Kinematics, Lie’s circle geometry, and the line-sphere transformation (1)

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Translated by D. H. Delphenich

I. Planar kinematics

One can, following STUDY (2), represent motions (e.g., rotations, parallel displacements) in the plane by four homogeneous parameters \(\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3\) with bilinear composition.

\[
\begin{align*}
(x_m, y_m) & \\
G_f(x) & \quad G_r(x') \\
G_r(x') & \quad \Omega
\end{align*}
\]

Figure 1. Figure 2.

If \((x_m, y_m)\) is the center of rotation and \(\omega\) is the angle of that rotation (Figure 1) then one will have:

\[
\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = -\cot \frac{\omega}{2} : x_m : y_m : 1.
\]

\(\alpha_3 = 0\) gives (Figure 2) parallel translation by the vector \((-2\alpha_2 / \alpha_0, 2\alpha_1 / \alpha_0)\) whose length is \(\Omega\). In the homogeneous, Cartesian coordinates \(x_1 : x_2 : x_3 = x : y : 1\), the equations of motion then read:

\[
\begin{align*}
(\alpha_0^2 + \alpha_3^2) x'_1 &= (\alpha_0^2 - \alpha_3^2) x_1 + 2 \alpha_0 \alpha_3 x_2 + 2 (\alpha_1 \alpha_3 - \alpha_0 \alpha_2) x_3, \\
(\alpha_0^2 + \alpha_3^2) x'_2 &= -2 \alpha_0 \alpha_3 x_1 + (\alpha_0^2 - \alpha_3^2) x_2 + 2 (\alpha_0 \alpha_1 + \alpha_2 \alpha_3) x_3,
\end{align*}
\]

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(1) Lecture presented on 19 September 1951 at the conference of the DMV in Berlin.

(2) EDUARD STUDY:
\[(\alpha_0^2 + \alpha_3^2) x_3' = (\alpha_0^2 + \alpha_3^2) x_3,\]

and the bilinear composition formulas are:

\[
\begin{align*}
\alpha_0'' &= \alpha_0' \alpha_0' + \alpha_2' \alpha_1', \\
\alpha_2'' &= \alpha_0' \alpha_2' + \alpha_1' \alpha_1' - \alpha_0' \alpha_2', \\
\alpha_1'' &= \alpha_0' \alpha_1' + \alpha_2' \alpha_0' + \alpha_2' \alpha_2' - \alpha_3' \alpha_0', \\
\alpha_3'' &= \alpha_0' \alpha_3' + \alpha_3' \alpha_0' + \alpha_3' \alpha_3' + \alpha_3' \alpha_3'.
\end{align*}
\]

One can, like STUDY, write these formulas in an especially elegant way, when one introduces a system of higher complex numbers that take the form:

\[
\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,
\]

or

\[
x = x_1 e_1 + x_2 e_2 + x_3 e_3,
\]

resp., which are a limiting case of Hamilton’s quaternions, and shall be referred to as the system of STUDY quaternions, whose four units \(e_0, e_1, e_2, e_3\) define the following product table:

\[
\begin{array}{c|cccc}
    & e_0 & e_1 & e_2 & e_3 \\
\hline
  e_0 & e_0 & e_1 & e_2 & e_3 \\
  e_1 & e_1 & 0 & 0 & -e_2 \\
  e_2 & e_2 & 0 & 0 & e_1 \\
  e_3 & e_3 & e_2 & -e_1 & -e_0
\end{array}
\]

The product formula then reads simply:

\[\alpha'' = \alpha \alpha',\]

and the equations of motion themselves read:

\[x' = \alpha^{-1} x \alpha,\]

in which:

\[\alpha^{-1} = \frac{\alpha_0 e_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3}{\alpha_0^2 + \alpha_3^2} = \frac{\bar{\alpha}}{\alpha \bar{\alpha}} = \frac{\bar{\alpha}}{N(\alpha)}\]

is the reciprocal quaternion to \(\alpha\).

\[N(\alpha) = \alpha_0^2 + \alpha_3^2 = 0\] characterizes the singular motions.

**II. Kinematic map of the line space and quasi-elliptic geometry**

If one interprets the four homogeneous parameters \(\alpha_i\) as homogeneous, Cartesian point coordinates in space, when one sets:
\[ \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 = z : x : y : 1, \]

such that \( \alpha_0 \) then corresponds to the (vertical) \( z \)-direction and \( \alpha_3 = 0 \) represents the plane at infinity then any motion \( \alpha \) in space will be associated with an image point \( A(\alpha) \). In Lie’s way of expressing things, space will be the parameter space of a planar motion.

