, x_3, x_4), then for a given α and for $s = 1, 2, 3, 4$, the four ${}_sh^{\alpha}$ are the projections on the α-axis of the Gaussian coordinates of four orthogonal unit vectors in a tangent

([1]) A. Einstein, Berlin Akad. Sitz. **1** (1930), 18.
([2]) A. Einstein and W. Mayer, *ibid.* **6** (1930), 110.
([3]) G. C. McVittie, Proc. Roy. Soc. (A) **124** (1929), 366.

Euclidian manifold, touching the Riemannian manifold at the point considered. It can be shown ([1]) to follow from this that:

$$g^{\mu\nu} = {}_s h^{\mu}\, {}_s h^{\nu}, \qquad g_{\mu\nu} = {}^s h_{\mu}\, {}^s h_{\nu}, \tag{2}$$
$${}^s h_{\mu}\, {}_s h^{\nu} = \delta^{\nu}_{\mu}, \qquad {}^s h_{\mu}\, {}_t h^{\mu} = \delta^{s}_{t},$$

where

$${}^s h_{\mu} = (\text{minor of } {}_s h^{\mu} \text{ in } |\, {}_s h^{\mu}\,|) \,/\, |\, {}_s h^{\mu}\,|, \tag{3}$$
$$\sqrt{g} = h = |\, {}_s h^{\mu}\,|,$$
$$\delta^{\nu}_{\mu} = \text{Kronecker's delta.}$$

A further restriction is placed on the ${}_s h^{\alpha}$, as follows: Imagine the four unit vectors defined by them set up at each point of the Riemannian manifold. We shall call this a "set of 4-vectors." Then, every set of 4-vectors which can be obtained from a given set by rotation – the same at every point – of the given set is to be considered equivalent to that set. This enables Einstein to define a connection with respect to the set of 4-vectors for which teleparallelism exists. The coefficients of the connection are:

$$\Delta^{\alpha}_{\mu\nu} = {}_s h^{\alpha} \frac{\partial\, {}^s h_{\mu}}{\partial x_{\nu}}, \tag{4}$$

and since they are not symmetrical in μ and ν, we put:

$$\Lambda^{\alpha}_{\mu\nu} = \Delta^{\alpha}_{\mu\nu} - \Delta^{\alpha}_{\nu\mu}, \tag{5}$$
$$\phi_{\mu} = \Lambda^{\alpha}_{\mu\alpha}. \tag{6}$$

The field equations given by Einstein are then:

$$g^{\mu\rho} g^{\nu\sigma} (\Lambda^{\alpha}_{\rho\sigma;\nu} - \Lambda^{\tau}_{\rho\sigma}\Lambda^{\alpha}_{\tau\nu}) = 0, \tag{7}$$
$$\Lambda^{\sigma}_{\mu\alpha;\sigma} = 0. \tag{8}$$

In (7) and (8), the semi-colon denotes that the covariant derivative with respect to the connection (4) has been taken.

The group (8) of equations can be replaced by:

$$\frac{\partial \phi_{\mu}}{\partial x_{\alpha}} - \frac{\partial \phi_{\alpha}}{\partial x_{\mu}} = 0. \tag{9}$$

The ${}^s h_{\mu}$ are interpreted physically by Einstein (in the first approximation only) as follows:

([1]) A. Einstein, Berlin Akad. Sitz. **17-19** (1928), 217. It should be observed that we use the summation convention regarding repeated suffixes, whether these are in Latin or Greek type.

If ${}^s h_\mu = \delta^s_\mu + \overline{h}_{s\mu}$, where $\overline{h}_{s\mu}$ is small compared to unity then $a_{s\mu} = \overline{h}_{s\mu} - \overline{h}_{\mu s}$ is the electromagnetic force tensor in this field, and the $g_{s\mu} = \overline{h}_{s\mu} + \overline{h}_{\mu s}$ are the gravitational potentials of the theory.

§ 3. The form of the field equations for axial symmetry. Let us denote by x_1 the coordinate along the axis of symmetry of the field and by x_2, x_3 the coordinates along the other two directions of space. Let x_4 denote the time. We consider fields which are statical and where, moreover, the ${}_s h^\alpha$ are functions of x_1 alone. In consequence of this, the metrical tensor $g_{\mu\nu}$ is, by (2), a function of x_1 alone. We may therefore take the geometry of the (x_2, x_3) "planes" to be Euclidian, and consider these two coordinates as analogous to Cartesian in plane geometry, so that x_2 and x_2 will enter symmetrically into our equations.

Furthermore, we contemplate fields containing continuous distributions of matter or energy, and assume that no singularities of our field variables will occur at the origin. We also take coordinates such that, at the origin, the ${}_s h^\alpha$ have Euclidian values.

We now proceed to show that under these conditions only six of the sixteen ${}_s h^\alpha$ are non-zero, and of these, only five are independent.

Consider, firstly, a spatial section of the four-dimensional manifold representing the field. Such a section is a three-dimensional continuum which is invariant under the transformation:

$$\overline{x}_1 = x_1, \qquad \overline{x}_\alpha = a_{\alpha\beta}\, x_\beta \qquad (\alpha, \beta = 2, 3), \tag{10}$$

where $((a_{\alpha\beta}))$ is any orthogonal matrix.

By hypothesis, all of the field-variables are functions of x_1 only; hence, we put:

$${}_s h^\alpha (x_1, x_2, x_3, x_4) = {}_s h^\alpha (x_1).$$

Since the geometry of the (x_2, x_3) planes is to be Euclidian, ${}_s h^\alpha$ must, for a fixed value of x_1 and for s, $\alpha = 2, 3$, be a constant multiple of δ^α_s; hence:

$${}_2 h^3 (x_1) = {}_3 h^2 (x_1) = 0 \quad \text{and} \quad {}_2 h^2 (x_1) = {}_3 h^3 (x_1).$$

Since our field variables are to have Euclidian values at the origin, we have:

$${}_s h^\alpha (0) = \delta^\alpha_s \qquad (s,\ \alpha = 1, 2, 3). \tag{11}$$

We now apply the condition that all sets of 3-vectors obtained from each other by simultaneous rotations at all points are to be equivalent. Perform the transformation (10) on a set of ${}_s h^\alpha$; we get:

$$\left.\begin{aligned} {}_s\overline{h}^1(\overline{x}_1) &= {}_s\overline{h}^1(x_1) && (s = 1, 2, 3), \\ {}_s\overline{h}^\alpha(\overline{x}_1) &= a_{\alpha\beta}\ {}_s\overline{h}^\beta(x_1) && (\alpha, \beta = 2, 3). \end{aligned}\right\} \tag{12}$$

If the new ${}_sh^\alpha$ are to be equivalent to the old, there must exist a unique orthogonal transformation $((A_{st}))$ – the same for each point of the three-space – such that the new set of 3-vectors, specified by the ${}_s\bar{h}^\alpha(\bar{x}_1)$ at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be rotated into the set of 3-vectors, specified by the ${}_sh^\alpha(\bar{x}_1)$ at the point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. That is to say:

$${}_s\bar{h}^\gamma(\bar{x}_1) = A_{st}\ {}_th^\gamma(\bar{x}_1) \qquad (s, t, \gamma = 1, 2, 3).$$

Hence, the functional equations for the ${}_sh^\nu$ are, by (10) and (12):

$$\left.\begin{aligned} &A_{st}\ {}_th^1(\bar{x}_1) = {}_sh^1(x_1) && (s, t = 1, 2, 3), \\ &A_{st}\ {}_th^\alpha(\bar{x}_1) = a_{\alpha\beta}\ {}_sh^\beta(x_1) && (\alpha, \beta = 2, 3). \end{aligned}\right\} \tag{13}$$

Since the $(A_{st}))$ is the same for each point, it is sufficient to calculate its value at one point. We take the origin. Applying (11), we get from the first group of equations (13):

$$\begin{aligned} &\delta_s^1 = A_{st}\ \delta_t^1, \\ &A_{11} = 1, \qquad A_{s1} = 0 \qquad \text{if } s \neq 1, \end{aligned}$$

and from the second group of (13):

$$a_{\alpha\beta}\ \delta_s^\beta = A_{st}\ \delta_t^\alpha \qquad (\alpha, \beta = 2, 3) \quad (s, t = 1, 2, 3).$$

Hence:

$$\begin{aligned} a_{\alpha s} &= A_{s\alpha} && (s, \alpha = 2, 3), \\ 0 &= A_{1\alpha} && (\alpha \quad = 2, 3). \end{aligned}$$

By substituting these values of the A_{st} into the first group of equations (13), they become:

$$\begin{aligned} {}_1h^1(x_1) &= {}_1h^1(x_1), \\ {}_2h^1(x_1) &= a_{22}\ {}_2h^1(x_1) + a_{32}\ {}_3h^1(x_1), \\ {}_3h^1(x_1) &= a_{23}\ {}_2h^1(x_1) + a_{33}\ {}_3h^1(x_1), \end{aligned}$$

but the a_{22}, a_{23}, a_{33} are the elements of *any* orthogonal matrix. Hence, we can only satisfy the last two equations if:

$${}_2h^1(x_1) = {}_3h^1(x_1) = 0.$$

By the same reason applied to the second group of equations (13), we get:

$${}_1h^2(x_1) = {}_1h^3(x_1) = 0.$$

Hence, we can describe any spatial section of our field by means of the three quantities ${}_1h^1(x_1)$, ${}_2h^2(x_1)$, and ${}_3h^3(x_1)$.

To extend this to four dimensions: The 4-space must now be invariant under the transformation:

$$\bar{x}_4 = x_4, \qquad \bar{x}_1 = x_1, \qquad \bar{x}_\alpha = a_{\alpha\beta}\, x_\beta, \quad (\alpha, \beta = 2, 3), \tag{14}$$

and such that:

$$_s h^\alpha (x_1, x_2, x_3, x_4) = {_s h^\alpha} (x_1, x_2, x_3) \qquad (s, \alpha = 1, 2, 3).$$

Hence, we have that the only non-zero $_s h^\alpha$ $(s, \alpha = 1, 2, 3)$ are $_1h^1(x_1)$, $_2h^2(x_1)$, $_3h^3(x_1)$. As before:

$$_s h^\alpha (x_1, x_2, x_3, x_4) = {_s h^\alpha} (x_1) \qquad \text{for } s, \alpha = 1, 2, 3, 4.$$

Applying (14) to the $_s h^\alpha$, we get:

$$_s\bar{h}^4(x_1) = {_s h^4} (x_1), \qquad {_s\bar{h}^1}(x_1) = {_s h^1} (x_1) \qquad (s = 1, 2, 3, 4),$$
$$_s\bar{h}^\alpha(x_1) = a_{\alpha\beta}\, {_s h^\beta} (x_1) \qquad (\alpha, \beta = 2, 3),$$

and, as before, there must be a unique orthogonal transformation (B_{st})) for all points, such that:

$$_s\bar{h}^\alpha(\bar{x}_1) = B_{st}\, {_s h^\alpha}(\bar{x}_1) \qquad (s, t, \alpha = 1, 2, 3, 4).$$

Thus, the functional equations for the $_s h^\alpha$ are now:

$$\left.\begin{array}{ll} _s h^\alpha(x_1) = B_{st}\, {_t h^\alpha}(x_1), & {_s h^1}(x_1) = B_{st}\, {_s h^1}(x_1), \\ a_{\alpha\beta}\, {_s h^\beta}(x_1) = B_{st}\, {_t h^\alpha}(x_1), & (s, t = 1, 2, 3, 4;\ \alpha, \beta = 2, 3). \end{array}\right\} \tag{15}$$

Applying (11), we get:

$$B_{44} = 1, \qquad B_{st} = 0 \quad (s \neq 4), \qquad B_{11} = 1, \qquad B_{s1} = 0 \quad (s \neq 1),$$

and

$$a_{\alpha\beta} = B_{st} \qquad (\alpha, \beta = 2, 3;\ s, t = 1, 2, 3, 4).$$

Substituting these into the equations (15), we prove, in the same manner as for $_1h^2$, $_2h^1$, $_3h^1$, $_1h^3$ that:

$$_4h^2 = {_4h^3} = {_2h^4} = {_3h^4} = 0.$$

Hence, finally:

$$\left.\begin{array}{l} \textit{The } {_s h^\nu} \textit{ appropriate to a field with axial symmetry are:} \\ _1h^1(x_1),\ {_4h^4}(x_1),\ {_4h^1}(x_1),\ {_1h^4}(x_1),\ {_2h^2}(x_1),\ {_3h^3}(x_1), \\ \textit{where} \quad {_2h^2}(x_1) = {_3h^3}(x_1) \\ \textit{and} \quad {_1h^4}(x_1) = {_4h^1}(x_1) = \delta_{14} = 0 \quad \textit{at the origin;} \\ \textit{all the other } {_s h^\nu} \textit{ are zero.} \end{array}\right\} \tag{16}$$

§ 4. The solution of the field equations. Before proceeding with the actual solution, we shall make the further restriction that the form (1) is indefinite, and to avoid the use of imaginaries in our calculation we shall introduce the numbers e_α, which are such that:

$$e_4 = 1,\ e_1 = e_2 = e_3 = -1$$

in our case.

The formulae (2) and (3) then become:

$$g^{\mu\nu} = e_s\, {}_sh^{\mu}\, {}_sh^{\nu}, \qquad g_{\mu\nu} = e_s\, {}^sh_{\mu}\, {}^sh_{\nu}, \tag{17}$$

$$\left.\begin{aligned} & e_s\, {}^sh_{\mu}\, {}_sh^{\nu} = \delta^{\nu}_{\mu},\ \ {}^sh_{\mu}\, {}_th^{\mu} = \delta^s_t,\ \ h = |\, e_s\, {}_sh^{\mu}\, |, \\ & {}^sh_{\mu} = (minor\ of\ {}_sh^{\mu}\ in\, |\, e_s\, {}_sh^{\mu}\, |) \,/\, |\, e_s\, {}_sh^{\mu}\, |, \end{aligned}\right\} \tag{18}$$

whilst

$$\Delta^{\alpha}_{\mu\nu} = e_s\, {}_sh^{\alpha}\, \frac{\partial\, {}^sh_{\mu}}{\partial x^{\nu}}, \tag{19}$$

and (5) and (6) remain unchanged in form.

Also, in virtue of (14), the form of (1) may now be written as:

$$ds^2 = g_{44}(x_1)\ dx_4^2 + g_{14}(x_1)\, dx^1\, dx^4 - g_{11}(x_1)\ dx_1^2 - g_{22}(x_1)\,(dx_2^2 + dx_3^2)\,. \tag{20}$$

Since we require both the ${}_sh^{\mu}$ and the ${}^sh_{\mu}$, we calculate the former in terms of the latter by means of:

$${}_sh^{\mu} = (minor\ of\ {}^sh_{\mu}\ in\ |\ e_s\, {}^sh_{\mu}\,|) \,/\, |\ e_s\, {}^sh_{\mu}\,|.$$

We get:

$$\left.\begin{aligned} & {}_4h^4 = {}^1h_1 / H, \quad {}_1h^1 = -\, {}^4h_4 / H, \qquad {}_4h^1 = -\, {}^1h_4 / H, \\ & {}_1h^4 = {}^4h_4 / H, \ \ {}_2h^2 = {}_2h^2 = 1/\, {}^2h_2, \\ & \qquad\qquad h = -{}^3h_3\, {}^2h_2\, H, \\ & \text{where} \qquad\quad H \equiv {}^4h_4\, {}^1h_1 - {}^4h_1\, {}^1h_4\,. \end{aligned}\right\} \tag{21}$$

The non-zero Δ^{α}_{kl} are:

$$\left.\begin{aligned} \Delta^4_{41} &= {}_4h^4\, \frac{d\, {}^4h_4}{dx_1} - {}_1h^4\, \frac{d\, {}^1h_4}{dx_1} = \left({}^1h_1\, \frac{d\, {}^4h_4}{dx_1} - {}^4h_1\, \frac{d\, {}^1h_4}{dx_1} \right) \Big/ H\,, \\ \Delta^4_{11} &= {}_4h^4\, \frac{d\, {}^4h_1}{dx_1} - {}_1h^4\, \frac{d\, {}^1h_1}{dx_1} = \left({}^1h_1\, \frac{d\, {}^4h_1}{dx_1} - {}^4h_1\, \frac{d\, {}^1h_1}{dx_1} \right) \Big/ H\,, \\ \Delta^3_{31} &= \Delta^2_{21} = {}_3h^3\, \frac{d\, {}^3h_3}{dx_1} = \frac{d \log {}^1h_1}{dx_1}, \\ \Delta^1_{11} &= {}_4h^1\, \frac{d\, {}^4h_1}{dx_1} - {}_1h^1\, \frac{d\, {}^1h_1}{dx_1} = \left({}^4h_4\, \frac{d\, {}^1h_1}{dx_1} - {}^1h_4\, \frac{d\, {}^4h_1}{dx_1} \right) \Big/ H\,, \\ \Delta^1_{41} &= {}_1h^1\, \frac{d\, {}^1h_4}{dx_1} + {}_4h^1\, \frac{d\, {}^4h_4}{dx_1} = \left({}^4h_4\, \frac{d\, {}^1h_4}{dx_1} - {}^1h_4\, \frac{d\, {}^4h_4}{dx_1} \right) \Big/ H\,. \end{aligned}\right\} \tag{22}$$

Hence, the non-zero Λ^{α}_{kl} are:

$$\Lambda^4_{41} = -\ \Lambda^4_{14} = \ \ \Lambda^4_{41},$$
$$\Lambda^3_{31} = \ \ \Lambda^2_{21} = -\Lambda^3_{13} = -\Lambda^2_{12} = \Lambda^3_{31} = \Lambda^2_{21}$$
$$\Lambda^1_{41} = -\ \Lambda^1_{14} = \ \ \Lambda^1_{41}.$$

The functions ϕ_μ are, by equation (6):

$$\left.\begin{aligned} \phi_4 = \Delta^4_{41}, \quad \phi_1 = -\Delta^4_{41} - \Delta^2_{21} - \Delta^3_{31}, \\ \phi_2 = \phi_3 = 0. \end{aligned}\right\} \tag{23}$$

We now proceed to substitute these values into the field equations (7) and (8). Take first the equations (8) or their equivalents (9); they reduce to the single one:

$$\frac{d\phi_4}{dx_1} = \frac{d}{dx_1}\Delta^1_{41} = 0.$$

Hence:

$$\Lambda^1_{41} = \alpha \qquad (\alpha = \text{constant}). \tag{24}$$

The equations (7) may be written in full as:

$$g^{\nu\rho}\left[\frac{\partial\Lambda^\alpha_{\sigma\rho}}{\partial x^\nu} - \Delta^j_{\sigma\nu}\Lambda^\alpha_{j\rho} - \Delta^j_{\rho\nu}\Lambda^\alpha_{\sigma j} + \Delta^\alpha_{\nu j}\Lambda^j_{\sigma\rho}\right] = 0.$$

Hence, the ones which do not vanish identically in our case are:

$$g^{11}\left[\frac{d}{dx_1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\Delta^4_{41} + \Delta^4_{11}\Delta^1_{41}\right] + g^{14}\Delta^1_{41}\Delta^4_{41} = 0, \tag{25}$$

$$g^{14}\left[\frac{d}{dx_1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\Delta^4_{41} + \Delta^4_{11}\Delta^1_{41}\right] + g^{44}\Delta^4_{41}\Delta^1_{41} = 0, \tag{26}$$

$$g^{11}\left[\frac{d}{dx_1}\Delta^4_{41} - \Delta^1_{41}\Delta^4_{41}\right] + g^{14}(\Delta^1_{41})^2 = 0, \tag{27}$$

$$g^{14}\left[\frac{d}{dx_1}\Delta^1_{41} - \Delta^1_{41}\Delta^4_{41}\right] + g^{44}\Delta^4_{41}\Delta^1_{41} = 0, \tag{28}$$

$$g^{11}\left[\frac{d}{dx_1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^2_{21}\Delta^1_{11}\right] - g^{14}\Delta^1_{41}\Delta^2_{21} = 0. \tag{29}$$

The six equations (24) to (29) now determine the five unknown ${}^s h_\nu$. These six equations are not, of course, independent; the identities existing between them have been given by Einstein ([1]).

We have, by (24) and (28), either:

$$g^{14} - g^{44} = 0$$

or

$$\Delta^4_{41} = 0.$$

The first alternative is impossible, for (by (11)):

$$g^{14} = {}_4h^4\, {}_1h^4 - {}_1h^1\, {}_1h^4 = ({}^4h_4\, {}^4h_1 - {}^1h_1\, {}^1h_4) / H^2 \to 0 \qquad \text{at the origin,}$$
$$g^{44} = ({}_4h^4)^2 - ({}_1h^4)^2 = \{({}^1h_1)^2 - ({}^4h_1)^2\} / H^2 \to 1 \qquad \text{at the origin.}$$

Hence, we have:

$$\Delta^4_{41} = 0.$$

(27) now gives:

$$\alpha^2 g^{14} = 0,$$

whilst (25) and (26) give:

$$\Delta^4_{11} = 0.$$

Hence, equations (24) to (29) are equivalent to:

$$\Delta^1_{41} = \alpha, \tag{30}$$
$$\Delta^4_{41} = 0, \tag{31}$$
$$\alpha^2 g^{14} = 0, \tag{32}$$
$$\Delta^4_{11} = 0, \tag{33}$$
$$g^{11}\left[\frac{d}{dx_1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^2_{21}\Delta^1_{11}\right] - g^{14}\alpha\,\Delta^2_{21} = 0. \tag{29}$$

By (22), the equation (33) is:

$${}^1h_1 \frac{d\,{}^4h_1}{dx_1} - {}^4h_1 \frac{d\,{}^1h_1}{dx_1} = 0.$$

Hence, ${}^4h_1$ is a constant multiple of ${}^1h_1$. But, ${}^4h_1 \to 0$ at the origin, whilst ${}^1h_1 \to 1$, so that the multiplier must be zero.

