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Theoretical studies in the elastic behavior of crystals

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I. Derivation of the basic equations from the assumption of molecules with polarity.

Using the hypothesis of discrete interacting molecules as a starting point was already well-known when the differential equations of elasticity were presented for the first time by Navier ¹⁾, Poisson ²⁾, and Cauchy ³⁾, but later on that sort of derivation lost its credibility due to the fact that an important result that the theory provided – viz., the numerical behavior of the two elastic constants for isotropic media – was not confirmed by experiment. Thus, since the time of Cauchy ⁴⁾ and Lamé ⁵⁾, one has employed the preferable approach to arriving at the basic equations that does not necessarily assume the molecular picture, but is nonetheless consistent with the dynamical hypotheses on the essence of matter.

Now, without a doubt, this new procedure is also so important that exploring and understanding the older one also deserves considerable interest. The molecular picture itself is not contradicted by the aforementioned experimental result, but only an arbitrary special assumption about the way that the molecules interact that is already improbable, in its own right. In fact, the cited papers by Navier, Poisson, and Cauchy assumed that the molecules act with equal strength in all directions, which is also not especially based in the assumption that the distances between the molecules are large when compared to their dimensions. When this is the case, the existence of the regular structure of the crystal remains a complete mystery.

¹⁾ Navier, *Mém. de l'Acad.*, **7** (1824), 374.

²⁾ Poisson, *Mém. de l'Acad.*, **8** (1828), 357.

³⁾ Cauchy, *Exerc. de Math.* **3** (1828), 188; **4** (1829), 129.

⁴⁾ Cauchy, *ibid.*, t. 4, pp. 293.

⁵⁾ Lamé, *Leçons sur la Théorie de l'Élasticité*, Paris, 1852 and 1866.

Poisson himself, in his last, incomplete paper ¹⁾, abandoned any arbitrary restriction and computed the elastic forces on the assumption of molecular interactions that varied with direction. However, even if the results that he obtained were, in fact, the most general, this theory was already contradicted by experiments, since the new Poisson formulas in the general case led to a symmetry with respect to three of the mutually normal planes, which was in contradiction to the experiments with calcite and quartz.

However, under closer scrutiny, one recognizes that Poisson only obtained this three-fold symmetry for the elementary interaction of two molecules with parallel axes by means of a fallacy ²⁾, and that any of the results were also based in this fallacy.

It is thus justified for us to repeat the argument on a more general foundation on these grounds already. In order to do this, we also discuss some other circumstances.

Indeed, Poisson allowed forces to act between the individual molecules that changed with direction – as we say, more briefly: *polar* forces – but he did not examine whether rotational moments could not or must not also appear along with such forces, and therefore such moments are no less probable than the polar forces. By their contribution, it can happen that under the growth of a crystal one molecular layer can lie with respect to the others in precisely the same way.

Furthermore, Poisson's presentation is, in my opinion, so cumbersome, and therefore, not especially clear, that it treats the elementary interaction between two parallel molecules as functions of four mutually independent arguments, although they actually depend upon only three; the final result is very difficult to discuss in that form. Here, a formal advance seems to be possible.

In what follows, I will give:

1. The derivation of some fundamental properties of the elementary interaction from the principle of energy.
2. Some theorem on the elastic stresses and rotational moments that act upon surfaces.
3. The calculation of these quantities from the elementary interactions.
4. The specialization of the results for the individual crystal systems.
5. A discussion of those results.
6. Consequences that follow for isotropic media.

1. We imagine a homogeneous, crystalline medium as consisting of a system of molecules that are in equilibrium under their mutual interaction ³⁾. These interactions are forces and rotational moments whose components vary with the relative positions of the molecules in an unknown way. Let the arrangement of the molecules be regular, in such a way that each of them is surrounded by the neighboring molecules in the same way. For the case in which this does not correspond to reality, since several types of ever-recurring groups of molecules are present, our theory requires an extension. Since, by our assumption, the molecules possess a polarity, one must treat them like finite rigid

¹⁾ Poisson, Mém. de l'Acad., **3** (1842), 3.

²⁾ Poisson, *ibid.*, pp. 41, *et. seq.*

³⁾ If one would like to employ the kinetic theory then one would have to calculate with the mean positions of the molecules.

bodies, and their positions must be determined by means of the coordinates of their centers of mass, as well as the attitude of a system of axes that is linked with them.

We denote the coordinates of the center of mass relative to the absolute fixed system of axes by x, y, z , the direction cosines of the axes that move with it compared to the fixed one by $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$, the displacements parallel to the fixed axes by u, v, w , and its rotation around them by l, m, n .

In order to explain the properties of the forces of interaction, we assume the principle of energy, which requires that the work that is done by any forces under an arbitrary variation of the system must be the complete variation of a function that depends only upon the configuration of the system; i.e., the relative position of its parts.

If we thus denote the components of the effect of one molecule m_k on another m_h by X_{hk}, Y_{hk}, Z_{hk} , and its moment by L_{hk}, M_{hk}, N_{hk} , and use $X_{hk}, Y_{hk}, Z_{hk}, L_{hk}, M_{hk}, N_{hk}$ in the same sense, then the elementary potential F_{hk} that is given by:

$$(1) \quad -dF_{hk} = X_{hk} dx_h + Y_{hk} dy_h + Z_{hk} dz_h + L_{hk} dl_h + M_{hk} dm_h + N_{hk} dn_h \\ + X_{hk} dx_k + Y_{hk} dy_k + Z_{hk} dz_k + L_{hk} dl_k + M_{hk} dm_k + N_{hk} dn_k$$

depends upon only the relative positions of the two molecules. Since the complete variation of the function F_{hk} must be on the right-hand side of this formula, it would appear that this would depend upon twelve arguments. Since the relative positions of two bodies are determined by six elements, one must convert it into a form in which only six differentials remain on the right-hand side.

The relative position of the system (m_h, m_k) does not change under a common displacement without rotation, so for:

$$dx_h = dx_k, \quad dy_h = dy_k, \quad dz_h = dz_k, \quad \text{and} \quad dl_h = dl_k = dm_h = dm_k = dn_h = dn_k = 0$$

one has:

$$dF_{hk} = 0,$$

from which, it follows that:

$$X_{hk} + X_{kh} = Y_{hk} + Y_{kh} = Z_{hk} + Z_{kh} = 0. \quad (2)$$

It also does not change under a common rotation, so for:

$$dl_h = dl_k = dl, \quad dm_h = dm_k = dm, \quad dn_h = dn_k = dn,$$

and

$$\begin{aligned} -dx_h &= y_h dn - z_h dm, & -dy_h &= z_h dl - x_h dn, & -dz_h &= x_h dm - y_h dl, \\ -dx_k &= y_k dn - z_k dm, & -dy_k &= z_k dl - x_k dn, & -dz_k &= x_k dm - y_k dl, \end{aligned}$$

one must then also have that:

$$dF_{hk} = 0,$$

from which, with hindsight of (2), in the event that one sets:

$$x_h - x_k = x_{hk}, \quad y_h - y_k = y_{hk}, \quad z_h - z_k = z_{hk},$$

it then follows that:

$$\begin{aligned} L_{hk} + L_{kh} + Z_{hk} y_{hk} - Y_{hk} z_{hk} &= 0, \\ M_{hk} + M_{kh} + X_{hk} z_{hk} - Z_{hk} x_{hk} &= 0, \\ N_{hk} + N_{kh} + Y_{hk} x_{hk} - X_{hk} y_{hk} &= 0. \end{aligned} \quad (3)$$

This system shows that when the direction of the interaction falls along the connecting line between the molecules; i.e., when one has:

$$X_{hk} = R_{hk} x_{hk}, \quad Y_{hk} = R_{hk} y_{hk}, \quad Z_{hk} = R_{hk} z_{hk},$$

one always has:

$$L_{hk} + L_{kh} = M_{hk} + M_{kh} = N_{hk} + N_{kh} = 0; \quad (4)$$

i.e., the rotational moments are opposite or zero, resp., and conversely.

In this case, when one also introduces:

$$l_h - l_k = l_{hk}, \quad m_h - m_k = m_{hk}, \quad n_h - n_k = n_{hk},$$

one simply has the relation:

$$\begin{aligned} -dF_{hk} &= X_{hk} dx_h + Y_{hk} dy_h + Z_{hk} dz_h + L_{hk} dl_h + M_{hk} dm_h + N_{hk} dn_h, \\ &= R_{hk} r_{hk} dr + L_{hk} dl_{hk} + M_{hk} dm_{hk} + N_{hk} dn_{hk}. \end{aligned} \quad (5)$$

Since the expression on the right must be the complete variation of F_{hk} , in the case in question it can depend upon only the four arguments that determine the mutual separation r_{hk} and relative rotations. If the rotational moments are equal to zero then all that remains is:

$$-dF_{hk} = R_{hk} r_{hk} dr_{hk},$$

and therefore, a potential that depends upon r_{hk} alone.

In the most general case, by the introduction of:

$$L_{hk} dl_h + L_{kh} dl_k = (L_{hk} + L_{kh}) d\left(\frac{l_h + l_k}{2}\right) + (L_{hk} - L_{kh}) d\left(\frac{l_h - l_k}{2}\right), \quad \text{etc.},$$

and the use of (3), one can easily arrive at the form:

$$\begin{aligned} -dF_{hk} &= X_{hk} \left(dx_{hk} + y_{hk} d\frac{n_h + n_k}{2} - z_{hk} d\frac{m_h + m_k}{2} \right) \\ (6) \quad &+ Y_{hk} \left(dy_{hk} + z_{hk} d\frac{l_h + l_k}{2} - x_{hk} d\frac{n_h + n_k}{2} \right) \\ &+ Z_{hk} \left(dz_{hk} + x_{hk} d\frac{m_h + m_k}{2} - y_{hk} d\frac{l_h + l_k}{2} \right) \end{aligned}$$

$$+ (L_{hk} - L_{kh}) d \frac{l_h - l_k}{2} + (M_{hk} - M_{kh}) d \frac{m_h - m_k}{2} + (N_{hk} - N_{kh}) d \frac{n_h - n_k}{2}.$$

If one thinks of a system of axes A_h, B_h, C_h as being fixed at m_h , and which was parallel to X, Y, Z before the displacement, and if a_{hk}, b_{hk}, c_{hk} are the coordinates of m_k relative to m_h then under a rotation of this system, or the molecule m_h itself, through dl_h, dm_h, dn_h , a system change of:

$$d'a_{hk} = y_{hk} dn_h - z_{hk} dm_h, \quad d'b_{hk} = z_{hk} dl_h - x_{hk} dn_h, \quad d'c_{hk} = x_{hk} dm_h - y_{hk} dl_h.$$

will be produced.

Likewise, for a system of axes that is fixed at m_k , one finds that:

$$-d'a_{kh} = y_{hk} dn_k - z_{hk} dm_k, \quad -d'b_{kh} = z_{hk} dl_k - x_{kh} dn_k, \quad -d'c_{kh} = x_{hk} dm_k - y_{hk} dl_k.$$

Since the axes A, B, C were originally X, Y, Z , one can also write:

$$dx_{hk} = \frac{d''a_{hk} - d''a_{kh}}{2}, \quad dy_{hk} = \frac{d''b_{hk} - d''b_{kh}}{2}, \quad dz_{hk} = \frac{d''c_{hk} - d''c_{kh}}{2},$$

in which d'' means the change of the relative coordinates under a displacement or rotation, and finally obtain:

$$(7) \quad -dF_{hk} = X_{hk} \frac{d(a_{hk} - a_{kh})}{2} + Y_{hk} \frac{d(b_{hk} - b_{kh})}{2} + Z_{hk} \frac{d(c_{hk} - c_{kh})}{2} \\ + (L_{hk} - L_{kh}) \frac{d(l_h - l_k)}{2} + (M_{hk} - M_{kh}) \frac{d(m_h - m_k)}{2} + (N_{hk} - N_{kh}) \frac{d(n_h - n_k)}{2},$$

if the da_{hk} , etc., mean the complete variation of the relative coordinates compared to the moving system that are produced by the displacement and rotation of the molecule. With that, one obtains a formula that shows how the function F_{hk} actually depends upon only the six arguments that determine the mutual positions of the two molecules.

If the two molecules, which are regarded as being of the same type, are parallel to each other in their homologous directions then their mutual position is determined by three arguments – viz., the three relative coordinates – that are $a_{hk} = -a_{kh}$, $b_{hk} = -b_{kh}$, $c_{hk} = -c_{kh}$. For this case, one must then have:

$$L_{hk} = L_{kh}, \quad M_{hk} = M_{kh}, \quad N_{hk} = N_{kh}, \quad (8)$$

and

$$-dF_{hk} = X_{hk} da_{hk} + Y_{hk} db_{hk} + Z_{hk} dc_{hk}.$$

In order for the right-hand side to be a complete differential, one must have:

$$-X_{hk} = \frac{\partial F_{hk}}{\partial a_{hk}}, \quad -Y_{hk} = \frac{\partial F_{hk}}{\partial b_{hk}}, \quad -Z_{hk} = \frac{\partial F_{hk}}{\partial c_{hk}}, \quad (9a)$$

while, along with (3), one must have:

$$2L_{hk} = Y_{hk} z_{hk} - Z_{hk} y_{hk}, \quad 2M_{hk} = Z_{hk} x_{hk} - X_{hk} z_{hk}, \quad 2N_{hk} = X_{hk} y_{hk} - Y_{hk} x_{hk}. \quad (9b)$$

2. Let a surface element ω_x that is normal to the X -axis be given at the location x, y, z in the interior of the elastic body in question, and construct a right cylinder over it that is parallel to the $+X$ -axis. When ω_x is quite small, but large compared to the sphere of action of the molecular forces, we can regard the sum of the (force) components that all molecules on the negative side of the plane of the surface element ω_x exert on the molecules in the cylinder as being proportional to ω_x , and after dividing by ω_x this is reduced to the unit of surface area. We denote the so-called elastic stress components that are so obtained by X_x, Y_x, Z_x , and analogously for the sums defined by the rotational moments that are exerted, which we denote by L_x, M_x, N_x . Just as a surface element ω_x can be regarded as normal to the X -axis, so can one be normal to the Y -axis, Z -axis, or any arbitrary axis n , which then leads to the definition of analogous components X_y, X_z, X_n, \dots and analogous moments L_y, L_z, L_n, \dots

We obtain the properties of these component and moment sums by considering the equilibrium conditions for volume elements in the interior or on the outer surface of the elastic body. We write them in the following form:

$$(10) \quad \begin{aligned} 0 &= \int \varepsilon dr X + \int do \bar{X}, \\ 0 &= \int \varepsilon dr Y + \int do \bar{Y}, \\ 0 &= \int \varepsilon dr Z + \int do \bar{Z}, \\ 0 &= \int \varepsilon dr (L + yZ - zY) + \int do (\bar{L} + \bar{y}\bar{Z} - \bar{z}\bar{Y}), \\ 0 &= \int \varepsilon dr (M + zX - xZ) + \int do (\bar{M} + \bar{z}\bar{X} - \bar{x}\bar{Z}), \\ 0 &= \int \varepsilon dr (N + xY - yX) + \int do (\bar{N} + \bar{x}\bar{Y} - \bar{y}\bar{X}). \end{aligned}$$

In this, ε denotes the density, dr , the space element, and do , the outer surface element of the body or sub-body considered. X, Y, Z are the components of the external force that is exerted on interior points, and L, M, N are the analogous moments, both of which are referred to the unit of mass. $\bar{X}, \bar{Y}, \bar{Z}$ are the external stress components that are exerted on the outer surface and $\bar{L}, \bar{M}, \bar{N}$ are the analogous rotational moments, both of which are referred to the unit of area.

We shall apply these formulas to spatial elements whose dimensions, in the sense described above, might be infinitely small, and consider only the terms of lowest order in them.

