"Sull'elettrodinamica di Minkowski," Rend. Circ. mat. Palermo (1) **30** (1910), 33-46.

# On Minkowski's electrodynamics

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## § 1.

## Introduction.

In a preceding paper ([1]), I developed a system of the electrodynamics of moving bodies that embraced the modern theories of **E. COHN**, **H. A. LORENTZ**, and **H. MINKOWSKI**, while conforming to the general principles of the theory of **MAXWELL** and **HERTZ**. For the special case of **MINKOWSKI**'s theory, an expression for the ponderomotive force resulted that differed from the one that was given by **MINKOWSKI**; I asserted that the expression satisfied the principle of relativity.

In the present article, I will prove that statement. In § 2, I will present some theorems that relate to four-dimensional vectors that are already essentially contained in **MINKOWSKI**'s paper ([2]) and will be applied in what follows. I consider it to be useful to give vector analysis a four-dimensional form that, when adapted to three-dimensional analysis, will permit one to immediately descend from the four-dimensional manifold of space and time to three-dimensional space.

In § 3, some quantities will be introduced that are called *four-dimensional tensors*. They are generalizations of the three-dimensional tensors ([3]), which characterize, for example, the state of tension in an elastic body. The ten components of the four-dimensional tensor that must be considered in electrodynamics contains six components of electromagnetic pressures, three components of the energy current, and the electromagnetic energy density. We will define a four-dimensional tensor whose components are identical with the values of the pressures, energy current, and electromagnetic energy density that were deduced from general principles of the electrodynamics of moving bodies in my cited paper.

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([1]) **M. ABRAHAM**, "Zur Elektrodynamik bewegter Körper," Rend. Circ. mat. Palermo **28** (2nd sem., 1909), 1-28.

([2]) **H. MINKOWSKI**, "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern," Nach. Kgl. Ges. Wiss. Göttingen (1908), 53-111.

([3]) **M. ABRAHAM**, "Geometrische Grundbegriffe," *Enzyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* IV, 2, pp. 3-47.

It will result that those principles are compatible with the postulate of relativity. The symmetry of the equations of the electromagnetic field for empty space, which is expressed by the **LORENTZ** transformation, can also be given to the electromagnetic equations for ponderable bodies – either in their **MINKOWSKI** form or in their equivalent **LORENTZ** form – without contradicting those principles.

**MINKOWSKI** already gave a form to the equations of motion for a material point that was invariant under **LORENTZ** transformations. Therefore, he believed that it was necessary to add an additional force to the ponderomotive electromagnetic force that was incompatible ([4]) with my system of electrodynamics. In § 4, I will write down equations of motion that satisfy the principle of relativity without introducing **MINKOWSKI**'s additional force. On the contrary, we will need to assume that the "rest density" of the mass is not constant, but increases whenever the electric current produces **JOULE** heat in the matter. That hypothesis was made already by **A. EINSTEIN** and **M. PLANCK** in the context of the principle of relativity.

Therefore, I find it dubious that the very concept of space and time that was developed by **MINKOWSKI** ([5]) might possibly have a basis in rational mechanics. Rather, the kinematics of rigid bodies that **M. BORN** ([6]) wanted to adapt to the **LORENTZ** group offers considerable difficulties, as **G. HERGLOTZ** ([7]) proved. The rigid body of **MINKOWSKI**'s "world" cannot be put into rotation.

## § 2.

## Four-dimensional vectors.

According to **MINKOWSKI**, a linear transformation of the four coordinates $x$, $y$, $z$, $u$ that leaves invariant:

$$x^2 + y^2 + z^2 + u^2$$

is called a **LORENTZ** *transformation*. In what follows, I shall confine myself to the group of orthogonal transformations – i.e., the rotations of four-dimensional space.

A system of four quantities that transform like the coordinates $x$, $y$, $z$, $u$ are called a *four-dimensional vector of the first kind* $(V_I^4)$. If one projects it onto the three-dimensional space of the $x$, $y$, $z$ then the first three components of $V_I^4$ will constitute a three-dimensional vector $(V^3)$, $\mathfrak{r}$, and the fourth one ($u$) will constitute a three-dimensional scalar ($S^1$).