Hence:

$${}^4h_1 = 0, \qquad \text{whilst } {}^1h_1 \text{ is arbitrary.} \tag{34}$$

Again, by (22) and (34), the equation (31) reduces to:

([1]) Berlin Akad. Sitz. **1** (1930), 18.

$$\frac{d\,{}^4h_4}{dx_1} = 0.$$

Hence:

$$^4h_4 = 1. \tag{35}$$

With regard to the equation (32), we have three possibilities:

$$\begin{array}{lll} (a) & \alpha \neq 0, & g^{14} = 0, \\ (b) & \alpha = 0, & g^{14} \neq 0, \\ (c) & \alpha = 0, & g^{14} = 0. \end{array}$$

Consider (a):

$$0 = g^{14} = ({}^4h_4\,{}^4h_1 - {}^1h_4\,{}^1h_1) / H^2 = -\,{}^1h_4 / ({}^4h_4)^2 \cdot ({}^1h_1),$$

by (34). Hence:

$$^1h_4 = 0. \tag{36}$$

But this is impossible if $a \neq 0$, since, by (24) and (22):

$$\alpha = \left({}^4h_4 \frac{d\,{}^1h_4}{dx_1} - {}^1h_4 \frac{d\,{}^4h_4}{dx_1} \right) \Big/ H\ ,$$

and, by (36) and (34), the right-hand side of this equation is zero whilst the left-hand side is not.

Hence, the alternative (a) is impossible. Similarly, it may be shown that (b) is impossible. We are thus left with (c), which, by (24) and the value of g^{14} given above, leads to:

$$^1h_4 = 0. \tag{36}$$

Again, the equation (29), by (22), (34), (36), becomes:

$$\frac{d^2}{dx_1^2}(\log\,{}^2h_2) - \left[\frac{d}{dx_1}(\log\,{}^2h_2)\right]^2 - \frac{d}{dx_1}(\log\,{}^1h_1)\cdot\frac{d}{dx_1}(\log\,{}^2h_2) = 0. \tag{37}$$

Now, by (34), ${}^1h_1$ is arbitrary. Hence, change the variable from x_1 to z by means of:

$$^1h_1\,dx_1 = dz.$$

(37) becomes:

$$\frac{d^2}{dz^2}(\log\,{}^2h_2) - \left[\frac{d}{dz}(\log\,{}^2h_2)\right]^2 = 0.$$

The solution of this equation, with suitable adjustment of the constants, is:

$$^2h_2 = \frac{1}{c(1-z)} \qquad (c = \text{constant}). \tag{38}$$

Hence, finally, putting $z = x_1 / c$, *we may write our solution in the form:*

$$\left.\begin{aligned} & ds^2 = dx_4^2 - c^{-2}dx_1^2 - (c - x_1)^{-2}(dx_2^2 + dx_3^2) \\ \text{with} \qquad & {}^4h_1 = {}^1h_4 = 0, \\ & {}^4h_1 = 1, \quad {}^1h_1 = c^{-1}, \quad {}^2h_2 = {}^3h_3 = (c - x_1)^{-1}. \end{aligned}\right\} \tag{39}$$

The condition ${}^4h_1 = {}^1h_4 = 0$ is important, since it enables us to say that there is no electromagnetic force in this field, according to the definition of this force in the theory of teleparallelism. For, referring to this definition given at the end of § 2, we see that for (39), in the first approximation, all the $a_{s\mu}$ are zero.

We see that the equations (24) to (29) are just sufficient to determine the field (39). This field is therefore the only one with the type of axial symmetry considered which can be obtained from the theory of teleparallelism, and it is a field not containing electromagnetic forces.

We should add that the metric given in (39) is that of a curved four-space, as may be seen by calculating a few components of the Riemann-Christoffel tensor belonging to it.

§ 5. Comparison with general relativity theory. It is interesting to note that the gravitational field of a uniform electric force ([1]), on the 1916 theory, has just the type of axial symmetry considered in this paper. The field is:

$$\left.\begin{aligned} & ds^2 = e^{\alpha x_1}dx_4^2 - e^{-2\alpha x_1}dx_1^2 - e^{-\alpha x_1}(dx_2^2 + dx_3^2) \\ \text{with} \qquad & F_{41} = \tfrac{1}{4}\alpha e^{\frac{1}{2}\alpha x_1}\pi^{-1/2} = -F_{41}, \quad F_{\mu\nu} = 0 \quad (\mu, \nu \neq 1, 4), \end{aligned}\right\} \tag{40}$$

where $F_{\mu\nu}$ is the electromagnetic force tensor.

Since (39) is the only solution of this type which will satisfy the equations of teleparallelism, the solution (40), which is not reducible to (39), will not satisfy them. The gravitational fields of electromagnetic forces on the two theories do not therefore agree.

If we calculate the energy tensor:

$$-8\pi\, T_\mu^\nu = G_\mu^\nu - \tfrac{1}{2}G\delta_\mu^\nu,$$

where $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor, for (39), we get:

([1]) G. C. McVittie, *loc. cit.*

$$\left.\begin{aligned} -8\pi T_4^4 &= 5\left(1-\frac{x_1}{c}\right)^{-2}, \\ -8\pi T_1^1 &= \left(1-\frac{x_1}{c}\right)^{-2}, \\ -8\pi T_2^2 = -8\pi T_3^3 &= 2\left(1-\frac{x_1}{c}\right)^{-2}. \end{aligned}\right\} \tag{41}$$

Since:

$$-8\pi T = -8\pi T_\nu^\nu = 10\left(1-\frac{x_1}{c}\right)^{-2} \neq 0,$$

the energy cannot be solely electromagnetic ([1]). The energy tensor (41) corresponds, in fact, to a distribution of matter whose density is zero at $x_1 = \pm\infty$ and infinite at $x_1 = c$. The hydrostatic pressure in the matter is such that, at any point, the pressure in the x_1 direction is half that in the x_2 and x_3 directions. Although theoretically possible, such a distribution can hardly be said to have any physical counterpart.

§ 6. Conclusion. The disagreement between the results, for the fields of electromagnetic forces, on the general relativity theory and the theory of teleparallelism, pointed out in the last paragraph, provides one reason for rejecting the latter in favour of the former. It is true, of course, that there is no direct experimental evidence in favour of the field (40), but this result was arrived at on the basis of general relativity, for which experimental evidence can be found in other directions. The theory of teleparallelism, on the other hand, has provided no results, as yet, which are in accordance with experiment.

Another disadvantage of this latter theory is its rigidity; one set of mathematical assumptions with regard to the field-variables leads to one result only; on general relativity, the same set of assumptions leads to more than one, corresponding to the solutions of more than one physical problem.

As far as the investigations in this paper go, we therefore conclude that the theory of teleparallelism is unsatisfactory.

([1]) See A. S. Eddington, *The Mathematical Theory of Relativity*, (1924), Ch. VI, § 77.

On the theory of spaces with a RIEMANN metric and teleparallelism

By A. EINSTEIN

Some time ago, a general property of such spaces was proved, in which the question of its physical interpretation was temporarily deferred ([1]).

Let ($T^{\mu\nu}$) be a tensor that can have other indices besides the contravariant indices μ and ν. The following commutation rule for differentiation is always true then:

$$T^{\mu\nu}{}_{;\,\sigma;\,\tau} - T^{\mu\nu}{}_{;\,\sigma;\,\tau} \equiv - T^{\mu\nu}{}_{;\,\alpha}\,\Lambda^{\alpha}_{\sigma\tau}\,. \tag{1}$$

By contraction, it arises that:

$$T^{\mu\nu}{}_{;\,\nu;\,\mu} - T^{\mu\nu}{}_{;\,\mu;\,\nu} \equiv - T^{\mu\nu}{}_{;\,\alpha}\,\Lambda^{\alpha}_{\mu\nu}\,. \tag{1a}$$

From this, a simple conversion gives:

$$[(T^{\mu\nu} - T^{\nu\mu})_{;\,\nu} - T^{\sigma\tau}\,\Lambda^{\mu}_{\sigma\tau}\,]_{;\,\mu} + T^{\sigma\tau}\,\Lambda^{\alpha}_{\sigma\tau;\alpha} \equiv 0. \tag{2}$$

Only the anti-symmetric part of the tensor T enters into (2). We may therefore assume, with no loss of generality, that the tensor T is anti-symmetric, as far as the indices in question are concerned. With that, (2) takes the form:

$$[T^{\mu\nu}{}_{;\,\nu} - \tfrac{1}{2}T^{\sigma\tau}\,\Lambda^{\mu}_{\sigma\tau}\,]_{;\,\mu} + \tfrac{1}{2}T^{\sigma\tau}\,\Lambda^{\mu}_{\sigma\tau;\mu} \equiv 0. \tag{2a}$$

This relation can be further converted by means of the identity that follows from the integrability of parallel translation:

$$\Lambda^{\mu}_{\sigma\tau;\mu} \equiv \phi_{\sigma,\tau} - \phi_{\tau,\sigma} \qquad\qquad (\phi_\sigma = \Lambda^{\alpha}_{\sigma\alpha}), \tag{3}$$

or

$$\Lambda^{\mu}_{\sigma\tau;\mu} \equiv \phi_{\sigma;\tau} - \phi_{\tau;\sigma} + \phi_\mu\,\Lambda^{\mu}_{\sigma\tau}\,. \tag{3a}$$

([1]) The contents of the paper "Die Kompatibilität…" in these Berichte, 1930, I, will be assumed as known in the present paper.

In fact, due to (3*a*), we have:

$$\tfrac{1}{2} T^{\sigma\tau} \Lambda^{\mu}_{\sigma\tau;\mu} \equiv (T^{\sigma\tau} \phi_\sigma)_{;\,\tau} - \phi_\sigma T^{\sigma\tau}{}_{;\,\tau} + \tfrac{1}{2} \phi_\mu T^{\sigma\tau} \Lambda^{\mu}_{\sigma\tau} .$$

If one inserts the right-hand side into (2*a*), into which one simultaneously introduces the divergence operator:

$$A^{\nu}{}_{/\nu} = A^{\nu}{}_{;\,\nu} - \phi_\nu A^{\nu}, \tag{4}$$

where A^ν is a tensor of arbitrary rank with a covariant index ν, then one gets:

$$\left.\begin{aligned} &U^{\mu}{}_{/\mu} \equiv 0, \\ &U^{\mu} = T^{\mu\nu}{}_{/\nu} - \tfrac{1}{2} T^{\sigma\tau} \Lambda^{\mu}_{\sigma\tau}. \end{aligned}\right\} \tag{5}$$

Thus, starting from any tensor T with an anti-symmetric pair of indices $\mu\nu$, a tensor U^μ of rank one lower, whose divergence then vanishes identically, can be obtained by a linear differential operation.

Therefore, by way of example, starting from the tensor:

$$L^{\alpha}_{\underline{\mu\nu}} = \Lambda^{\alpha}_{\underline{\mu\nu}} + a(\phi_{\underline{\mu}} g^{\nu\alpha} - \phi_{\underline{\nu}} g^{\mu\alpha}) + b S^{\alpha}_{\underline{\mu\nu}} , \tag{6}$$

in which *a*, *b* are arbitrary constants, and we have set:

$$S^{\alpha}_{\underline{\mu\nu}} = \Lambda^{\alpha}_{\underline{\mu\nu}} + \Lambda^{\mu}_{\underline{\nu\alpha}} + \Lambda^{\nu}_{\underline{\alpha\mu}} , \tag{7}$$

we can derive the tensor:

$$G^{\mu\alpha} = L^{\alpha}_{\underline{\mu\nu}/\nu} - \tfrac{1}{2} L^{\alpha}_{\underline{\sigma\tau}} \Lambda^{\mu}_{\sigma\tau} , \tag{8}$$

whose /-divergence, when taken over μ, vanishes identically:

$$G^{\mu\alpha}{}_{/\mu} \equiv 0. \tag{8a}$$

From this, it follows that the system of equations:

$$G^{\mu\alpha} = 0 \tag{9}$$

is a compatible system of equations for the h^{ν}_{s} that might also be affected with the constants *a* and *b*.

ABSOLUTE PARALLELISM AND UNITARY FIELD THEORY

E. CARTAN

The first attempts to construct a unitary theory of the gravitational field and the electromagnetic field returned to the ideas that spawned general relativity. The theory of Einstein, when reduced to its essentials, reduces the physical theory of gravitation *in vacuo* to geometry. Spacetime is described by a four-dimensional Riemannian manifold whose curvature expresses the apparent deviations of the principle of inertia that are produced by gravitation; as for the laws of gravitation themselves, they are expressed by certain geometric restrictions that are imposed on the general schema of the four-dimensional Riemannian manifold, restrictions that analytically translate into the ten celebrated Einstein equations.

In this theory, there is no place for the electromagnetic field, electricity, or matter. One arrives at the truth by considering matter to be the generator of the gravitational field, but in the form of a point-like singularity of spacetime. As far as electromagnetism is concerned, the Maxwell equations may not be associated, either locally or asymptotically, with any geometric property of a Riemannian manifold; only the electromagnetic energy tensor is susceptible to an interpretation in terms of Riemannian geometry.

The success itself that was obtained by Einstein in the theory of the pure gravitational field must lead to the search for a more complete theory. Everyone that attacked the problem took essentially the same viewpoint as Einstein: find a geometric schema that realizes all that general relativity has realized for just the gravitational field *in vacuo*, but also for the electromagnetic field, electricity, and matter. H. Weyl was the first to imagine a metric manifold in which there exists no absolute unit of length, or rather, in which the units of length that are chosen by different observers may be compared locally, and for two given observers the result of the comparison varies with the choice of intermediate observers. Our intention here is neither to study the theory of Weyl nor to describe the history of the work that followed. We direct our attention to the latter attempts of Einstein that were founded on the notion of a Riemannian space with absolute parallelism. Moreover, some of the observations that we will be led to formulate depend on the principle of the theories of the geometrization of physics itself.

I

As one knows, Riemannian geometry is a generalization of elementary, or Euclidean, geometry. Riemann founded it by detaching the notion of distance from geometry and, in order to define the distance between infinitely close points, by giving it an analytical expression, *a priori,* that is analogous to the one that provided the theorem of Pythagoras, but more general. The possibility of constructing a geometric theory on such foundations that preserves at least some of the Euclidean geometric notions is guaranteed by the following remark: in elementary geometry, the given of the distance between two infinitely close points suffices to reconstruct the entire edifice of that theory. The same procedures that one uses to locally reconstruct a given Euclidean space by its ds^2 ([1]) in curvilinear coordinates may also be employed, in part, when one is concerned with an arbitrary ds^2; from this one may arrive at the very important notion of *parallelism* that was introduced by Levi-Civita; thanks to that notion, it is possible to say that two directions with *infinitely close* origins are or are not parallel, and form this or that angle. One knows the physical importance of this notion in general relativity: when a material point of very small mass is placed in a vacuum inside a gravitational field it moves in such a way that the world-vector that represents its momentum and energy remains constantly parallel – or rather, *equipollent* – to itself; in other words, it obeys the law of inertia. The parallelism of Levi-Civita is related to this (*vincolato*), in the sense that if a vector is displaced by parallelism in such a manner that its origin goes from a point A to a point B then the final position of the vector depends on the path followed from A to B; parallel transport is not integrable.

Riemannian geometry, when completed with the discovery of Levi-Civita, was used that way by Einstein in his general theory of relativity ([2]). One may nevertheless remark that nothing obligates us to think that this geometry corresponds to physical reality. Indeed, we start with the hypothesis that is quite difficult to not admit that our space, without being Euclidean, may be reduced to a Euclidean space in any sufficiently small region. Imagine some physicists of the Euclidean mentality; each of them makes his observations in his immediate neighborhood, and will naturally adopt a rectangular coordinate system and place himself at its origin. If two neighboring physicists want to coordinate their observations, then they must localize the reference system of the second with respect to that of the first. They carry out some physical procedure, into whose nature we shall not enter, that permits us to say:

1. That the origin of the second triad has such-and-such coordinates with respect to the first;
2. That the axes of the second triad make such-and-such angles with the axes of the first.

Physics therefore gives us:

1. The distance between two infinitely close points, in other words, the ds^2 of the space.
2. The angle between two directions that issue from two infinitely close points, in other words, the law by which a vector may be locally transported parallel to itself.

([1]) ds^2 is the expression that gives the square of the distance between two infinitely close points.
([2]) In reality, the discovery of Levi-Civita came after general relativity, but the notion of parallelism has served to make it much more intuitive.

We thus recover the two fundamental notions of distance and direction here, but these two notions appear independently of each other here. Indeed, there is no reason, *a priori,* other than the reason of geometric simplicity, to think that the parallel transport that is provided by physical observation coincides with the parallel transport that is deduced from the ds^2 of space according to the law of Levi-Civita. The geometric schema that logic itself imposes upon us in order to recover the laws of physics must therefore be more general than that of classical Riemannian geometry, because once one is given the ds^2 of space one may imagine an infinitude of distinct laws of parallel transport.

II

In his latter work on unitary field theory, Einstein does not take a viewpoint that is as general as in the foregoing. He admits that the final position of a vector that is transported by parallelism in such a manner that its origin goes from a point A to a point B does not depend on the intermediate path that was followed, in other words, that parallel transport is integrable, or furthermore, that the angle between two vectors whose origins are arbitrary has an absolute significance: it is that of *absolute parallelism* (Fernparallelismus).

It is easy to describe the most general manner of defining an absolute parallelism in a given Riemannian space. Indeed, attach reference systems or rectangular frames to the various points of space according to some arbitrary law; it then suffices to agree that two vectors with arbitrary origins, A and B, are parallel or rather, equipollent if they have the same projections on the axes of the reference systems with their origins at A and B; these reference systems themselves will be called parallel. There are thus an infinitude of possible absolute parallelisms in a given Riemannian space since the by which one attaches a rectangular frame to a point of space is completely arbitrary; however, it is important to remark that if one makes all of the frame turn about their origins in the same manner then one obtains the same absolute parallelism; as a result, one may give the frame that is attached to a particular point of space once and for all.

One may also arrive at the notion of a Riemannian space with absolute parallelism by following a path that is the opposite of the preceding one. One first defines an absolute parallelism in an n-dimensional *non-metric* manifold by attaching Cartesian frames to the different points M of this manifold, that are formed from n vectors whose origin is at M, and agreeing to say that two vectors with their origins at M and M' are equipollent if they have the same projections on the coordinate vectors that are attached to at M and M'. One then introduces the metric by agreeing, for example, that the square of the distance between two *infinitely close* points M and M' is equal to the sum of the squares of the projections of the vector MM' onto the coordinate vectors whose origin is at M. Naturally, one may obtain a different metric if one attaches another system of Cartesian frames that are mutually equipollent to the different points of space, while preserving the previously defined parallelism.

From the foregoing, one sees that the metric and the parallelism are dependent on each other, but each of them may be defined arbitrarily; once the metric is given, there are an infinitude of absolute parallelisms that are compatible with that metric; once the absolute parallelism is given, there are an infinitude of metrics that compatible with that absolute parallelism.

In classical Riemannian geometry the notion of *Riemannian curvature* plays a fundamental role; it is related to the deviation that a vector experiences when one transports it by parallelism by making it describe a closed circuit, or cycle, about its origin. This notion, when envisioned from the preceding viewpoint, disappears in the new Einsteinian schema since parallelism has an absolute significance; one may say that a *Riemannian space with absolute parallelism has no curvature.* Meanwhile, there is something that differentiates such a space from Euclidean space, and that is its *torsion.*

In order to make this new notion neatly comprehensible, recall some well-known properties. One knows that in ordinary the coordinates of a point M with respect to a rectangular system of axes with origin O are the projections of the vector $\overrightarrow{OM}$ on these axes; one may also obtain them by connecting O to M by a broken line and summing the projections of the different parts of that line. One may likewise take a curved line, which is to be regarded as the limit of a broken line. Now imagine an observer that is placed in a Riemannian space with absolute parallelism, but has a Euclidean mentality. If the observer that was placed at O and has adopted a rectangular system of axes with origin O wants to calculate the coordinates that one must attribute to a point M then he connects O to M by a continuous line and proceeds as we have always done: he regards the line OM as the geometric sum of a very large number of small vectors. He transports them to O parallel to themselves and forms their geometric sum. He thus finds a vector with its origin at O that he considers to be equipollent to the line OM, and whose projections on the axes will be the desired coordinates. However, if the observer joins O to M by another line, he will be led to consider it as equipollent to a second vector, *which not be the same as the first vector, in general.* In other words, the various lines that join O to M are not all equipollent to the same vector.

One may present other things. If one considers a closed contour or cycle C that is traversed in a certain sense, in the context of Euclidean geometry, then it is equipollent to a null vector, from a fundamental theorem of the theory of vectors; in a Riemannian space with absolute parallelism, this is no longer the case: the cycle C is equipollent to a certain vector that one calls the *torsion vector* of the cycle. It is only in Euclidean space that all vectors have a null torsion vector.

The notion of torsion may also be introduced in a Riemannian space with a parallelism that is not absolute, but it is more difficult to explain in the general case. We content ourselves by pointing out that the classical Riemannian space, which has Levi-Civita parallelism, is endowed with curvature, but not torsion; on the contrary, the new Einsteinian space is endowed with torsion, but not curvature.

One imagines that the analytical expression for the torsion of a space involves a tensor with three indices. Indeed, any cycle may be decomposed into elementary parallelograms; on the other hand, the torsion vector of such a parallelogram involves three directions, those of the edges of the parallelogram and those of the torsion vector itself; a series of indices corresponds to each of these three directions. In reality, the torsion vector of an infinitely small parallelogram is proportional to the area of that parallelogram, and the factor of proportionality appears in the torsion tensor Λ_{ij}^{k}.

The various components of torsion are not absolutely arbitrary functions; they satisfy certain identities, and it will suffice for us to indicate their geometric significance. Consider a three-dimensional volume in space; decompose the closed surface that bounds it into a large number of small areas, which are bounded by cycles that are all described in the same sense. The geometric sum of the torsion vectors of all of these cycles is null. This theorem is a particular case of the general theorem of the conservation of curvature and torsion.