If the volume element is a cylinder that lies inside the system completely, and whose height we assume to be either infinitely small of higher order than the cross-sectional dimensions or independent of them, then when we either neglect terms of higher order or

set the terms that are independent of the height of the cylinder equal to zero, in the event that we denote the interior normals to the two base surfaces by $+n$ and $-n$, we obtain:

$$(11) \quad \begin{aligned} X_n + X_{-n} = Y_n + Y_{-n} &= Z_n + Z_{-n} = 0, \\ L_n + L_{-n} = M_n + M_{-n} &= N_n + N_{-n} = 0. \end{aligned}$$

If one of the base surfaces of the cylinder is an element of the free outer surface, on which the components \bar{X} , \bar{Y} , \bar{Z} and the moments \bar{L} , \bar{M} , \bar{N} might act (we preserve the notation of (10) in this sense) then one has:

$$(12) \quad \begin{aligned} \bar{X}_n + \bar{X} &= \bar{Y}_n + \bar{Y} = \bar{Z}_n + \bar{Z} = 0, \\ \bar{L}_n + \bar{L} &= \bar{M}_n + \bar{M} = \bar{N}_n + \bar{N} = 0, \end{aligned}$$

in which, we understand n to mean the exterior normal to the body. Since, as it seems, we have no means in practice to exert the outer surface element rotational moment on the molecule, but can operate only with stresses in different directions, in practice, one sets:

$$(12a) \quad \bar{L}_n = \bar{M}_n = \bar{N}_n = 0.$$

If we further take a spatial element that is bounded by three surface elements that are normal to the X , Y , and Z axes and a normal to a direction n (which is regarded as positive away from the coordinate origin and is defined by the angles (n, x) , (n, y) , (n, z)) then, upon restricting to the lowest order, one has:

$$(13) \quad \begin{aligned} -X_{-n} = X_n &= X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z), \\ -Y_{-n} = Y_n &= Y_x \cos(n, x) + Y_y \cos(n, y) + Y_z \cos(n, z), \\ -Z_{-n} = Z_n &= Z_x \cos(n, x) + Z_y \cos(n, y) + Z_z \cos(n, z), \\ -L_{-n} = L_n &= L_x \cos(n, x) + L_y \cos(n, y) + L_z \cos(n, z), \\ -M_{-n} = M_n &= M_x \cos(n, x) + M_y \cos(n, y) + M_z \cos(n, z), \\ -N_{-n} = N_n &= N_x \cos(n, x) + N_y \cos(n, y) + N_z \cos(n, z). \end{aligned}$$

Finally, if one considers an infinitely small prism that is parallel to the coordinate planes then one gets:

$$(14) \quad \begin{aligned} 0 &= \varepsilon X - \frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z}, \\ 0 &= \varepsilon Y - \frac{\partial Y_x}{\partial x} - \frac{\partial Y_y}{\partial y} - \frac{\partial Y_z}{\partial z}, \\ 0 &= \varepsilon Z - \frac{\partial Z_x}{\partial x} - \frac{\partial Z_y}{\partial y} - \frac{\partial Z_z}{\partial z}, \end{aligned}$$

$$\begin{aligned}
0 &= \varepsilon L - \frac{\partial L_x}{\partial x} - \frac{\partial L_y}{\partial y} - \frac{\partial L_z}{\partial z} - Z_y + Y_z, \\
0 &= \varepsilon M - \frac{\partial M_x}{\partial x} - \frac{\partial M_y}{\partial y} - \frac{\partial M_z}{\partial z} - X_z + Z_x, \\
0 &= \varepsilon N - \frac{\partial N_x}{\partial x} - \frac{\partial N_y}{\partial y} - \frac{\partial N_z}{\partial z} - Y_x + X_y.
\end{aligned}$$

In this, as we said, X, Y, Z, L, M, N denote the components and rotational moments that are exerted on the point x, y, z , perhaps from the outside, both of which are referred to the unit of mass. In practice, L, M, N , like $\bar{L}, \bar{M}, \bar{N}$, are, as a rule, equal to zero.

Equations (11) to (14) are the most general ones in the X_x, \dots , and L_x, \dots that are derivable from the fundamental equilibrium conditions (10); one can then get to any other equation from them.

When the first equation in (14) is integrated over an arbitrary piece of the elastic body, this gives, in fact:

$$0 = \int \varepsilon X \, dr - \int do [\bar{X}_x \cos(n, x) + \bar{X}_y \cos(n, y) + \bar{X}_z \cos(n, z)],$$

from which, using (12) and (13), it follows that:

$$0 = \int \varepsilon X \, dr - \int do \bar{X};$$

one has analogous expressions for the Y and Z components.

From the fourth of equations (14), it follows from the same operation that:

$$0 = \int \varepsilon L \, dr - \int do [\bar{L}_x \cos(n, x) + \bar{L}_y \cos(n, y) + \bar{L}_z \cos(n, z)] - \int dr (Z_y - Y_x);$$

in this, one can write:

$$-\int dr (Z_y - Y_x) = - \iint dx dz [y Z_y] + \int dr y \frac{\partial Z_y}{\partial y} + \iint dx dy [z Y_z] + \int dr z \frac{\partial Y_z}{\partial y},$$

or, from (14):

$$\begin{aligned}
&= - \int do [\bar{y} \bar{Z}_y \cos(n, y) - \bar{z} \bar{Y}_y \cos(n, z)] + \int dr \left[y \left(\varepsilon Z - \frac{\partial Z_x}{\partial x} - \frac{\partial Z_z}{\partial z} \right) - z \left(\varepsilon Y - \frac{\partial Y_x}{\partial x} - \frac{\partial Y_y}{\partial y} \right) \right] \\
&= - \int do \{ \bar{y} [\bar{Z}_y \cos(n, y) + \bar{Z}_z \cos(n, z)] \\
&\quad - \bar{z} [\bar{Y}_x \cos(n, y) + \bar{Y}_y \cos(n, z) + \bar{Y}_z \cos(n, z)] \} + \int z \, dr (yZ - zY),
\end{aligned}$$

and when one substitutes this, while using formulas (12) and (13), this yields:

$$0 = \int \varepsilon L dr + \int \varepsilon dr (yZ - zY) + \int do \bar{L} + \int do (\bar{y} \bar{Z} - \bar{z} \bar{Y}),$$

and similarly for the other formulas.

However, with that, the general equilibrium conditions (10) are included in the consideration of the extended fundamental formula.

3. Now, we shall calculate the most general values for the elastic stresses and moments at an arbitrary location. We construct a very small surface element ω_x at any location that is perpendicular to the X -axis and parallel to the positive X -axis of a right cylinder over it. The sum of the effects of all molecules on the negative side of ω_x on the molecules in any cylinder provides us with the definition of the components X_x, Y_x, Z_x and the moments L_x, M_x, N_x . As we said, the surface element ω_x in this shall be quite small compared to the sphere of influence of the molecular forces, and the latter, in turn, might be large compared to the distances between the neighboring molecules. Therefore, a layer of molecules over ω_x in the sphere of influence would indeed possess only a vanishing thickness, in the usual sense of the term, but would still include a very large number of individual layers.

We now refer the molecules to the absolute fixed coordinate system X, Y, Z , and call the coordinates of the attracting molecules x_k, y_k, z_k , while the coordinates of the attracted ones are x_h, y_h, z_h . These coordinates still do not determine their positions completely, since, as we have seen, not only displacements, but also rotations, come into consideration. We imagine that the latter are expressed with respect to a system of axes A_h, B_h, C_h , with each molecule fixed in it in the same way and moving with it; we assume that in the natural state of the crystal all of these systems are subject to that assumption and parallel to the fixed X, Y, Z system.

If the displacement components of the points x_h, y_h, z_h and x_k, y_k, z_k are u_h, v_h, w_h and u_k, v_k, w_k , resp., then the relative displacement of two points is given by:

$$u_h - u_k = u_{hk}, \quad v_h - v_k = v_{hk}, \quad w_h - w_k = w_{hk}. \quad (15)$$

Since the u, v, w vary continuously with position, this relative displacement can be developed in powers of the relative coordinates of the two points, and we assume that the deformations are such that one can truncate this development with the first-order terms for molecules that act upon each other appreciably, so *in the domain of the molecular sphere of influence, the dilatations can be regarded as constant*.

Therefore, when one further sets $x_h - x_k = x_{hk}, y_h - y_k = y_{hk}, z_h - z_k = z_{hk}$, one has:

$$(16) \quad \begin{aligned} u_{hk} &= x_{hk} \frac{\partial u}{\partial x} + y_{hk} \frac{\partial u}{\partial y} + z_{hk} \frac{\partial u}{\partial z}, \\ v_{hk} &= x_{hk} \frac{\partial v}{\partial x} + y_{hk} \frac{\partial v}{\partial y} + z_{hk} \frac{\partial v}{\partial z}, \\ w_{hk} &= x_{hk} \frac{\partial w}{\partial x} + y_{hk} \frac{\partial w}{\partial y} + z_{hk} \frac{\partial w}{\partial z}. \end{aligned}$$

The rotations, like the displacements, are also different for the variation molecules, but they vary infinitely little along an infinitely small length. We will then also satisfy ourselves with the lowest degree of approximation, and regard the rotations, like the dilatations, as constant inside the molecular domain of influence. Accordingly, for the region in which our considerations on the determination of the elastic stresses and moments apply, all of the molecules are to be considered as having their axes parallel, and we place a single system A, B, C through the coordinate origin of X, Y, Z in order to single out their directions from all of the other ones. The aforementioned two molecules (h) and (k) might have the coordinates $a_h, b_h, c_h, a_k, b_k, c_k$ when referred to this system A, B, C , which are coupled to x_h, y_h, z_h and x_k, y_k, z_k by means of linear relations that likewise apply to the relative coordinates $a_h - a_k = a_{hk}, b_h - b_k = b_{hk}, c_h - c_k = c_{hk}$, and we write, accordingly:

$$(17) \quad \begin{aligned} a_{hk} &= \alpha_1 x_{hk} + \alpha_2 y_{hk} + \alpha_3 z_{hk}, & x_{hk} &= \alpha_1 a_{hk} + \beta_1 b_{hk} + \gamma_1 c_{hk}, \\ b_{hk} &= \beta_1 x_{hk} + \beta_2 y_{hk} + \beta_3 z_{hk}, & y_{hk} &= \alpha_2 a_{hk} + \beta_2 b_{hk} + \gamma_2 c_{hk}, \\ c_{hk} &= \gamma_1 x_{hk} + \gamma_2 y_{hk} + \gamma_3 z_{hk}, & z_{hk} &= \alpha_3 a_{hk} + \beta_3 b_{hk} + \gamma_3 c_{hk}. \end{aligned}$$

Since we bring only infinitely small deformations into consideration, we also must expect only infinitely small rotations of the molecules; i.e., since the system A, B, C coincides with the system X, Y, Z in the natural state – hence, $\alpha_1 = \beta_2 = \gamma_3 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_3 = \gamma_1 = \gamma_2 = 0$ – we will be able to write down:

$$(18) \quad \begin{aligned} a_{hk} &= x_{hk} + y_{hk} n - z_{hk} m, & x_{hk} &= a_{hk} - b_{hk} n + c_{hk} m, \\ b_{hk} &= y_{hk} + z_{hk} l - x_{hk} n, & y_{hk} &= b_{hk} - c_{hk} l + a_{hk} n, \\ c_{hk} &= z_{hk} + x_{hk} m - y_{hk} l, & z_{hk} &= c_{hk} - a_{hk} m + b_{hk} l, \end{aligned}$$

for the deformed state.

In this, l, m, n denote the infinitely small rotations around the X, Y, Z axes by which the system A, B, C is brought from its original position to the displaced one parallel to the X, Y, Z system.

We would now like to determine the stress on the surface element ω_x normal to the X -axis at the location x, y, z ; i.e., to sum the X, Y , and Z components of the effects of all molecules that lie on the negative side of the plane of ω_x on the ones that are found in the cylinder that is constructed over ω_x . When these sums are divided by ω_x this then gives the components X_x, Y_x, Z_x , referred to the unit of area. We thus imagine that the crystal is already found in the deformed state.

We can then write:

$$(19) \quad \begin{aligned} X_x &= \frac{1}{\omega_x} \sum_{(h)} \sum_{(k)} X_{hk}, \\ Y_x &= \frac{1}{\omega_x} \sum_{(h)} \sum_{(k)} Y_{hk}, \\ Z_x &= \frac{1}{\omega_x} \sum_{(h)} \sum_{(k)} Z_{hk}. \end{aligned}$$

In this, X_{hk} , Y_{hk} , Z_{hk} denote the components of the effect of the molecule (k) on (h). Indeed, the sums are taken over all (k) – i.e., all molecules on the negative side of the element ω_x – and all (h) – i.e., all molecules in the cylinder over ω_x . However, due to the infinitely small sphere of influence for the molecular forces, only the molecules that are negligibly far from both sides of ω_x contribute anything to the sum.

The six-fold sum can be reduced to a three-fold one.

Since, by our assumption, the molecules in a homogeneous crystal are regularly arranged in such a way that any of them is surrounded by neighboring molecules in the same way, and this regular arrangement, by our assumption, will not be altered in the infinitely small region around ω_x that we are considering during the deformation either, molecule pairs with equal relative coordinates x_{hk} , y_{hk} , z_{hk} will appear in the sum above as often as the section of the cylinder of height x_{hk} contains molecules; i.e., when one calls the number of them that lie in a unit volume v , this number is $v x_{hk} \omega_x$. If one considers this and imagines that, from what was said above, the components X_{hk} , Y_{hk} , Z_{hk} depend upon only the relative coordinates x_{hk} , y_{hk} , z_{hk} , in addition to the angles between the axes A , B , C and X , Y , Z , which are constant in the entire region, then one can write the sums as:

$$(19a) \quad \begin{aligned} X_x &= v \sum_{x'=0}^{+\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x' X', \\ Y_x &= v \sum_{x'=0}^{+\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x' Y', \\ Z_x &= v \sum_{x'=0}^{+\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x' Z'. \end{aligned}$$

From this, one can say that for a definite, but arbitrary, molecule on the surface ω_x , the sum shall be taken over all the components that are exerted on the molecules that lie on the negative side of ω_x , each of which is multiplied by the relative coordinate of the one that lies on ω_x . Therefore, in particular, x' , y' , z' enters in place of x_{hk} , y_{hk} , z_{hk} , and the notation X' , Y' , Z' shall suggest that the particular relative coordinates x' , y' , z' are likewise substituted in the values of the components X_{hk} , Y_{hk} , Z_{hk} .

Since the crystal is found in the deformed state, the axes of the molecule A , B , C are not parallel to the fixed axes X , Y , Z . If we call the components of the elementary effects, when calculated parallel to the former, A' , B' , C' then from (18) we get:

$$X' = A' - n B' + m C', \quad Y' = B' - l C' + n A', \quad Z' = C' - m A' + l B'. \quad (19)^*$$

The components A' , B' , C' refer to the deformed state – i.e., to the relative coordinates a' , b' , c' of the interacting molecules that come about by increasing the original ones (a), (b), (c) through $\delta a'$, $\delta b'$, $\delta c'$. Since A' , B' , C' depend upon only a' , b' , c' , we can express their values in terms that relate to the original state, and might be denoted in brackets, when we develop:

* Translator's note: the equation numbering is inconsistent.

$$\begin{aligned}
A' &= (A') + \left(\frac{\partial A'}{\partial a'}\right) \delta a' + \left(\frac{\partial A'}{\partial b'}\right) \delta b' + \left(\frac{\partial A'}{\partial c'}\right) \delta c', \\
B' &= (B') + \left(\frac{\partial B'}{\partial a'}\right) \delta a' + \left(\frac{\partial B'}{\partial b'}\right) \delta b' + \left(\frac{\partial B'}{\partial c'}\right) \delta c', \\
C' &= (C') + \left(\frac{\partial C'}{\partial a'}\right) \delta a' + \left(\frac{\partial C'}{\partial b'}\right) \delta b' + \left(\frac{\partial C'}{\partial c'}\right) \delta c'.
\end{aligned} \tag{20}$$

In this, when one neglects terms of second order, such as $u' \cdot n$, and the like, from (18), since the displacements take place starting from the natural state, one has:

$$\delta a' = u' + (b') n - (c') m, \quad \delta b' = v' + (c') l - (a') n, \quad \delta c' = w' + (a') m - (b') l,$$

or, upon considering (16), also ¹⁾:

$$\begin{aligned}
\delta a' &= (a') \frac{\partial u}{\partial x} + (b') \left(\frac{\partial u}{\partial y} + n \right) + (c') \left(\frac{\partial u}{\partial z} - m \right), \\
\delta b' &= (a') \left(\frac{\partial v}{\partial x} - n \right) + (b') \frac{\partial v}{\partial y} + (c') \left(\frac{\partial v}{\partial z} + l \right), \\
\delta c' &= (a') \left(\frac{\partial w}{\partial x} + m \right) + (b') \left(\frac{\partial w}{\partial y} - l \right) + (c') \frac{\partial w}{\partial z}.
\end{aligned} \tag{21}$$

Finally, one has:

$$x' = (a') \left(1 + \frac{\partial u}{\partial x} \right) + (b') \frac{\partial u}{\partial y} + (c') \frac{\partial u}{\partial z}.$$

If one substitutes this then, with the repeated restriction to terms of first order, this yields:

$$\begin{aligned}
+ X_x &= \nu \sum_{a'=-\infty}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} \left\{ (A'a') \left(1 + \frac{\partial u}{\partial x} \right) + (A'b') \frac{\partial u}{\partial y} + (A'c') \frac{\partial u}{\partial z} - (B'a') n + (C'a') m \right. \\
&\quad + \left(\frac{\partial A'}{\partial a'} a' \right) \left[(a') \frac{\partial u}{\partial x} + (b') \left(\frac{\partial u}{\partial y} + n \right) + (c') \left(\frac{\partial u}{\partial z} - m \right) \right] \\
&\quad + \left(\frac{\partial A'}{\partial b'} a' \right) \left[(a') \left(\frac{\partial v}{\partial x} - n \right) + (b') \frac{\partial v}{\partial y} + (c') \left(\frac{\partial v}{\partial z} + l \right) \right] \\
&\quad \left. + \left(\frac{\partial A'}{\partial c'} a' \right) \left[(a') \left(\frac{\partial w}{\partial x} + m \right) + (b') \left(\frac{\partial w}{\partial y} - l \right) + (c') \frac{\partial w}{\partial z} \right] \right\},
\end{aligned}$$

¹⁾ Despite the fact that $(a') = (x')$, $(b') = (y')$, $(c') = (z')$ for the original state, I prefer to choose a' , b' , c' to be the summation variables, in order to emphasize that they are calculated parallel to the axes of the molecules.

$$\begin{aligned}
(22) \quad + Y_x = v \sum_{a'=0}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} & \left\{ (B'a') \left(1 + \frac{\partial u}{\partial x} \right) + (B'b') \frac{\partial u}{\partial y} + (B'c') \frac{\partial u}{\partial z} - (C'a')l + (A'a')n \right. \\
& + \left(\frac{\partial B'}{\partial a'} a' \right) \left[(a') \frac{\partial u}{\partial x} + (b') \left(\frac{\partial u}{\partial y} + n \right) + (c') \left(\frac{\partial u}{\partial z} - m \right) \right] \\
& + \left(\frac{\partial B'}{\partial b'} a' \right) \left[(a') \left(\frac{\partial v}{\partial x} - n \right) + (b') \frac{\partial v}{\partial y} + (c') \left(\frac{\partial v}{\partial z} + l \right) \right] \\
& \left. + \left(\frac{\partial B'}{\partial c'} a' \right) \left[(a') \left(\frac{\partial w}{\partial x} + m \right) + (b') \left(\frac{\partial w}{\partial y} - l \right) + (c') \frac{\partial w}{\partial z} \right] \right\}, \\
+ Z_x = v \sum_{a'=0}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} & \left\{ (C'a') \left(1 + \frac{\partial u}{\partial x} \right) + (C'b') \frac{\partial u}{\partial y} + (C'c') \frac{\partial u}{\partial z} - (A'a')m + (B'a')l \right. \\
& + \left(\frac{\partial C'}{\partial a'} a' \right) \left[(a') \frac{\partial u}{\partial x} + (b') \left(\frac{\partial u}{\partial y} + n \right) + (c') \left(\frac{\partial u}{\partial z} - m \right) \right] \\
& + \left(\frac{\partial C'}{\partial b'} a' \right) \left[(a') \left(\frac{\partial v}{\partial x} - n \right) + (b') \frac{\partial v}{\partial y} + (c') \left(\frac{\partial v}{\partial z} + l \right) \right] \\
& \left. + \left(\frac{\partial C'}{\partial c'} a' \right) \left[(a') \left(\frac{\partial w}{\partial x} + m \right) + (b') \left(\frac{\partial w}{\partial y} - l \right) + (c') \frac{\partial w}{\partial z} \right] \right\}.
\end{aligned}$$

In this, one must remember that v , as the number of molecules that found in a unit volume, will possess a different value after the deformation that it did before. Namely, one has:

$$(23) \quad v = (v) \left(1 - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right);$$

however, the factor of (v) would only be considered in the first terms of the formulas above.