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([4]) *See* the discussion in **G. NORDSTRÖM** and **M. ABRAHAM**, Phys. Zeit. **10** (1909), 681-687, 737-741.

([5]) **H. MINKOWSKI**, *Raum und Zeit*, Leipzig, Teubner, 1909.

([6]) **M. BORN**, "Die Theorie des starren Elektrons in der Kinematik des Relativitätsprincips," Ann. Phys. (Leipzig) **30** (1909), 1-56.

([7]) **G. HERGLOTZ**, "Über den vom Standpunkt des Relativitätsprincips aus als "starr" zu bezeichnen den Körper," Ann. Phys. (Leipzig) **31** (1910), 393-415.

*A four-dimensional vector of the second kind* $(V_{II}^4)$ is a system of six quantities that transform like the following expressions that are constructed from the components $x_1$, $y_1$, $z_1$, $u_1$ and $x_2$, $y_2$, $z_2$, $u_2$ of two $V_I^4$:

$$(1)\qquad \begin{cases} \mathfrak{a}_x = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad \mathfrak{a}_y = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad \mathfrak{a}_z = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \\ \mathfrak{b}_x = \begin{vmatrix} x_1 & u_1 \\ x_2 & u_2 \end{vmatrix}, \quad \mathfrak{b}_y = \begin{vmatrix} y_1 & u_1 \\ y_2 & u_2 \end{vmatrix}, \quad \mathfrak{b}_z = \begin{vmatrix} z_1 & u_1 \\ z_2 & u_2 \end{vmatrix}. \end{cases}$$

Obviously, if one projects onto three-dimensional space then the resulting $V_{II}^4$ will be composed of two $V^3$'s that are written:

$$(1_a)\qquad \mathfrak{a} = [\mathfrak{r}_1\,\mathfrak{r}_2], \qquad \mathfrak{b} = \mathfrak{r}_1\,u_2 - \mathfrak{r}_2\,u_1$$

in the symbolism of ordinary vector analysis.

If one is given two $V_I^4$:

$$\mathfrak{r},\ u \qquad \text{and} \qquad \mathfrak{r}_1,\ u_1$$

then one can compose a *four-dimensional scalar* $(S^4)$ in the following way:

$$(2)\qquad S = xx_1 + yy_1 + zz_1 + uu_1 = \mathfrak{r}\mathfrak{r}_1 + uu_1\,.$$

Conversely, if one is given an arbitrary four-dimensional scalar $\varphi\,(x, y, z, u)$ then one will obtain a $V_I^4$ by differentiating with respect to the coordinates:

$$(3)\qquad X = \frac{\partial\varphi}{\partial x}, \qquad Y = \frac{\partial\varphi}{\partial y}, \qquad Z = \frac{\partial\varphi}{\partial z}, \qquad U = \frac{\partial\varphi}{\partial u}.$$

Therefore, the operators:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial u}$$

transform like the components of a $V_I^4$. **MINKOWSKI** called those operators the components of the "lor" operator.

One can compose an $S^4$ from four $V_I^4$ that determines the space of the parallelepiped of the four vectors:

(4)
$$\varphi = \begin{vmatrix} x & y & z & u \\ x_1 & y_1 & z_1 & u_1 \\ x_2 & y_2 & z_2 & u_2 \\ x_3 & y_3 & z_3 & u_3 \end{vmatrix}.$$

If one applies the operator (3) to this $S^4$ then one will obtain a $V_I^4$ that is composed of three other $V_I^4$ – namely, $\mathfrak{r}_1\, u_1$, $\mathfrak{r}_2\, u_2$, $\mathfrak{r}_3\, u_3$ – whose components will be:

(5)
$$\begin{cases} X = \dfrac{\partial\varphi}{\partial x} = \begin{vmatrix} y_1 & z_1 & u_1 \\ y_2 & z_2 & u_2 \\ y_3 & z_3 & u_3 \end{vmatrix}, \quad Y = \dfrac{\partial\varphi}{\partial y} = \begin{vmatrix} z_1 & x_1 & u_1 \\ z_2 & x_2 & u_2 \\ z_3 & x_3 & u_3 \end{vmatrix}, \\ Z = \dfrac{\partial\varphi}{\partial z} = \begin{vmatrix} x_1 & y_1 & u_1 \\ x_2 & y_2 & u_2 \\ x_3 & y_3 & u_3 \end{vmatrix}, \quad U = \dfrac{\partial\varphi}{\partial u} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \end{cases}$$