III

We first approach the problem of the unitary theory by basing on the notion of a Riemannian space with absolute parallelism. From the general ideas of Einstein, there is nothing to stop us from passing to a rigorously Euclidean universe. Such a universe is physically impossible: its metric may be produced only by the presence of material bodies, and the existence of these bodies is sufficient to make the universe no longer Euclidean. However, *all of the intrinsic geometric properties that characterize a Riemannian space with absolute parallelism are derivable from its torsion,* and are expressed analytically by means of the components of the torsion tensor and their covariant derivatives of various orders. Therefore, if physics is geometrizable, then it must be true that all of the physical laws are expresses by partial differential equations between the components of the torsion. On the other hand, it is natural to admit that all of the physical laws are logical consequences of a finite number of them. The problems that the unitary theory poses are thus the following:

PROBLEM A. – *By what partial differential equations E must the general schema of a Riemannian space with absolute parallelism be restricted in order to obtain a faithful image of the physical universe?*

PROBLEM B. – *Integrate the equations E and recover matter, electricity, and the gravitational-electromagnetic field in the solutions so obtained, and in the various manifestations that experiment reveals to us.*

IV

We first occupy ourselves with problem A. Apparently, it may be solved only if we have prior knowledge of the physical laws. This is true, but to some extent much less than one would think, *a priori*. Indeed, from the logical conditions that are imposed the nature of the question itself and conditions of analytical simplicity that are reasonable to accept, it suffices to add just one condition, which comes from physical determinism, in order that Problem A admit only a very restricted number of solutions, in such a way that the physicist, if the attempts of Einstein were not in vain, will only have to choose between a small number of universes that are constructed by a purely deductive method.

We briefly review the conditions to which we alluded that the equations E must verify.

1. *Logical conditions.* – Equations E must obviously express the intrinsic geometric properties of space. In order to describe them effectively, one may – and this is the simplest procedure – attach rectangular frame to the various points of space that are mutually equipollent. Equations E are then expressed by relations between the components of torsion, as referred to these frames, and their various derivatives. However, these relations must remain the same if one chooses the frames that are attached to the points of space in another manner, with the frames still mutually equipollent, since otherwise the equations E express particular properties of the chosen frames, rather than intrinsic properties of space.

The latter stated condition may be extended in a large or restricted way. As one knows, in ordinary space there exist two distinct categories of triads: direct triads and inverse triads. An inverse triad may be obtained as the mirror image of a direct triad. There is one analogous distinction in spaces of arbitrary dimension. One may then imagine a system of equations E that retain their form in all direct rectangular systems of reference, but which change their form for the inverse frames. Such a system will correspond to one universe, in which the set of laws for the gravitational-electromagnetic field enjoy a type of polarization: for example, if one considers a system of electric charges and their evolution in a certain interval of time, and this evolution will be impossible if one reverses the sense of the interval; physics will be *irreversible.* There is no correlate to this in classical theory; however, it is not forbidden to think that the irreversibility of physics eludes our experiments due to the feebleness of the fields that enter our domain of immediate observation.

One may also demand that the equations E be independent of the choice of the unit of length; in this case, they must satisfy certain supplementary conditions of homogeneity. One is or is not constrained to restrict this homogeneity depending on whether one does or does not admit, *a priori,* that there exists no unit of length - or rather, of *interval* - that plays a privileged role in the universe.

We must add another observation. Equations E must depend on both the metric and the parallelism, because physical laws must obviously involve the metric, and we know that the metric alone is not sufficient to specify them. For example, one recognizes that equations E depend only upon parallelism if they preserve their form upon replacing all of the chosen rectangular frames by another system of *Cartesian* frames, rectangular or not, that are equipollent to them. Similarly, the old equations of general relativity may be written by introducing an absolute parallelism into a classical Riemannian space, but it is clear that they express properties of space that are independent of this parallelism; for this reason alone, they must be rejected.

2. *Conditions of analytical simplicity.* – From a purely logical viewpoint, these conditions present a great degree of arbitrariness. It is natural to admit, with Einstein, that equations E must involve only the first order derivatives of torsion, and these linearly, while reserving the possibility that there are other terms that contain the components of torsion; these terms will be quadratic if the equations are homogenous in the sense that was indicated above.

3. *Compatibility conditions and conditions that come from physical determinism.* – The compatibility conditions of a differential system are in the domain of mathematical

technique. At this point in analysis, the actual theory of partial differential equations allows us to decide the compatibility of a system only if one occupies oneself with *analytic* solutions of that system, which is itself assumed to be formed from *analytic* equations ([1]). *A priori*, there is no reason to assume that the laws of physics are expressed by means of analytic functions; this is a hypothesis that we will be obliged to make – for lack of anything better! If one admits that, then one has the means to decide whether such a system of equations *E* that satisfies the previously stated conditions is compatible or not. To say that it is compatible is to simply affirm the existence of *locally* defined *analytic* solutions, i.e., solutions that are defined in a sufficiently small neighborhood of a point in spacetime. If the compatibility, in this sense, is necessary in order for the equations *E* to lead to the desired image of spacetime, then it is obviously not sufficient; this is an important point that we shall return to later on.

It is not sufficient that equations *E* be compatible; one must further have that they are not in disaccord with physical determinism.

This is an extremely important point that has not been given enough attention in the various discussions that followed the creation of general relativity.

In the ordinary sense of the word, we affirm that physical determinism gives the state of the universe at a completely determined given moment in its ultimate evolution. Of course, one must specify what one means by the *state* of the universe. The classical mechanics of material points conforms to determinism, with the condition that we call the state of a point at a given instant the set of its position *and velocity*. In any physical theory that is based on partial differential equations, one imagines that one may precisely define what one means by *state* in order for this theory to conform to determinism.

What complicates a few things is precisely the fact that the theory of relativity tells us that time is inseparable from space; to speak of the state of the universe at a given instant does not have an absolute sense. In reality, one must speak of the state of the universe in a three-dimensional section of spacetime.

But then some other difficulties present themselves, which Hadamard has drawn attention to. In reality, there is mathematical determinism and there is physical determinism. It may happen that the state of the universe in a three-dimensional section of spacetime involves the state of the universe in the neighboring sections *or the physicist would have to confirm this;* this amounts to saying that a very weak variation of the state of the universe in the given section may, in a certain case, involve enormous variations in a section that is as close as one wants to the first one; the dependency of the states on the two sections is therefore completely masked from the physicists. In classical electromagnetism, there is mathematical determinism for almost all of the three-dimensional sections of spacetime, but there is physical determinism only for sections that do not penetrate the interior of the time cone at each of their points.

Of course, equations *E* on which the unitary field theory will be founded, will be too complicated for one to study anything but mathematical determinism, but they must conform to this determinism. If one confines his ambitions to that much, then the actual state of analysis permits us to decide whether this or that system of equations *E* conforms

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([1]) In this phrase, the word *analytic* has a very precise technical sense. A function is analytic if it is developable into a power series.

to determinism ([1]). For example, one may be tempted to think that the notion itself of a Riemannian space with absolute parallelism expresses all of the laws of the field, and that no restrictive equation is necessary; however, this will be contrary to determinism. The geometric schema will be too general.

One may apply these latter considerations to the old theory of general relativity. In the formulations that one habitually makes, any four-dimensional Riemannian space is likely to represent a possible universe (with no electromagnetic field). The regions of space in which the ten Einstein equations are verified are the ones in which no matter exists; as for the others, the state of matter, which is composed of its density, the velocities of its particles, and its elastic pressures is only the physical manifestation of a purely geometric spacetime tensor. One must reject this formulation, even in the absence of any electromagnetic field, because it does not conform to determinism. Indeed, to know the evolution of this material fluid, one must know not only the state of this fluid at a given instant, but also the distribution of its elastic pressures at all instants of the time interval. This is not to say that gravitation and matter do not obey the laws that are indicated by that theory, but only that these laws are not the only ones; one may say that one has a theory of phenomena that is not *explanatory*, but at most *descriptive.*

V

We return to the problem of the unitary field. Thanks to more-or-less artificial restrictions that convenience or the insufficiency of our knowledge demands that we state, the conditions of various types that we would like to use to restrict the equations E permit us to solve problem A completely. Einstein has indicated a solution that involves twenty-two equations. There are several others, some of them with fifteen equations, others with perhaps sixteen, and still others twenty-two equations. One may think that there is good reason to prefer the systems that contain the largest number of equations; for the most part, this is a matter of personal taste. In the presence of the system of Einstein, there only remains one other system that consists of the same number of equations and involves two absolute numerical constants, which are arbitrary, *a priori.* Such a system will correspond to irreversible physics, at least if one of the constants is non-null, but the irreversibility is not obvious in the first approximation. We must, moreover, reject that system for the following reason: it contains the original ten equations of gravitation, which involve only the metric. This is quite improbable, although it is logically possible. By definition there remains only the system of Einstein, which was found by almost miraculous intuition. It is therefore upon this system that the destiny of the new unitary theory rests.

VI

We now arrive at problem B. It amounts to the problem of integrating the twenty-two Einstein equations and recovering the field, matter and electricity. In the program that Einstein discussed at the two conferences he addressed in November, 1929 at l'Institut

([1]) Of course, one is constrained to consider only the *analytic* solutions of equations E and the states that are expressed by means of analytic functions. Under these restrictions, the mathematical problem becomes absolutely intractable.

Henri-Poincaré, he proposed to look for the physical laws in the *non-singular* solutions of his equations; matter and electricity therefore exist only in the *continuous* state. We place ourselves in the terrain that he chose, without being too surprised that the path we follow seems to be in opposition to the one that contemporary physicists have followed with success.

A first difficulty of an exclusively mathematical nature presents itself. Indeed, not only does one have no method for finding the non-singular solutions of a differential equation, but furthermore, there is no reason to admit that such a system, which is compatible in the *local* sense that was indicated above, will also be compatible in the *integral* sense. In truth, one may easily indicate some non-singular solutions of the new system of Einstein, but these are isolated solutions that are too small in number for one to hope to found a physical theory upon them.

The problem is further complicated here as a result of the following circumstance. The four variables that permit us to localize a point in spacetime do not appear in the equations *E* such as were considered above; however, in practice one must express the functions (16 in number) that define the metric and the absolute parallelism of the universe by means of these four variables. Now, when presented in this form, the statement of the problem involves a gratuitous hypothesis, namely, that one may establish a bijective point-like correspondence between spacetime and a four-dimensional Euclidean space. However, there is no reason to assume that spacetime enjoys the same topological properties as a Euclidean space. One may envision many other hypotheses, for example, that spacetime is closed. All that one has a right to demand is that the points of a sufficiently small region of the universe may be framed by four numbers, without which there would necessarily exist a framing that is valid for the entire universe. As long as one retains the viewpoint of *local* integration, the topological properties of the universe do not enter into the discussion, but they must necessarily play an important role, and preponderate when one seeks a solution without singularities that exists in all of space.

One sees as well that the search for the local laws of physics may not be disassociated from problem of cosmogony. Moreover, one may not say that the one precedes the other; they are inextricably linked to each other.

One confirms the preceding viewpoints by considering a system that is analogous to the system of Einstein, which Einstein has imagined, but immediately rejected, and with good reason, moreover. It is the system of equations that expresses that a space with absolute parallelism has constant torsion; this signifies that two equipollent cycles have equipollent torsion vectors, and, analytically, that the components A_{ij}^k of torsion, when referred to frames that are mutually equipollent, are constants. This system is, moreover, independent of the metric of space. The theorem of the conservation of torsion shows that the constants A_{ij}^k are not arbitrary, but are linked by certain algebraic relations. The search for spaces with constant torsion is only a well-known problem of analysis, but in a new geometric form, because these spaces are none other than representation spaces of the transformations of a finite continuous group. Now, the integration that provides the spaces with given constant torsion leads to one or several non-singular solutions, depending on the case. When there are several, they each correspond to topologically distinct spaces.

VII

Whereas one may be stopped in the solution of problem B by the mathematical difficulties that we just pointed out, one may nevertheless infer some important physical consequences from the continuity hypotheses that were formulated by Einstein, which are, moreover, in agreement with the concepts that contemporary physics tends to make on matter. We have said above that equations *E* must conform to mathematical determinism, i.e., the state of the universe in neighboring sections. Now, this may not be the case for certain particular sections that one calls *characteristic*. These characteristic sections play an important role in physics; for example, the equation of the propagation of light admits characteristic sections that are three-dimensional sections that are tangent to each of their points at the light-cone relative to this point. Now, in the unitary theory that is founded on the notion of a Riemannian space with absolute parallelism, it is quite easy to account for the fact, *a priori* - and this is true precisely because of the invariant character of equations *E* with respect to a rotation of the frames – that the only possible characteristic sections are the ones that are tangent at each of their points to the light-cone at that point, and these characteristic sections will exist essentially when the equations *E* include the metric of space, as is necessary. One therefore dramatically recovers the classical laws of propagation for light as a logical consequence of the metric character of space.

This conclusion is only natural. However, there is something much more disconcerting. In the classical theories that concerned matter in a continuous state, for example, hydrodynamics, there are other characteristic sections than the ones that refer to the propagation of light, which are the ones that are generated by the world-lines of the various material points that comprise the fluid considered; for these characteristic sections, these world-lines play the same role as the light rays in the latter, and they are obviously completely distinct. Since any characteristic section of this type may present itself in the unitary theory of Einstein, one is led to think that this theory will be obliged to deny the physical individuality of the various points of the fluid that comprise the material or electrical fluid that is assumed to be in the continuous state. The material point is a mathematical abstraction that we have assumed, as is the custom, and concluded by attributing a physical reality to it. It is furthermore an illusion that we must abandon if the unitary theory of fields is to be established.

If one wants to discuss the preceding, one sees the variety of aspects that one must envision for the unitary theory of fields, and also the difficulty of the problems that it raises. However, Einstein is not one to shrink from difficulties, and likewise, if his attempts are not successful, then we shall be forced to reflect upon the great questions that lie at the foundations of science.

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Direction fields and teleparallelism in n-dimensional manifolds

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Introduction

1. The n-dimensional manifolds that will be considered in this paper will be closed and continuously differentiable ([1]). The question of whether a non-singular, continuous direction field exists on such a manifold is answered by the following well-known theorem ([2]):

Theorem A_1 . *A singularity-free, continuous direction field exists on the manifold M^n iff the Euler characteristic of M^n has the value* 0 *(§ 5, no. 2).*

Therefore, on the one hand, amongst all closed and orientable surfaces, the ones with the topological type of the torus are the only ones that admit the existence of a continuous direction field ([3]); on the other hand, one can endow any manifold of odd dimension – in particular, any three-dimensional manifold – with a continuous direction field (§ 6, no. 1).

However, since one would not expect that all manifolds of odd dimension behave precisely the same way in relation to the continuous direction fields that exist on them, the contradiction that was formulated just now (e.g., between $n = 2$ and $n = 3$) compels one to look for a refinement of the original question. The following question is closely related: Let an n-dimensional manifold M^n and a number m from the sequence 1, 2, …, n be given. *Is there a system of m direction fields on M^n that are linearly independent at every point of M^n?*

This question, which is answered by Theorem A_1 for $m = 1$, and which commands special and self-evident interest for $m = n - 1$ and $m = n$ (cf., no. 5 of this introduction), defines the subject of the present paper. Indeed, the question will not be answered completely, in the sense of presenting the generalization of Theorem A_1 to a necessary and sufficient condition for the existence of a system of m independent direction fields – in the sequel, referred to briefly as an "m-field." Rather, some theorems will be proved that, on the one hand, serve to resolve the problem in many special cases, and which, on the other hand, represent new contributions to the general topology of closed manifolds.

([1]) Cf., chap. XIV, § 4 of *Topologie* (v. 1) of Alexandroff and Hopf (J. Springer, Berlin, 1935). This book, whose terminology we will follow in this paper, will be briefly referred to as "AH" in the sequel.

([2]) AH: chap. XIV, § 4, Theorem III.

([3]) Poincaré, Journal de Liouville (4) **I**, pp. 203-208.

2. Before we formulate the most important theorem, we recall a theorem that is related to Theorem A_1 and is likewise well-known ([1]):

Theorem B_1 . *There exists a direction field on any manifold M^n that is singular (i.e., discontinuous) at no more than finitely many points. The number of these singularities, when counted with the correct multiplicities ("indices"), is independent of the particular field: It is always equal to the characteristic of M^n (§ 5, no. 2).*

We shall prove the following generalization of this theorem:

Theorem B_m . *For any m* ($1 \leq m \leq n$), *there exist m-fields on any M^n whose singularities (i.e., points of discontinuity for the individual direction fields or points of linear dependency for the various fields) define a complex of dimension at most* $m - 1$. *With a correct enumeration of the multiplicities of the singularities, it is a cycle, and the homology class of this cycle is independent of the particular m-field: It is a distinguished element of the $(m - 1)^{th}$ Betti group* ([4a]) *of M^n (§ 4, no. 4, 5).*

We shall call this homology class F^{m-1} the "*m^{th} characteristic class*" of M^n. In the case of $m = 1$, it is the zero-dimensional homology class that consists of a point of M^n, multiplied by the Euler characteristic.

Theorem A_1 will now be generalized, in a certain sense, by way of the following theorem:

Theorem A_m . *There exists an m-field on M^n whose singularities define a complex of dimension at most* $m - 2$ *iff* $F^{m-1} = 0$ *(i.e., the zero element of the $(m - 1)^{th}$ Betti group of M^n) (§ 4, no. 5).*

It follows from this immediately that:

Theorem A'_m. *In order for a singularity-free m-field to exist on M^n, it is necessary that:*

$$F^0 = F^1 = \ldots = F^{m-1} = 0.$$

However, this condition might not be sufficient.

3. This suggests the problem of determining the characteristic classes F^{m-1} ($m = 1, 2, \ldots$) for a given M^n. In the case $m = 1$, the determination of F^{m-1} is equivalent to the determination of the Euler characteristic of M^n, and on the basis of the Euler-Poincaré formula:

$$\sum (-1)^r a^r = \sum (-1)^r p^r,$$

([1]) AH: chap. XIV, § 4, Theorem I.

([4a]) The coefficient domain to which these Betti groups relates is defined in § 4, no. 3 (cf., also AH: chap. V).

in which the a^r refer to the numbers of r-dimensional cells in a decomposition of M^n and p^r means the r^{th} Betti number of M^n, one can express it in two different ways: namely, in terms of the a^r and in terms of the p^r.

The first of these two possibilities seems to be capable of being carried over to an arbitrary m (§ 5, no. 3, footnote 22); however, the more important question is whether one can also represent the class F^{m-1} in a way that corresponds to the representation of the characteristic on the right-hand side of the Euler-Poincaré formula, and thus in terms of known topological invariants of M^n. Moreover, if the answer to this question, which was unknown to us up till now, is in the negative then that would teach us something new: F^{m-1} would be a *new topological invariant* of a manifold.

There exists yet another relationship between the class F^{m-1} and the Euler characteristic, in another regard: The intersection number of F^{m-1} with an $(n - m + 1)$-dimensional manifold that is embedded in M^n is congruent (mod 2) to the characteristic of that manifold, as long as the embedding fulfills certain requirements that are formulated in § 6, no. 2.

4. The determination of F^{m-1} for a given manifold is achieved in some cases with the help of Theorem B_m alone; on the basis of that theorem, one indeed needs to construct only a *special m*-field that is constructed so neatly that one can specify the complex by means of its singularities. In this way, we will treat the $(4k + 1)$-dimensional projective spaces as an example; it will be shown that:

Theorem *C. For the (4k + 1)-dimensional real projective space P^{4k+1}, F^1 is the class that contains the projective line, so it is therefore non-zero (§ 6, no. 3).*

This theorem, as well as in the fact that there is a continuous direction field on any odd-dimensional manifold, includes the fact that:

Theorem *C′. There is a continuous direction field on P^{4k+1}, so for any pair of fields there exist points at which the directions of the two fields are either equal or opposite.*

This property of projective spaces allows one to prove certain algebraic theorems whose proofs seem to be unknown, up to now, when one works with the usual algebraic lemmas (§ 6, no. 3).

5. The question of whether an n-field exists on an M^n deserves a special and self-evident interest; namely, the existence of such a field is equivalent to the idea that one can introduce a *teleparallelism* on M^n, or, as we also say, that M^n is "*parallelizable.*" Therefore, we call M^n parallelizable when one can decompose the totality of *all* directions in M^n into mutually disjoint, single-valued, and continuous direction fields that we call "parallel fields," such that the following condition is fulfilled: If $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_k$ are directions at a point p of M^n and $\mathfrak{v}_1', \mathfrak{v}_2', \ldots, \mathfrak{v}_k'$ are the same directions at another arbitrary point p', as deduced from some parallel fields, then the linear independence of the $\mathfrak{v}_i'$

follows from the linear independence of the $\mathfrak{v}_i$. We will briefly call directions "parallel" when they are taken from the same parallel field.

In fact, one easily sees that parallelizability is identical to the existence of an *n*-field: If an *n*-field exists then one calls two directions $\mathfrak{v}$, $\mathfrak{v}'$ at the points p and p', resp., "parallel" in the event that their components relative to the directions of the *n*-field at p and p', resp., agree with each other, up to a positive factor; one has then introduced a teleparallelism. On the other hand, if a teleparallelism is defined then one distinguishes n linearly-independent directions at a fixed point; the directions that are parallel to these directions at the remaining points of M^n then define an *n*-field.

Non-orientable manifolds are not parallelizable. On the other hand, one easily shows that the existence of an *n*-field on an orientable manifold already follows from the existence of an $(n - 1)$-field. With that, the examination of parallelizability is completely converted into the examination of $(n - 1)$-fields. It is therefore no restriction when we assume that $m < n$ in what follows. Theorem A'_m yields:

Theorem *D*. *The vanishing of all characteristic classes $F^0, F^1, \ldots, F^{n-2}$ is necessary for the parallelizability of M^n.*

Here, as well, – confer Theorem A'_m – one should not assume that the condition is sufficient.