In 1911, W. BLASCKE \(^{(1)}\) and J. GRÜNWALD \(^{(2)}\) realized this map of planar motions to spatial points, which E. MÜLLER and E. KRUPPA \(^{(3)}\) later referred to as the kinematic map, by a simple geometric construction.

\[ \begin{align*}
\alpha_0 & \quad \alpha_1 & \quad \alpha_2 & \quad \alpha_3 = z & \quad x & \quad y & \quad 1,
\end{align*} \]

In fact, if one draws (Figures 3 and 4) the \( \infty^2 \) lines \( g \) of a bundle through the image point \( A(\alpha) \) in space, and one cuts it with the planes \( z = -1 \) and \( z = +1 \) at the points \( G_- \) and \( G_+ \), and one further looks for their base planes \( G'_- \), \( G'_+ \) on the plane \( \pi \) (viz., \( z = 0 \)), and one pivots these points \( G'_- \), \( G'_+ \) around their midpoint \( G \) through \( +\pi/2 \) then one will get two points \( G_l \) and \( G_r \) as the “kinematic image” of the line \( g \), which correspond precisely to the ones whose image point was \( A(\alpha) \) under the motion \( \alpha \).


It is occasionally convenient to think of the plane $p$, which is regarded as the locus of the left image points $G_l$ and the right image points $G_r$, as being divided into two sheets, and then speaking of the plane $\pi_l$ of the left image points and the plane $\pi_r$ of the right ones.

It then follows that the left and right images of two lines $g$, $h$ that intersect at $A(\alpha)$ will have equal distances between them:

$$G_l H_l = G_r H_r.$$ 

A pencil of rays will have two congruent, linear point sequences as its kinematic image, a point (i.e., a pencil of rays) will have a motion, and a plane (i.e., a ray field) will have a transfer $G_l \rightarrow G_r$. The image of a real point $A$ (i.e., $\alpha_3 \neq 0$) will be a rotation, and the image of a point at infinity (i.e., $\alpha_3 = 0$) will be a parallel displacement. The point at infinity $O$ of the vertical $\alpha_0$-axis will have the identity motion $G_l \equiv G_r$ for its image.

The singular motions $\alpha_0^2 + \alpha_3^2 = 0$ – i.e., $\alpha_0 \pm i\alpha_3 = 0$ – correspond (Figure 5) to the points of a conjugate complex pair $\iota^+, \iota^-$ of planes: $z = \pm i$, the singular transfers likewise correspond to planes through the absolute points $J^+, J^-$ of the line at infinity $s$ of the image plane $\pi$ (i.e., $z = 0$), such that a self-dual singular structure in the kinematic parameter space will be distinguished that consists, in total, of two conjugate complex planes $i^+$, $i^-$, and conjugate complex points $J^+, J^-$, and that (when regarded as a locus of lines) will carry two distinguished pairs of restricted pencils of rays of “generators,” namely, the left generators $(J^+, \iota^+)$, $(J^-, \iota^-)$, and the right generators $(J^+, \iota^-)$, $(J^-, \iota^+)$. If one distinguishes this structure as the absolute structure of a projective metric then the space will take on a quasi-elliptic structure. It consists of a limiting case of elliptic space – viz., the so-called quasi-elliptic space – whose geometry is very similar to that of elliptic space. For example, there are also Clifford parallels, Clifford displacement, etc., here.

The lines $g$, $h$, for example, are left-parallel in the Clifford sense when they have the same left kinematic image points $G_l = H_l$ in common; they then intersect (Figure 6) the metric structure at points with the same pairs of left generators.

Lines that are right-parallel in the Clifford sense are defined analogously. All of the mutually left- (right-, resp.) parallel lines define a ray net, namely a:

<table>
<thead>
<tr>
<th>Left net</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Right net</td>
<td></td>
</tr>
</tbody>
</table>
with the representation:

\[
\begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
p_{01} + p_{23} & p_{02} + p_{31} & p_{03}
\end{pmatrix} = 0,
\]

and

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
p_{01} - p_{23} & p_{02} - p_{31} & p_{03}
\end{pmatrix} = 0,
\]

that is generally elliptic, while in the case \( \beta_3 = 0 \) (\( \alpha_3 = 0 \), resp.) it is parabolic, and its guiding lines are two conjugate complex generators of the:

<table>
<thead>
<tr>
<th>Left family</th>
<th>Right family</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>

of the absolute structure, which will coincide with the line at infinity \( s \) in the parabolic case.