Since they contain only the variables a' , b' , c' , the sums that enter into them have the interpretation that for the point x , y , z one should sum all components of the effects (or their differential quotients) that start from all of the molecules that lie on the negative side of a plane through this point that is normal to the X -axis, while each one of them is multiplied by one or more relative coordinates of the point x , y , z of the molecule in question.

These sums can be expressed in yet another way.

From equations (2), one has:

$$\begin{aligned}
A'(a', b', c') &= -A'(-a', -b', -c'), \\
B'(a', b', c') &= -B'(-a', -b', -c'), \\
C'(a', b', c') &= -C'(-a', -b', -c'),
\end{aligned} \tag{24}$$

and since the arrangement of the molecules is determined by the forces that act upon them, one must assume that they come about in opposite directions in the same way.

However, from this, it follows that:

$$\begin{aligned}
 \sum_{a'=0}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'a' &= \frac{1}{2} \sum_{a'=-\infty}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'a', \\
 \sum_{a'=0}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'b' &= \frac{1}{2} \sum_{a'=-\infty}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'b', \\
 \sum_{a'=0}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'c' &= \frac{1}{2} \sum_{a'=-\infty}^{+\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} A'c',
 \end{aligned} \tag{25}$$

and analogous statements are true for B' and C' . Accordingly, the coefficients of the first five terms of the formulas above get converted.

From (24), the first differential quotients will keep their signs when one inverts the signs of all three arguments. However, since they appear in all sums multiplied with two coordinates, the sums that they contain will also admit the replacement of the summation over a' from 0 to ∞ with one from $-\infty$ to $+\infty$ when one puts the factor $1/2$ in front of the sums.

If one applies these formulas to the natural state and imagines that all elastic stresses vanish for it, then one gets, when one abbreviates the triple sums taken from $-\infty$ to $+\infty$ with a single one:

$$0 = \Sigma (A' a') = \Sigma (B' a') = \Sigma (C' a').$$

The consideration of a surface element that is perpendicular to the Y and Z axis will yield, analogously:

$$\begin{aligned}
 0 &= \Sigma (A'b') = \Sigma (B'b') = \Sigma (C'b') \\
 0 &= \Sigma (A'c') = \Sigma (B'c') = \Sigma (C'c').
 \end{aligned} \tag{26}$$

A noticeable simplification of the formulas above enters into this, and when one considers the fact that:

$$A' = -\frac{\partial F}{\partial a'}, \quad B' = -\frac{\partial F}{\partial b'}, \quad C' = -\frac{\partial F}{\partial c'},$$

they become:

$$-X_x =$$

$$\begin{aligned}
 &\frac{\partial u}{\partial x} \frac{(v)}{2} \Sigma \left(a'^2 \frac{\partial^2 F}{\partial a'^2} \right) + \left(\frac{\partial u}{\partial y} + n \right) \frac{(v)}{2} \Sigma \left(a'b' \frac{\partial^2 F}{\partial a'^2} \right) + \left(\frac{\partial u}{\partial z} - m \right) \frac{(v)}{2} \Sigma \left(a'c' \frac{\partial^2 F}{\partial a'^2} \right) \\
 &+ \left(\frac{\partial v}{\partial x} - n \right) \frac{(v)}{2} \Sigma \left(a'^2 \frac{\partial^2 F}{\partial a' \partial b'} \right) + \frac{\partial v}{\partial y} \frac{(v)}{2} \Sigma \left(a'b' \frac{\partial^2 F}{\partial a' \partial b'} \right) + \left(\frac{\partial v}{\partial z} + l \right) \frac{(v)}{2} \Sigma \left(a'c' \frac{\partial^2 F}{\partial a' \partial b'} \right)
 \end{aligned}$$

$$+ \left(\frac{\partial w}{\partial x} + m \right) \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial a' \partial c'} \right) + \left(\frac{\partial w}{\partial y} - l \right) \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial a' \partial c'} \right) + \frac{\partial w}{\partial z} \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial a' \partial c'} \right)$$

(27)

– $Y_x =$

$$\begin{aligned} & \frac{\partial u}{\partial x} \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial a' \partial b'} \right) + \left(\frac{\partial u}{\partial y} + n \right) \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial a' \partial b'} \right) + \left(\frac{\partial u}{\partial z} - m \right) \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial a' \partial b'} \right) \\ & + \left(\frac{\partial v}{\partial x} - n \right) \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial b'^2} \right) + \frac{\partial v}{\partial y} \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial b'^2} \right) + \left(\frac{\partial v}{\partial z} + l \right) \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial b'^2} \right) \\ & + \left(\frac{\partial w}{\partial x} + m \right) \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial b' \partial c'} \right) + \left(\frac{\partial w}{\partial y} - l \right) \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial b' \partial c'} \right) + \frac{\partial w}{\partial z} \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial b' \partial c'} \right) \end{aligned}$$

– $Z_x =$

$$\begin{aligned} & \frac{\partial u}{\partial x} \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial a' \partial b'} \right) + \left(\frac{\partial u}{\partial y} + n \right) \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial a' \partial b'} \right) + \left(\frac{\partial u}{\partial z} - m \right) \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial a' \partial b'} \right) \\ & + \left(\frac{\partial v}{\partial x} - n \right) \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial b' \partial c'} \right) + \frac{\partial v}{\partial y} \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial b' \partial c'} \right) + \left(\frac{\partial v}{\partial z} + l \right) \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial b' \partial c'} \right) \\ & + \left(\frac{\partial w}{\partial x} + m \right) \frac{(v)}{2} \sum \left(a'^2 \frac{\partial^2 F}{\partial c'^2} \right) + \left(\frac{\partial w}{\partial y} - l \right) \frac{(v)}{2} \sum \left(a' b' \frac{\partial^2 F}{\partial c'^2} \right) + \frac{\partial w}{\partial z} \frac{(v)}{2} \sum \left(a' c' \frac{\partial^2 F}{\partial c'^2} \right). \end{aligned}$$

We would now like to abbreviate the sums that enter into these expressions, and the corresponding ones for X_y , Y_y , Z_y , X_z , Y_z , Z_z , which depend upon only the nature of the crystalline substance, and therefore must have a certain connection with its elastic constants, with the symbols D_{hk}^m , and interpret the upper indices to refer to the factors, while the lower indices refer to the denominators, in the differential quotients that occur in the sums, such that one will then set:

$$(28) \quad \frac{(v)}{2} \sum_{-\infty}^{+\infty} \left(a' c' \frac{\partial^2 F}{\partial b' \partial c'} \right) = D_{23}^{13},$$

and similarly for the other ones.

We then obtain the following system of coefficients in the nine components that appear in the first column under each other and the nine arguments that appear in the upper row:

| | $\frac{\partial u}{\partial x} \left(\frac{\partial u}{\partial y} + n \right) \left(\frac{\partial u}{\partial z} - m \right)$ | $\left(\frac{\partial v}{\partial x} - n \right) \frac{\partial v}{\partial y} \left(\frac{\partial v}{\partial z} + l \right)$ | $\left(\frac{\partial w}{\partial x} + m \right) \left(\frac{\partial w}{\partial y} - l \right) \frac{\partial w}{\partial z}$ |
|--------|---|---|---|
| $-X_x$ | $D_{11}^{11} \quad D_{11}^{12} \quad D_{11}^{13}$ | $D_{12}^{11} \quad D_{12}^{12} \quad D_{12}^{13}$ | $D_{13}^{11} \quad D_{13}^{12} \quad D_{13}^{13}$ |
| $-X_y$ | $D_{11}^{21} \quad D_{11}^{22} \quad D_{11}^{23}$ | $D_{12}^{21} \quad D_{12}^{22} \quad D_{12}^{23}$ | $D_{13}^{21} \quad D_{13}^{22} \quad D_{13}^{23}$ |
| $-X_z$ | $D_{11}^{31} \quad D_{11}^{32} \quad D_{11}^{33}$ | $D_{12}^{31} \quad D_{12}^{32} \quad D_{12}^{33}$ | $D_{13}^{31} \quad D_{13}^{32} \quad D_{13}^{33}$ |
| $-Y_x$ | $D_{21}^{11} \quad D_{21}^{12} \quad D_{21}^{13}$ | $D_{22}^{11} \quad D_{22}^{12} \quad D_{22}^{13}$ | $D_{23}^{11} \quad D_{23}^{12} \quad D_{23}^{13}$ |
| $-Y_y$ | $D_{21}^{21} \quad D_{21}^{22} \quad D_{21}^{23}$ | $D_{22}^{21} \quad D_{22}^{22} \quad D_{22}^{23}$ | $D_{23}^{21} \quad D_{23}^{22} \quad D_{23}^{23}$ |
| $-Y_z$ | $D_{21}^{31} \quad D_{21}^{32} \quad D_{21}^{33}$ | $D_{22}^{31} \quad D_{22}^{32} \quad D_{22}^{33}$ | $D_{23}^{31} \quad D_{23}^{32} \quad D_{23}^{33}$ |
| $-Z_x$ | $D_{31}^{11} \quad D_{31}^{12} \quad D_{31}^{13}$ | $D_{32}^{11} \quad D_{32}^{12} \quad D_{32}^{13}$ | $D_{33}^{11} \quad D_{33}^{12} \quad D_{33}^{13}$ |
| $-Z_y$ | $D_{31}^{21} \quad D_{31}^{22} \quad D_{31}^{23}$ | $D_{32}^{21} \quad D_{32}^{22} \quad D_{32}^{23}$ | $D_{33}^{21} \quad D_{33}^{22} \quad D_{33}^{23}$ |
| $-Z_z$ | $D_{31}^{31} \quad D_{31}^{32} \quad D_{31}^{33}$ | $D_{32}^{31} \quad D_{32}^{32} \quad D_{32}^{33}$ | $D_{33}^{31} \quad D_{33}^{32} \quad D_{33}^{33}$ |

(29)

Since $D_{hk}^{mn} = D_{kh}^{mn} = D_{hk}^{nm} = D_{kh}^{nm}$, 36 of these 81 coefficients are different from each other.

In order to also obtain the rotational moments L_x, M_x, N_x , and their constants in a form that is analogous to that of the X_x, \dots , we would like to employ the fact that from equations (9b) the rotational moments around the coordinate axes L_{hk}, M_{hk}, N_{hk} , which are exerted from the molecule (k) to the parallel one (h), are coupled with the corresponding components X_{hk}, Y_{hk}, Z_{hk} , and the relative coordinates $x_h - x_k = x_{hk}, y_h - y_k = y_{hk}, z_h - z_k = z_{hk}$ by means of the relations:

$$Z_{hk} y_{hk} - Y_{hk} z_{hk} + 2L_{hk} = 0, \quad X_{hk} z_{hk} - Z_{hk} x_{hk} + 2M_{hk} = 0, \quad Y_{hk} x_{hk} - X_{hk} y_{hk} + 2N_{hk} = 0.$$

Just as we did in connection with the arguments that were presented for the determination of X_x, Y_x, Z_x , we can therefore write:

$$\begin{aligned}
 L_x &= \nu \sum_{x'=0}^{\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x'(Y'z' - Z'y'), \\
 M_x &= \nu \sum_{x'=0}^{\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x'(Z'x' - X'z'), \\
 N_x &= \nu \sum_{x'=0}^{\infty} \sum_{y'=-\infty}^{+\infty} \sum_{z'=-\infty}^{+\infty} x'(X'y' - Y'x').
 \end{aligned}
 \tag{30}$$

In this, we go over to the quantities that relate to the moving axes A, B, C when we set:

$$\begin{aligned}
 Y'z' - Z'y' &= B'c' - C'b' - n(C'a' - A'c') + m(A'b' - B'a'), \\
 Z'x' - X'z' &= C'a' - A'c' - l(A'b' - B'a') + n(B'c' - C'b'), \\
 X'y' - Y'x' &= A'b' - B'a' - m(B'c' - C'b') + l(C'a' - A'c'), \\
 x' &= a' - b'n + c'm.
 \end{aligned}
 \tag{31}$$

However, since we consider the crystal in the deformed state, the relative coordinates are:

$$a' = (a') + \delta a', \quad b' = (b') + \delta b', \quad c' = (c') + \delta c',$$

and the components are:

$$\begin{aligned} A' &= (A') + \left(\frac{\partial A'}{\partial a'}\right) \delta a' + \left(\frac{\partial A'}{\partial b'}\right) \delta b' + \left(\frac{\partial A'}{\partial c'}\right) \delta c', \\ B' &= (B') + \left(\frac{\partial B'}{\partial a'}\right) \delta a' + \left(\frac{\partial B'}{\partial b'}\right) \delta b' + \left(\frac{\partial B'}{\partial c'}\right) \delta c', \\ C' &= (C') + \left(\frac{\partial C'}{\partial a'}\right) \delta a' + \left(\frac{\partial C'}{\partial b'}\right) \delta b' + \left(\frac{\partial C'}{\partial c'}\right) \delta c', \end{aligned}$$

in which the brackets, in turn, mean that the values in question are to be taken *in the original state*.

By substitution, one gets, e.g.:

$$\begin{aligned} L_x &= \nu \sum_{a'=0}^{\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} [(B'c') - (C'b')][(a') + u'] + (B'a') \delta c' - (C'a') \delta b' \\ (32) \quad &+ (a') \left\{ \left[\left(\frac{\partial B'}{\partial a'} c' \right) - \left(\frac{\partial C'}{\partial a'} b' \right) \right] \delta a' + \left[\left(\frac{\partial B'}{\partial b'} c' \right) - \left(\frac{\partial C'}{\partial b'} b' \right) \right] \delta b' + \left[\left(\frac{\partial B'}{\partial c'} c' \right) - \left(\frac{\partial C'}{\partial c'} b' \right) \right] \delta c' \right. \\ &\quad \left. - n[(C'a') - (A'c')] + m[(A'b') - (B'a')] \right\}, \end{aligned}$$

and similarly for the remaining ones.