Since a *group manifold* ([1]) is certainly parallelizable, Theorem *D* yields a necessary condition for a given manifold M^n to be able to be made into a *group space.*

6. All manifolds for which the Euler characteristic is non-zero are certainly non-parallelizable – like, e.g., the spheres of even dimension – so one indeed also has $F^0 \neq 0$; neither are the projective spaces of dimension $4k + 1$ that were mentioned in Theorem *C*. By a product construction, one can further prove:

Theorem *E*. *For any dimension n that is different from 1 and 3, there are n-dimensional (closed and orientable) manifolds that are non-parallelizable (§ 6, no. 2).*

For $n = 1$, there is a single closed manifold, namely, the circle; it is trivially parallelizable. The question of parallelizability is then first open only for $n = 3$, and there one has:

Theorem *F*. *Any three-dimensional closed and orientable manifold* ([5a]) *is parallelizable (§ 5, no. 3).*

This remarkable special position of dimension three once again points to the difficulty in the search for a classification of three-dimensional manifolds; the attempt to divide the orientable three-dimensional manifolds into parallelizable and non-parallelizable ones would then fail.

([1]) AH: Introduction, § 3, no. 17; there, you will also find references.
([5a]) In addition, the manifold must fulfill certain differentiability assumptions (cf., § 5 and Appendix I).

7. The theorems that were stated in this introduction will be formulated and proved in §§ 4-6; §§ 1 and 2 have a preparatory character. In § 1, only the definition in no. 1 and the results of no. 4 are important for the remaining part of the paper. In Appendix I, the determination of the class F^1 for three-dimensional, orientable manifolds will be discussed in detail that was only suggested in § 5, no. 3. Appendix II subsequently arises; in it, it will be proved that a manifold with an odd characteristic that lies in Euclidian space cannot be represented by regular equations ([1]).

I have already reported on the individual partial results of this paper in other places (Verh. der schw. naturf. Gesellschaft, 1934, pp. 270; furthermore, Enseignement mathématique, 1934, 1, pp. 6).

At this point, I would like to thank Herrn Prof. H. Hopf for the impetus to do this work and for his enduring interest in its progress, as well as for his worthwhile advice at decisive moments.

§ 1. The manifolds $V_{n,m}$.

1. Definitions. In the sequel, we shall call an ordered, normalized orthogonal system $\sigma_{n,m}$ of m vectors $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_m$ that contact a point in n-dimensional Euclidian space R^n an *m-system in R^n*. In this, let m be constrained by the inequalities:

$$0 < m < n. \tag{1}$$

$V_{n,m}$ is defined to be the set of all m-systems $\sigma_{n,m}$ at a fixed point of R^n. If one introduces a notion of neighborhood into this set in a natural way then $V_{n,m}$ becomes a topological space whose points v are the m-systems $\sigma_{n,m}$.

$V_{n,1}$ is homeomorphic to the $(n-1)$-dimensional sphere S^{n-1} that it traced out by the endpoints of the vector $\mathfrak{v}_1$. However, if $m > 1$ then we displace the vectors $\mathfrak{v}_2, \ldots, \mathfrak{v}_m$ of $\sigma_{n,m}$ parallel to the endpoint of the vector $\mathfrak{v}_1$. Therefore, $V_{n,m}$ can also be described as the set of all $(m-1)$-systems in R^n that are tangential to S^{n-1}. In particular, $V_{n,2}$ is the set of directed line elements on S^{n-1}.

One can arrive at another representation of the space $V_{n,m}$ by *stereographic projection*, which we will briefly denote by V in what follows: If one projects S^{n-1} from its North Pole onto its equatorial space R^{n-1} then a system $\sigma_{n,m-1}$ that contacts the sphere at a point p goes to an $(m-1)$-system $\sigma_{n-1,m-1}$ in R^{n-1} that contacts the image point p_1 to p. $\sigma_{n-1,m-1}$ is established uniquely by its contact point p_1 and the $(m-1)$-system that is parallel to $\sigma_{n-1,m-1}$ of a $V_{n-1,m-1} = V_1'$ that is embedded in R^{n-1}. A point v of V is thus given by a point p_1 of R^{n-1} and a point v_1 of V_1'. We briefly write:

$$v = p_1 \times v_1 . \tag{2}$$

([1]) One can also confer AH: Introduction, § 1, no. 7.

This representation breaks down only for those systems $\sigma_{n,\,m-1}$ that contact the North Pole. In order to also treat these systems, we project S^{n-1} onto R^{n-1} from the South Pole. Analogous to (2), one gets:

$$v = p_2 \times v_2\,. \tag{2}$$

v_2 is a point of the set V_2' that features in place of V_1' under the second projection. If we denote the equatorial sphere of S^{n-1} by S^{n-2} then the two points p_1 and p_2 go to each other under the transformation by means of reciprocal radii in S^{n-2}.

Formula (2) describes a relationship between V and V_1'; i.e., between $V_{n,\,m}$ and $V_{n-1,m-1}$. By iteration, we obtain a relation between spaces of the sequence:

$$V_{n,\,m}\,,\ V_{n-1,\,m-1}\,,\ \ldots,\ V_{n-k,\,m-k},\ \ldots,\ V_{n-m+1,\,1} = S^{n-m}. \tag{4}$$

One can infer the following conclusions from this:

I. Any point of $V_{n,\,m}$ possesses a neighborhood that is homeomorphic to the interior of a Euclidian ball.

II. $V_{n,\,m}$ is connected. (Due to (1), S^{n-m} is connected.)

III. One has the recursion formula for the dimension $\mu_{n,\,m}$ of $V_{n,\,m}$:

$$\mu_{n,\,m} = \mu_{n-1,\,m-1} + (n-1), \tag{5}$$

so

$$\mu_{n,\,m} = m \cdot \left(n - \frac{m+1}{2}\right). \tag{6}$$

2. Decomposition of $V_{n,\,m}$. For our first projection, S^{n-2} bounds the closed ball E_1 in R^{n-1}. We define:

$$K_1 = E_1 \times V_1'\,. \tag{7}$$

Analogously, for the second projection, one has:

$$K_2 = E_2 \times V_2'\,. \tag{8}$$

V is then the set union of K_1 and K_2:

$$V = K_1 + K_2\,. \tag{9}$$

If one iterates this decomposition of $V_{n,\,m}$ for the sequence (4) then it follows inductively that:

VI. $V_{n,\,m}$ is a polyhedron.

It now follows from I-IV that:

Theorem 1. *$V_{n,m}$ is a closed manifold.*

We call the manifolds of the sequence (4) the *manifolds that are associated with $V_{n,m}$*. For the intersection of K_1 and K_2, one gets:

$$\text{For the first projection:} \qquad K_1 \cdot K_2 = S^{n-2} \times V_1', \tag{10}$$

$$\text{For the second projection:} \qquad K_1 \cdot K_2 = S^{n-2} \times V_2'. \tag{11}$$

We would like to derive the properties of the Betti groups of V from our decomposition (9) of the manifold V by induction on the sequence of associated manifolds. For $r > 0$, we understand $B^r(K)$ to mean the r-dimensional Betti group of the complex K, while for $r = 0$, it is the group of 0-dimensional integer homology classes that contain only reducible cycles. (A 0-dimensional cycle is reducible when the sum of its coefficients vanishes ([1])). We call algebraic subcomplexes of:

$$\begin{array}{cccccc} V = K_1 + K_2, & K_1, & K_2, & K_1 \cdot K_2, & V_1', & V_2' \\ C, & C_1, & C_2, & C_{12}, & C_1', & C_2', \text{ resp.} \end{array}$$

Cycles will always be denoted by z or Z.

We now make the following basic assumption:

$$\text{let } B^r(V_{n-1,\,m-1}) = 0 \text{ for a fixed } r \text{ with } 0 \le r < n-2. \tag{$\mathbf{J}_1$}$$

One then has ([2]), for an arbitrary $(r + 1)$-dimensional sub-cycle z^{r+1} of $V_{n,m}$:

$$z^{r+1} = z_1^{r+1} + z_2^{r+1}. \tag{12}$$

(z_1^{r+1} is a sub-cycle of K_1 and z_2^{r+1} is a sub-cycle of K_2 .)

Proof: It follows from ($\mathbf{J}_1$) that $B^r(V_1') = 0$, so one also has ([3]) $B^r(S^{n-2} \times V_1') = 0$; it then follows from (10) that:

$$B^r(K_1 \cdot K_2) = 0. \tag{13}$$

Now let $z^{r+1} = C_1 - C_2$ be any decomposition of z^{r+1} into two algebraic $(r + 1)$-dimensional sub-complexes of K_1 and K_2 . Taking the boundary yields $\dot{C}_1 = \dot{C}_2$; this common boundary lies in K_1, as well as in K_2 , so it is a z_{12}^r. It follows from (13) that z_{12}^r

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([1]) AH: chap. IV, § 4, no. 7, and furthermore, chap. V, § 1, no. 5.

([2]) This theorem is a special case of an addition theorem in combinatorial topology; cf., AH: chap. VII, § 2, especially no. 5.

([3]) For Betti groups of product complexes, see AH: chap. VII, § 3.

$= \dot{C}_{12} \cdot C_1 - C_{12}$ and $C_2 - C_{12}$ are cycles z_1, z_2, resp., and one has $z^{r+1} = z_1 - z_2$, with which (12) is proved.

Under the sharper assumption:

$$\text{Let } B^r(V_{n-1,\,m-1}) = 0 \text{ for a fixed } r \text{ with } 0 \leq r < n - 3, \tag{$\mathbf{J}_2$}$$

one then obtains the isomorphism:

$$B^r(V_{n,\,m}) \approx B^r(V_{n-1,\,m-1}). \tag{14}$$

Proof: From the theorem on the Betti groups of product complexes, it follows that:

$$B^r(K_1) = B^{r+1}(E_1 \times V_1') = B^{r+1}(E_1 \times V_{n-1,\,m-1}) \approx B^{r+1}(V_{n-1,\,m-1}).$$

Analogously, one obtains, with consideration of the fact that $r + 1 < n - 2$:

$$B^{r+1}(K_1 \cdot K_2) = B^{r+1}(S^{n-2} \times V_1') = B^{r+1}(S^{n-2} \times V_{n-1,\,m-1}) \approx B^{r+1}(V_{n-1,\,m-1}), \tag{15}$$

and therefore:

$$B^{r+1}(K_1 \cdot K_2) \approx B^{r+1}(K_1).$$

This isomorphism can be realized if one associates a homology class of $K_1 \cdot K_2$, whose representative cycle is z_{12}^{r+1}, with the homology class of z_{12}^{r+1} in K_1. From that, we infer the following conclusions:

a) A cycle of $K_1 \cdot K_2$ is contained in any $(r + 1)$-dimensional homology class of K_1 (or K_2).

b) From the homology $z_{12}^{r+1} \sim 0$ in K_1 (or K_2), it follows that:

$$z_{12}^{r+1} \sim 0 \text{ in } K_1 \cdot K_2 .$$

If one associates a homology class of $K_1 \cdot K_2$, whose representative cycle is Z_{12}^{r+1}, with the homology class of Z_{12}^{r+1} in $K_1 + K_2$ then a homomorphic map of $B^{r+1}(K_1 \cdot K_2)$ into $B^{r+1}(K_1 + K_2)$ comes about. This map is an isomorphism, in the event that:

1. A cycle of $K_1 \cdot K_2$ is contained in any $(r + 1)$-dimensional homology class of $K_1 + K_2$.

2. The homology $Z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$ follows from the homology $Z_{12}^{r+1} \sim 0$ in $K_1 + K_2$.

1. follows from (12) and a).

2. is verified in the following way:
$Z_{12}^{r+1} \sim 0$ in $K_1 + K_2$ means that $Z_{12}^{r+1} = \dot{C}$. A decomposition $C = C_1 - C_2$ of C gives $Z_{12}^{r+1} = \dot{C}_1 - \dot{C}_2$. This possible only when $\dot{C}_1 = z_{12}^{r+1}$ and $\dot{C}_2 = \overline{z}_{12}^{r+1}$. Since $z_{12}^{r+1} \sim 0$ in K_1, one gets from b) that $z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$, and likewise $\overline{z}_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$, and therefore also $Z_{12}^{r+1} \sim 0$ in $K_1 \cdot K_2$. With that, we have $B^{r+1}(K_1 + K_2) \approx B^{r+1}(K_1 \cdot K_2)$.

Our proof then gives:

Lemma. *Under the assumption* ($\mathbf{J}_2$), *an (r + 1)-dimensional homology basis for $K_1 \cdot K_2$ is also a homology basis for $V = K_1 + K_2$.*

The following theorem can now be proved easily:

Theorem 2. *For* $0 \leq r < n - m - 1$, one has $B^r(V_{n,\,m}) = 0$.

The proof proceeds by complete induction on the sequence of associated manifolds; thus, let it be already proved that:

$$B^{r+1}(V_{n-1,\,m-1}) = 0 \qquad \text{for } 0 \leq r < n - m - 1.$$

It further follows from Theorem 1 that $B^0(V_{n-1,\,m-1}) = 0$, so one also has $B^r(V_{n-1,\,m-1}) = 0$. Since $m > 1$ was assumed, ($\mathbf{J}_2$) is true, and therefore (14), and therefore Theorem 2. The induction will be anchored on the manifold $V_{n-m+1,\,1} = S^{n-m}$, for which Theorem 2 is trivial.

Theorem 3. *For $m > 2$, one has $B^{n-m}(V_{n,\,m}) \approx B^{n-m}(V_{n-1,\,m-1})$.*

Proof: From Theorem 2, ($\mathbf{J}_2$) is true for $r = n - m - 1$. (14) then gives the assertion.

3. Topology of $V_{n,2}$. $B^{n-m}(V_{n,\,m})$ can be determined from Theorem 3 when $B^{n-m}(V_{n-m+2,\,2})$ is known; therefore, the $(n - 2)$-dimensional Betti group of a manifold $V_{n,\,2}$ shall be calculated in this section. The sequence of associated manifolds consists of only an $(n - 2)$-dimensional sphere in this case. We use our first projection for the representation of $V_{n,\,2}$; V_1' is then a sphere $S_1'^{n-2}$. Let the two spheres S^{n-2} and $S_1'^{n-2}$ be equally oriented, so we also denote the cycles that are provided by these orientations by S^{n-2} and $S_1'^{n-2}$. If s is an arbitrary, but chosen once and for all, point of S^{n-2}, and s_1' is a point of $S_1'^{n-2}$ then, from (10), the two cycles $z_{12} = s \times S_1'^{n-2}$ and $S^{n-2} \times s_1'$ define an $(n - 2)$-dimensional homology basis for $K_1 \cdot K_2$. (The case of $n = 3$ is represented in Fig. 1) Any $(n-2)$-dimensional cycle Z_{12} of $K_1 \cdot K_2$ thus satisfies a homology:

$$Z_{12} \sim \alpha\, z_{12} + \beta\, \overline{z}_{12} \qquad \text{in } K_1 \cdot K_2, \tag{17}$$

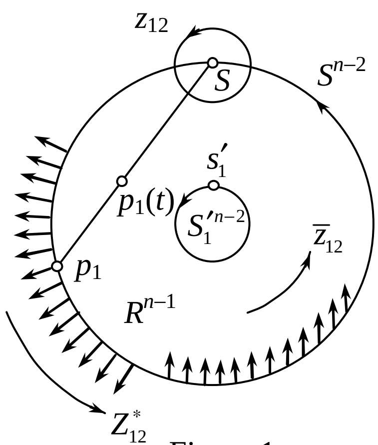

Figure 1.

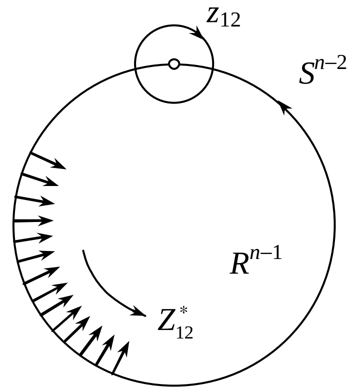

Figure 2.

where α and β are well-defined numbers. We now pose the problem of determining the homologies (17) that Z_{12} fulfills in K_1 or K_2. We first solve this problem for a special cycle Z_{12}^* that is defined in the first projection as the field of exterior normal vectors on S^{n-2}. For this cycle, (17) reads:

$$Z_{12}^* \sim z_{12} + \overline{z}_{12} \qquad \text{in } K_1 \cdot K_2. \tag{17*}$$

Proof: Z_{12}^* fulfills a homology:

$$Z_{12}^* \sim \alpha^* z_{12} + \beta^* \overline{z}_{12} \quad \text{in } K_1 \cdot K_2. \tag{17**}$$

The determination of the unknowns α^* and β^* is achieved in the following way: One associates a point $p_1 \times v_1$ of $K_1 \cdot K_2$ [see (2)] with the point v_1 of $V_1' = S_1'^{n-2}$; this continuous map f of $K_1 \cdot K_2$ into $S_1'^{n-2}$ induces a homomorphic map of the Betti groups of $K_1 \cdot K_2$ into the Betti groups of $S_1'^{n-2}$ that transforms (17**) into the homology $f(Z_{12}^*) \sim \alpha^* \cdot f(z_{12}) + \beta^* \cdot f(\overline{z}_{12}) = \alpha^* \cdot S_1'^{n-2}$. The fact that $f(Z_{12}^*) \sim S_1'^{n-2}$ yields $\alpha^* = 1$; one finds that $\beta^* = 1$ in an analogous way.

Relative to K_1, Z_{12}^* fulfills the homology:

$$Z_{12}^* \sim z_{12} \qquad \text{in } K_1. \tag{18*}$$

The proof is by continuous variation of Z_{12}^*: One lets an arbitrary point $p_1 \times v_1$ of Z_{12}^* run through the path that is suggested by the following schema:

$$p_1 \times v_1, \qquad p_1(t) \times v_1, \qquad s \times v_1. \tag{\textbf{D}}$$

In this, t is a deformation parameter that ranges from 0 to 1; $p_1(t)$ moves uniformly and rectilinearly from p_1 to the point s of S^{n-2}.

By performing the transformation through reciprocal radii on S^{n-2}, Figure 1 becomes Figure 2, where one finds, in an analogous way:

$$Z^*_{12} \sim (-1)^n \cdot z_{12} \qquad \text{in } K_2 . \tag{19*}$$

For the arbitrary cycle Z_{12} , we now have, from (17) and (17*), that $Z_{12} \sim (\alpha - \beta) \cdot z_{12} + \beta \cdot Z^*_{12}$ in $K_1 \cdot K_2$, and thus also in K_1 ; it then follows from (18*) that $Z_{12} \sim \alpha \cdot z_{12}$ in K_1 . Analogously, with the use of (19*), one gets: $Z_{12} \sim [\alpha - \beta + (-1)^n \cdot \beta] \cdot z_{12}$ in K_2 . This then yields the following solution to our problem:

From the fact that:

$$Z_{12} \sim \alpha z_{12} + \beta \bar{z}_{12} \qquad \text{in } K_1 \cdot K_2 ,$$

it follows that:

$$Z_{12} \sim \alpha z_{12} \qquad \text{in } K_1 \tag{18}$$

and

$$Z_{12} \sim [\alpha - \beta + (-1)^n \beta]\, z_{12} \text{ in } K_2. \tag{19}$$

We now infer some consequences from these formulas:

Theorem 4. *The* $(n - 2)$*-dimensional Betti group of* $V_{n,2}$ *is cyclic and has order* 0 *for even n and order* 2 *for odd n*.

In this, we understand a cyclic group of order 0 to mean a free cyclic group.

Proof: From (7), our cycle z_{12} defines an $(n - 2)$-dimensional homology basis in K_1 ; however, since K_1 and K_2 are mapped to each other topologically by our transformation through reciprocal radii, z_{12} is also a homology basis for K_2 . Furthermore, from (12) [the assumption $(\mathbf{J}_1)$ is fulfilled for $r = n - 3$], any $(n - 2)$-dimensional cycle of $V_{n,2}$ can be written as the sum of a cycle in K_1 and a cycle in K_2 . From these facts, it follows that the homology class of z_{12} in $V_{n,2}$ generates the group $B^{n-2}(V_{n,\,2})$, so that group is cyclic; in order to establish its order, we must determine the order of z_{12} . Thus, let, say, $\gamma \cdot z_{12} \sim 0$ in $V_{n,\,2}$ – i.e., $\gamma \cdot z_{12} = \dot{C}$. A decomposition $C = C_1 + C_2$ of C then gives $\gamma \cdot z_{12} = \dot{C}_1 + \dot{C}_2$. This is possible only for $\dot{C}_1 = Z_{12}$ and $\dot{C}_2 = \bar{Z}_{12}$. We then find that:

$$\gamma \cdot z_{12} = Z_{12} + \bar{Z}_{12} \qquad \text{with } Z_{12} \sim 0 \text{ in } K_1 \text{ and } \bar{Z}_{12} \sim 0 \text{ in } K_2 . \tag{20}$$

If we assume that n is perhaps odd then it follows from $Z_{12} \sim 0$ in K_1, by means of (18), that $Z_{12} \sim \beta \cdot \bar{z}_{12}$ in $K_1 \cdot K_2$. By substituting this into (20), we find the homology $\gamma \cdot z_{12} \sim 2\bar{\beta} \cdot z_{12} + (\beta + \bar{\beta}) \cdot \bar{z}_{12}$ in $K_1 \cdot K_2$. This homology is possible only for $\gamma = 2 \cdot \bar{\beta}$; it then follows that γ is even from the fact that $\gamma z_{12} \sim 0$ in $V_{n,2}$. The order of z_{12} is then at least 2; the fact that it is exactly 2 follows from a consideration of $-\bar{z}_{12}$. Namely, from (18), one has $-\bar{z}_{12} \sim 0$ in $V_{n,2}$, and from (19), $-\bar{z}_{12} \sim 2 \cdot z_{12}$ in $V_{n,2}$. One then has, in fact, that $2z_{12} \sim 0$ in $V_{n,2}$. Since the case of even n can be examined analogously, Theorem 4 is proved.