For this net, there is a one-parameter continuous group of collineations, under which the point \( x \) in space will be displaced along the rays of the net (and therefore rectilinearly!), and which, because they will thus necessarily leave the absolute structure of the quasi-elliptic space fixed, one will then refer to as quasi-elliptic Clifford displacements. More precisely, one speaks of left-displacements (right-displacements, resp.) according to whether the path-lines of the displacement are left-parallel (right-parallel, resp.).

If one also composes the homogeneous coordinates \( x_i \) of the Study quaternions in space, when one sets:

\[
x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,
\]

and if:

\[
\alpha = \alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,
\]

and

\[
\beta = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3
\]

mean arbitrary Study quaternions then these quasi-elliptic Clifford:

<table>
<thead>
<tr>
<th>Left displacements</th>
<th>Right displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x' = \beta x )</td>
<td>( x' = x \alpha )</td>
</tr>
</tbody>
</table>

They will define the commutative, three-parameter groups:

\[
\mathfrak{G}_3^l \quad \mathfrak{G}_3^r
\]

which will collectively yield the \( \mathfrak{G}_6 \) of quasi-elliptic motions:

\[
x' = \beta x \alpha.
\]

Now, how does one express such a Clifford displacement when it is applied to the line \( (g) \) in space in the planes \( \pi_l \) and \( \pi_r \) (which cover the plane \( \pi \)) of the left and right kinematic image points \( (G_l) \) and \( (G_r) \)?

The answer gives the so-called fundamental theorem of the kinematic map:
The left image field $\pi_l$ will experience a motion under a left displacement, while the right one $\pi_r$ will remain fixed.

The right image field $\pi_r$ will experience a motion under a right displacement, while the left one $\pi_l$ will remain fixed.

Application: the bundle of vertical lines $\bar{g}$ through the infinitely-distant point $O$ on the $x_0$-axis will be mapped to the identical image pair $\bar{G}_l = \bar{G}_r$. (Figure 7). It will then follow that:

If one brings (Figure 7) the point $O (x = e_0)$ to $A (x = e_0 \alpha = \alpha)$ by a right displacement ($\alpha$) then $\bar{g}$ will go to $g$, so the left image field $\pi_l$ will remain fixed ($G_l = \bar{G}_l$), although the right one $\pi_r$ will experience a motion ($G_r = \bar{G}_r$), namely, the one that belongs to the point $A(\alpha)$ and is the image of the right displacement $\overrightarrow{OA}$.

The kinematic image of the point $A$ (e.g., rotation, translation) is then identical with the image of the right displacement that takes $O$ to $A$. 

Figure 7.

Figure 8.

Figure 9.

Figure 10.
The rotational angle $\omega$ (the translation segment $\Omega$, resp.) is therefore equal to twice the quasi-elliptic displacement length $2\overrightarrow{OA}$.

III. Kinematic map of lines to turbines. Lie’s circle geometry.

One can (Figure 8) characterize the position of the plane $\pi$ by the position of an oriented line element $\gamma$ with respect to a given basic element $\gamma_0$ (i.e., an Ur-element). STUDY called $\gamma$ a “soma,” and $\gamma_0$, the “Ur-soma” (or “proto-soma”). Any motion $\alpha$ (viz., $\pi_l \rightarrow \pi_r$) takes the basic element $\gamma_0$ (which should lie in $\pi_l$) to an oriented line element $\gamma$ (in $\pi_r$), which, conversely, determines the motion $\alpha$ uniquely (by its position with respect to $\gamma_0$), and thus the motion $\alpha$ is mapped to the image point $A(\alpha)$ in a one-to-one way. This invertible, single-valued “kinematic map of the oriented line element $\gamma$ to the spatial point $A(\alpha)$” is based upon the following (1):

From the fundamental theorem, the point groups $(x)$, $(x)$ in space, which arise from each other by a right displacement $\alpha$, will thus have images $(\gamma)$, $(\gamma)$, resp., in the plane $\pi$ figure of oriented line elements that emerge from each other by a motion, and indeed by the motion $\alpha$ of $\pi_l$ to $\pi_r$ that corresponds to right displacement.

Which figures of oriented line elements correspond to the point $A$ of a line $g$ under our kinematic map?