A further development of the values is not necessary, if one recognizes that sums of the form:

$$\sum_{a'=0}^{\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} (B'a'c') \quad \text{and} \quad \sum_{a'=0}^{\infty} \sum_{b'=-\infty}^{+\infty} \sum_{c'=-\infty}^{+\infty} \left(\frac{\partial B'}{\partial a'} a'b'c' \right)$$

appear here as coefficients of the arguments $\partial u / \partial x, \dots$, and l, m, n , which one can regard as being of equal order of magnitude in which, in fact, the components themselves seem to be multiplied in a product of two, and their differential quotients in a product of three coordinates. However, since the elements of the sums give noticeable values only insofar as the variables a', b', c' are unnoticeably small, the coefficients that occur in the expressions for the rotational moments are regarded as infinitely small compared to the ones that appear in the components X_x, \dots

This has the effect that in the second triple of the equilibrium conditions (14), in all cases where the rotational moments do not vary with exceptionally quickly with the coordinates x, y, z , their differential quotients can be neglected along with the remaining terms – in agreement with the fact that $\bar{L}_n, \bar{M}_n, \bar{N}_n$ are set to zero on the outer surface of elastic bodies for all known problems – and that each equation then reads:

$$\begin{aligned}
0 &= \varepsilon X - \frac{\partial X_x}{\partial x} - \frac{\partial X_y}{\partial y} - \frac{\partial X_z}{\partial z}, \\
0 &= \varepsilon Y - \frac{\partial Y_x}{\partial x} - \frac{\partial Y_y}{\partial y} - \frac{\partial Y_z}{\partial z}, \\
0 &= \varepsilon Z - \frac{\partial Z_x}{\partial x} - \frac{\partial Z_y}{\partial y} - \frac{\partial Z_z}{\partial z}, \\
0 &= \varepsilon L - Z_y + Y_x, \quad 0 = \varepsilon M - X_z + Z_x, \quad 0 = \varepsilon N - Y_x + X_y.
\end{aligned} \tag{33}$$

If one multiplies these equations by $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$, $\frac{\partial l}{\partial t}$, $\frac{\partial m}{\partial t}$, $\frac{\partial n}{\partial t}$, resp., and integrates over the entire elastic body then the result is:

$$\begin{aligned}
0 &= \int \varepsilon dr \left(X \frac{\partial u}{\partial t} + Y \frac{\partial v}{\partial t} + Z \frac{\partial w}{\partial t} + L \frac{\partial l}{\partial t} + M \frac{\partial m}{\partial t} + N \frac{\partial n}{\partial t} \right) \\
&+ \int do \left(\bar{X} \frac{\partial \bar{u}}{\partial t} + \bar{Y} \frac{\partial \bar{v}}{\partial t} + \bar{Z} \frac{\partial \bar{w}}{\partial t} \right) \\
&+ \int dr \left[X_x \frac{\partial^2 u}{\partial t \partial x} + Y_y \frac{\partial^2 v}{\partial t \partial y} + Z_z \frac{\partial^2 w}{\partial t \partial z} + Y_z \left(\frac{\partial v}{\partial z} + l \right) + Z_y \left(\frac{\partial w}{\partial y} - l \right) \right. \\
&\left. + Z_x \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial x} + m \right) + X_z \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial z} - m \right) + X_y \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} + n \right) + Y_x \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial x} - n \right) \right].
\end{aligned} \tag{34}$$

The last integral is the work done by the internal forces, which must have a potential, by our basic assumption of the validity of the energy principle. One must therefore be able to set:

$$(34a) \quad X_x = - \frac{\partial \Phi}{\partial \left(\frac{\partial u}{\partial x} \right)}, \quad X_y = - \frac{\partial \Phi}{\partial \left(\frac{\partial u}{\partial y} + n \right)}, \text{ etc.},$$

or one must have:

$$\frac{\partial X_x}{\partial \left(\frac{\partial u}{\partial y} + n \right)} = \frac{\partial X_y}{\partial \left(\frac{\partial u}{\partial x} \right)}, \text{ etc.}$$

The agreement of our results with this requirement proves the symmetric form of the system of coefficients (29) with respect to the diagonal.

For the further analysis, it will be useful to write out at least the second triple of equations in (33) explicitly:

$$\begin{aligned}
0 = \varepsilon L + \frac{\partial u}{\partial x}(D_{31}^{21} - D_{21}^{31}) + \left(\frac{\partial u}{\partial y} + n\right)(D_{31}^{22} - D_{21}^{22}) + \left(\frac{\partial u}{\partial z} - m\right)(D_{31}^{23} - D_{21}^{23}) \\
+ \left(\frac{\partial v}{\partial x} - n\right)(D_{32}^{21} - D_{22}^{31}) + \frac{\partial u}{\partial y}(D_{32}^{22} - D_{22}^{22}) + \left(\frac{\partial u}{\partial z} + l\right)(D_{32}^{23} - D_{22}^{23}) \\
+ \left(\frac{\partial w}{\partial x} + m\right)(D_{33}^{21} - D_{23}^{31}) + \left(\frac{\partial w}{\partial y} - l\right)(D_{33}^{22} - D_{23}^{22}) + \frac{\partial w}{\partial z}(D_{33}^{23} - D_{23}^{23}),
\end{aligned}
\tag{35}$$

$$\begin{aligned}
0 = \varepsilon M + \frac{\partial u}{\partial x}(D_{11}^{31} - D_{31}^{11}) + \left(\frac{\partial u}{\partial y} + n\right)(D_{11}^{32} - D_{31}^{12}) + \left(\frac{\partial u}{\partial z} - m\right)(D_{11}^{33} - D_{31}^{13}) \\
+ \left(\frac{\partial v}{\partial x} - n\right)(D_{12}^{31} - D_{32}^{11}) + \frac{\partial u}{\partial y}(D_{12}^{32} - D_{32}^{12}) + \left(\frac{\partial u}{\partial z} + l\right)(D_{12}^{33} - D_{32}^{13}) \\
+ \left(\frac{\partial w}{\partial x} + m\right)(D_{13}^{31} - D_{33}^{11}) + \left(\frac{\partial w}{\partial y} - l\right)(D_{13}^{32} - D_{33}^{12}) + \frac{\partial w}{\partial z}(D_{13}^{33} - D_{33}^{13}),
\end{aligned}$$

$$\begin{aligned}
0 = \varepsilon N + \frac{\partial u}{\partial x}(D_{21}^{11} - D_{11}^{21}) + \left(\frac{\partial u}{\partial y} + n\right)(D_{21}^{12} - D_{11}^{22}) + \left(\frac{\partial u}{\partial z} - m\right)(D_{21}^{13} - D_{11}^{23}) \\
+ \left(\frac{\partial v}{\partial x} - n\right)(D_{22}^{11} - D_{12}^{21}) + \frac{\partial u}{\partial y}(D_{22}^{12} - D_{12}^{22}) + \left(\frac{\partial u}{\partial z} + l\right)(D_{22}^{13} - D_{12}^{23}) \\
+ \left(\frac{\partial w}{\partial x} + m\right)(D_{23}^{11} - D_{13}^{21}) + \left(\frac{\partial w}{\partial y} - l\right)(D_{23}^{12} - D_{13}^{22}) + \frac{\partial w}{\partial z}(D_{23}^{13} - D_{13}^{23}).
\end{aligned}$$

These formulas are obtained relatively simply and are derived for completely general values of the l, m, n .

If one summarizes the terms in each equation that are free of l, m, n with the symbol f and briefly denotes the factors of l, m, n by λ, μ, ν then these equations take on the form:

$$\begin{aligned}
0 &= \lambda_1 l + \mu_1 m + \nu_1 n + f_1, \\
0 &= \lambda_2 l + \mu_2 m + \nu_2 n + f_2, \\
0 &= \lambda_3 l + \mu_3 m + \nu_3 n + f_3.
\end{aligned}
\tag{35'}$$

In this, one has:

$$\begin{aligned}
\lambda_1 &= 2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22}), & \mu_2 &= 2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33}), & \nu_3 &= 2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11}), \\
\nu_3 = \mu_3 &= (D_{11}^{23} + D_{23}^{11}) - (D_{31}^{12} + D_{12}^{31}), & \lambda_3 = \nu_1 &= (D_{22}^{31} + D_{31}^{22}) - (D_{12}^{23} + D_{23}^{13}), \\
\mu_1 &= \lambda_2 = (D_{33}^{12} + D_{12}^{33}) - (D_{23}^{31} + D_{31}^{23}).
\end{aligned}
\tag{35''}$$

If one sets the determinant:

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = \Pi$$

then one obtains the formulas:

$$\begin{aligned}
 -l \Pi &= f_1 (\mu_2 \nu_3 - \mu_3 \nu_2) + f_2 (\mu_3 \nu_1 - \mu_1 \nu_3) + f_3 (\mu_1 \nu_2 - \mu_2 \nu_1), \\
 -m \Pi &= f_1 (\nu_2 \lambda_3 - \nu_3 \lambda_2) + f_2 (\nu_3 \lambda_1 - \nu_1 \lambda_3) + f_3 (\nu_1 \lambda_2 - \nu_2 \lambda_1), \\
 -n \Pi &= f_1 (\lambda_2 \mu_3 - \lambda_3 \mu_2) + f_2 (\lambda_3 \mu_1 - \lambda_1 \mu_3) + f_3 (\lambda_1 \mu_2 - \lambda_2 \mu_1)
 \end{aligned} \tag{36}$$

for the determination of the rotations l , m , n when the dilatations $\partial u / \partial x$, ... are determined by these and the given momenta L , M , N .

However, what is more important than the determination of these quantities is their elimination from the X_z , Y_y , ... that enter into the equilibrium conditions (33).

To that end, we add to the three equations (35') above the one that (29) yields for the value of an arbitrary one of the stress components, which might be denoted by K , in a form that is analogous to (35'), and thus obtain the system:

$$\begin{aligned}
 -K &= \lambda l + \mu m + \nu n + f, \\
 0 &= \lambda_1 l + \mu_1 m + \nu_1 n + f_1, \\
 0 &= \lambda_2 l + \mu_2 m + \nu_2 n + f_2, \\
 0 &= \lambda_3 l + \mu_3 m + \nu_3 n + f_3.
 \end{aligned} \tag{37}$$

When one multiplies the f by an arbitrary quantity p and divides, one can now solve the four equations for p and then obtain, since p again falls out:

$$-K \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = -K \Pi = \begin{vmatrix} f & \lambda & \mu & \nu \\ f_1 & \lambda_1 & \mu_1 & \nu_1 \\ f_2 & \lambda_2 & \mu_2 & \nu_2 \\ f_3 & \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}, \tag{38}$$

and with it, the form of $-K$ that is free of l , m , n . The f are linear functions of $\partial u / \partial x$, ... If one denotes the coefficients of any of these arguments in the f, f_1, f_2, f_3 by $\kappa, \kappa_1, \kappa_2, \kappa_3$, resp., then the coefficient k of this same argument be given in the definitive form of $-K$ by:

$$(38') \quad +k \Pi = \begin{vmatrix} \kappa & \lambda & \mu & \nu \\ \kappa_1 & \lambda_1 & \mu_1 & \nu_1 \\ \kappa_2 & \lambda_2 & \mu_2 & \nu_2 \\ \kappa_3 & \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

The number of coefficients of the differential expressions $\partial u / \partial x$, ... thus obtained is again 81, since nine components with nine terms are present. However, they do not all vanish, by any means.

Next, since Y_z and Z_y , Z_x and X_z , X_y , and Y_x differ from each other only by εL , εM , εN , resp., the nine coefficients of the $\partial u / \partial x$, ... that appear twice in these component pairs must be equal; with that, the total number of them reduces to 54.

Moreover, the terms:

$$\frac{\partial v}{\partial z}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x}, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$$

occur only in the combinations:

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

since the elastic components can depend upon only deformations and, perhaps on the rotations of the volume elements, as well.

We would like to prove that our theory actually yields this result, and thus further reduce the number of factors in the dilatations $\partial u / \partial x$, ... to 36.

Let $\kappa'_1, \kappa'_2, \kappa'_3$ and $\kappa''_1, \kappa''_2, \kappa''_3$ be the factors of two terms of pairs of differential expressions in the equations of eliminations above, let κ' and κ'' be the factors in the value of any of the components X_x, \dots , and let k' and k'' be the factors that appear in the same formula after the elimination of l, m, n . The statement above then demands that one have:

$$k' = k''.$$

This condition reduces to:

$$\begin{vmatrix} (\kappa' - \kappa'') & \lambda & \mu & \nu \\ (\kappa'_1 - \kappa''_1) & \lambda_1 & \mu_1 & \nu_1 \\ (\kappa'_2 - \kappa''_2) & \lambda_2 & \mu_2 & \nu_2 \\ (\kappa'_3 - \kappa''_3) & \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0.$$

However, the λ, μ, ν are the coefficients of l, m, n in the same four equations that we mentioned. If one observes that they occur in (29) only in the combinations:

$$\frac{\partial v}{\partial z} + l, \quad \frac{\partial w}{\partial y} - l, \quad \frac{\partial w}{\partial x} + m, \quad \frac{\partial u}{\partial y} - m, \quad \frac{\partial u}{\partial y} + n, \quad \frac{\partial v}{\partial x} - n$$

then one sees that no matter which alternative one chooses, in the determinant in question, one of the last three columns always equals the first column, which must then be all zeroes.

Thus, the values of the elastic stresses that are obtained by the elimination of l, m, n will, in fact, depend upon just the six arguments:

$$\frac{\partial u}{\partial x} = x_x, \quad \frac{\partial u}{\partial y} = y_y, \quad \frac{\partial u}{\partial z} = z_z, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = y_z, \quad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = z_x, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x_y.$$

However, from the relations for the components X_x, \dots that are free of l, m, n , one finally has:

$$\frac{\partial X_x}{\partial x_y} = \frac{\partial X_y}{\partial x_x}, \quad \frac{\partial X_x}{\partial y_z} = \frac{\partial Y_x}{\partial x_z}, \quad \text{etc.}$$

For this, it is requisite that when the same thing that is written in terms of the sequence $X_x, X_y, X_z, Y_x, Y_y, Y_z, Z_x, Z_y, Z_z$ is associated with the arguments in the sequence $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ it exhibits a system of coefficients that is symmetric with respect to the diagonal.

This is easily proved with hindsight of the system (29) and (35).

If one takes two corresponding terms (perhaps k and h) in two arbitrary components (thus, h and k) then the coefficients will be determined by (38') thus:

$$k_h^k \Pi = \begin{vmatrix} \kappa_h^k & \lambda_h & \mu_h & \nu_h \\ \kappa_1^k & \lambda_1 & \mu_1 & \nu_1 \\ \kappa_2^k & \lambda_2 & \mu_2 & \nu_2 \\ \kappa_3^k & \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}, \quad k_k^h \Pi = \begin{vmatrix} \kappa_k^h & \lambda_k & \mu_k & \nu_k \\ \kappa_1^h & \lambda_1 & \mu_1 & \nu_1 \\ \kappa_2^h & \lambda_2 & \mu_2 & \nu_2 \\ \kappa_3^h & \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

However, from (29), one has:

$$\kappa_h^k = \kappa_k^h$$

in it, and from (29) and (35):

$$\begin{aligned} -\lambda_h &= \kappa_1^h, & -\mu_h &= \kappa_2^h, & -\nu_h &= \kappa_3^h, \\ -\lambda_k &= \kappa_1^k, & -\mu_k &= \kappa_2^k, & -\nu_k &= \kappa_3^k. \end{aligned}$$

Moreover, since, from (35), one has:

$$\mu_1 = \lambda_2, \quad \nu_2 = \mu_3, \quad \lambda_3 = \nu_1,$$

the above two determinants can be brought into a form in which the rows of one of them agree with the columns of the others by inverting the signs in their first row and last column: They are therefore equal.

With that, the number of distinct coefficients of the $\partial u / \partial x, \dots$ ultimately reduces to 21 in the most general case, in agreement with the result of the theories that are not based upon the molecular hypothesis.

As far as the rotational moments L, M, N are concerned, they always have the factor zero in the first of equations (37), and in the following ones only one appears with the factor ε . One can then readily write down the coefficients k', k'', k''' that they are afflicted with in the value of $-K$; they are, in fact:

$$(38'') \quad k' \Pi = -\varepsilon \begin{vmatrix} \lambda & \mu & \nu \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}, \quad k'' \Pi = -\varepsilon \begin{vmatrix} \lambda & \mu & \nu \\ \lambda_3 & \mu_3 & \nu_3 \\ \lambda_1 & \mu_1 & \nu_1 \end{vmatrix}, \quad k''' \Pi = -\varepsilon \begin{vmatrix} \lambda & \mu & \nu \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix}.$$

Therefore, all three moments L , M , N enter into each of the stress components, in general.

The result of this general examination is that values of the elastic components that are obtained by the elimination of l , m , n contain a system of constants that shows the following arrangement:

| | x_x | y_y | z_z | y_x | z_x | x_y | $-\varepsilon L$ | $-\varepsilon M$ | $-\varepsilon N$ |
|--------|----------|----------|----------|----------|----------|----------|------------------|------------------|------------------|
| $-X_x$ | D_{11} | D_{12} | D_{13} | D_{14} | D_{15} | D_{16} | E_{11} | E_{12} | E_{13} |
| $-Y_y$ | D_{21} | D_{22} | D_{23} | D_{24} | D_{25} | D_{26} | E_{21} | E_{22} | E_{23} |
| $-Z_z$ | D_{31} | D_{32} | D_{33} | D_{34} | D_{35} | D_{36} | E_{31} | E_{32} | E_{33} |
| $-Y_z$ | D_{41} | D_{42} | D_{43} | D_{44} | D_{45} | D_{46} | E_{41} | E_{42} | E_{43} |
| $-Z_y$ | D_{41} | D_{42} | D_{43} | D_{44} | D_{45} | D_{46} | $1 + E_{41}$ | E_{42} | E_{43} |
| $-Z_x$ | D_{51} | D_{52} | D_{53} | D_{54} | D_{55} | D_{56} | E_{51} | E_{52} | E_{53} |
| $-X_z$ | D_{51} | D_{52} | D_{53} | D_{54} | D_{55} | D_{56} | E_{51} | $1 + E_{52}$ | E_{53} |
| $-X_y$ | D_{61} | D_{62} | D_{63} | D_{64} | D_{65} | D_{66} | E_{61} | E_{62} | E_{63} |
| $-X_x$ | D_{61} | D_{62} | D_{63} | D_{64} | D_{65} | D_{66} | E_{61} | E_{62} | $1 + E_{63}$ |

(39)

In this, D_{hk} equals D_{kh} , but E_{hk} does not equal E_{kh} .