It is likewise shown that z_{12} is a basis cycle for the group $B^{n-2}(V_{n,2})$. (This will be important later.) We shall then given a definition of z_{12} that is independent of the decomposition of $V_{n,2}$. To this end, one considers all 2-systems $\sigma_{n,2}$ of $V_{n,2}$ (no. 1) that coincide in their first vector. The endpoints of the second vectors of this system will run through an $(n – 2)$-dimensional sphere, which we think of as oriented. The system $\sigma_{n,2}$ then defines an $(n – 2)$-dimensional cycle that call $z_{n,2}$. It is clear that $z_{n,2}$ can be identified with z_{12} ; we then find the following:

Lemma: *The cycle $z_{n,2}$ is the basis element for the $(n – 2)$-dimensional Betti group of* $V_{n,2}$.

The manifold $V_{n,2}$ is orientable. We will prove this later. From Theorems 2 and 4, one can then determine all Betti groups of $V_{n,2}$ with the help of the Poincaré duality theorem. One then obtains the following result:

Theorem 5. *For even n, the non-zero Betti numbers of $V_{n,2}$ are:* $p^0 = p^{n-2} = p^{n-1} = p^{2n-3} = 1$; *no torsion is present. For odd n, one also has* $p^{n-2} = p^{n-1} = 0$, *but an* $(n – 2)$-*dimensional torsion of order* 2 *also enters in.*

Furthermore, the relations (18) and (19) allow us to determine the continuous maps of an at most $(n – 2)$-dimensional sphere into $V_{n,2}$. One has, in fact:

Theorem 6. *Two continuous maps of an at most* $(n – 2)$-*dimensional sphere into* $V_{n,2}$ *are homotopic if they have the same homology type* ([1]) ([10a]).

We preface the proof with some preliminary considerations. Let, perhaps, f be a given continuous map of the sphere S_0^r $(r \leq n – 2)$ into $V_{n,2}$, and let v_0 be an arbitrary point of S_0^r . If, as in no. 1, we think of $V_{n,2}$ as the set of all vectors in R^n that are tangent to S^{n-1} then we can assume for all homotopy investigations that the image vector of point v_0 does not contact S^{n-1} at the North Pole. (If this were not true then, since $r < n – 1$, one could always make it so by a continuous change in f.) No image vectors are then lost under the transition to our first projection, and one has, from (2), that $f(v_0) = p_1 \times v_1$. Furthermore, one can actually assume that only the points $s \times v_1$ (see Fig. 1) can appear as image points. (In fact, the continuous map $v_0 \to p_1 \times v_1$ can be changed into a map that has the desired property by the deformation process (**D**) (beginning of this no.)) We then assume that:

$$f(v_0) = s \times v_1. \tag{21}$$

We call the map $\varphi(v_0) = v_1$ of S_0^r into the associated manifold $S_1'^{\,n-2}$ to $V_{n,2}$ the *associated map* φ to the map f. Now, if $\bar{f}$ is a second map of S_0^r into $V_{n,2}$ and $\bar{\varphi}$ is its associated map then one has:

([1]) AH: chap. VIII, § 3.

([10a]) This theorem is a generalization of the theorem on the classification of sphere maps (AH: chap. XIII, § 2).

$$\text{The homotopy of } f \text{ and } \bar{f} \text{ follows from the homotopy of } \varphi \text{ and } \bar{\varphi}. \tag{22}$$

This follows simply from the fact that multiplication by the fixed point s is a topological map of $S_1'^{n-2}$ into $V_{n,2}$.

We now go on to the proof of Theorem 6. There are three cases to consider:

Case 1. $r < n - 2$. From Theorem 2, we must show that any map of S_0^r into $V_{n,2}$ is homotopic to zero, so the image of S_0^r can be contracted to a point. However, from (22), this is a consequence of the fact that since $r < n - 2$, the associated map is homotopic to zero.

Case 2. $r = n - 2$ and n is even. Let f and $\bar{f}$ be the two maps of which we spoke in Theorem 6. If we understand $S_0^r = S_0^{n-2}$ to also mean the cycle that this sphere represents with a chosen orientation then the assumption of Theorem 6 says that $f(S_0^{n-2}) \sim \bar{f}(S_0^{n-2})$ in $V_{n,2} = K_1 + K_2$. From (21), $f(S_0^{n-2})$ and $\bar{f}(S_0^{n-2})$ are cycles in $K_1 \cdot K_2$, so they fulfill the homologies (17): $f(S_0^{n-2}) \sim \alpha z_{12}$, $\bar{f}(S_0^{n-2}) \sim \bar{\alpha}\, z_{12}$ in $K_1 \cdot K_2$; one then has $\alpha \cdot z_{12} \sim \bar{\alpha}\, z_{12}$ in $V_{n,2}$. From Theorem 4, this is possible only if $\alpha = \bar{\alpha}$, and one finally gets that $f(S_0^{n-2}) \sim \bar{f}(S_0^{n-2})$ in $K_1 \cdot K_2$. We map this homology to $S_1'^{n-2}$ by assigning the point $p_1 \times v_1$ in $K_1 \cdot K_2$ to the point v_1. One thus finds that $\varphi(S_0^{n-2}) \sim \bar{\varphi}(S_0^{n-2})$ in $S_1'^{n-2}$. The two maps φ and $\bar{\varphi}$ of S_0^{n-2} into $S_1'^{n-2}$ thus have the same mapping degree, from which their homotopy follows. (22) concludes the proof.

Case 3. $r = n - 2$ and n is odd. Theorem 4 then gives only that $\alpha \equiv \bar{\alpha}$ (mod 2). Let $\bar{\alpha} = \alpha - 2k$, perhaps. The proof above will also work in this case if we can show that our map f with $f(S_0^{n-2}) \sim \alpha\, z_{12}$ in $K_1 \cdot K_2$ can be changed continuously into a map f_2 with $f_2(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in $K_1 \cdot K_2$ that satisfies the condition (21). To that end, let F be an arbitrary map of S_0^{n-2} into S^{n-2} of degree k. Next, f will be changed into a map f_1 according to the following schema:

$$f(v_0) = s \times v_1\,, \qquad F(v_0, 1 - t) \times v_1\,, \qquad F(v_0) \times v_1 = f_1(v_0).$$

$F(v_0, t)$ again moves uniformly and rectilinearly from $F(v_0)$ to s. The cycle $f_1(S_0^{n-2})$ again lies in $K_1 \cdot K_2$ and satisfies the homology $f_1(S_0^{n-2}) \sim \alpha\, z_{12} + k\,\bar{z}_{12}$ there, which one proves analogously to (17*). From (19), one has $f_1(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in K_2. One now goes to Figure 2 by means of the transformation through reciprocal radii, and changes f_1 there by the deformation process that is analogous to (**D**). The result is a map f_2 with $f_2(S_0^{n-2}) \sim (\alpha - 2k)\, z_{12}$ in K_2 and $f_2(S_0^{n-2}) \sim \delta\, z_{12}$ in $K_1 \cdot K_2$ that satisfies the condition (21). As for the unknown δ, one easily finds from (19) that $\delta = \alpha - 2k$. With that, Theorem 6 is proved completely.

It then follows from Theorems 2 and 6 that:

Theorem 7. *For* $n > 3$, *the manifold* $V_{n,2}$ *is simply-connected, and thus orientable*.

As a non-simply-connected manifold, the manifold $V_{3,2}$ then occupies a special place in the $V_{n,2}$, which we will later (§ 5, no. 3) exploit in our investigation of the parallelizability of three-dimensional manifolds. We mention that $V_{3,2}$ is homeomorphic to the three-dimensional projective space P^3. To prove this, one observes that $V_{3,2}$, as the set of line elements on a two-dimensional sphere, is homeomorphic to the group of Euclidian rotations of that sphere. Such a rotation is, however, determined uniquely by four homogeneous parameters.

4. Topology of $V_{n,m}$. The union of the results of sections 2 and 3 allows the derivation of further topological properties of the $V_{n,m}$. One proves the following theorem by induction on the sequence (4) of associated manifolds – which is now, however, broken by the manifold $V_{n-m+2,\,2}$ – in which one always assumes that $m > 1$:

1. The Betti group $B^{n-m}(V_{n,m})$ is cyclic of order 0 for even $n - m$ and of order 2 for odd $n - m$.

The proof follows from Theorems 3 and 4. In order to find a basis cycle for $B^{n-m}(V_{n,m})$, one considers all m-systems $\sigma_{n,m}$ in $V_{n,m}$ (no. 1) whose first $(m - 1)$ vectors are given as fixed. The endpoints of the latter vectors of this system run through an $(n - m)$-dimensional sphere that we regard as being oriented. The systems $\sigma_{n,m}$ then define an $(n - m)$-dimensional cycle $z_{n,m}$.

2. $z_{n,m}$ is a basis cycle for $B^{n-m}(V_{n,m})$.

The proof follows from the two lemmas in no. 2 and no. 3.

3. Two continuous maps of an at most $(n - m)$-dimensional sphere into $V_{n,m}$ are homotopic when they have the same homology type.

To prove this, if f and $\bar{f}$ are two maps then one defines the associated maps φ and $\bar{\varphi}$ into $V_{n-1,m-1}$ in a manner that is analogous to no. 3. The homotopy of f and $\bar{f}$ then follows from the homotopy of the associated maps.

From 3, $V_{n,m}$ is simply-connected for $m < n - 1$, so it is also orientable. $V_{n,n-1}$ is homeomorphic to the group of Euclidian rotations of an $(n - 1)$-dimensional sphere and, as a group manifold, it is therefore orientable. For this manifold, one has, moreover:

4. The fundamental group of $V_{n,n-1}$ is a cyclic group of order 2 $(n > 2)$.

The proof of this differs from that of 3 in only inessential ways. (In order to anchor the induction, one observes that 4. follows for $V_{3,2}$ from its homeomorphism with projective space.)

In conclusion, we would like to derive some properties of $V_{n,m}$ from these theorem that will be needed in what follows:

Theorem 8. *The continuous image of an at most* $(n - m - 1)$*-dimensional sphere in* $V_{n,m}$ *(m arbitrary) can be contracted to a point.*

Proof is from 3. and Theorem 2.

Theorem 9. *If f is a continuous map of an orientable sphere* S_0^{n-m} *into* $V_{n,m}$ *then one has the homology:*

$$f(S_0^{n-m}) \sim \alpha z_{n,m} \qquad \text{in } V_{n,m}\,.$$

If $n - m$ *is even or* $m = 1$ *then* α *is determined uniquely, and two maps with the same value of* α *are homotopic.*

However, if $n - m$ *is odd and m is different from* 1 *then* α *is determined only* (mod 2) ([1]); *two maps that are associated with values of* α *that are congruent* (mod 2) *are homotopic.*

§ 2. The open manifolds $V_{n,m}^*$.

1. Definitions. In this section, we would like to freely make the restriction to orthogonal and normalized *m*-systems. We define: An ordered system $\sigma_{n,m}^*$ of m linearly-independent vectors $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_m$ that contact a point of R^n is called an *affine m-system* in R^n. We now call the systems $\sigma_{n,m}$ of § 1 *orthogonal m-systems*, in order to distinguish them from the affine *m*-systems; *m* again fulfills the inequalities:

$$0 < m < n. \tag{1}$$

The set of all affine *m*-systems that contact R^n at a fixed point is called $V_{n,m}^*$. A system $\sigma_{n,m}^*$ is given by the $n \cdot m$ components of its vectors, so it can be regarded as a point in an $(n \cdot m)$-dimensional numerical space. In this way of looking at things, $V_{n,m}^*$ becomes a sub-domain of the numerical space, so it is an open manifold.

2. Retraction mapping. For any *m*-system of $V_{n,m}^*$, we replace the vector $\mathfrak{v}_i$ with the vector:

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(1) Therefore, we can assume in what follows that α has the value 0 or 1 in this case.

$$\mathfrak{v}_i' = \mathfrak{v}_i - (\mathfrak{v}_i \cdot \mathfrak{v}_j)\,\mathfrak{v}_j\,. \tag{2}$$

i and j are chosen to be fixed, but different from each other; $(\mathfrak{v}_i \cdot \mathfrak{v}_j)$ means the scalar product of $\mathfrak{v}_i$ and $\mathfrak{v}_j$. This produces a continuous map f of $V_{n,m}^*$ into itself; we denote the image set by $f(V_{n,m}^*)$. By considering the family of maps:

$$\mathfrak{v}_i'(t) = \mathfrak{v}_i - t\,(\mathfrak{v}_i \cdot \mathfrak{v}_j)\,\mathfrak{v}_j \qquad (0 \le t \le 1),$$

one recognizes that f is a deformation; i.e., it belongs to the class of the identity. If one replaces the vector $\mathfrak{v}_k$ in any system $\sigma_{n,m}^*$ in $V_{n,m}^*$ for a definite value of k with the vector:

$$\mathfrak{v}_k' = \frac{\mathfrak{v}_k}{|\mathfrak{v}_k|} \tag{3}$$

then this gives another continuous map g of $V_{n,m}^*$ into itself. g is also a deformation, as the family of maps:

$$\mathfrak{v}_k'(t) = [t + (1-t)\cdot|\mathfrak{v}_k|]\frac{\mathfrak{v}_k}{|\mathfrak{v}_k|} \qquad (0 \le t \le 1)$$

yields. The two maps f and g leave the manifold $V_{n,m}$ invariant, which is indeed a subset of $V_{n,m}^*$.

One can once more perform a deformation of type (2) [(3), resp.] with $f(V_{n,m}^*)$ [$g(V_{n,m}^*)$, resp.], and ultimately construct a deformation that maps $V_{n,m}^*$ onto $V_{n,m}$ continuously by composing finitely many deformations of this type. This follows from the well-known fact of analytic geometry that any affine system $\sigma_{n,m}^*$ in $V_{n,m}^*$ can be orthogonalized by finitely many steps of type (2) and (3). We call the deformation F the *retraction mapping* ([1]) of $V_{n,m}^*$ onto $V_{n,m}$.

3. Topology of $V_{n,m}^*$. With the help of our retraction, we can now carry over the results of § 1, no. 4 to the open manifold $V_{n,m}^*$:

Theorem 10. *$V_{n,m}^*$ is completely homology-equivalent to $V_{n,m}$; i.e., one has for an arbitrary r: $B^r(V_{n,m}^*) \approx B^r(V_{n,m})$; furthermore, all of the results that were proved for $V_{n,m}$ in § 1, no. 4 are also true for the manifold $V_{n,m}^*$.*

([1]) This concept goes back to K. Borsuk; cf., AH: chap. VIII, § 6.

Proof: The retraction map F induces a map of $B^r(V^*_{n,m})$ to $B^r(V_{n,m})$. In order to prove that this homomorphism is an isomorphism, it suffices (since any r-dimensional homology class of $V_{n,m}$ appears trivially as an image class) to show that its kernel consists of only the zero class. Therefore, let, say, z^r be a cycle of $V^*_{n,m}$ and $F(z^r) \sim 0$ in $V_{n,m}$, hence, also in $V^*_{n,m}$. Since $F(z^r)$ goes to z^r under deformation, one has $F(z^r) \sim z^r$ in $V^*_{n,m}$, so, in fact, $z^r \sim 0$ in $V^*_{n,m}$.

The second assertion of Theorem 10 can now be proved easily with the help of our retraction.

Remark. All *positively-oriented* n-systems that contact a fixed point of R^n define a manifold that is homeomorphic to the group A_n of all proper affine maps of R^n. From our analysis, it easily follows that A_n is completely homology-equivalent to $V_{n,\,n-1}$ and that the fundamental group of A_n is a cyclic group of order 2 for $n > 2$.

§ 3. Vector fields in Euclidian space. Characteristic.

1. Characteristic of an *m*-field on a sphere. In this section, we understand E^{r+1} to mean an $(r + 1)$-dimensional curved cell that is embedded in the Euclidian space R^n and S^r to mean the boundary sphere of E^{r+1}. If we denote a point of S^r by p then we can establish the points of the cell E^{r+1} by means of a polar coordinate system ρ, p. (ρ is a number that runs from 0 to 1, the point $(0, p)$ is the origin of the coordinate system, and $(1, p)$ is identical with p.)

If an *affine* m-system $\sigma(p)$ of R^n is attached to every point of S^r then we speak of an *m-field* $\mathfrak{F}$ on S^r. The examination of this field is the objective of this paragraph. To that end, we choose a set of vectors $V^*_{n,m}$ that is embedded in R^n and associate the point p of S^r with the m-system of $V^*_{n,m}$ that is parallel to $\sigma(p)$. A map f of the sphere S^r into the manifold $V^*_{n,m}$ is given by this association that we call a *mapping by parallel m-systems.* We further call the field $\mathfrak{F}$ *continuous* when f is continuous; this will always be assumed in what follows. We define a continuous field on the cell E^{r+1} and the associated mapping by parallel m-systems in an analogous way.

Now, this immediately suggests the question: Under what conditions can a continuous field $\sigma(p)$ that is given on S^r be extended to a continuous field $\sigma(\rho, p)$ on E^{r+1}? [i.e., $\sigma(1, p) = \sigma(p)$.] If the dimension r of our sphere is less than $n - m$ then Theorem 8 (10) shows that this process is always possible. In fact, if $f(S^r)$ is then homotopic to zero in $V^*_{n,m}$ then ([1]) f can be extended to a continuous map of E^{r+1} into $V^*_{n,m}$. However, if $r = n - m$ then the sphere is $(n - m)$-dimensional (and oriented), and it follows from Theorem 9 (10) that the desired process is possible iff the number α that is associated with our map f by parallel m-systems vanishes.

([1]) AH: chap. XIII, § 1, Lemma II.

This number α is called the *characteristic* of the *m*-field $\mathfrak{F}$ on the sphere $S^r = S^{n-m}$. One then finds that:

Theorem 11. *A continuous m-field that is given on the boundary of a cell can be continuously extended into its interior:*

a) *If the dimension of the sphere is less than n – m.*
b) *If the sphere is* $(n - m)$*-dimensional and the characteristic of the field on it is* 0.

Extension through central projection:

A boundary field can always be extended into the interior of the cell E^{r+1} by the definition: "$\sigma(\rho, p)$ is parallel to $\sigma(p)$." We call this process *extension through central projection* from the point $(0, p)$. However, the continuity of the extended field will then generally break down at the center of projection. Moreover, if an arbitrary, not-necessarily-continuous *m*-field is given on the boundary sphere S^r, and we denote the set of its discontinuities by *M*, then the field that is extended by central projection into the cell E^{r+1} is discontinuous at all points of the cone over *M* with the center of projection for its vertex.

2. Remarks on the calculation of the characteristic. In many cases, it proves to be useful to calculate the characteristic in some other way than by means of the mapping by parallel *m*-systems: Let a continuous field $\mathfrak{B}$ of *positively-oriented n*-systems $\beta(\rho, p)$ be given on the cell E^{n-m+1}. Such a field is called a *basis field* on the cell E^{n-m+1}. ("Positively-oriented" means oriented the same as the system $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ of basis vector in R^n.) In order to calculate the characteristic of an *m*-field $\sigma(p)$ that is given on S^{n-m}, we proceed as follows: Let $\mathfrak{v}_\mu$ ($\mu = 1, 2, \ldots, m$) be a vector of $\sigma(p)$ and let $v_{\mu i}$ ($i = 1, 2, \ldots, n$) be its components *relative to the basis* $\beta(1, p)$. If one now associates every vector $\mathfrak{v}_\mu$ with the vector that contacts the origin of R^n and has the components $v_{\mu i}$ relative to $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ then this produces a continuous map f' of S^{n-m} into the $V^*_{n,m}$ at the origin of R^n. From Theorem 9 (10), a number α' is associated with this map; we prove that α' is the characteristic of the given *m*-field on S^{n-m}.

To that end, we construct a continuous family $\beta_t(p)$ $(0 \leq t \leq 1)$ of basis fields on S^{n-m} such that $\beta_0(p) = \beta(1, p)$ and $\beta_1(p)$ is parallel to $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$. (To construct this family, one defines, say, for $0 \leq t \leq \frac{1}{2}$: $\beta_t(p)$ is parallel to $\beta(1 - 2t, p)$; the systems $\beta_{1/2}(p)$ are then parallel to each other and can easily be made parallel to $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ by a deformation in the interval $\frac{1}{2} \leq t \leq 1$.) A map f'_t of S^{n-m} into $V^*_{n,m}$ that is continuous and continuously varying in *t* belongs to every basis field $\beta_t(p)$. f'_0 is our f', while f'_1 is identical with the map *f* through parallel *m*-systems. *f* and f' are then homotopic; the assertion the follows from this.

Calculation of the characteristic by recursion:

Our new method of calculation of the characteristic is very useful when one is dealing with the following situation:

a) The cell E^{n-m+1} lies in an n'-dimensional plane $R^{n'}$ of R^n. ($n' < n$) $R^{n'}$ will be spanned by, perhaps, the basis vectors $\mathfrak{e}_{n-n'+1}, \ldots, \mathfrak{e}_n$.

b) Suppose that the vectors $\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_{n-n'}$ of the system $\sigma(p)$ are not contained in $R^{n'}$; they then define an $(n - n')$-system in R^n, and all of these systems define an $(n - n')$-field on S^{n-m}. We assume that this field can be extended to an $(n - n')$-field $\bar{\sigma}(\rho, p)$ on E^{n-m+1}.

c) Let the vectors $\mathfrak{v}_{n-n'+1}, \ldots, \mathfrak{v}_m$ of $\sigma(p)$ be contained in $R^{n'}$; they then define an m'-system $\sigma'(p)$ in $R^{n'}$. ($m' = m - n + n'$).

$\sigma(p)$ and $\sigma'(p)$ then possess characteristics α and α' on S^{n-m}. One then has:

$$\alpha \equiv \alpha' \pmod 2.$$

(One can actually prove the equality of α and α' for certain orientation assumptions; for our purposes, however, it suffices to have congruence mod 2.)