1. Let the line $\overline{g}$ contain the point $O$, and:
   a) Let there be a real – thus vertical – line (Figure 9). One will then have $\overline{G}_l = \overline{G}_r$, so the points of $g$ will map to the rotations of $\pi_l$ to $\pi_r$ around the fixed point $\overline{G}_l = \overline{G}_r = A'$, by which the oriented basic element $\gamma_0$ will describe a rotational family of oriented line

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(1) The line elements $\gamma$, as representatives of planar motions, in the sense of EDUARD STUDY’s Geometrie der Dynamen (Teubner, Leipzig, 1903), are referred to as (positive) somas. Since the kinematic map analogously takes the planes in quasi-elliptic space to transfers of the plane $\pi$ in a one-to-one way, which, following STUDY, one can identify with the negative somas $\gamma'$ in the plane $\pi$ (after choosing the Ur-soma $\gamma_0$), that will yield a similar map of the negative somas of $p$ to the planes of quasi-elliptic space. From a presentation of FRANK LÖBELL, one can represent positive and negative somas by line elements that are oriented merely on their left (i.e., positive) or right (i.e., negative) edge by a half-arrow, with which, figures will arise that LÖBELL referred to as right (left, resp.) hooks. However, due to the required brevity, we cannot go further into this important set of circumstances.
elements (γ – viz., a Kasner turbine through γ₀) – (Figure 10), for which the rotational angle \( \omega = 2OA \), so it is equal to twice the quasi-elliptic displacement distance.

b) If the line \( g \) is a line at infinity through \( O \) (Figure 11) then its points \( \bar{A} \) will map to a fixed direction of translations, whereby the oriented basic element \( \gamma_0 \) will describe a line turbine through \( \gamma_0 \), and the translation distance \( \Omega = 2\bar{OA} \) will again be equal to twice the quasi-elliptic displacement distance. If \( A' = [\bar{g}s] \) is the (infinitely distant) intersection point of the line \( \bar{g} \) with the image plane \( \pi \) then the translation direction will be normal to the direction of the point at infinity \( A' \).

\[ G_l = G_i = \bar{G}_r \]

\[ G_r \]

\[ g \]

\[ \gamma \]

Figure 12.

2. If \( g \) is an arbitrary line:

a) That does not meet the absolute line \( s \) then one can (Figure 7) convert it into such a line \( \bar{g} \) through \( O \) by a right displacement. The left image point \( G_l \) of \( g \) thus remains unchanged \( (G_l = \bar{G}_l) \), while the right one suffers a motion (viz., a rotation) \( G_r \rightarrow \bar{G}_r \) that, from the fundamental theorem, will take the image figure of the oriented line elements of \( g \) to those of \( \bar{g} \). It then follows (Figure 12):

The image of an arbitrary line \( g \) (that does not intersect the absolute line \( s = [\tau' \tau'] \)) is a Kasner turbine with the right kinematic image point \( G_r \) of \( g \) as its midpoint that is congruent to that turbine that the basic element \( \gamma_0 \) describes under a rotation around the left kinematic image points \( G_l \) of \( g \).

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b) If \( g \) is an arbitrary line that intersects the absolute line \( s = [I^t I] \) (i.e., it is horizontal) then one will analogously obtain an arbitrary line turbine as the kinematic image of the point \( A \) of \( g \). (Figure 13).

If \((G_l, G_r)\) are the two (infinitely-distant) kinematic image points of the horizontal line \( g \), and if \( \gamma \) is the image element of an arbitrary point of the line \( g \) then the image line turbine of \( g \) will then arise by translating \( \gamma \) normal to the direction \( G_r \), and is thus congruent to the line turbine that arises when one displaces the basic element \( \gamma_0 \) normal to the direction \( G_l \).

![Figure 13.](image)

*Intersecting lines* \( g_1, g_2 \) always correspond to contacting turbines – i.e., ones that have an oriented line element (viz., the image of the intersection point \( A = [g_1 g_2] \)) in common (Figure 14).

![Figure 14.](image)

*Special case* (Figure 15): If \( G_l \) lies on the normal \( n_l \) of the basic element \( \gamma_0 \) then the line \( g \) will belong to a thread (viz., a “left thread”) \( \mathcal{G}_l \). If \( \gamma_0 \) lies on the zero point of the \( x \)-axis then the equation of this so-called “auxiliary thread” will be:
\[ p_{01} + p_{23} = 0. \]

One gets the oriented line elements of *cycles* as the kinematic images of the (points of) lines \( g \) of this auxiliary thread \( G_l \). *Intersecting lines* \( g, h \) of the auxiliary thread \( G_l \) will have *contacting cycles* as their images.