As long as L , M , N are non-zero, Y_z and Z_y , Z_x and X_z , X_y and Y_x are not equal, and one must consistently keep these values distinct when substituting into the equilibrium equations.

We keep the rotational moments L , M , N in the formulas, regardless of whether there is no means of exerting them on molecules directly in applications up to now, because the recent hypothesis of Prof. Riecke ¹⁾ of a permanent electrical polarity in molecules suggests that it does not seem unthinkable that such moments would be preserved by electrical interactions; this gives rise to interesting theoretical speculations, moreover.

4. We have previously carried out the completely general proof that the theory of elasticity that is based upon the interaction of polarized molecules leads to the same form the differential equations and the same values of the elastic stress components for ordinary applications as the theories that make no assumption about the causes of elastic phenomena; however, we have not summarized the comparison between the general final form and the older one. The values of the elastic components that are obtained by eliminating the molecular rotations l , m , n are, in the form that only includes the sums D_{hk}^{mn} , very complicated and, for the time being, lacking in practical significance.

¹⁾ E. Riecke, Gött. Nachrichten 7 (1887), 194.

By contrast, from now on, they, as well as the values of the molecular rotations themselves, shall be derived and communicated for the crystal systems that are distinguished by their symmetries. Meanwhile, only the most important ones shall be given for the summary; the less hemi-hedral and hemi-morphic structures that, to judge from the crystallographic symmetries, diverge from the elastic behavior of holohedral forms shall only be mentioned, but not treated.

As in the previous theory for the elastic potential, so also here for the behavior of the potential F of the elementary action between molecules, which appears in the sums that define the coefficients D_{hk}^{mn} , we shall consider the *symmetry properties of the crystalline form* as definitive, because observations have shown that in all known physical properties (e.g., concerned with light and heat), the crystal must possess at least the symmetry of the form, and at most still higher symmetries. Therefore, it seems appropriate to read off the most general symmetry law of the crystalline substance from the crystal form and to assume that it represents the same thing as the comprehensive law for all physical properties. On the same grounds, however, it does seem permissible to employ other physical properties of the crystal that obey more particular laws than the crystal form alone would require for the specialization of the elastic potentials ¹⁾, except for the fact that the less numerous previous observations on the elastic phenomena in crystals have already shown that the same general and complicated symmetry properties feature in the optical and thermal phenomena. The restriction to this as a means of specialization thus seems so prudent that it still remains hypothetical and in need of confirmation by experiment.

The symmetry elements that determine the crystal form are the symmetry center, symmetry axis, and symmetry plane; their definitions are well-known to be the following ones ²⁾.

A *symmetry center* is a point that bisects all lines that go through it such that both ends terminate on the crystal polyhedron.

A *symmetry axis* is any line, around which one can rotate the polyhedron through an aliquot part of a complete rotation in such a way that all of its points coincide with the points of its initial position. If $2p/n$ is the smallest rotation angle that belongs to the axis then the symmetry axis is called *n-fold*. The only crystallographic possibilities are the cases $n = 2, 3, 4, 6$. Two symmetry axes are called equal when the arrangement of surfaces and edges for the one is the same for the other one. Accordingly, if the two sides of one and the same axis are equal then one calls the symmetry axis *two-sided*.

A *symmetry plane* is any plane that divides the crystal polyhedron in such a way that one half is the mirror image of the other relative to the symmetry plane. Two symmetry planes are called equal when the arrangement of surfaces and edges is identical for both of them.

For our special purpose of the application to the specialization of the values of elastic potentials and stress forces, the following question has special importance: Which of the simultaneously appearing equal or different kinds of mutually independent symmetry elements should one consider, in particular?

There exist a number of easily-proved theorems in regard to this.

¹⁾ For that reason, I cannot agree with the procedure of Minnigerode [Götting. Nachr. **6** (1884), 219], that calls upon the optical behavior of crystals in certain cases.

²⁾ Cf., Liebisch, *Krystallographie*, Leipzig, 1881, pp. 191, *et seq.*

For symmetry axes alone, one has the following:

If two equal two-fold symmetry axes are present then they must make angles of $2\pi/3$, $2\pi/4$, or $2\pi/6$ with each other, and there simultaneously exist 1, 2, or 4 equal axes with them ¹⁾ that lie in the same plane and which define the same angles with the neighboring axes. The bisectors of this angle are likewise equal, and for the former distinct two-fold symmetry axes, the direction that is normal to their plane is a two-sided 3, 4, or 6-fold symmetry axis, resp. This theorem can also be inverted in such a way that the existence of two-sided 5, 4, or 6-fold symmetry axes has the two-fold neighboring axis as a consequence.

Several equal three-fold symmetry axes are possible only in the case in which they lie on the four vertex diagonals of a regular hexahedron. Three four-fold equal parallels to the edges of the hexahedron and six two-fold equal parallels to the surface diagonals then appear with them.

For symmetry planes, one further has:

If several symmetry planes intersect in a line then they are either all equal or divide into two groups of mutually equal ones that appear alternately. Neighboring symmetry planes therefore intersect with equal angles and the planes of the one group bisect the angle that is defined by those of the other group. The line of intersection is an n -fold symmetry axis when n symmetry planes intersect in it.

If a center of the symmetry is present then it follows from the presence of two normal symmetry planes that a third one exists that is normal to the first two.

All three symmetry elements are linked to the theorem: A center of symmetry, an even-numbered symmetry axis, and a symmetry plane that is normal to it are three things, the presence of any two of which necessarily implies the third one.

From this, in any case, one can select from the number of symmetry elements that are perceived for a crystal form, the independent ones, and therefore the only ones that are essential for applications.

Yet another simplification is provided by the result that emerges from the utilization of equation (2) or the principle of the equality of action and reaction, that opposite directions are elastically equivalent for arbitrary, homogeneous, elastic media in full generality, so a center of symmetry always exists in them. From this, it follows that the theorems above that are connected with the existence of a symmetry center are always fulfilled when one considers the elastic behavior of the crystal. In this sense, a symmetry plane is therefore perpendicular to any two-fold symmetry axis, and conversely, and so forth. In general, one can state the rule that the symmetry behavior that is definitive for elastic behavior is not that of the crystal form itself, but that of a permutation or completion that one obtains from them when one adds opposite face to any face of the form, in the event that is absent. From this, it follows immediately that all of the hemihedral and hemimorphic forms that one can obtain holohedrally by letting one of the pairs of opposite faces in the latter vanish must behave the same as holohedral crystals do elastically.

Of the symmetry elements, only the symmetry axes come under consideration as definitive for the specialization of the elastic potential or the potential for the elementary interaction, from their basic existence. All symmetry planes are then necessarily linked

¹⁾ In the last two cases, any two axes lie in opposite directions and together they define a two-sided symmetry axis.

with certain symmetry axes here, and seem to be their consequence, which is not conversely true. The general hypothesis that was already stated above that any symmetry of the form corresponds to the same symmetry of the elastic behavior will then take on the following special form:

If the crystal form (when “completed” as we said above) possesses a symmetry axis – i.e., if there is a direction, around which the crystal form rotates as a rotational axis through an aliquot part of 2π (hence, also the point system that gives the positions of the molecules) so that it comes to cover its original position at all points – then the potential must again completely assume the original form when transformed to a correspondingly rotated coordinate system.

Meanwhile, for our applications, it is often more convenient to employ, in place of the property that a even-number symmetry axis be present, that of the presence of a normal symmetry plane that is coupled with it. The latter yields, without computation, the value zero for all of the sums D_{hk}^{mm} whose arguments possess opposite values on the two sides of the symmetry plane.

The *monoclinic* system is characterized ¹⁾ by the existence of a two-fold crystallographic symmetry axis, which, from the foregoing, always has a normal elastic symmetry plane.

If one chooses the *YZ*-plane to be the symmetry plane then all of the coefficients D_{hk}^{mm} must vanish in which the index 1 appears once or three times. Accordingly, the X_x, Y_y, Z_z, Z_y in system (29) will be free of:

$$\left(\frac{\partial u}{\partial y} + n\right), \quad \left(\frac{\partial v}{\partial x} - n\right), \quad \left(\frac{\partial w}{\partial x} + m\right), \quad \left(\frac{\partial u}{\partial z} - m\right),$$

and the X_y, Y_x, Z_z, Z_x will be free of:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z}, \quad \left(\frac{\partial v}{\partial z} + l\right), \quad \left(\frac{\partial w}{\partial y} - l\right).$$

Accordingly, one will have:

$$\begin{aligned} \lambda_1 &= 2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22}), \quad \mu_2 = 2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33}), \quad \nu_3 = 2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11}), \\ \nu_2 = \mu_3 &= (D_{11}^{23} + D_{23}^{11}) - (D_{31}^{12} + D_{12}^{31}), \quad \lambda_3 = \nu_1 = 0, \quad \mu_1 = \lambda_2 = 0; \end{aligned} \tag{40}$$

$$f_1 = \epsilon L + \frac{\partial u}{\partial x} (D_{31}^{12} - D_{12}^{31}) + \frac{\partial v}{\partial y} (D_{23}^{22} - D_{22}^{23}) + \frac{\partial w}{\partial z} (D_{33}^{23} - D_{23}^{33}) + \frac{\partial v}{\partial z} (D_{23}^{23} - D_{22}^{33}) + \frac{\partial w}{\partial y} (D_{33}^{22} - D_{23}^{23}),$$

¹⁾ Liebisch, *loc. cit.*, pp. 212 and 380; Minnigerode, *loc. cit.*, pp. 216.

$$f_2 = \varepsilon M + \frac{\partial u}{\partial y} (D_{11}^{23} - D_{31}^{12}) + \frac{\partial u}{\partial z} (D_{11}^{33} - D_{31}^{31}) + \frac{\partial v}{\partial x} (D_{12}^{31} - D_{23}^{11}) + \frac{\partial w}{\partial x} (D_{31}^{31} - D_{33}^{11}),$$

$$f_3 = \varepsilon N + \frac{\partial u}{\partial y} (D_{12}^{12} - D_{11}^{22}) + \frac{\partial u}{\partial z} (D_{12}^{31} - D_{11}^{23}) + \frac{\partial v}{\partial x} (D_{22}^{11} - D_{12}^{12}) + \frac{\partial w}{\partial x} (D_{23}^{11} - D_{31}^{12});$$

one then has:

$$\Pi = \lambda_1 (\mu_2 \nu_3 - \nu_2 \mu_3),$$

and thus, from (36):

$$\begin{aligned} -l\lambda_1 &= f_1, \\ -m(\mu_2\nu_3 - \nu_2\mu_3) &= f_2\nu_3 - f_3\nu_2, \\ -n(\mu_2\nu_3 - \nu_2\mu_3) &= -f_2\mu_3 + f_3\mu_2. \end{aligned} \quad (41)$$

Likewise, (38') gives, more simply:

$$k\lambda_1 (\mu_2 \nu_3 - \nu_2 \mu_3) = \begin{vmatrix} \kappa & \lambda & \mu & \nu \\ \kappa_1 & \lambda_1 & 0 & 0 \\ \kappa_2 & 0 & \mu_2 & \nu_3 \\ \kappa_3 & 0 & \mu_3 & \nu_2 \end{vmatrix} \quad (42)$$

and (38'') gives:

$$k'\lambda_1 = -\varepsilon\lambda, \quad k''(\mu_2 \nu_3 - \nu_2 \mu_3) = -\varepsilon(\mu\nu_3 - \nu\mu_3), \quad k'''(\mu_2 \nu_3 - \nu_2 \mu_3) = -\varepsilon(\mu_2 \nu - \nu_2 \mu).$$

One then easily calculates the values of the components that are free of l, m, n :

$$\begin{aligned} -X_x &= D_{11} x_x + D_{12} y_y + D_{13} z_z + D_{14} y_z & -E_{11} \varepsilon L, \\ -Y_y &= D_{21} x_x + D_{22} y_y + D_{23} z_z + D_{24} y_z & -E_{21} \varepsilon L, \\ -Z_z &= D_{21} x_x + D_{32} y_y + D_{33} z_z + D_{34} y_z & -E_{31} \varepsilon L, \\ -Y_x &= D_{41} x_x + D_{42} y_y + D_{43} z_z + D_{44} y_z & -E_{41} \varepsilon L, \\ -Z_y &= D_{41} x_x + D_{42} y_y + D_{43} z_z + D_{44} y_z & -(1 + E_{41}) \varepsilon L, \end{aligned} \quad (43)$$

$$\begin{aligned} -Z_x &= & D_{55} z_x + D_{56} x_y & -E_{11} \varepsilon L - E_{53} \varepsilon N, \\ -X_y &= & D_{55} x_x + D_{56} x_y & -E_{21} \varepsilon L - E_{53} \varepsilon N, \\ -X_z &= & D_{65} x_x + D_{66} x_y & -E_{31} \varepsilon L - E_{63} \varepsilon N, \\ -Y_x &= & D_{65} x_x + D_{66} x_y & -E_{41} \varepsilon L - (1 + E_{63}) \varepsilon N. \end{aligned}$$

In this, the D_{hk} and E_{hk} denote the following aggregate of D_{hk}^{mn} :

$$\begin{aligned} \lambda_1 D_{11} &= \lambda_1 D_{11}^{11} - (D_{12}^{13} - D_{13}^{12})^2, \quad \lambda_1 D_{22} = \lambda_1 D_{22}^{23} - (D_{22}^{23} - D_{23}^{22})^2, \quad \lambda_1 D_{23} = \lambda_1 D_{33}^{33} - (D_{32}^{33} - D_{33}^{32})^2, \\ \lambda_1 D_{23} &= \lambda_1 D_{32} = \lambda_1 D_{23}^{23} - (D_{22}^{23} - D_{23}^{22})(D_{32}^{33} - D_{33}^{32}), \\ \lambda_1 D_{31} &= \lambda_1 D_{13} = \lambda_1 D_{31}^{31} - (D_{32}^{33} - D_{33}^{32})(D_{12}^{13} - D_{13}^{12}), \\ \lambda_1 D_{12} &= \lambda_1 D_{12} = \lambda_1 D_{12}^{12} - (D_{12}^{13} - D_{13}^{12})(D_{22}^{23} - D_{23}^{22}), \end{aligned}$$

$$\begin{aligned}
\lambda_1 D_{14} &= \lambda_1 D_{41} = D_{12}^{13} (D_{33}^{22} - D_{23}^{23}) + D_{13}^{12} (D_{22}^{33} - D_{23}^{23}), \\
\lambda_1 D_{24} &= \lambda_1 D_{42} = D_{22}^{23} (D_{33}^{22} - D_{23}^{23}) + D_{23}^{22} (D_{22}^{33} - D_{23}^{23}), \\
\lambda_1 D_{34} &= \lambda_1 D_{43} = D_{32}^{33} (D_{33}^{22} - D_{23}^{23}) + D_{33}^{32} (D_{22}^{33} - D_{23}^{23}), \\
\lambda_1 D_{44} &= D_{22}^{33} (D_{23}^{23} - D_{33}^{22}) + D_{23}^{23} (D_{23}^{23} - D_{22}^{33}) = D_{23}^{23} D_{23}^{23} - D_{22}^{33} D_{33}^{22},
\end{aligned}$$

$$\begin{aligned}
(\mu_2 \nu_3 - \nu_2 \mu_3) D_{55} &= \begin{vmatrix} D_{31}^{13} & (D_{33}^{11} - D_{31}^{13}) & (D_{31}^{12} - D_{32}^{11}) \\ -(D_{11}^{33} - D_{31}^{13}) & \mu_2 & \nu_2 \\ -(D_{33}^{11} - D_{31}^{13}) & \mu_3 & \nu_3 \end{vmatrix}, \\
(\mu_2 \nu_3 - \nu_2 \mu_3) D_{66} &= \begin{vmatrix} D_{11}^{22} & (D_{13}^{21} - D_{11}^{23}) & (D_{11}^{22} - D_{12}^{21}) \\ -(D_{11}^{32} - D_{31}^{12}) & \mu_2 & \nu_2 \\ -(D_{21}^{12} - D_{11}^{22}) & \mu_3 & \nu_3 \end{vmatrix}, \\
(\mu_2 \nu_3 - \nu_2 \mu_3) D_{56} &= \begin{vmatrix} D_{11}^{23} & (D_{13}^{21} - D_{11}^{23}) & (D_{11}^{22} - D_{12}^{21}) \\ -(D_{11}^{33} - D_{31}^{13}) & \mu_2 & \nu_2 \\ -(D_{21}^{13} - D_{11}^{23}) & \mu_3 & \nu_3 \end{vmatrix},
\end{aligned}
\tag{44}$$

$$\begin{aligned}
\lambda_1 E_{11} &= (D_{12}^{13} - D_{13}^{12}), \quad \lambda_1 E_{21} = (D_{22}^{23} - D_{23}^{22}), \quad \lambda_1 E_{31} = (D_{32}^{33} - D_{33}^{32}), \quad \lambda_1 E_{41} = (D_{22}^{33} - D_{23}^{32}), \\
E_{52} (\mu_2 \nu_3 - \nu_2 \mu_3) &= [(D_{33}^{11} - D_{31}^{13}) \nu_3 - (D_{31}^{12} - D_{32}^{11}) \mu_3], \\
E_{53} (\mu_2 \nu_3 - \nu_2 \mu_3) &= [(D_{31}^{12} - D_{32}^{11}) \mu_2 - (D_{33}^{11} - D_{31}^{13}) \nu_2], \\
E_{62} (\mu_2 \nu_3 - \nu_2 \mu_3) &= [(D_{13}^{21} - D_{11}^{23}) \nu_3 - (D_{11}^{22} - D_{12}^{21}) \mu_3], \\
E_{63} (\mu_2 \nu_3 - \nu_2 \mu_3) &= [(D_{11}^{22} - D_{21}^{21}) \mu_2 - (D_{13}^{21} - D_{11}^{23}) \nu_2].
\end{aligned}$$

These formulas are always exceedingly complicated to discuss, so here they shall also constitute the starting point for simpler forms.