Outline of proof: One chooses a basis field $\beta'(r, p)$ on the cell E^{n-m+1} in $R^{n'}$. This basis field will be extended by $\bar{\sigma}(\rho, p)$ to a basis field $\beta(\rho, p)$ in R^n. One calculates the desired characteristics relative to this basis field, where one suitably lets the basic cycle $z_{n,m}$ of $V^*_{n,m}$ (§ 1, no. 4) run through the orthogonal m-systems that contact the origin of R^n whose first $(m - 1)$ vectors are $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_{m-1}$.

3. Characteristic of a field-pair on a cell. If two continuous m-fields $\sigma_0(\rho, p)$ and $\sigma_1(\rho, p)$ are given on our cell E^{r+1}, and if, moreover, a continuous family $\sigma_t(p)$ of m-fields in constructed on the boundary sphere S^r for $0 \leq t \leq 1$ that satisfies the boundary conditions $\sigma_0(p) = \sigma_0(1, p)$ and $\sigma_1(p) = \sigma_1(1, p)$ then we speak of a *field-pair* in E^{r+1}. A field-pair thus consists of two fields on a cell that are coupled on the boundary by a continuous family.

We would now like to examine the conditions under which this continuous coupling can be extended into the interior. A continuous family of m-fields $\sigma_t(\rho, p)$ shall then be constructed in E^{r+1} that satisfies the requirement $\sigma_t(\rho, p) = \sigma_t(p)$. This investigation can be carried out with the help of Theorems 8 and 9, with consideration given to Theorem 10, if the dimension $r + 1$ of the cell is at most $n - m$:

Let T be, say, the (oriented) unit interval that the parameter t runs through. We then construct the cylinder Z, *in abstracto*, which is defined as the topological product $T \times E^{r+1}$, and we denote its points by $t \times (\rho, p)$. We further associate the point $0 \times (\rho, p)$ of Z

with the system that is parallel to $\sigma_0(\rho, p)$ that contacts the origin in R^n [and analogously for $1 \times (\rho, p)$] and associate the point $t \times (\rho, p)$ with the system that is parallel to $\sigma_t(p)$ that contacts the origin in R^n. With that, a continuous map of the boundary of Z into $V^*_{n,m}$ is given. If $r + 1 < n - m$ then f can be extended to a continuous map of the entire cylinder into $V^*_{n,m}$, from which, the extension of our continuous coupling is also constructed. However, if $r + 1 = n - m$ then the extension is possible iff the number α that is associated with f according to Theorem 9 (10) vanishes, so we call it the *characteristic of the field-pair* on E^{r+1}. In order to calculate this characteristic, the cylinder boundary must be oriented; since Z is a product, an orientation can be given by an orientation of the cell E^{r+1}. One then has:

Theorem 12. *The boundary family that belongs to a field pair can be extended into the interior:*

a) *If the dimension of the cell on which the pair lies is less than* $(n - m)$.

b) *If this dimension is* $(n - m)$ *and the characteristic of the field-pair on the (oriented) cell is* 0.

We then give a relation between the characteristic of a field and a field-pair. Let two arbitrary continuous m-fields $\mathfrak{F}$ and $\mathfrak{F}'$ be given on the sphere S^{n-m} with the characteristics α and α', resp. Furthermore, let S^{n-m} be decomposed into the cells E_i^{n-m}, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be coupled by a continuous family of fields on the complex K of the $(n - m - 1)$-dimensional cells of this cell decomposition. With that, a field-pair is given on any cell E_i^{n-m}, whose characteristic we denote by α_i. (Let the cells E_i^{n-m} be coherently oriented with respect to the orientation of S^{n-m} that was employed for the calculation of α and α'.) One then has:

$$\alpha' = \alpha + \sum_{(i)} \alpha_i \qquad \text{for even } n - m \text{ or } m = 1.$$

$$\alpha' \equiv \alpha + \sum_{(i)} \alpha_i \pmod 2 \qquad \text{for odd } n - m \text{ and } m \neq 1. \tag{C}$$

These formulas define the foundation for the following analysis; it is easy to prove:

One constructs the orientated product complex $T \times S^{n-m} = T \times \sum E_i^{n-m} = \sum Z_i$, where the Z_i are constructed over E_i^{n-m} and with the cylinder that was employed in the proof of Theorem 12. Taking the boundary gives the relation:

$$(1 \times S^{n-m}) - (0 \times S^{n-m}) = \sum \dot{Z}_i \,. \tag{R}$$

The cylinders Z_i define a cell decomposition of $T \times S^{n-m}$; if one maps each $\dot{Z}_i$ into $V^*_{n,m}$, as in the proof of Theorem 12 then a continuous map F is given from the complex of $(n - m)$-dimensional cells of this cell decomposition into $V^*_{n,m}$, and it follows from (R) that:

$$F(1 \times S^{n-m}) - F(0 \times S^{n-m}) = \sum F(\dot{Z}_i) \text{ in } V^*_{n,m},$$

and from the definition of α, α', and α_i that:

$$\alpha' z_{n,m} - \alpha z_{n,m} \sim \sum \alpha_i z_{n,m} \quad \text{in } V^*_{n,m}.$$

The assertion follows from this homology and Theorems 9 and 10.

Here, we must mention the following special case of a field-pair: We call a field-pair with $\sigma_0(1, p) = \sigma_t(p) = \sigma_1(1, p)$ a *field-pair with rigid boundary values;* it consists of two continuous m-fields that are given on the cell E^{r+1} and coincide on the boundary S^r. (The connecting boundary family coincides with the common boundary values of the two fields.) It now follows from Theorem 12 that: The first field of a given field-pair with rigid boundary values on E^{n-m} can be deformed into the second field *while preserving its boundary values* iff the characteristic of the pair vanishes on E^{n-m}.

4. Fields and field-pairs with given characteristics. We need a topological lemma for what follows:

Let S^k be a k-dimensional sphere that is decomposed into the two k-dimensional cells E and E', and let P be a connected polyhedron. A continuous map f_1 of E into the polyhedron P can be extended to a continuous map of S^k that belongs to a given mapping class of S^k into the polyhedron P.

Proof: Let F_0 be any map of S^k into the polyhedron P that belongs to the given class, and let f_0 be the map that F_0 induces on E. We construct a continuous family of maps f_t $(0 \leq t \leq 1)$ that connects f_0 to f_1. (Such a family can be found, since P is connected.) The family f_t can be extended to a family of maps F_t of S^k into the polyhedron P ([1]); F_1 is our desired map.

If one identifies S^k with our cylinder Z over a cell E^{n-m} (no. 3) in this lemma and identifies P with the manifold $V^*_{n,m}$ then this easily yields:

Theorem 13. *A continuous m-field that is given on the cell E^{n-m} can be extended through a second field on that cell to a field-pair with rigid boundary values (no. 3) and a given characteristic.*

([1]) AH: chap. XIII, § 1, lemma Ia.

§ 4. Vector fields on manifolds

1. ***m*-fields, parallelizability.** We now move on to the study of m-fields on a closed n-dimensional and differentiable manifold M^n. For this, we must temporarily make the case distinction of Theorem 9:

Case 1. $n - m$ is even or $m = 1$. M^n is then orientable.

Case 2. $n - m$ is odd and $m \neq 1$. M^n can then also be non-orientable.

We call M^n *differentiable* if the following condition is fulfilled: M^n is endowed with a system of neighborhoods that is chosen once and for all, and which we will call *elements* in the sequel. Each element is homeomorphic to a Euclidian space R^n and is equipped with a Cartesian coordinate system. The coordinate transformation that is induced on the overlap of two coordinate systems shall be continuously differentiable and possess a nowhere-vanishing, and in Case 1, positive functional determinant.

With these assumptions, one can define vectors on M^n and apply the conceptual structures and theorems of § 3 to it; One must only replace the Euclidian space with an element in M^n, which is reasonable.

If an m-system is attached to every point of M^n then we speak of an *m-field* on M^n; this field is called *continuous* in the event that it is continuous on every element. If there are continuous μ-fields on M^n but no continuous $(\mu + 1)$-fields then we call μ the *degree of parallelizability* of M^n; A manifold with $\mu = n$ will be referred to as a *parallelizable manifold.* The basis for this terminology is easy to see: If $\mu = n$ then there is a continuous basis field (§ 3, no. 2) on M^n. If we establish an arbitrary vector on M^n with the contact point p by its components relative to the basis that is given at p then two vectors are called *parallel* when the possess positively-proportional components. With that, a continuous teleparallelism is constructed on M^n, from which it follows, for example, that the manifold of directed line elements in M^n is homeomorphic to the topological product of M^n with an $(n - 1)$-dimensional sphere. Examples of parallelizable manifolds are easy to give: The product of two parallelizable manifolds is again parallelizable, so the n-dimensional torus (i.e., the product of n circles) provides an example of a parallelizable M^n. We further remark that one can calculate characteristics by parallel translation of all the distributed vectors to a fixed point of M^n, precisely as one does in Euclidian spaces (§ 3, no. 2).

The central problem of this paper, towards whose solution some steps will be made in what follows, is the determination of the degree μ of a given manifold. We are justified in calling this problem a topological one, since two manifolds that correspond by means of a map that is one-to-one and differentiable in both directions will obviously have the same degree.

2. Frameworks and framework-pairs. Let a fixed cell decomposition of M^n be established for the following considerations; we denote an r-dimensional, oriented cell by

x^r and the cell that is dual to x^r in the dual decomposition ([1]) by ξ^{n-r}. Let the cell decomposition be sufficiently fine that the star of x^r (which is the totality of all cells that have points in common with x^r) lies completely in some element of M^n. In Case 1 (no. 1), we would further like to orient the dual cell ξ^{n-r} to x^r as is customary in orientable manifolds ([15]); in Case 2, orientations will play no role whatsoever.

Now, a *framework* is a continuous m-field that is defined on all cells of a sub-complex K of the *dual* cell-decomposition. If K is homogeneously ρ-dimensional ([2]) then we also briefly speak of a *ρ-dimensional framework*. In the case that is most important for us, K is the complex of all ρ-dimensional cells of the dual cell-decomposition; a framework that belongs to this complex is called an *r-dimensional framework that is defined everywhere on the manifold M^n*. In the sequel, it will always be assumed that the cells of K are at most $(n - m)$-dimensional.

One then has:

Theorem 14. *Any framework on M^n can be extended to an $(n - m)$-dimensional framework that is defined on all of the manifold M^n.* $(0 < m < n)$.

Proof: Let ξ^ρ be the cells of K and let $\bar{\xi}^\rho$ be the cells of the dual cell-decomposition that do not belong to K. One now attaches an arbitrary m-system to every vertex $\bar{\xi}^0$. With that, an m-field is given on the boundary of every cell $\bar{\xi}^1$, which, from Theorem 11, can be extended continuously into the interior. Now, the m-field that is given on the boundary of every cell $\bar{\xi}^2$ can again (in the event that $m < n - 1$) be extended into the interior of the cell. (Theorem 11) One proves the theorem by pursuing the construction further. It follows from this that:

Corollary. *There exists an $(n - m)$-dimensional framework that is defined on all of M^n.*

Such a framework will always be denoted by $\mathfrak{G}$.

We will understand $\dot{\mathfrak{G}}$ to mean the framework that $\mathfrak{G}$ induces on the complex of $(n - m - 1)$-dimensional cells of the dual cell decomposition, while an arbitrary $(n - m - 1)$-dimensional framework that is defined on all of M^n will be denoted by $\mathfrak{g}$.

Two frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ define a *framework-pair* when a continuous family $\mathfrak{g}_t$ $(0 \leq t \leq 1)$ of frameworks $\mathfrak{g}$ is given with $\mathfrak{g}_0 = \dot{\mathfrak{G}}_0$ and $\mathfrak{g}_1 = \dot{\mathfrak{G}}_1$. $\mathfrak{G}_0$ and $\mathfrak{G}_1$ are then connected to each other on the complex of $(n - m - 1)$-dimensional cells of M^n by a continuous family.

Theorem 15. *Two arbitrary frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ can always be combined into a framework-pair.*

([1]) cf., Seifert-Threlfall: *Lehrbuch der Topologie* (B. G. Teubner, 1934), and furthermore, AH: chap. XI, § 1, § 68.

([2]) cf., AH: chap. IV, § 1, no. 2.

The proof proceeds analogously to that of Theorem 14. One then connects the two m-systems that are given by $\mathfrak{G}_0$ and $\mathfrak{G}_1$ at a vertex ξ^0 of the dual cell decomposition by means of a continuous family of m-systems and then extends this connection to the higher-dimensional cells using Theorem 12.

3. Preliminary remarks on characters in Λ^{n-r}. We next choose a *coefficient ring* J that will serve for the definition of algebraic complexes in M^n ([1]), and, in fact, let J be the *ring of whole numbers* in Case 1 (no. 1) and the *ring of residue classes* (mod 2) in Case 2. We denote algebraic sub-complexes of the x-cell decomposition by C and algebraic sub-complexes of the ξ-cell decomposition by Γ. All $(n - m)$-dimensional complexes Γ^{n-r} define a group Λ^{n-r} that contains the group Z^{n-r} of $(n - r)$-dimensional cycles and the group H^{n-r} of $(n - r)$-dimensional boundaries as subgroups. The difference group $Z^{n-r} - H^{n-r}$ is, as is well-known, the $(n - r)$-dimensional Betti group B^{n-r} of M^n.

A *character* χ in Λ^{n-r} is a homomorphic map from Λ^{n-r} to the coefficient ring J. Therefore, if Γ_1 and Γ_2 are complexes in Λ^{n-r} and α is an element of J then one has:

a) $\chi(\Gamma_1 + \Gamma_2) = \chi(\Gamma_1) + \chi(\Gamma_2)$; b) $\chi(\alpha\Gamma_1) = \alpha\chi(\Gamma_1)$.

From these two facts, it follows that:

c) A character χ is known when its values for the complex defined by a basis of Λ^{n-r} are given. (e.g., all cells ξ^{n-r} define such a basis.)

d) If C is an r-dimensional sub-complex of the x-cell decomposition that is chosen to be fixed then a character χ in Λ^{n-r} will be generated by setting:

$$\chi(\Gamma) = \phi(C, \Gamma).$$

(In this, Γ is an arbitrary complex of Λ^{n-r} and ϕ means the intersection number of the complexes in parentheses.)

e) Any character in Λ^{n-r} can be generated by a complex C in the way that is suggested by d). C is determined uniquely, and is called the *complex that is associated with* χ.

The proof best proceeds by giving C explicitly. One has:

$$C = \sum_{(j)} \chi(\xi_j^{n-r})\, x_j^r .$$

In this, the summation is extended over all r-dimensional cells x_j^r.

([1]) AH: chap, IV.

Our next objective is to determine the properties of the generated complex C from the properties of the characters χ:

f) C is a cycle iff χ vanishes in the group H^{n-r}; i.e., if for every $(n - r + 1)$-dimensional complex Δ of the ξ-cell decomposition one has:

$$\chi(\dot{\Delta}) = 0. \tag{I}$$

The proof follows from the fact that for any character χ and an arbitrary Δ one has the relation:

$$\chi(\dot{\Delta}) = \phi(C, \dot{\Delta}) = \pm\, \phi(\dot{C}, \Delta).$$

g) Between two characters χ_0 and χ_1 in Λ^{n-r} and a character $\dot{\chi}$ in Λ^{n-r-1}, there exists the following relation:

$$\chi_1(\Gamma) - \chi_0(\Gamma) = \dot{\chi}(\dot{\Gamma}), \tag{II}$$

so between the associated complexes C_0, C_1, and D, there exists the relation:

$$C_1 - C_0 = \pm\, \dot{D}.$$

Proof: For an arbitrary $(n - r)$-dimensional complex Γ, one has:

$$\begin{aligned}\phi(C_1 - C_0, \Gamma) &= \phi(C_1, \Gamma) - \phi(C_0, \Gamma) = \chi_1(\Gamma) - \chi_0(\Gamma)\\ &= \dot{\chi}(\dot{\Gamma}) = \phi(D, \dot{\Gamma}) = \pm\, \phi(\dot{D}, \Gamma).\end{aligned}$$

Since Γ was arbitrary, the assertion follows from this that:

h) Let a set of characters χ_i in the group Λ^{n-r} be given, each of which satisfies the condition (I), and any two of which fulfill a relation of the form (II). One then has:

α) The given set determines a character χ^* in the Betti group B^{n-r} whose elements (these are homology classes) we denote by Ξ.

β) The complexes that are associated with χ_i are, from f), cycles and lie in a single r-dimensional homology class A.

γ) One has: $\chi^*(\Xi) = \phi(A, \Xi)$.

Proof:

Of α): From the existence of (II), it then follows that all characters χ_i in the cycle group Z^{n-r} coincide, and thus induce a single character in that group. Due to (I), this character has the same value for homology cycles, moreover, so it actually determines a single character χ^* in the Betti group B^{n-r}.

Of β): Due to (II), one has the assertion g), from which, it follows that the cycles that are associated with two characters of the given set are homologous.

Of γ): This follows directly from the definitions of χ^* and A.

4. The characters that are determined by frameworks; main theorems. Now, let an $(n - m)$-dimensional framework $\mathfrak{G}$ that is defined on all of the manifold M^n be given. (cf., corollary to Theorem 14.) We now define a character χ in Λ^{n-m+1} by giving the values of χ for the cells ξ^{n-m+1}, as in no. 3, c): Let $\chi(\xi^{n-m+1})$ be the characteristic of the continuous m-field that is given by $\mathfrak{G}$ on the boundary sphere $\dot{\xi}^{n-m+1}$ of ξ^{n-m+1}. (Naturally, the orientation $\dot{\xi}^{n-m+1}$ is the one that was employed in the calculation of this characteristic. One further observes that the characteristic is an element of J.) The character thus defined is called the *character χ that is associated with* $\mathfrak{G}$.

In addition, we consider all $(n - m)$-dimensional frameworks $\mathfrak{G}_i$ that are also defined on all of M^n. The characters χ_i that are associated with them define a set like the one that we considered in no. 3, h). We assert that this set fulfills the assumption of no. 3, h).

Proof: Let, say, $\mathfrak{G}_0$ and $\mathfrak{G}_1$ be two frameworks, and let χ_0 and χ_1, resp., be the associated characters. We next show that a character $\dot{\chi}$ exists in Λ^{n-m} such that χ_0 , χ_1, and $\dot{\chi}$ fulfill the relation (II) of no. 3. Due to no. 3, c), it suffices to define $\dot{\chi}$ for the cells ξ^{n-m}. To that end, we couple the frameworks $\mathfrak{G}_0$ and $\mathfrak{G}_1$ into a framework-pair using Theorem 15; let $\dot{\chi}(\xi^{n-m})$ be the characteristic of the field-pair that is induced on ξ^{n-m} by this framework-pair. Due to formula (C) of § 3, no. 3, one has:

$$\chi_1(\xi^{n-m+1}) - \chi_0(\xi^{n-m+1}) = \dot{\chi}(\dot{\xi}^{n-m+1}) .$$

The relation (II) now follows from no. 3a) and b), in fact. Furthermore, we have to show that each of our characters satisfies the condition (I) of no. 3. If we apply the relation (II) that we just proved to the complex $\dot{\Delta}$ then this yields:

$$\chi_1(\dot{\Delta}) = \chi_0(\dot{\Delta}),$$

so it suffices to prove (I) for a single character that is induced by a special framework $\mathfrak{G}_0$. Moreover, due to a) and b), it suffices that Δ be a cell ξ^{n-m+2}. We now construct $\mathfrak{G}_0$ as follows: Let the m-systems of $\mathfrak{G}_0$ be parallel to each other on the boundary $\dot{\xi}^{n-m+2}$. (This definition makes sense, since ξ^{n-m+2} lies in an element (no. 1).) From Theorem 14, such a framework can always be found. For the associated character χ_0, one now has, trivially: $\chi_0(\xi^{n-m+2}) = 0$, with which all parts of (I) are proved.

From the assertion of no. 3, h), it now follows that:

The character χ that is associated with a framework $\mathfrak{G}$ has a cycle for its associated complex, which will be called the *singular cycle* of $\mathfrak{G}$, and from no. 3, e), it is given by:

$$z = \sum_{(j)} \chi(\xi_j^{n-m+1})\, x_j^{m-1}\,.$$

All of the characters χ_i determine a character χ^* in the $(n - m + 1)$-dimensional Betti group B^{n-m+1}, which we will call χ^{n-m+1} in the sequel ([1]). One further has:

Theorem 16. (First main theorem). *The singular cycles of all* $(n - m)$*-dimensional frameworks* $\mathfrak{G}$ *that consist of m-systems and can be defined on the entire manifold* M^n *lie in a single (m – 1)-dimensional homology class; it is called the* characteristic homology class F^{m-1}. *If* Ξ *is an arbitrary* $(n - m + 1)$*-dimensional homology class then one has:*

$$\chi^{n-m+1}(\Xi) = \phi(F^{m-1}, \Xi).$$

In the next paragraph, we shall see that the character χ^{n-m+1} represents a *generalization of the Euler characteristic.*

To these immediate consequences of the discussion in no. 3, we must add a somewhat deeper theorem:

Theorem 17 (Second main theorem). *Any cycle that is contained in the characteristic class* F^{m-1} *is the singular cycle of a framework.*

Proof: Let, say, z be the given cycle in the class F^{m-1}. We choose an arbitrary, but fixed, initial framework $\mathfrak{G}_0$ with the singular cycle z_0. From Theorem 16, z_0 also lies in F^{m-1}, so one has $z \sim z_0$, and therefore $z - z_0 = \dot{D}$. Our framework $\mathfrak{G}_0$ induces an m-field $\mathfrak{F}_0$ on the cell ξ^{n-m}, which we extend by means of another field $\mathfrak{F}_1$ to a field-pair with rigid boundary values (§ 3, no. 3) whose characteristic on ξ^{n-m} possesses the value $\phi(D, \xi^{n-m})$ (Theorem 13). The m-field $\mathfrak{F}_1$ that is thus constructed on all cells ξ^{n-m} combines into a framework $\mathfrak{G}_1$. $\mathfrak{G}_0$ and $\mathfrak{G}_1$ together define a framework-pair that gives rise to a character $\dot{\chi}$ as in the beginning of this section. By construction, one has $\dot{\chi}(\xi^{n-m}) = \phi(D, \xi^{n-m})$; i.e., the complex that is associated with $\dot{\chi}$ is the complex D.