We thus have thus arrived at an exceptionally simple constructive (descriptive-geometric) presentation of any of SOPHUS LIE’s celebrated contact transformations, by which the rays \( g \) of a thread will be mapped to the oriented Lie circle (i.e., cycle) (\(^{1}\)). The ten-parameter continuous group \( \mathfrak{g}_{10} \) of projective automorphisms of the auxiliary thread \( G_l \) will thus correspond to the \( \mathfrak{g}_{10} \) of Lie circle transformations.

\[ \text{Figure 15.} \quad \text{Figure 16.} \]

*Special case* (Figure 16): The rays \( g \) of the auxiliary thread \( G_l \) that cut the absolute structure \( s \) (which likewise lies in \( G_l \)), and which define a parabolic net (“auxiliary net” \( \mathfrak{N} \)), correspond to a *line cycle* – i.e., a *spear*. The seven-parameter continuous group \( \mathfrak{g}_7 \) of the projective automorphisms of the auxiliary net will then taken to the \( \mathfrak{g}_7 \) of Laguerre spear transformations.

### IV. Euclidian line-sphere transformations

*Cyclography* (\(^{2}\)) teaches us (Figure 17 and Figure 18) that oriented line elements \( \gamma \) in the plane \( \pi \) should be regarded as the images of isotropic lines \( a \) or, after a reality displacement (viz., multiplication of the \( z \)-coordinate by \( i \)), as the images of lines that are inclined above the image plane \( \pi \) by a rotation of \( 45^\circ \) “to the left, as seen from above,” and thus cut a certain one-piece circle at infinity “\( C, \)” with the equation:

\[ t = 0 \quad \text{Plane at infinity} \]

\(^{1}\) SOPHUS LIE, Geometrie der Berührungstransformationen (Teubner, Leipzig, 1896).

The oriented line elements of a turbine are then cyclographic images of the generators of a family of such “spheres,” which is oriented by distinguishing a family of generators. Contacting turbines (Figure 14) are cyclographic images of contacting, oriented “spheres.”

By composing the kinematic map of the lines in space to turbines and the cyclographic map of turbines to oriented (C)-spheres, we have thus obtained, all tolled, a conceivably simple descriptive-geometric construction of Lie’s celebrated contact transformations (1) that maps the lines g in (quasi-elliptic) space to the oriented spheres k of (quasi-Euclidian) space.

Intersecting lines then correspond to contacting spheres.

Lines of the auxiliary thread \( \mathcal{G}_l \) correspond, first, to a cycle, then (Figure 17) to a cyclographically-isotropic cone (tangent cone to the conic section C of the so-called “C-cone”). Lines g that cut the absolute line s correspond, first, kinematically to turbines, and then cyclographically to non-isotropic planes that are oriented by distinguishing one of their isotropic families. In particular, lines of the auxiliary net \( \mathcal{H} \) first correspond kinematically to spears and the (Figure 18) cyclographically to isotropic planes (viz., C-planes) that admit only one orientation, like the isotropic cone (i.e., C-cone).

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LIE himself had still not oriented the spheres – i.e., his line-sphere-transformation was still not one-to-one. The two isotropic families of generators of the sphere correspond cyclographically to “polar turbines” whose oriented line elements lie symmetrically with respect to the carrier circle of the turbine (Figure 19); i.e., kinematically, they are lines $g, \overline{g}$ whose left image points $G_l, \overline{G}_l$ lie symmetrically with respect to the image line $n_l$ of the auxiliary thread $\mathcal{G}_l$ – i.e., lines $g, \overline{g}$ that are null polar with respect to the auxiliary thread $\mathcal{G}_l$.

The necessity of orienting the spheres was first recognized (1897) by E. STUDY (1), who was also the first (1926) to give a complete analytic presentation (2) of the (Euclidian) line-sphere-transformation that was free of objections, and which is in agreement with our geometric model.

![Figure 19](image-url)

One can base Lie’s circle geometry on the sphere, and the non-Euclidian line-sphere-transformation on geometric constructions in the same way when one appeals to the kinematics of the sphere and its (elliptic) parameter space, as I already showed in 1930 (3).

The Euclidian model is already found in a Vienna dissertation of A. E. MAYER (4) that originated at the same time, which is still not available, and which was also not published.

In the winter and summer semesters of 1935/36, I myself have presented the situation thoroughly in a Vienna lecture on “New Kinematics.” In the year 1948, W. BLASCHKE (5) published on it in the “Münchener Sitzungsberichten” and in 1949 in the “Rendiconti di matematica.”

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