In crystals of the *rhombic* systems (with the exception of the rare “hemi-morphic forms of the second kind”), there is elastic behavior in three mutually normal symmetry planes¹⁾, so all of the D_{hk}^{mn} in whose indices any one of the numbers 1, 2, 3 appears just once will vanish.

With that, one has:

$$\begin{aligned}
\lambda_1 &= 2D_{23}^{32} - (D_{22}^{33} + D_{33}^{22}), & \mu_1 &= 0, & \nu_1 &= 0, \\
\lambda_2 &= 0, & \mu_2 &= 2D_{31}^{13} - (D_{33}^{11} + D_{11}^{33}), & \nu_2 &= 0, \\
\lambda_3 &= 0, & \mu_3 &= 0, & \nu_3 &= 2D_{12}^{21} - (D_{11}^{22} + D_{22}^{11}),
\end{aligned}
\tag{45}$$

¹⁾ Liebisch, *loc. cit.*, pp. 212 and 365; Minningerode, *loc. cit.*, pp. 215.

$$\begin{aligned}
f_1 &= \varepsilon L + \frac{\partial v}{\partial z} (D_{32}^{23} - D_{22}^{33}) + \frac{\partial w}{\partial y} (D_{33}^{22} - D_{23}^{32}), \\
f_2 &= \varepsilon M + \frac{\partial w}{\partial x} (D_{13}^{31} - D_{33}^{11}) + \frac{\partial u}{\partial z} (D_{11}^{33} - D_{31}^{13}), \\
f_3 &= \varepsilon N + \frac{\partial u}{\partial y} (D_{21}^{12} - D_{11}^{22}) + \frac{\partial v}{\partial x} (D_{22}^{11} - D_{12}^{21}),
\end{aligned}$$

and all of the results simplify to a considerable degree.

One next has:

$$-l\lambda_1 = f_1, \quad -m\mu_2 = f_2, \quad -n\nu_3 = f_3;$$

one can also write this as:

$$\begin{aligned}
l &= \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - \frac{\varepsilon L + \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) (D_{33}^{22} - D_{22}^{33})}{2D_{23}^{32} - (D_{22}^{33} + D_{33}^{22})} = \varphi - \frac{\varepsilon L + \frac{1}{2} y_z (D_{33}^{22} - D_{22}^{33})}{2D_{23}^{32} - (D_{22}^{33} + D_{33}^{22})}, \\
m &= \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) - \frac{\varepsilon M + \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) (D_{11}^{33} - D_{33}^{11})}{2D_{31}^{13} - (D_{33}^{11} + D_{11}^{33})} = \psi - \frac{\varepsilon M + \frac{1}{2} z_x (D_{11}^{33} - D_{33}^{11})}{2D_{31}^{13} - (D_{33}^{11} + D_{11}^{33})}, \\
n &= \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{\varepsilon N + \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (D_{22}^{11} - D_{11}^{22})}{2D_{12}^{21} - (D_{11}^{22} + D_{22}^{11})} = \chi - \frac{\varepsilon N + \frac{1}{2} x_y (D_{22}^{11} - D_{11}^{22})}{2D_{12}^{21} - (D_{11}^{22} + D_{22}^{11})},
\end{aligned} \tag{46}$$

in which one lets φ, ψ, χ denote the rotational angles of the entire volume element at the location x, y, z .

Furthermore, one has:

$$\begin{aligned}
-X_x &= D_{11} x_x + D_{12} y_y + D_{13} z_z = D_{11}^{11} x_x + D_{12}^{12} y_y + D_{13}^{13} z_z, \\
-Y_y &= D_{21} x_x + D_{22} y_y + D_{23} z_z = D_{21}^{21} x_x + D_{22}^{22} y_y + D_{23}^{23} z_z, \\
-Z_z &= D_{31} x_x + D_{32} y_y + D_{33} z_z = D_{31}^{31} x_x + D_{32}^{32} y_y + D_{33}^{33} z_z, \\
-Y_y &= D_{44} y_x - E_{41} \varepsilon L = \frac{(D_{23}^{23} D_{23}^{23} - D_{33}^{33} D_{33}^{22}) y_z - (D_{22}^{33} - D_{23}^{23}) \varepsilon L}{2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22})}, \\
-Z_y &= D_{44} y_z - (1 + E_{41}) \varepsilon L = \frac{(D_{23}^{23} D_{23}^{23} - D_{22}^{33} D_{33}^{22}) y_z - (D_{22}^{23} - D_{23}^{33}) \varepsilon L}{2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22})}, \\
-Z_x &= D_{55} z_x - E_{52} \varepsilon M = \frac{(D_{31}^{31} D_{31}^{31} - D_{33}^{11} D_{11}^{33}) z_x - (D_{33}^{11} - D_{31}^{31}) \varepsilon M}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},
\end{aligned} \tag{47}$$

$$\begin{aligned}
-X_z = D_{55} z_x - (1 + E_{52}) \varepsilon L &= \frac{(D_{31}^{31} D_{31}^{31} - D_{33}^{11} D_{11}^{33}) z_x - (D_{31}^{31} - D_{11}^{33}) \varepsilon M}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})}, \\
-X_y = D_{66} x_y - E_{63} \varepsilon N &= \frac{(D_{12}^{12} D_{12}^{12} - D_{11}^{22} D_{22}^{11}) x_y - (D_{11}^{22} - D_{12}^{12}) \varepsilon N}{2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11})}, \\
-Y_z = D_{66} x_y - (1 + E_{63}) \varepsilon N &= \frac{(D_{12}^{12} D_{12}^{12} - D_{11}^{22} D_{22}^{11}) x_y - (D_{12}^{12} - D_{22}^{11}) \varepsilon N}{2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11})}.
\end{aligned}$$

The crystals of the *quadratic* systems, with the exception of the pyramidal hemihedral, and the rhomboid and sphenoid tetrahedral forms, possess two equivalent normal symmetry planes in their elastic behavior ¹); in the event that one chooses the Z-axis as the distinguished one, and thus regards the Y and Z directions as indistinguishable, the following sequence of values will result ²):

$$l = \varphi - \frac{\varepsilon L + \frac{1}{2} y_z (D_{33}^{11} - D_{11}^{33})}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$m = \psi - \frac{\varepsilon M + \frac{1}{2} z_x (D_{11}^{33} - D_{33}^{11})}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$n = \chi - \frac{\varepsilon N}{2(D_{12}^{12} - D_{11}^{22})},$$

$$\begin{aligned}
-X_x = D_{11}^{11} x_x + D_{12}^{12} y_y + D_{13}^{13} z_z, & \quad -Y_y = D_{12}^{12} x_x + D_{11}^{11} y_y + D_{13}^{13} z_z, \\
-Z_z = D_{13}^{13} x_x + D_{13}^{13} y_y + D_{33}^{33} z_z, &
\end{aligned}$$

$$-X_x = \frac{(D_{31}^{31} D_{31}^{31} - D_{11}^{33} D_{33}^{11}) y_z - (D_{11}^{33} - D_{31}^{31}) \varepsilon L}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$-Z_y = \frac{(D_{31}^{31} D_{31}^{31} - D_{11}^{33} D_{33}^{11}) y_z - (D_{31}^{31} - D_{33}^{11}) \varepsilon L}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$-Z_x = \frac{(D_{31}^{31} D_{31}^{31} - D_{11}^{33} D_{33}^{11}) z_x - (D_{33}^{11} - D_{31}^{31}) \varepsilon M}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$-X_z = \frac{(D_{31}^{31} D_{31}^{31} - D_{11}^{33} D_{33}^{11}) z_x - (D_{31}^{31} - D_{11}^{33}) \varepsilon M}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})},$$

$$-X_y = \frac{(D_{12}^{12} + D_{11}^{22}) x_y + \varepsilon N}{2}, \quad -Y_x = \frac{(D_{12}^{12} + D_{11}^{22}) x_y - \varepsilon N}{2}.$$

¹) Liebisch, *loc. cit.*, pp. 212 and 339. Minningerode, *loc. cit.*, pp. 213.

²) Indeed, this indistinguishability has the consequence that $D_{11}^{11} = D_{22}^{22}$ and $D_{11}^{22} = D_{22}^{11}$, but not, by contrast, $D_{11}^{11} = D_{11}^{22}$, which is easy to show precisely.

Whereas the rhombic and lower symmetry systems, even when no external rotational moments L, M, N act, yield an autonomous rotation of the molecule – namely, the l, m, n are different from the φ, ψ, χ , which rotate the volume elements – this then shows that for quadratic systems these autonomous rotations vanish around the Z -axis when it is distinguished.

For the regular systems ¹⁾, where all three normal symmetry planes are indistinguishable, one can introduce an abbreviated notation to good effect. One sets:

$$\begin{aligned} D_{11}^{11} = D_{22}^{22} = D_{33}^{33} = A, & \quad D_{23}^{23} = D_{31}^{31} = D_{12}^{12} = B, \\ D_{33}^{22} = D_{22}^{33} = D_{11}^{33} = D_{33}^{11} = D_{22}^{11} = D_{11}^{22} = C \end{aligned}$$

so one has for all forms of the system:

$$\begin{aligned} l = \varphi - \frac{\varepsilon L}{2(B-C)}, \quad m = \psi - \frac{\varepsilon M}{2(B-C)}, \quad n = \chi - \frac{\varepsilon N}{2(B-C)}, \\ -X_x = Ax_x + By_y + Bz_z, \quad -Y_y = Bx_x + Ay_y + Bz_z, \quad -Z_z = Bx_x + By_y + Az_z, \\ -Y_x = \frac{B+C}{2} y_z + \frac{\varepsilon L}{2}, \quad -Z_y = \frac{B+C}{2} y_z - \frac{\varepsilon L}{2}, \\ -Z_x = \frac{B+C}{2} z_x + \frac{\varepsilon L}{2}, \quad -X_z = \frac{B+C}{2} y_x - \frac{\varepsilon M}{2}, \\ -X_y = \frac{B+C}{2} x_y + \frac{\varepsilon N}{2}, \quad -Y_x = \frac{B+C}{2} x_y - \frac{\varepsilon N}{2}. \end{aligned} \tag{49}$$

Here, one thus finds that under the influence of only stress forces and vanishing moments L, M, N there never exists an autonomous rotation of the molecule in the volume element, so one always has $l = \varphi, m = \varphi, n = \chi$.

The formulas for the crystals of hexagonal systems – apart from the rhombohedral forms, which we will treat separately – one gets quite easily, when one uses the fact that the symmetry elements that are present in them, when completed in the way carried out above (pp. 25), collectively give a six-fold symmetry axis, and thus necessarily couple twice six to their normal equivalent two-fold symmetry axes or twice three to their parallel equivalent symmetry planes, which subtend equal angles with each other ²⁾. If one puts the six-fold (principal) axis along the Z -axis then when one rotates the coordinate system around it in six positions that are inclined at 60° to each other, the elementary potential must exhibit the property of three-fold symmetry with respect to the coordinate planes, and thus take on the identical form.

¹⁾ Liebisch, *loc. cit.*, pp. 211 and 223. Minnigerode, *loc. cit.*, pp. 209.

²⁾ Liebisch, *loc. cit.*, pp. 211 and 279. Minnigerode, *loc. cit.*, pp. 379, *et seq.*

Let such a position, as the initial position, be chosen to be the coordinate system, so the system (46) and (47) is valid for it; a rotation of the X and Y -axes around the Z -axis gives the substitution:

$$\begin{aligned} x = a &= \xi\alpha + \eta\beta, & \xi &= a\alpha - b\beta, \\ y = b &= -\xi\beta + \eta\alpha, & \eta &= a\beta + b\alpha, \end{aligned}$$

we now have to introduce the sums that define D_{hk}^{mn} .

This yields, e.g.:

$$D_{11}^{11} = \sum a^2 \frac{\partial^2 F}{\partial \xi^2} = \sum (\alpha^2 \xi^2 + \beta^2 \eta^2 + 2\alpha\beta\xi\eta) \left(\alpha^2 \frac{\partial^2 F}{\partial \xi^2} + \beta^2 \frac{\partial^2 F}{\partial \eta^2} + 2\alpha\beta \frac{\partial^2 F}{\partial \xi \partial \eta} \right),$$

or when one introduces the abbreviation Δ_{hk}^{mn} for the new system in the same sense as D_{hk}^{mn} for the old one:

$$D_{11}^{11} = \alpha^4 \Delta_{11}^{11} + \beta^4 \Delta_{22}^{22} + 2\alpha^2 \beta^2 (\Delta_{11}^{22} + \Delta_{22}^{11} + 4\Delta_{12}^{12}) + \alpha^3 \beta (2\Delta_{12}^{11} + 2\Delta_{11}^{12}) + \alpha \beta^3 (2\Delta_{12}^{22} + 2\Delta_{22}^{12}).$$

In order for the new system be equivalent to the old one, the Δ_{hk}^{mn} must equal the D_{hk}^{mn} , so, since the coordinate planes should be symmetry planes, one must then have:

$$D_{11}^{11} = \alpha^4 D_{11}^{11} + \beta^4 D_{22}^{22} + \alpha^2 \beta^2 (D_{11}^{22} + D_{22}^{11} + 4D_{12}^{12}).$$

One likewise finds that:

$$\begin{aligned} D_{22}^{22} &= \beta^4 D_{11}^{11} + \alpha^4 D_{22}^{22} + \alpha^2 \beta^2 (D_{11}^{22} + D_{22}^{11} + 4D_{12}^{12}), \\ D_{11}^{22} &= \alpha^2 \beta^2 (D_{11}^{11} + D_{22}^{22} - 4D_{12}^{12}) + \alpha^4 D_{11}^{22} + \beta^4 D_{22}^{11}, \\ D_{22}^{11} &= \alpha^2 \beta^2 (D_{11}^{11} + D_{22}^{22} - 4D_{12}^{12}) - \beta^4 D_{11}^{22} + \alpha^4 D_{22}^{11}, \\ D_{12}^{12} &= \alpha^2 \beta^2 (D_{11}^{11} + D_{22}^{22} - D_{11}^{22} - D_{22}^{11}) + (\alpha^2 - \beta^2)^2 D_{12}^{12}, \end{aligned} \tag{50}$$

$$\begin{aligned} D_{13}^{13} &= \alpha^2 D_{13}^{13} + \beta^2 D_{23}^{23}, & D_{23}^{23} &= \beta^2 D_{13}^{13} + \alpha^2 D_{23}^{23}, \\ D_{33}^{11} &= \alpha^2 D_{33}^{11} + \beta^2 D_{33}^{22}, & D_{33}^{22} &= \beta^2 D_{33}^{11} + \alpha^2 D_{33}^{22}, \\ D_{11}^{33} &= \alpha^2 D_{11}^{33} + \beta^2 D_{22}^{33}, & D_{22}^{33} &= \beta^2 D_{11}^{33} + \alpha^2 D_{22}^{33}. \end{aligned}$$

It follows from this that for any α and β that deviate from 1 and 0:

$$(51) \quad D_{11}^{11} = D_{22}^{22}, \quad D_{11}^{22} = D_{22}^{11}, \quad 2D_{12}^{12} = D_{12}^{12} - D_{11}^{22}, \quad D_{13}^{13} = D_{23}^{23}, \quad D_{33}^{11} = D_{33}^{22}, \quad D_{11}^{33} = D_{22}^{33}.$$

These relations must then also be true for hexagonal systems, and together with (46) and (47) they yield the following values:

$$\begin{aligned}
l &= \varphi - \frac{\varepsilon L + \frac{1}{2} y_z (D_{33}^{11} - D_{11}^{33})}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, & m &= \psi - \frac{\varepsilon M + \frac{1}{2} z_x (D_{11}^{33} - D_{33}^{11})}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
n &= \chi - \frac{\varepsilon N}{2(D_{12}^{12} - D_{11}^{11})}, \\
-X_z &= D_{11}^{11} x_x + D_{12}^{12} y_y + D_{13}^{13} z_z, & -Y_y &= D_{12}^{12} x_x + D_{11}^{11} y_y + D_{13}^{13} z_z, \\
-Z_z &= D_{13}^{13} x_x + D_{13}^{13} y_y + D_{33}^{33} z_z, \\
-Y_z &= \frac{(D_{13}^{13} D_{13}^{13} - D_{11}^{33} D_{33}^{11}) y_z - (D_{11}^{33} - D_{13}^{13}) \varepsilon L}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
-Z_y &= \frac{(D_{13}^{13} D_{13}^{13} - D_{11}^{33} D_{33}^{11}) y_z - (D_{13}^{13} - D_{33}^{11}) \varepsilon L}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
-Z_x &= \frac{(D_{13}^{13} D_{13}^{13} - D_{11}^{33} D_{33}^{11}) z_x - (D_{33}^{11} - D_{13}^{13}) \varepsilon M}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
-X_z &= \frac{(D_{13}^{13} D_{13}^{13} - D_{11}^{33} D_{33}^{11}) z_x - (D_{13}^{13} - D_{33}^{11}) \varepsilon M}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
-X_y &= \frac{D_{11}^{11} + D_{12}^{12}}{2} x_y + \frac{\varepsilon N}{2}, & -Y_x &= \frac{D_{11}^{11} + D_{13}^{13}}{2} x_y - \frac{\varepsilon N}{2}.
\end{aligned}$$

This system agrees precisely with long-known formulas, up to the values of l , m , n , which the older theory does not give, and the coefficients of L , M , N , which no one considered.