We have seen that the relation (II) of no. 3 exists between the characters χ_0 of $\mathfrak{G}_0$ and χ_1 of $\mathfrak{G}_1$ and the character $\dot{\chi}$, so the assertion of no. 3, g) is true; i.e., $z_1 - z_0 = \pm\dot{D}$, if we denote the singular cycle of $\mathfrak{G}_1$ by z_1. The given cycle z is then a singular cycle of $\mathfrak{G}_1$, with which, Theorem 17 is proved.

([1]) The character χ^{n-m+1} is, *ex definitone*, independent of the choice of framework; it is given by the geometric properties of M^n.

The meaning of the characteristic class F^{m-1} for the problem of this paper is based in the following Theorem:

Theorem 18 (Existence theorem). *The exists an* $(n - m + 1)$*-dimensional framework that is defined on the entire manifold* M^n *iff the characteristic class* F^{m-1} *is the zero class.*

Proof: a) Let an $(n - m + 1)$-dimensional framework that is defined on all of M^n be given. It induces a framework $\mathfrak{G}$ on the complex of cells ξ^{n-m}, and thus an m-field $\mathfrak{F}$ on each cell boundary $\dot{\xi}^{n-m+1}$. Since $\mathfrak{F}$ is extended into the interior of the cell ξ^{n-m+1}, its characteristic vanishes on $\dot{\xi}^{n-m+1}$, so the character χ that is associated with $\mathfrak{G}$ also vanishes, and one has $z = 0$ for the singular cycle z of $\mathfrak{G}$, so $z \sim 0$ precisely.

b) Let the characteristic class F^{m-1} be the zero class. From Theorem 17, there is a framework $\mathfrak{G}$ whose singular cycle is the zero cycle. The characteristic χ that is associated with $\mathfrak{G}$ then vanishes; however, from Theorem 11, the field that is induced by $\mathfrak{G}$ on $\dot{\xi}^{n-m+1}$ can be extended into the interior.

5. Fields with singularities. Our endeavors to construct a continuous m-field on the manifold M^n step-wise by frameworks are obstructed by the existence of the class F^{m-1}; however, we can always find m-fields whose continuity is broken at certain "singular" points. In order to not go into dimension-theoretic difficulties, we would like to consider only m-fields that satisfy the following assumption: If a cell x^{r-1} of our x-cell decomposition contains a singular point in its interior then it consists of nothing but singular points. All of these cells define an absolute complex K^{r-1} – viz., the *singularity complex* of the field in question. [The number $(r - 1)$ means the dimension of the highest-dimensional cell in this complex.] Now, a field with the singularity complex K^{r-1} obviously induces an $(n - r)$-dimensional framework that is defined on all of M^n. However, the converse is also true: Every $(n - r)$-dimensional framework that is defined on all of M^n is associated with an m-field on M^n with a singularity complex K^{r-1}. In order to see this, one extends the m-field that is given by the framework on the cells ξ^{n-m} by central projection (§ 3, no. 11) into the higher-dimensional cells ξ^{n-m+k}. If one then chooses the projection center to be the intersection point of ξ^{n-m+k} with the dual cell x^{r-k} then the necessary cone construction can be performed simplicially on a common subdivision U of the x and ξ-cell decompositions. With this relationship between frameworks and singular fields, it now follows from the corollary to Theorem 14 and from Theorem 18 that:

Theorem 19. *There always exists an m-field with an* $(m - 1)$*-dimensional singularity complex on a manifold* M^n*; the necessary and sufficient condition for the existence of an m-field with an at most* $(m - 2)$*-dimensional singularity complex is the vanishing of the characteristic class* F^{m-1}.

Since any singular m-field an $(m - 1)$-dimensional singularity complex on M^n uniquely determines an $(n - m)$-dimensional framework $\mathfrak{G}$ that is defined on all of M^n, we can briefly call the singular cycle that is associated with $\mathfrak{G}$ (no. 14) the *singular cycle* of the given field. One then has:

Theorem 19a. *The singular cycle of an m-field with the* $(m - 1)$*-dimensional singularity complex* K^{m-1} *is an algebraic sub-complex of* K^{m-1} *in the subdivision that it induces through U; it measures the multiplicities of the* $(m - 1)$*-dimensional singularities and represents the characteristic class* F^{m-1}.

In order to prove this, one employs the explicit representation of the singular cycle $z = \sum_{(j)} \chi(\xi_j^{n-m+1})\, x_j^{m-1}$ and Theorem 11.

§ 5. Determination of the characteristic classes in special cases.

1. Differential simplicial decompositions. A simplicial decomposition K of a given manifold is called *differentiable* when any simplex of K, along with its perimeter, lies in an element of M^n (§ 4, no. 1) and is either a Euclidian simplex ([1]) or the image of a Euclidian simplex by means of a topological map that is continuously differentiable in both directions in this element.

For what follows, we will need the barycentric subdivision ([2]) $\bar{K}$ of such a simplicial decomposition K. If we denote the center of mass of an r-dimensional simplex of K by a_r then the simplexes $x^s = (a_{r_0}, a_{r_1}, \ldots, a_{r_s})$ are the simplexes of $\bar{K}$. $(r_0 < r_1 < \ldots < r_s)$ and $(s = 0, 1, \ldots, n)$. Now, let $\bar{K}$ be our x-cell decomposition of § 4; we denote the dual cell ξ^{n-s} of x^s by $\xi^{n-s} = \xi_{(r_0 r_1 \ldots r_s)}$.

2. Single vector fields. In this number, we concern ourselves with the theory of 1-fields (in the sequel, we briefly refer to them as *vector fields*) on a manifold M^n. This theory has already been developed for some time ([3]), and the concluding results go back to H. Hopf.

Theorem 19 then shows that there is always a vector field $\mathfrak{F}$ with a 0-dimensional singularity complex in M^n; $\mathfrak{F}$ is then singular at only finitely many vertices x_i^0 of the x-cell decomposition. We understand the *index* j_i of the singularity x_i^0 to mean the characteristic of the 1-field that is given by $\mathfrak{F}$ on the boundary $\dot{\xi}_i^n$ of the cell ξ_i^n that is dual to x_i^0. (We find ourselves in Case 1 of § 4, no. 1; M^n is therefore *orientable*, and the

([1]) AH: chap. III, § 1, no. 1.
([2]) AH: chap. III, § 2, no. 3.
([3]) AH: chap. XIV, § 4.

cells ξ_i^n are *coherently oriented*.) A simple argument gives the singular cycle of $\mathfrak{F}$ (§ 4, no. 5) as:

$$z = \sum_{(i)} j_i \, x_i^0 \, . \tag{1}$$

The characteristic class F^0 will then be represented by the cycle $x^0 \sum j_i$ (x^0 is an arbitrary, but fixed, vertex of the x-cell decomposition). The index sum $\sum j_i$ is called the *algebraic number of singularities.* If one denotes the n-dimensional homology class that is represented by the sum of all cells ξ^n by Ξ^n then, from Theorem 16, this yields for the character χ^n in the Betti group B^n :

$$\chi^n(\Xi^n) = \phi(F, \Xi^n) = \sum j_i \, . \tag{2}$$

Since Ξ^n is the single basis element for B^n, (2) determines the character χ^n completely.

It now follows from Theorem 16 and 18 that:

Theorem 20. *The algebraic number of singularities is the same for all vector fields on M^n; one then has vector fields that are continuous at all points of M^n iff this number vanishes.*

One further has:

Theorem 20*a*. *For a suitable orientation of M^n, the algebraic number of singularities of any vector field on M^n is equal to the Euler characteristic $\chi(M^n)$ of M^n.*

This theorem is equivalent to the following assertion:

The characteristic class F^0 can be represented by $x^0 \cdot \chi(M^n)$. Moreover, the formula:

$$\chi^n(\Xi^n) = \chi(M^n) \tag{3}$$

also says precisely the same thing. We will prove the theorem for the simplest case of $n = 2$ in this latter form. We carry out the proof under the assumption that M^2 possesses a differentiable simplicial decomposition. (Theorem 20*a* is still true without this assumption.) We then construct a *special* one-dimensional framework $\mathfrak{G}$ that consists of 1-systems on the barycentric subdivision of the dual cell decomposition whose associated character χ we determine. The part of $\mathfrak{G}$ that lies in a simplex (a_0, a_1, a_2) (no. 1) of the barycentric subdivision $\bar{K}$ is depicted in Fig. 3. From this figure, it is clear that the vectors of $\mathfrak{G}$ that lie on the boundary of a cell of type $\xi_{(0)}$ (no. 1) point to the exterior of $\xi_{(0)}$, and on the boundary of a cell of type $\xi_{(2)}$, they point to the interior of $\xi_{(2)}$. [In Fig. 3, the parts of three cells that lie in (a_0, a_1, a_2) are suggested by $\xi_{(0)}$, $\xi_{(1)}$, $\xi_{(2)}$.] For a suitable orientation, one finds that the characteristic of the field that is induced by $\mathfrak{G}$ on the boundary of a cell $\xi_{(r)}$ has the value $(-1)^r$, so one has:

$$\chi(\xi_{(r)}) = (-1)^r \qquad (r = 0, 1, 2), \tag{4}$$

and the singular cycle z of $\mathfrak{G}$ will be:

$$z = \sum(-1)^r\, a_r, \tag{4a}$$

where the summation is taken over all vertices of $\bar{K}$. If one denotes the number of cells of type $\xi_{(r)}$ by a_r then that would yield for the character $\chi^n = \chi^2$:

$$\chi^2(\Xi^2) = \chi(\Sigma\, \xi^2) = \Sigma\, \chi(\xi^2) = \Sigma\, \chi(\xi_{(0)}) + \Sigma\, \chi(\xi_{(1)}) + \Sigma\, \chi(\xi_{(2)}) = a_0 - a_1 + a_2\,.$$

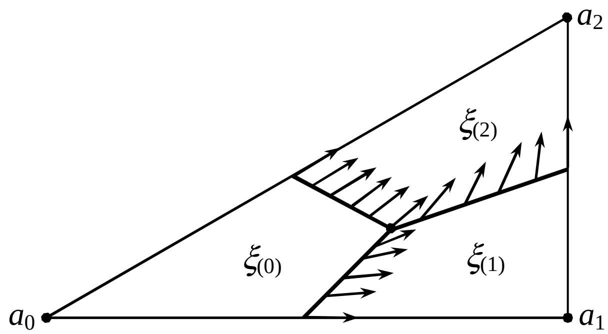

Figure 3.

However, by definition, $a_0 - a_1 + a_2$ is the Euler characteristic $\chi(M^2)$; with that, (3) is proved in the special case $n = 2$. Theorem 20*a*) can be proved for n-dimensional manifolds in an analogous way.

Formula (3) confirms the fact that was mentioned in § 4 that the character χ^{n-m+1} can be regarded as a generalization of the Euler characteristic.

If follows from Theorems 20 and 20*a*) that:

Corollary. *There exists a continuous vector field on the manifold* M^n *iff the Euler characteristic* $\chi(M^n)$ *vanishes* ([21a]).

This theorem is true for *non-orientable manifolds*, but this is not directly provable by our methods. Our argument can also be carried out for non-orientable manifolds in the event that we introduce the ring of residue classes (mod 2) in place of the ring of whole numbers (§ 4, no. 3). If we understand Ξ^n in this case to mean the n-dimensional homology class that is represented by the sum of the (unoriented) cells ξ^n then one has:

$$\chi^n(\Xi^n) \equiv \chi(M^n) \qquad (\text{mod } 2). \tag{3a}$$

([21a]) Cf., AH: chap. XIV, § 4, Theorem 3.

3. Three-dimensional manifolds. We now examine the parallelizability (§ 4, no. 1) of three-dimensional manifolds. One has the important result:

Theorem 21. *Any orientable, three-dimensional, closed manifold that admits a differentiable simplicial decomposition is parallelizable.*

Before we give the proof of this theorem, we mention that it follows from the considerations of § 4, no. 1 that:

Corollary. *If a three-dimensional manifold M^3 fulfills the assumptions of Theorem 21 then the manifold of its directed line elements is homeomorphic to the topological product of M^3 with a two-dimensional sphere.*

The proof of Theorem 21 proceeds in four steps:

I. Determination of the characteristic class F^1.

We can satisfy ourselves with the following hints for the solution of this problem, since in Appendix I we have rigorously determined the characteristic class F^1 for three-dimensional, orientable manifolds under somewhat different assumptions and by other methods.

F^1 is the characteristic class of the 2-fields, so we must set $m = 2$ and $n = 3$. We are then in Case 2 of § 4, no. 1; J is then the ring of residue classes (mod 2). In order to determine F^1, one can, in analogy to no. 2 (Fig. 3), construct a *special* 1-dimensional framework $\mathfrak{G}$ that is defined on all of M^3, and which is coupled with the barycentric subdivision $\bar{K}$. I will not go into the somewhat tedious construction of this framework that is composed of 2-systems here; one finds for the associated character χ that:

$$\chi(\xi_{(r_0, r_1)}) = 1, \tag{5}$$

such that the singular cycle z of $\mathfrak{G}$ is given by ([1]):

$$z = \sum\left(a_{r_0}, a_{r_1}\right). \tag{5a}$$

This cycle (mod 2) thus consists of *all edges of the barycentric subdivision* $\bar{K}$. One can now show that z always bounds in an *orientable* manifold M^3, while this does not

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([1]) Formulas (4a) and (5a) are closely related to the conjecture that for arbitrary n and m the characteristic class F^{m-1} can be represented:

in Case I of § 4, no. 1 by $\Sigma\,(-1)^{r_0+r_1+\cdots+r_{m-1}}\left(a_{r_0}, a_{r_1}, \ldots, a_{r_{m-1}}\right)$

and in Case II, by $\Sigma\left(a_{r_0}, a_{r_1}, \ldots, a_{r_{m-1}}\right)$.

The summation is therefore taken over all $(m-1)$-dimensional cells of $\bar{K}$; the complexes above are, in fact, cycles of the coefficient ring J.

necessarily need to be the case in a non-orientable manifold ([1]). This then yields that in an orientable M^3 the characteristic class F^1 is always the zero class.

II. There exists a framework $\mathfrak{H}$ that is defined on all of M^3 and consists of 2-systems.

Since, from I, the class F^1 vanishes in our orientable M^3, this fact is a direct consequence of the existence Theorem 16.

III. There exist continuous 2-fields on M^3.

In order to prove this, we show that the 2-field $\mathfrak{F}$ that is given by $\mathfrak{H}$ on the boundary $\dot{\xi}^3$ of a cell ξ^3 can be continuously extended into the interior of ξ^3. Since ξ^3 lies in an element (§ 4, no. 1), we must therefore prove the following theorem: A continuous 2-field $\mathfrak{F}$ that is given on the boundary sphere S^2 of a 3-dimensional cell E^3 that lies in Euclidian space R^3 can be continuously extended into the interior of E^3.

The following statement is equivalent to this theorem: The map of S^2 into the manifold $V_{3,2}^*$ by parallel 2-systems (§ 3, no. 1) that is associated with $\mathfrak{F}$ is homotopic to zero. Our statement III can thus be expressed in the following form: Any continuous map of a 2-dimensional sphere S^2 into $V_{3,2}^*$ is homotopic to zero. Now, since, from § 2, no. 2, the closed manifold $V_{3,2}$ is a deformation retract of $V_{3,2}^*$, it suffices to prove this assertion for maps of S^2 into $V_{3,2}$. However, since $V_{3,2}$ is homeomorphic to the projective space P^3 (§ 1, no. 3), and since any map of S^2 into P^3 is, in fact, homotopic to zero, we have proved the assertion III.

IV. There exist continuous 3-fields on M^3.

The fact that the existence of continuous 3-fields follows from the existence of continuous 2-fields on an *orientable* M^3 is easily proved.

§ 6. Theorems on characteristic cohomology classes. Applications.

1. Order of the characteristic class. In this section, we pose the problem of determining the order of a non-vanishing characteristic class. This problem is meaningful only in Case 1 of § 4, no. 1, for which the coefficient ring J is the ring of whole numbers. We will solve it for even $(n - m)$.

We preface the following analysis with a subsidiary consideration that relates to the manifolds $V_{n,m}^*$ (§ 2) for which $n - m$ is even. Namely, we shall examine the topological map φ of $V_{n,m}^*$ to itself that comes about when one replaces the m^{th} vector $\mathfrak{v}_m$ in any m-system of $V_{n,m}^*$ with its opposite vector $-\mathfrak{v}_m$. On the $(n - m)$-dimensional sphere that is

([1]) Cf., problem 187 in the Jahresbericht der deutschen Mathematikervereinigung, Band 45, pp. 22.

provided by the basis cycle $z_{n,\,m}$ of § 1, no. 4 for a fixed orientation, φ is the diameteral map; since this sphere possesses an even dimension, this yields:

$$\varphi(z_{n,\,m}) = - z_{n,\,m}\,. \tag{1}$$

With those preparations, a framework $\mathfrak{G}$ that consists of m-systems will be constructed on the given manifold M^n by employing the notations and assumptions of § 4, and we let $n - m$ be even. We thus find ourselves in Case 1 of § 4, no. 1, and the coefficient ring J is therefore the ring of whole numbers. The framework $\mathfrak{G}$ induces an m-field on the boundary of any $(n - m + 1)$-dimensional cell ξ whose characteristic $\chi(\xi)$ is established by means of the map f of $\dot{\xi}$ into $V_{n,m}^{*}$ by parallel m-systems (§ 3, no. 1).

If one now replaces the m^{th} vector on any m-system of $\mathfrak{G}$ with its opposite vector then a new framework $\bar{\mathfrak{G}}$ arises that is associated with the characteristic $\bar{\chi}(\xi)$ and the map $\bar{f}$. Obviously, $\bar{f}$ arises from the composition of f and φ; it then follows from (1) that: $\bar{\chi}(\xi) = - \chi(\xi)$. The relation:

$$\bar{\chi} = - \chi \tag{2}$$

then exists between the characters χ and $\bar{\chi}$ that belong to $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp. χ, as well as $\bar{\chi}$, then induce the character χ^{n-m+1} in the $(n - m + 1)$-dimensional Betti group; it then follows from (2) that: $\chi^{n-m+1} = - \chi^{n-m+1}$, so ultimately $\chi^{n-m+1} = 0$.

It would be incorrect to conclude the vanishing of the characteristic class F^{m-1} from the vanishing of χ^{n-m+1}; this conclusion is only permissible when no $(m - 1)$-dimensional torsion is present in M^n.

If we set, say, $m = 1$ then we find that $\chi^n = 0$ for manifolds of odd dimension; however, from § 5, formula (3), it follows from this that the characteristic of an orientable manifold of odd dimensions vanishes ([1]). From the corollary to Theorem 20 it then follows, moreover, that any orientable manifold of odd dimension possesses a continuous vector field.

Theorem 22. *If M^n is orientable, $(n - m)$ is even, and the class F^{m-1} is not the zero class then that class has order 2.*

Proof: We have to show: For even $(n - m)$, one always has $2 \cdot F^{m-1} = 0$. Now, from (2), the relation $z = - \bar{z}$ exists between the singular cycles z and $\bar{z}$ of the frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp., that were employed above. Since both of these cycles lie in F^{m-1}, one has $F^{m-1} = - F^{m-1}$; this was to be proved.

Corollary. *If $(n - m)$ is even and no $(m - 1)$-dimensional torsion is present in M^n then F^{m-1} is the zero class.*

([1]) To my knowledge, J. Hadamard was the first to derive the vanishing of the Euler characteristic of a manifold of odd dimension from the theory of vector fields. Cf., Tannery: *Introduction à la théorie des fonctions* (Paris, Hermann, 1910), t. II, note by Hadamard, no. 42, pp. 475.

2. An intersection theorem. In what follows, Cases 1 and 2 of § 4, no. 1 will no longer be distinct; *all consideration will be based upon the ring of residue classes* (mod 2) *as the coefficient ring J*, and M^n can be either an orientable or non-orientable manifold.

In order to bring our theory to a definite conclusion, we must find manifolds in which non-zero characteristic classes exist; only then will the theorems of § 4 contain non-trivial statements. The analysis of this section will serve to resolve this problem.

We call a ν-dimensional manifold M^ν that is embedded in the given manifold M^n a *hypersurface* when the following conditions are fulfilled:

a) Let M^ν be the image of a differentiable parameter manifold by means of a topological and continuously-differentiable map of this parameter manifold into M^n.

b) M^ν admits a cell decomposition that is a sub-complex of the ξ-cell decomposition (§ 4, no. 2) of the manifold M^n.

Due to a), vectors on M^ν are also vectors on M^n, and the totality of all vectors on M^ν that contact a point p of M^ν defines a ν-dimensional vector structure on M^n. If the vectors in a $(n - \nu)$-system on M^n that contact p do not belong to this structure then we call the system *foreign* to M^ν. If a continuous field of $(n - \nu)$-systems exists on M^n that are foreign to M^ν then we say that M^ν possesses an *external* $(n - \nu)$-field ([1]). If $\nu = n - 1$ then this simply means that M^ν is two-sided in M^n.