The number of mutually independent D_{hk}^{mn} in it is six.

Finally, in order to address the *rhombohedral* forms, we employ the fact that, at least when one recalls the previous general remarks, they possess a three-fold axis in the context of elasticity, and therefore, twice three equivalent neighboring symmetry axes to any normal or three equivalent parallel symmetry planes that subtend equal angles with each other¹⁾. Only the rhombohedral tetartohedra (Tetartoëdrie) and the fourth hemimorphy are excluded as forms. If one again puts the principal axis in the Z -direction then the elementary potential must exhibit the property of symmetry with respect to a plane that goes through the Z -axis – say, the YZ -plane – under rotation of the X and Y -directions around it through three positions that are separated by 120° , and thus exhibit an identical form.

Let such a position be chosen to be the initial position, so the system (43) is valid for it. A rotation of the X and Y around the Z -axis by way of the substitution:

$$\begin{aligned}
x = a &= \xi\alpha + \eta\beta, & \xi &= a\alpha - b\beta, \\
y = b &= -\xi\beta + \eta\alpha, & \eta &= a\beta + b\alpha,
\end{aligned}$$

¹⁾ Liebis, *loc. cit.*, pp. 211 and 299; Minnigerode, *loc. cit.*, pp. 379, *et seq.*

gives D_{11}^{11} , D_{22}^{22} , D_{22}^{11} , D_{11}^{22} , D_{12}^{12} , D_{13}^{13} , D_{23}^{23} , D_{33}^{11} , D_{11}^{33} , D_{33}^{22} , D_{22}^{33} for the sums of the values that are assembled into the system (50), and therefore gives the same relations (51) between them for the rhombohedral system as for the hexagonal system. The sums that appear in (43) along with these are, with consideration for the fact that the HZ symmetry plane is equivalent to the YZ symmetry plane, found to be determined as follows:

$$(53) \quad \begin{aligned} D_{11}^{23} &= D_{11}^{23} \alpha^3 + D_{23}^{23} \alpha \beta^2 - D_{12}^{13} 2\alpha \beta^2, & D_{23}^{11} &= D_{23}^{11} \alpha^3 + D_{23}^{22} \alpha \beta^2 - D_{13}^{12} 2\alpha \beta^2, \\ D_{22}^{23} &= D_{11}^{23} \alpha \beta^2 + D_{22}^{23} \alpha^3 + D_{12}^{13} 2\alpha \beta^2, & D_{23}^{22} &= D_{23}^{11} \alpha \beta^2 + D_{23}^{22} \alpha^3 + D_{13}^{12} 2\alpha \beta^2, \\ D_{12}^{13} &= D_{12}^{13} \alpha(\alpha^2 - \beta^2) + \alpha \beta^2 (D_{22}^{23} - D_{11}^{23}), & D_{13}^{12} &= D_{13}^{12} \alpha(\alpha^2 - \beta^2) + \alpha \beta^2 (D_{23}^{22} - D_{23}^{11}), \\ D_{23}^{33} &= \alpha D_{23}^{33}, & D_{33}^{23} &= \alpha D_{33}^{23}. \end{aligned}$$

The foregoing formulas lead to the following results:

$$\begin{aligned} D_{11}^{23} + D_{22}^{23} &= 0, & D_{12}^{13} &= \frac{2D_{11}^{22} \alpha \beta^2}{\alpha(\alpha^2 - \beta^2) - 1}, & D_{23}^{33} &= 0, \\ D_{23}^{11} + D_{23}^{22} &= 0, & D_{13}^{12} &= \frac{2D_{23}^{11} \alpha \beta^2}{\alpha(\alpha^2 - \beta^2) - 1}, & D_{33}^{23} &= 0. \end{aligned}$$

Therefore, only two of these relations depend upon the rotation angle. The latter subtends 120° here, so one has $\alpha = -1/2$, $\beta = -\sqrt{3}/2$, and the values that pertain to this thus read completely:

$$(54) \quad \begin{aligned} D_{11}^{11} &= D_{22}^{22}, \quad D_{11}^{22} = D_{22}^{11}, \quad 2D_{12}^{12} = D_{11}^{11} - D_{11}^{22}, \quad D_{13}^{13} = D_{23}^{23}, \quad D_{33}^{11} = D_{33}^{22}, \quad D_{33}^{33} = D_{22}^{33} \\ D_{11}^{23} &= -D_{22}^{23} = D_{12}^{13}, \quad D_{23}^{11} = -D_{23}^{22} = D_{13}^{12}, \quad D_{23}^{33} = D_{33}^{23} = 0. \end{aligned}$$

From this, one finally gets the following system that is valid for rhombohedral crystals:

$$\begin{aligned} l &= \varphi - \frac{\varepsilon L + \frac{1}{2} y_z (D_{33}^{11} - D_{11}^{33})}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\ m &= \psi - \frac{\varepsilon M + \frac{1}{2} z_x (D_{11}^{33} - D_{33}^{11})}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\ n &= \chi - \frac{\varepsilon N}{2(3D_{12}^{12} - D_{11}^{11})}, \end{aligned}$$

$$\begin{aligned} -X_x &= D_{11} x_x + D_{12} y_y + D_{13} z_z + D_{14} y_z & -E_{11} \varepsilon L, \\ -Y_y &= D_{12} x_x + D_{11} y_y + D_{13} z_z - D_{14} y_z & +E_{11} \varepsilon L, \\ -Z_z &= D_{13} x_x + D_{13} y_y + D_{33} z_z, \\ -Y_z &= D_{14} x_x - D_{14} y_y & + D_{44} y_z & -E_{41} \varepsilon L, \\ -Z_y &= D_{14} x_x - D_{14} y_y & + D_{44} y_z & - (1 + E_{41}) \varepsilon L, \end{aligned}$$

$$\begin{aligned}
-Z_x &= D_{55} z_x - D_{14} x_y - E_{52} \varepsilon M, \\
-X_z &= D_{55} z_x - D_{14} x_y - (1 + E_{52}) \varepsilon M, \\
-X_y &= -D_{14} z_x + \frac{D_{11} - D_{12}}{2} x_y - E_{62} \varepsilon M + \frac{1}{2} \varepsilon N, \\
-Y_x &= -D_{14} z_x + \frac{D_{11} - D_{12}}{2} x_y - E_{62} \varepsilon M - \frac{1}{2} \varepsilon N.
\end{aligned}$$

In this, the coefficients have the following values:

$$\begin{aligned}
D_{11} &= D_{11}^{11} - \frac{(D_{11}^{23} - D_{23}^{11})^2}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, & D_{33} &= D_{33}^{33}, & D_{12} &= D_{12}^{12}, & D_{13} &= D_{13}^{13}, \\
D_{14} &= \frac{D_{11}^{23}(D_{23}^{11} - D_{13}^{13}) + D_{23}^{11}(D_{11}^{33} - D_{13}^{13})}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, & D_{11} &= \frac{D_{13}^{13}D_{12}^{13} - D_{11}^{33}D_{33}^{11}}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
E_{11} &= \frac{D_{11}^{23} - D_{23}^{11}}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, & E_{41} &= \frac{D_{11}^{23} - D_{13}^{13}}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, \\
E_{52} &= \frac{D_{33}^{11} - D_{23}^{11}}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}, & E_{62} &= \frac{D_{13}^{12} - D_{11}^{23}}{2D_{13}^{13} - (D_{11}^{33} + D_{33}^{11})}.
\end{aligned} \tag{56}$$

The values deviate substantially from the values that are valid for hexagonal crystals in many respects.

5. The foregoing formulas for the various crystal systems differ formally from the ones that are ordinarily employed only by the appearance of the rotational moments L , M , N , and the rotational angles l , m , n , which relate to the individual molecules, not to the volume element of the crystal. We would like to next discuss the relevant terms.

Rotations l , m , n of the molecule, which obviously do not involve the entire volume element, do not appear as a result of the usual mechanical influences ¹⁾ in the regular systems at all, and in the quadratic and hexagonal ones they appear only around the neighboring axes. We would therefore like to inquire of the formulas for the *rhombic* systems (46) and (47), as the lowest-order ones that exhibit the general phenomena at all, whether they actually permit the calculation of the molecular rotations that appear for an arbitrary deformation; in particular, whether the constants that determine their quantities are derivable from observations with mechanical influences.

From (47), these methods of observation lead to a knowledge of the nine constants:

$$D_{11}^{11}, D_{22}^{22}, D_{33}^{33}, D_{23}^{23}, D_{31}^{31}, D_{12}^{12}, \tag{56}$$

¹⁾ By the term “mechanical” influences, we understand this to mean ones that leave the L , M , N equal to zero, and therefore exclusively distant forces of attraction and surface tensions, in contrast to the forces of electric or magnetic origin, which can give finite values to the L , M , N .

$$\frac{D_{23}^{23}D_{23}^{23} - D_{22}^{33}D_{33}^{22}}{2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22})}, \frac{D_{31}^{31}D_{11}^{31} - D_{33}^{11}D_{11}^{33}}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})}, \frac{D_{12}^{12}D_{12}^{12} - D_{11}^{22}D_{22}^{11}}{2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11})},$$

while in equations (46), only the following three appear for l, m, n :

$$\frac{D_{33}^{22} - D_{22}^{33}}{2D_{23}^{23} - (D_{22}^{33} + D_{33}^{22})}, \frac{D_{11}^{33} - D_{33}^{11}}{2D_{31}^{31} - (D_{33}^{11} + D_{11}^{33})}, \frac{D_{22}^{11} - D_{11}^{22}}{2D_{12}^{12} - (D_{11}^{22} + D_{22}^{11})}.$$

One recognizes with no calculation that the latter aggregate is independent of the former one and one can therefore express the following assertion for the rhombic, and all the more for the crystal systems of lower symmetry – but also, as is easily seen, also for higher symmetries:

“Observations of deformations that are produced by purely mechanical forces do not lead to knowledge of the constants upon which the autonomous molecular rotations depend.”

However, these constants would be determined when one is concerned with measuring the deformations that are produced by the rotational moments L, M, N that the molecules are subjected to. Such rotational moments are perhaps the result of external electrostatic forces in crystals or also the influence of galvanic currents in permanently magnetized steel, in the event that one regards its molecules as being endowed with polarizations, similar to those of a crystal.

Since the problem still has no immediate practical significance, I would like to appeal to only a special case of this to show how one of the aforementioned influences in fact leads to the determination of a new aggregate, and therefore to a means for the resolution of the question.

Let a rectangular prism be given in a rhombic crystal whose boundaries are parallel to the principal crystallographic axes. A constant moment L around the X -axis acts on it, and tangential forces on the surfaces that are parallel to the X -axis. Let the givens be chosen such that the rotation of the entire prism φ is equal to zero. In the event that the force P is parallel to the $\pm Y$ -axis on the surface that is normal to the $\pm Z$ -axis, one then has: $-Y_z = P, Z_y = 0$, and therefore, from (33):

$$P = \varepsilon L,$$

and, from (47):

$$y_z = \frac{\varepsilon L(D_{23}^{23} - D_{33}^{22})}{D_{23}^{23}D_{23}^{23} - D_{33}^{22}D_{22}^{33}}.$$

By contrast, if the force Q acts parallel to the $\pm Z$ -axis on the surface that is normal to the $\pm Y$ -axis, one has, analogously, $-Y_z = 0, Z_y = 0$, and therefore:

$$Q = \varepsilon L,$$

$$y_z = \frac{\varepsilon L(D_{22}^{33} - D_{23}^{23})}{D_{23}^{23}D_{23}^{23} - D_{33}^{22}D_{22}^{33}}.$$

If it were thus possible to measure the change of angle $y_z / 2$ between the surfaces that are normal to the Y and Z axes then one would arrive at a new way of determining a constant.

Observations of a similar kind with moments M and N allow one to present three relations that allow one to calculate, along with the known constants (56), in total, all twelve D_{hk}^{mm} . The molecular rotation is thus also completely determined by them.

For the further considerations, we would now like to assume purely mechanical influences, and thus set L, M, N equal to zero in the values of X_x, \dots , from which one likewise has $Y_z = Z_y, Z_x = X_z, X_y = Y_x$.

We thus obtain the form of the elastic tension components that the observations on crystal elasticity have been based upon up to now, and this begs the question of how one is to evaluate theoretically the numerical values that are obtained by measurements of the constants that appear in it.

One next remarks that for the case of an *unpolarized* elementary interaction – i.e., for interactions between the molecules that depend upon only the relative separations – the potential F can only be a function of the relative separation r' , which is given by:

$$r'^2 = a'^2 + b'^2 + c'^2.$$

Here, one then has, e.g.:

$$\frac{\partial F}{\partial a'} = \frac{a'}{r'} \frac{\partial F}{\partial r'},$$

thus:

$$\frac{\partial^2 F}{\partial a'^2} = \frac{1}{r'} \frac{\partial F}{\partial r'} + \frac{a'^2 \partial}{\partial r'} \frac{1}{r'} \frac{\partial F}{\partial r'}, \quad \frac{\partial^2 F}{\partial a' \partial b'} = \frac{a' b'}{r'} \frac{\partial F}{\partial r'} + \frac{\partial}{\partial r'} \frac{1}{r'} \frac{\partial F}{\partial r'}, \quad \frac{\partial^2 F}{\partial a' \partial c'} = \frac{a' c'}{r'} \frac{\partial F}{\partial r'} + \frac{\partial}{\partial r'} \frac{1}{r'} \frac{\partial F}{\partial r'}.$$

Now since, from (26), the nine sums of the form:

$$\Sigma (a' A') = - \Sigma \left(a' \frac{\partial F}{\partial a'} \right), \quad \Sigma (a' B') = - \Sigma \left(a' \frac{\partial F}{\partial b'} \right)$$

vanish, the six sums of the form:

$$\Sigma \frac{a'^2}{r'} \frac{\partial F}{\partial r'}, \quad \Sigma \frac{a' b'}{r'} \frac{\partial F}{\partial r'},$$

which are identical to them, also equal zero for unpolarized forces, and from the above we get:

$$\Sigma \left(a'^2 \frac{\partial^2 F}{\partial a'^2} \right) = \Sigma \frac{a'^4}{r'} \frac{\partial}{\partial r'} \frac{1}{r'} \frac{\partial F}{\partial r'}, \quad \Sigma \left(b' c' \frac{\partial^2 F}{\partial a'^2} \right) = \Sigma \frac{a'^2 b' c'}{r'} \frac{\partial}{\partial r'} \frac{1}{r'} \frac{\partial F}{\partial r'},$$

etc. The characteristic conditions:

$$D_{hk}^{mn} = D_{mn}^{hk} = D_{hm}^{kn} = D_{kn}^{hm}, \text{ etc.},$$

for unpolarized forces follow from this, or the rule that the four indices of any D_{hk}^{mn} can be permuted arbitrarily.

A consideration of them lets one recognize that the molecular rotations vanish completely for the general system (29), and the system of coefficients for triclinic crystals is given by the matrix:

$$\begin{array}{c|cccccc} & x_x & y_y & z_z & y_z & z_x & x_y \\ \hline -X_x & D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ -Y_y & D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ -Z_z & D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ -Y_z & D_{14} & D_{24} & D_{34} & D_{23} & D_{36} & D_{25} \\ -Z_x & D_{15} & D_{25} & D_{35} & D_{36} & D_{31} & D_{14} \\ -X_y & D_{16} & D_{26} & D_{36} & D_{25} & D_{14} & D_{12} \end{array}, \quad (57)$$

which contains 15 different constants. It differs from the system (39), not only by the absence of terms in L, M, N , but by the validity of the six conditions:

$$D_{44} = D_{23}, \quad D_{55} = D_{31}, \quad D_{66} = D_{12}, \quad D_{56} = D_{14}, \quad D_{64} = D_{25}, \quad D_{45} = D_{36}. \quad (57')$$

One can also conclude that in crystals for which observations show that these conditions are fulfilled the molecular interactions are found to be close to uniform in all directions, so the molecule possesses only very slight polarity.