Due to b), M^ν is a cycle (mod 2) of the ξ-cell decomposition that represents a ν-dimensional homology class Ξ^ν of M^n and a ν-dimensional homology class $\bar{\Xi}^\nu$ in M^ν. One has:

Theorem 23. *If a hypersurface M^ν that lies in M^n possesses an external $(n - \nu)$-field then the intersection number of the characteristic class $F^{n-\nu}$ of M^n with M^ν is the (mod 2) reduced Euler characteristic of M^ν.*

Before we prove this theorem, we introduce the following relations: Let $\bar{\xi}$ be the cells of the ξ-cell decomposition that induce a cell decomposition of M^ν using b); a $(\nu - 1)$-dimensional framework that is defined on all of M^n and consists of $(n - \nu + 1)$-systems will be denoted by $\mathfrak{G}$, and associated character in the group Λ^ν of M^n (§ 4, no. 4), by χ. A $(\nu - 1)$-dimensional framework that is defined on all of M^ν and consists of 1-systems will be denoted by $\bar{\mathfrak{G}}$, and the associated character in the group Λ^ν of M^ν, by $\bar{\chi}$. The characters χ determine the character χ^ν (§ 4, no. 4) in the ν-dimensional Betti group of M^n, while the characters $\bar{\chi}$ determine the character $\bar{\chi}^\nu$ in the ν-dimensional Betti group of M^ν in an analogous way.

We then prove the following:

([1]) A hypersurface with an external $(n - \nu)$-field that lies in an orientable manifold is orientable.

Lemma. If there exist two frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ such that for every cell $\bar{\xi}^\nu$ the relation:

$$\chi(\bar{\xi}^\nu) \equiv \bar{\chi}(\bar{\xi}^\nu) \qquad (\text{mod } 2) \tag{2}$$

is fulfilled then the assertion of Theorem 23 is true.

Proof: By summing over all cells $\bar{\xi}^\nu$, one gets from (2) that:

$$\chi^\nu(\Xi^\nu) \equiv \bar{\chi}^\nu(\bar{\Xi}^\nu) \qquad (\text{mod } 2). \tag{3}$$

From Theorem 16, the left-hand side of (3) is the intersection number of $\phi(F^{n-\nu}, \Xi^\nu)$, while, from § 5, formula (3a), the right-hand side is congruent to the Euler characteristic of M^ν. With that, we have proved the lemma.

In order to prove Theorem 23 now, we have to construct the frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ that satisfy the assumption of the lemma: First, $\bar{\mathfrak{G}}$ is chosen arbitrarily. Furthermore, the system of $\mathfrak{G}$ on the cells $\bar{\xi}^{\nu-1}$ shall be the system of external $(n - \nu)$-fields, extended by the vectors of $\bar{\mathfrak{G}}$; in the remaining part of M^n, $\mathfrak{G}$ will be constructed arbitrarily with the use of Theorem 14. (2) is, in fact, fulfilled with this choice of $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, as one easily confirms by applying the process of calculating the characteristic by recursion (§ 3, no. 2).

We shall not go into the closely-related generalizations of Theorem 23, but merely apply this theorem to the solution of the problem that was posed at the start of this paragraph:

Theorem 24. *For a given n and m with $n \equiv m - 1$ (mod 2), there exists a manifold M^n in which the characteristic class F^{m-1} is not the zero class.*

Addendum. *If $n \equiv m - 1$ (mod 4) then there is indeed an orientable M^n in which F^{m-1} does not vanish.*

The following remarks suffice for the proof of these theorems:

1. The assumption of Theorem 25 is fulfilled when M^n is the topological product of M^ν and an arbitrary $(n - \nu)$-dimensional manifold.

2. If the assumption of Theorem 23 is fulfilled and if M^ν possesses an odd Euler characteristic then it follows from this theorem that the class $F^{n-\nu}$ does not vanish in M^n.

3. There exist manifolds of even dimension that have odd characteristics, and there exist orientable manifolds with dimensions that are divisible by 4 and have odd characteristics. One now sets $m - 1 = n - n$ and constructs M^n as a product manifold.

By a special choice of m, it follows easily from the Addendum that:

Theorem 25. *For any dimension n that is not equal to 1 or 3, there exists an orientable, but not parallelizable, n-dimensional manifold.*

(One observes that, from Theorem 19, the vanishing of all characteristic classes is a necessary condition for parallelizability.)

3. Examples and applications. Let $x_0, x_1, x_2, \ldots, x_n$ be coordinates in an $(n + 1)$-dimensional number-space R^{n+1}, and let $\mathfrak{p}$ mean the position vector $(x_0, x_1, x_2, \ldots, x_n)$ in that space. Let m vector fields $\mathfrak{v}^\mu$ ($\mu = 1, 2, \ldots, m$) be given in R^{n+1}, and for every μ, let the components v_i^μ ($i = 0, 1, 2, \ldots, n$) of the vector $\mathfrak{v}^\mu$ be homogeneous functions of first degree of the independent variables $x_0, x_1, x_2, \ldots, x_n$. We project this vector field from the origin of R^{n+1} onto the n-dimensional projective space P^n that completes R^{n+1} into an $(n + 1)$-dimensional projective space. From our homogeneity condition, it follows that in order for m vector fields in P^n to define an m-field in the sense of § 4, no. 1, the $(m + 1)$ vectors $\mathfrak{p} = \mathfrak{v}^0, \mathfrak{v}^1, \mathfrak{v}^2, \ldots, \mathfrak{v}^m$ would have to be linearly independent at all points of R^{n+1}, except for the origin.

We shall employ this convenient representation for the vector fields in projective spaces in the sequel in order to discuss the characteristic classes of n-dimensional projective spaces. So, for example, for $n = 3$ and $m = 3$, the vectors:

$$\begin{array}{llll}
\mathfrak{v}^0(\;\; x_0, & x_1, & x_2, & x_3) \\
\mathfrak{v}^1(-x_1, & x_0, & -x_3, & x_2) \\
\mathfrak{v}^2(-x_2, & x_2, & x_0, & -x_1) \\
\mathfrak{v}^3(-x_3, & -x_2 & x_1, & x_0)
\end{array} \tag{I}$$

provide a continuous 3-field in 3-dimensional projective space P^3, with which the parallelizability of P^3, and therefore the 3-dimensional sphere, is established by example. One can also find an analogous example in dimension 7 that parallelizes P^7 and the 7-dimensional sphere ([1]).

We now examine the case $n = 5$, $m = 2$, so we concern ourselves with 2-fields in P^5. The three vectors:

$$\begin{array}{llllll}
\mathfrak{v}^0(\;\; x_0, & x_1, & x_2, & x_3, & x_4, & x_5) \\
\mathfrak{v}^1(-x_1, & x_0, & -x_3, & x_2, & -x_5, & x_4) \\
\mathfrak{v}^2(-x_2, & x_2, & x_0, & -x_1, & 0, & 0)
\end{array} \tag{II}$$

are linearly-independent only for $x_0 = x_1 = x_2 = x_3 = 0$, so except for the projective line P^1 that is given by $x_0 = x_1 = x_2 = x_3 = 0$, they provide *two* linearly-independent vector fields on P^5 that that we again denote by $\mathfrak{v}^1$ and $\mathfrak{v}^2$, for the sake of simplicity. We now construct a ξ-cell decomposition of P^5, with the use of the notations of no. 2, in which the

([1]) Cf., H. Hurwitz: "Über die Komposition der quadratischen Formen von beliebig vielen Variabeln" (Math. Werke, Band II, pp. 565-571, especially pp. 570, where one finds the matrix that is analogous to I.)

4-dimensional projective space P^4 lies as the hypersurface $x_4 = 0$. The intersection point P of P^1 and P^4 lies in the interior of a cell $\bar{\xi}_0^4$ of the cell decomposition of P^4. Furthermore, two frameworks $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ shall be constructed that satisfy the assumptions of the lemma in no. 2: Let the vectors of $\bar{\mathfrak{G}}$ be the vectors $\mathfrak{v}^2$ on the cells $\bar{\xi}^3$, while the 2-systems of $\mathfrak{G}$ shall be the system $\mathfrak{v}^1$, $\mathfrak{v}^2$ on the cells $\bar{\xi}^3$; $\mathfrak{G}$ is arbitrary on the remaining cells ξ^3 of P^5 and can be constructed using Theorem 14. The characters χ and $\bar{\chi}$ that are associated with $\mathfrak{G}$ and $\bar{\mathfrak{G}}$, resp., actually fulfill the congruence (2) that was required in the lemma:

$$\chi(\bar{\xi}^4) \equiv \bar{\chi}(\bar{\xi}^4) \qquad (\mathrm{mod}\ 2).$$

In order to prove this, one observes that for any cell $\bar{\xi}^4$, except $\bar{\xi}_0^4$, the relation $\chi(\bar{\xi}^4) \equiv \bar{\chi}(\bar{\xi}^4) = 0$ exists, since $\mathfrak{G}$, as well as $\bar{\mathfrak{G}}$, can be continuously extended into the interior of the cell. One verifies the assertion for the cell $\bar{\xi}_0^4$ by calculating the characteristic by recursion (§ 3, no. 2); in order to be able to apply this method, it suffices that the cell $\bar{\xi}_0^4$ be foreign to the projective space $x_5 = 0$; the vectors $\mathfrak{v}^1$ whose contact points are points of $\bar{\xi}_0^4$ do not lie in P^4 then.

From the statement of the lemma, it now follows that the intersection number of the class F^1 of P^5 with the hypersurface P^4 is the (mod 2) reduced characteristic of P^4; however, this characteristic has the value 1. Therefore, the class F^1 is not the zero class, and will be represented by a projective line.

One achieves the determination of the class F^1 in projective spaces of dimension $4k + 1$ ($k > 0$) with the help of analogous vector fields; one finds:

Theorem 26. *The one-dimensional characteristic class in a real projective space of dimension* $(4k + 1)$ $(k > 0)$ *will be represented by a projective line; it is therefore impossible to find two linearly-independent continuous vector fields in these spaces.*

An algebraic application. We would like to relate our investigation of projective spaces to an algebraic problem that has a close connection with the older investigations ([1]).

We call $(m + 1)$ linearly-independent quadratic $(n + 1)$-sequences of real matrices:

$$A^{(\mu)} = \left(a_{ik}^{\mu}\right) \qquad \begin{pmatrix} \mu = 0,1,2,\ldots,m \\ i,k = 0,1,2,\ldots,n \end{pmatrix} \tag{1}$$

linearly-independent when any matrix $\sum A^{(\mu)} y_\mu$ that comes about through linear combination is non-singular, as long as only one of the real numbers y_μ is non-zero. One then has the following:

([1]) Cf., Hurwitz: Werke, Band II, pp. 565-571 and pp. 641-666; furthermore, Radon: Abh. math. Seminar der Univ. Hamburg, Band I, pp. 1-14.

Lemma. *If there are (m + 1) linearly-independent matrices (1) then there exists an everywhere-continuous m-field in projective space P^n.*

Proof: If B is any non-singular $(n + 1)$-rowed matrix then obviously the matrices $B A^{(\mu)}$ ($\mu = 0, 1, \ldots, m$) are also linearly-independent; since we can choose $B = (A^{(0)})^{-1}$, we can assume from now on that:

$$a_{ik}^{0} = \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k. \end{cases} \tag{2}$$

We now understand $\mathfrak{v}^{\mu}$, for $\mu = 0, 1, \ldots, m$, to mean the vectors of R^{n+1} whose i^{th} component ($i = 0, 1, 2, \ldots, n$) is given by:

$$v_i^{\mu} = \sum_{k=0}^{n} a_{ik}^{\mu} x_k \, ; \tag{3}$$

if one recalls (2) then $\mathfrak{v}^0$ is the position vector $\mathfrak{p} = (x_0, x_1, \ldots, x_n)$ in R^{n+1}. From no. 3, it follows that the statement of the lemma will be proved, as long as one can show that the $(m + 1)$ vectors $\mathfrak{v}^{\mu}$ are linearly-independent for $\mathfrak{p} \neq 0$.

Therefore, let $\sum_{\mu=0}^{m} y_{\mu} \mathfrak{v}^{\mu} = 0$ for a certain vector $\mathfrak{p} \neq 0$; i.e.:

$$\sum_{k,\mu} a_{ik}^{\mu} x_k y_{\mu} = 0 \qquad (i = 0, 1, 2, \ldots, n).$$

Since $\mathfrak{p} \neq 0$, the rank of the matrix $\left(\sum_{\mu} a_{ik}^{\mu} y_{\mu}\right)$ is less than $(n + 1)$. Since the matrices $A^{(\mu)} = \left(a_{ik}^{\mu}\right)$ are linearly independent, this is possible only when all $y_{\mu} = 0$. This was to be proved.

The lemma now permits the following algebraic formulation of Theorem 26:

Theorem 27. *Any three quadratic* $(4k + 2)$*-rowed matrices are linearly independent* $(k \geq 0)$.

APPENDIX I

The one-dimensional characteristic class of an orientable three-dimensional manifold

In § 5, no. 3, we saw that that for a three-dimensional manifold M^3, the vanishing of the one-dimensional characteristic class F^1 is a necessary and sufficient condition for parallelizability. We further mentioned that for an orientable M^3 with a differentiable simplicial decomposition, F^1 is always the zero class, but left the reader responsible for the proof of this fact. It shall now be returned to under somewhat different differentiability assumptions.

1. A combinatorial lemma. The following lemma is interesting in its own right and is useful for the study of three-dimensional manifolds.

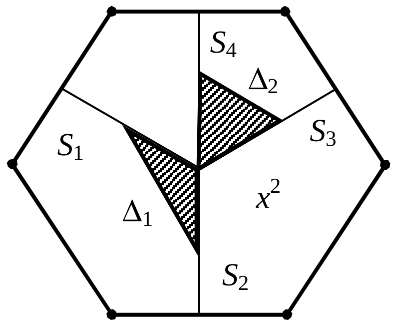

Figure 4.

Lemma. *Any cell decomposition of a three-dimensional manifold M^3 can be refined to a subdivision U such that any two-dimensional homology class (mod 2) of M^3 can be represented by a sub-cycle of U that consists of one or more disjoint two-dimensional manifolds.*

One must then show that any two-dimensional cycle z^2 of the given cell decomposition in U gives one or more disjoint surfaces that collectively define a cycle that is homologous to z^2. The proof proceeds in two steps:

1. z^2 is a cycle (mod 2), so an even number of polygons of z^2 meet along an edge of z^2. We now consider an edge ξ^1 of z^2 at which more than two (say, $2n$) polygons meet. Let ξ_1^0 and ξ_2^0 be the boundary points of ξ^1 and let x^2 be the dual cell to ξ^1 in the given cell decomposition of M^3. We denote the intersecting line segments of x^2 with the $2n$ polygons that meet at ξ^1 by $s_1, s_2, \ldots, s_{2n}$, where the numbering shall be given by the natural cyclic ordering of these line segments (see Fig. 4 for $n = 2$). Between two successive line segments s_{2k-1} and s_{2k} ($k = 1, 2, \ldots, n$), we now interpolate a small triangle Δ_k and construct the cone K_{k1} over the boundary of Δ_k that has its vertex at ξ_1^0. Analogously, K_{k2} will be constructed with its vertex at ξ_2^0. $K_{k1} + K_{k2}$ is a two-dimensional cycle that is homologous to zero, so $z^2 + \sum_{(k)}(K_{k1}+K_{k2})$ is a cycle homologous to z^2, in which ξ^1 is replaced with edges, each of which is incident with precisely two polygons of this new cycle. One naturally introduces a suitable subdivision of the given cell decomposition by carrying out this construction.

If all edges of z^2 at which more than two polygons met were removed by this construction then one would obtain a cycle $\bar{z}^2$ that would be homologous to z^2 and would consist of one or more disjoint pseudo-manifolds.

2. Let ξ^0 be an arbitrary vertex of $\bar{z}^2$. We construct a sub-division U in which the stars of the vertices ξ^0 are disjoint. Let S^2 be the boundary sphere of the star of ξ^0. The intersection of $\bar{z}^2$ with S^2 consists of some disjoint closed polygon perimeters that bound a sub-complex C^2 of S^2. We construct the cone K^2 that has its vertex at ξ^0 over the boundary C^2. $C^2 + K^2$ is a two-dimensional cycle that is homologous to zero, so $\bar{z}^2 + C^2 + K^2$ is a cycle that is homologous to $\bar{z}^2$, which we replace $\bar{z}^2$ with.

If one carries out this construction for every vertex then a cycle arises that is homologous to $\bar{z}^2$, as well as z^2, that consists of some disjoint two-dimensional surfaces.

3. Determination of the class F^1. We now determine the class F^1 of a given *orientable* manifold M^3 by comparing M^3 to a "standard manifold" M_0^3. M_0^3 is either the three-dimensional projective space P^3 or the topological product $T^3 = S^2 \times S^1$ of a sphere and a circle. Both standard manifolds are parallelizable. (The parallelizability of P^3 was proved in § 6, no. 3; from Theorem 23, the class F^1 is the zero class in T^3, so T^3 is parallelizable. One can, moreover, also give a continuous 3-field on T^3 directly.) The given manifold M^3 now fulfills the following *assumption:*

Any two-dimensional manifold that is embedded in M^3 without singularities possesses a neighborhood that can be mapped into a standard manifold topologically and continuously differentiably.

This assumption is only a differentiability assumption, since any two-dimensional manifold F that is embedded in M^3 without singularities possesses a neighborhood that can be mapped topologically into one of the standard manifolds. In order to show this, one constructs a manifold without singularities F' in P^3 or T^3 that is homeomorphic to F. (Three cases must be distinguished in the process of making this construction: a) F is orientable; F' can then be constructed in P^3 or T^3. b) F is not orientable and possesses an odd Euler characteristic; F' can then be constructed in P^3. c) F is not orientable and possesses an even Euler characteristic; F' can be constructed in T^3.) Now, since M^3 is orientable, F' is two-sided ([1]) in M_0^3, as long as F is two-sided in M^3, and likewise F' is one-sided in M_0^3 when F is one-sided in M^3; a topological map of F onto F' can then always be extended to a topological map of a neighborhood of F to a neighborhood of F'. With that, our assertion is proved.

We now consider the cell decomposition U of M^3 that was mentioned in the lemma, whose cells we denote by ξ^r; furthermore, let F now be a sub-cycle (mod 2) of U, in particular, that consists of the cells $\bar{\xi}^3$ of U. If we imagine that a continuous 2-field is constructed on the standard manifold M_0^3 then the map of a neighborhood of F into M_0^3, which exists by assumption, induces a continuous 2-field $\mathfrak{F}$ on that neighborhood. The 2-systems of $\mathfrak{F}$ that contact the points of the cells $\bar{\xi}^1$ define a one-dimensional framework (§ 4, no. 2) that, from Theorem 14, can be extended to a one-dimensional framework $\mathfrak{G}$ that is defined on all of M^3 and consists of 2-systems. The character χ (§ 4, no. 4) that is

([1]) On the relationships between the concepts of "orientable" and "two-sided," cf., Seifert-Threlfall, § 76.

associated with $\mathfrak{G}$ has the value 0 for every cell $\bar{\xi}^2$ if the 2-field that is induced by $\mathfrak{G}$ on $\dot{\xi}^2$ is continuously extended into the interior of $\bar{\xi}^2$. One then has $\chi(F) = 0$. In other words: The characteristic class F^1 has intersection number zero with F. Now, since F^1 has intersection number zero with any surface F, and on the other hand, from our lemma, any two-dimensional homology class (mod 2) can be represented by one or more two-dimensional manifolds F, F^1 has intersection number zero with any two-dimensional homology class, so from the Poincaré-Veblen duality theorem, it is the zero class (mod 2).

APPENDIX II

On the representation of hypersurfaces in Euclidian space by systems of equations ([1])

In this appendix, we deduce a consequence of the intersection theorem 23. In analogy to § 6, no. 2, we understand a ν-dimensional hypersurface that is embedded in n-dimensional Euclidian space to mean a sub-complex of the cell decomposition of R^n that is the topological image of a ν-dimensional parameter manifold by means of a topological continuously-differentiable map ($1 < \nu < n$).

Now, let $x_1, x_2, \ldots, x_n$ be Cartesian coordinates in R^n and let $(n - \nu)$ continuously-differentiable functions $f_i(x_1, x_2, \ldots, x_n)$ ($i = 1, 2, \ldots, n - \nu$) of these coordinates be given. Now, the equations:

$$f_i(x_1, x_2, \ldots, x_n) = 0 \tag{1}$$

define a ν-dimensional hypersurface M^ν, and if the functional matrix of the functions f_i has rank $(n - \nu)$ at every point of M^ν then we will call M^ν a "hypersurface that is regularly representable by equations."

Theorem 28. *Any hypersurface that is regularly representable by equations has an even Euler characteristic.*

Proof: The gradients grad f_i of the functions f_i that contact the points of M^ν are disjoint to M^ν (§ 6, no. 2), and the gradients that contact a point of M^ν are, by assumption, linearly independent, so they define an $(n - \nu)$-system. Since this system varies continuously with its contact point, moreover, M^ν possesses an external $(n - \nu)$-field, in the sense of § 6, no. 2.

We close the Euclidian space R^n into the n-dimensional sphere S^n with an infinitely distant point. Our hypersurface M^ν that lies in S^n fulfills the assumption of Theorem 23, so, from that theorem, its characteristic is congruent (mod 2) to the intersection number of the characteristic class $F^{n-\nu}$ of S^n with M^ν. Since $F^{n-\nu}$ is trivially the zero class in S^n, this intersection number vanishes, with which our assertion is proved.

([1]) This Appendix came about as a follow-up to a question of H. Seifert.

It follows, in particular, from Theorem 28 that a hypersurface that is regularly representable by equations and homeomorphic to a real or complex plane cannot lie in any Euclidian space of any dimension ([1]).

([1]) The Euler characteristic of the real projective plane is 1, while that of the complex projective plane is 3 (cf., B. L. van der Waerden: "Topologische Begründung des Kalkuls der abzählenden Geometrie," Math. Ann. **102** (1929), 337-362, especially pp. 361.) The fact that the real projective plane cannot be regularly represented by equations in any R^n follows from the general theorem that any manifold that is regularly representable in R^n is orientable. (For the proof, cf., footnote 25.) This theorem was already proved by Poincaré (J. Ec. poly. (2), **I**, pp. 3). The representation of the projective plane in R^4 that was given in pp. 301 of the book by Hilbert and Cohn-Vossen on intuitive geometry (Berlin, J. Springer, 1932) is not regular.