If observations do not agree with these relations then one can, if need be, infer a conclusion about the behavior of the elementary interaction from the sense of the deviation.

For the sake of example, let the elementary potential F be constant around the Z -axis, so it is a function of only $e'^2 = a'^2 + b'^2$, so one can arbitrarily permute the indices 1 and 2 in those D_{hk}^{mn} that contain only those two, and thus set $D_{11}^{22} = D_{22}^{11} = D_{12}^{12}$, $D_{11}^{12} = D_{12}^{11}$, $D_{22}^{12} = D_{12}^{22}$, and likewise when one or the other upper index is 3, one then sets $D_{11}^{32} = D_{12}^{31}$, $D_{22}^{31} = D_{12}^{32}$, $D_{32}^{11} = D_{31}^{12}$, $D_{31}^{22} = D_{32}^{12}$, $D_{31}^{32} = D_{32}^{31}$; however, the remaining relations are not valid.

From (51), for a hexagonal crystal with the Z -axis as its principal axis this would yield $3D_{12}^{12} = D_{11}^{11}$, so D_{66} , which is the factor of x_y in the equation for $-X_y$ in (52), is equal to D_{12}^{12} .

6. Up to now, from the procedure of Navier, Poisson, and Cauchy, one has always, as far as I know, specialized the general elastic equations that are obtained by calculating the molecular interactions in isotropic media, in which one introduces the assumption that any medium possesses molecules that exhibit no polarity and are uniformly distributed in any direction. One then arrives at the relation between the two elasticity constants for

isotropic media that has provoked so many objections that it is in contradiction with experiment.

However, any hypothesis that is introduced is basically quite arbitrary if a substance that possesses polarized molecules in a crystallized modification can scarcely be ascribed unpolarized molecules in an amorphous modification, and if, as we know, the purely mechanical effects are too coarse to act upon the individual molecules, so it is therefore entirely plausible to assume that the formation of amorphous bodies also produces smaller or larger crystal fragments, and they would originate in the mechanical forces between them. The fact that we know any number of examples for any sort of rock – which can be resolved into aggregates of crystal fragments, whether to the naked eye or under great magnification – confirms this intuition.

However, its theoretical consequences with regard to elastic behavior produce entirely different results from the assumption of unpolarized molecules.

A very small surface element in a homogeneous substance is attached to crystal fragments with all possible orientations, and the resulting values of the stress components will be obtained, accordingly, as the *mean values* of the ones that are yielded by a regularly crystallized substance for all possible orientations of the surface elements in the crystal.

We now calculate these mean values. For this, one best employs the potential of the elastic force, which is defined by:

$$-X_x = \frac{\partial F}{\partial x_x}, \quad -X_y = -Y_x = \frac{\partial F}{\partial x_x}, \text{ etc.},$$

and in the most general case of a triclinic crystal it is a function of degree two in the x_x , ... with 21 constants.

To abbreviate, we set:

$$x_x = x_1, \quad y_y = x_2, \quad z_z = x_3, \quad y_z = x_4, \quad z_x = x_5, \quad x_y = x_6,$$

so one can briefly write:

$$2F = \sum_m \sum_n D_{mn} x_m x_n, \quad (58)$$

where the sums are taken from 1 to 6.

Now, let a second coordinate system Ξ, H, Z be given by its attitude with respect to X, Y, Z :

$$\begin{aligned} x &= \xi\alpha_1 + \eta\beta_1 + \zeta\gamma_1, & \xi &= x\alpha_1 + y\alpha_2 + z\alpha_3, \\ y &= \xi\alpha_2 + \eta\beta_2 + \zeta\gamma_2, & \eta &= x\beta_1 + y\beta_2 + z\beta_3, \\ z &= \xi\alpha_3 + \eta\beta_3 + \zeta\gamma_3, & \zeta &= x\gamma_1 + y\gamma_2 + z\gamma_3, \end{aligned} \quad (59)$$

and the deformation quantities that relate to this are similarly abbreviated:

$$\xi_\xi = \xi_1, \quad \eta_\eta = \xi_2, \quad \zeta_\zeta = \xi_3, \quad \eta_\zeta = \xi_4, \quad \zeta_\xi = \xi_5, \quad \xi_\eta = \xi_6,$$

so one must write:

$$2F = \sum_{\mu} \sum_{\nu} \Delta_{\mu\nu} \xi_{\mu} \xi_{\nu} \quad (60)$$

in terms of them, where the sums are likewise extended from 1 to 6. The $\Delta_{\mu\nu}$ in them are the “derived elasticity constants” of the substance for the system Ξ , H, Z.

The latter form for the potential must follow from the former by means of equations (59). On the same grounds, we next conclude the relations:

$$\begin{aligned} x_1 &= \alpha_1^2 \xi_1 + \beta_1^2 \xi_2 + \gamma_1^2 \xi_3 + \beta_1 \gamma_1 \xi_4 + \gamma_1 \alpha_1 \xi_5 + \alpha_1 \beta_1 \xi_6 = \sum_{\nu} c_{1\nu} \xi_{\nu}, \\ x_2 &= \alpha_2^2 \xi_1 + \beta_2^2 \xi_2 + \gamma_2^2 \xi_3 + \beta_2 \gamma_2 \xi_4 + \gamma_2 \alpha_2 \xi_5 + \alpha_2 \beta_2 \xi_6 = \sum_{\nu} c_{2\nu} \xi_{\nu}, \\ x_3 &= \alpha_3^2 \xi_1 + \beta_3^2 \xi_2 + \gamma_3^2 \xi_3 + \beta_3 \gamma_3 \xi_4 + \gamma_3 \alpha_3 \xi_5 + \alpha_3 \beta_3 \xi_6 = \sum_{\nu} c_{3\nu} \xi_{\nu}, \\ x_4 &= 2\alpha_2 \alpha_3 \xi_1 + 2\beta_2 \beta_3 \xi_2 + 2\gamma_2 \gamma_3 \xi_3 + (\beta_2 \gamma_3 + \gamma_2 \beta_3) \xi_4 + (\gamma_2 \alpha_3 + \alpha_2 \gamma_3) \xi_5 + (\alpha_2 \beta_3 + \beta_2 \alpha_3) \xi_6 = \sum_{\nu} c_{4\nu} \xi_{\nu}, \\ x_5 &= 2\alpha_3 \alpha_1 \xi_1 + 2\beta_3 \beta_1 \xi_2 + 2\gamma_3 \gamma_1 \xi_3 + (\beta_3 \gamma_1 + \gamma_3 \beta_1) \xi_4 + (\gamma_3 \alpha_1 + \alpha_3 \gamma_1) \xi_5 + (\alpha_3 \beta_1 + \beta_3 \alpha_1) \xi_6 = \sum_{\nu} c_{5\nu} \xi_{\nu}, \\ x_6 &= 2\alpha_1 \alpha_2 \xi_1 + 2\beta_1 \beta_2 \xi_2 + 2\gamma_1 \gamma_2 \xi_3 + (\beta_1 \gamma_2 + \gamma_1 \beta_2) \xi_4 + (\gamma_1 \alpha_2 + \alpha_1 \gamma_2) \xi_5 + (\alpha_1 \beta_2 + \beta_1 \alpha_2) \xi_6 = \sum_{\nu} c_{6\nu} \xi_{\nu}, \end{aligned} \quad (61)$$

which can be briefly summarized in:

$$x_n = \sum_{\nu} c_{n\nu} \xi_{\nu}.$$

The introduction of this into (58) gives:

$$\begin{aligned} 2F &= \sum_m \sum_n D_{mn} \sum_{\nu} c_{n\nu} \xi_{\nu} \sum_{\mu} c_{m\mu} \xi_{\mu} \\ &= \sum_{\mu} \sum_{\nu} \xi_{\nu} \xi_{\mu} \sum_{\nu} \sum_{\mu} D_{mn} c_{m\mu} c_{n\nu}, \end{aligned}$$

from which, by comparing with (60), it follows that:

$$\Delta_{\mu\nu} = \sum_{\nu} \sum_{\mu} D_{mn} c_{m\mu} c_{n\nu}.$$

The $\Delta_{\mu\nu}$ depend upon the position of the coordinate system. If we take their mean values ($\Delta_{\mu\nu}$) over all possible positions then, from the above, we obtain those coefficients that appear with either the products $x_{\mu} x_{\nu}$ or $\xi_{\mu} \xi_{\nu}$ in the form of the potential, which corresponds to an isotropic substance in the picture that we developed, in the event that it can be regarded as an aggregate of very many different oriented crystal fragments.

Since we know from the symmetry behavior that only the coefficients:

$$(\Delta_{11}) = (\Delta_{22}) = (\Delta_{33}) = A, \quad (\Delta_{23}) = (\Delta_{31}) = (\Delta_{12}) = B, \quad (62)$$

$$= (\Delta_{44}) = (\Delta_{55}) = (\Delta_{66}) = C = \frac{A-B}{2}$$

can be non-zero, the problem then reduces to the calculation of only three (two, resp.) of them.

One next has:

$$\begin{aligned} \Delta_{11} = & D_{11} \alpha_1^4 + 2D_{12} \alpha_1^2 \alpha_2^2 + 2D_{13} \alpha_1^2 \alpha_3^2 + 4D_{14} \alpha_1^2 \alpha_2 \alpha_3 + 4D_{15} \alpha_1^2 \alpha_3 \alpha_1 + 4D_{16} \alpha_1^2 \alpha_1 \alpha_2 \\ & + 2D_{22} \alpha_2^4 + 2D_{23} \alpha_1^2 \alpha_3^2 + 4D_{24} \alpha_2^2 \alpha_2 \alpha_3 + 4D_{25} \alpha_2^2 \alpha_3 \alpha_1 + 4D_{26} \alpha_2^2 \alpha_1 \alpha_2 \\ & + D_{33} \alpha_3^4 + 4D_{34} \alpha_3^2 \alpha_2 \alpha_3 + 4D_{35} \alpha_3^2 \alpha_3 \alpha_1 + 4D_{36} \alpha_3^2 \alpha_1 \alpha_2 \\ & + 4D_{44} \alpha_2^2 \alpha_3^2 + 8D_{45} \alpha_2 \alpha_3 \alpha_3 \alpha_1 + 8D_{46} \alpha_2 \alpha_3 \alpha_1 \alpha_2 \\ & + 4D_{55} \alpha_2^2 \alpha_3^2 + 8D_{56} \alpha_3 \alpha_1 \alpha_1 \alpha_2 \\ & + 4D_{66} \alpha_1^2 \alpha_2^2. \end{aligned}$$

It is clear that when one takes the mean value of this all terms must vanish that contain an odd power of one of the direction cosines $\alpha_1, \alpha_2, \alpha_3$. We do not write out these terms, and one has, more briefly:

$$(63') \quad \Delta_{11} = (D_{11} \alpha_1^4 + D_{22} \alpha_2^4 + D_{33} \alpha_3^4) + 4(D_{44} \alpha_2^2 \alpha_3^2 + D_{55} \alpha_3^2 \alpha_1^2 + D_{66} \alpha_1^2 \alpha_2^2) \\ + 2(D_{23} \alpha_2^2 \alpha_3^2 + D_{31} \alpha_3^2 \alpha_1^2 + D_{12} \alpha_1^2 \alpha_2^2) + \dots$$

If one likewise omits the terms in the other $\Delta_{\mu\nu}$ that drop out from symmetry considerations then one obtains:

$$\begin{aligned} \Delta_{12} = & (D_{11} \alpha_1^2 \beta_1^2 + D_{22} \alpha_2^2 \beta_2^2 + D_{33} \alpha_3^2 \beta_3^2) + 4(D_{44} \alpha_2 \alpha_3 \beta_2 \beta_3 + D_{55} \alpha_3 \alpha_1 \beta_3 \beta_1 + D_{66} \alpha_1 \alpha_2 \beta_1 \beta_2) \\ & + [D_{23} (\alpha_2^2 \beta_3^2 + \alpha_3^2 \beta_2^2) + D_{31} (\alpha_3^2 \beta_1^2 + \alpha_1^2 \beta_3^2) + D_{12} (\alpha_1^2 \beta_2^2 + \alpha_2^2 \beta_1^2)] + \dots \end{aligned} \quad (63'')$$

$$\begin{aligned} \Delta_{44} = & (D_{11} \beta_1^2 \gamma_1^2 + D_{22} \beta_2^2 \gamma_2^2 + D_{33} \beta_3^2 \gamma_3^2) + [D_{44} (\beta_2 \gamma_3 + \beta_3 \gamma_2)^2 + D_{55} (\beta_3 \gamma_1 + \beta_1 \gamma_3)^2 + D_{66} (\beta_1 \gamma_2 + \beta_2 \gamma_1)^2] \\ & + 2(D_{23} \beta_2 \gamma_3 \beta_3 \gamma_3 + D_{31} \beta_3 \gamma_3 \beta_1 \gamma_1 + D_{12} \beta_1 \gamma_1 \beta_2 \gamma_2) + \dots \end{aligned}$$

The determination of the mean value of these $\Delta_{\mu\nu}$ comes down to the calculation of the five mean values that are denoted by brackets:

$$(\alpha_h^4), \quad (\alpha_h^2 \alpha_k^2), \quad (\alpha_h^2 \beta_h^2), \quad (\alpha_h^2 \beta_k^2), \quad (\alpha_h \beta_h \alpha_k \beta_k),$$

so, from symmetry, all of the terms that are present coincide with one of these five when one takes the arithmetic mean.

For the calculation of these quantities, we would especially like to consider the cases:

$$(\alpha_3^4), \quad (\alpha_1^2 \alpha_2^2), \quad (\alpha_1^2 \beta_1^2), \quad (\alpha_1^2 \beta_2^2), \quad (\alpha_1 \beta_1 \alpha_2 \beta_2).$$

If the angle between the Ξ direction and the Z -axis is called φ and the angle between the XZ -plane and ΞZ is called χ then we can set:

$$\alpha_3 = \cos \varphi, \quad \alpha_1 = \sin \varphi \cos \chi, \quad \alpha_2 = \sin \varphi \sin \chi.$$

If one further sets the angle between the planes ΞX and ΞY equal to ψ and the one between ΞX and ΞH equal to ω then one has:

$$\beta_1 = \cos \omega \sqrt{1 - \alpha_1^2}, \quad \beta_2 = \cos(\psi - \omega) \sqrt{1 - \alpha_2^2}.$$

The first two mean values are to be taken over the direction cosines of *only one* of the axes Ξ , H , Z , so they are obtained by simple integration over a spherical surface and division by 4π .

$$\begin{aligned} (\alpha_3^4) &= \frac{1}{4\pi} \int_0^{2\pi} d\chi \int_0^\pi d\varphi \cos^4 \varphi \sin \varphi = \frac{1}{5} \\ (\alpha_1^2 \alpha_2^2) &= \frac{1}{4\pi} \int_0^{2\pi} d\chi \cos^2 \chi \sin^2 \chi \int_0^\pi \sin^5 \varphi d\varphi = \frac{1}{15}. \end{aligned} \tag{64'}$$

The latter refers to the direction cosine of two axes. In order to obtain all possible positions for them, we next rotate the H -axis around the Ξ -axis by integrating over ω and then move the Ξ -axis while we integrate over a spherical surface. Here, the denominator is then $8\pi^2$. One then obtains:

$$\begin{aligned} (\alpha_1^2 \beta_1^2) &= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi d\varphi d\chi \sin^3 \varphi \cos^2 \chi (1 - \sin^2 \varphi \cos^2 \chi) \int_0^{2\pi} d\omega \cos^2 \omega = \frac{1}{15} \\ (\alpha_1^2 \beta_2^2) &= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi d\varphi d\chi \sin^3 \varphi \cos^2 \chi (1 - \sin^2 \varphi \sin^2 \chi) \int_0^{2\pi} d\omega \cos^2(\psi - \omega) = \frac{2}{15} \\ (\alpha_1 \beta_1 \alpha_2 \beta_2) &= \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi d\varphi d\chi \sin^3 \varphi \cos \chi \sin \chi \sqrt{1 - \sin^2 \varphi \cos^2 \chi} \sqrt{1 - \sin^2 \varphi \sin^2 \chi} \int_0^{2\pi} d\omega \cos \omega \cos(\psi - \omega) \\ &= -\frac{1}{30}. \end{aligned} \tag{64''}$$

If one briefly sets:

$$D_{11} + D_{22} + D_{33} = 3A, \quad D_{23} + D_{31} + D_{12} = 3B, \quad D_{44} + D_{55} + D_{66} = 3\Gamma$$

then, from (63), this yields:

$$\begin{aligned}
 (65) \quad A &= (\Delta_{11}) = \frac{1}{5}(3A + 2B + 4\Gamma), \\
 B &= (\Delta_{12}) = \frac{1}{5}(A + 4B - 2\Gamma), \\
 C &= (\Delta_{44}) = \frac{1}{5}(A - B + 3\Gamma).
 \end{aligned}$$

From this, it follows that:

$$C = \frac{A - B}{2},$$

in agreement with the results of all other theories. However, the objectionable relation:

$$A = 3B$$

is true for only the special value $B = \Gamma$, which, as we have seen, must indeed necessarily appear when one assumes molecules with no polarity, which however, are not produced by the general theory.

By means of the assumption of polarized molecules, which seems to be required on many other grounds, the contradiction between molecular elasticity theory and experiment will then be lifted completely.