When this is written in a vectorial way, one will have:

($5_a$)
$$\begin{cases} \mathfrak{R} = u_1[\mathfrak{r}_2\,\mathfrak{r}_3] + u_2[\mathfrak{r}_3\,\mathfrak{r}_1] + u_3[\mathfrak{r}_1\,\mathfrak{r}_2], \\ U = -\mathfrak{r}_1[\mathfrak{r}_2\,\mathfrak{r}_3]. \end{cases}$$

If one cancels the index 3 then the $V_I^4$ that one obtains can be written:

$$\mathfrak{R} = u\,[\mathfrak{r}_1\,\mathfrak{r}_2] + [\mathfrak{r},\, \mathfrak{r}_1\, u_2 - \mathfrak{r}_2\, u_1],$$
$$U = -\,\mathfrak{r}\,[\mathfrak{r}_1\,\mathfrak{r}_2].$$

If one introduces, instead of the $V_I^4$'s $\{\mathfrak{r}_1, u_1\}$ and $\{\mathfrak{r}_2, u_2\}$, the $V_{II}^4$ $\{\mathfrak{a}, \mathfrak{b}\}$ that is composed of them according to the rule ($1_a$) then what will result is:

(6)
$$\begin{cases} \mathfrak{R} = u\,\mathfrak{a} + [\mathfrak{r}\mathfrak{b}], \\ U = -\,\mathfrak{r}\,\mathfrak{a}; \end{cases}$$

i.e., a $V_I^4$ that is composed of a $V_I^4$ and a $V_{II}^4$.

One can obtain another $V_I^4$ by permuting the $V^3$'s $\mathfrak{a}$ and $\mathfrak{b}$ in the expressions (6). In order to prove that, form the two $S^4$'s:

$$\mathfrak{r}\mathfrak{r}_1 + uu_1 \qquad \text{and} \qquad \mathfrak{r}\mathfrak{r}_2 + uu_2\,.$$

If one multiplies them by:

$$-\,\mathfrak{r}_2\,, -\,u_2\,, \qquad \text{and} \qquad -\,\mathfrak{r}_1\,, -\,u_1\,,$$

respectively, and sums then one will define the $V_I^4$:

$$\mathfrak{R}' = \mathfrak{r}_1\,(\mathfrak{r}\,\mathfrak{r}_2 + uu_2) - \mathfrak{r}_2\,(\mathfrak{r}\,\mathfrak{r}_1 + uu_1),$$
$$U' = u_1\,(\mathfrak{r}\,\mathfrak{r}_2 + uu_2) - u_2\,(\mathfrak{r}\,\mathfrak{r}_1 + uu_1),$$

which can be written:

$$\mathfrak{R}' = u\,(\mathfrak{r}_1\,u_2 - \mathfrak{r}_2\,u_1) + [\mathfrak{r}\,[\mathfrak{r}_1\,\mathfrak{r}_2],$$
$$U' = -\,(\mathfrak{r},\,\mathfrak{r}_1\,u_2 - \mathfrak{r}_2\,u_1).$$

If one introduces the $V_{II}^4$ $\{\mathfrak{a},\,\mathfrak{b}\}$, by means of ($1_a$) then the following formula will result, which is analogous to (6):

$$(6_a)\qquad \left\{\begin{array}{l} \mathfrak{R}' = u\mathfrak{b} + [\mathfrak{r}\mathfrak{a}], \\ U' = -\mathfrak{r}\mathfrak{b}, \end{array}\right.$$

in which $\mathfrak{a}$ and $\mathfrak{b}$ have changed places.

In **MINKOWSKI**'s electrodynamics, four $V^3$ intervene – i. e., the electric and magnetic excitations $\mathfrak{D}$ and $\mathfrak{B}$, and two auxiliary vectors $\mathfrak{E}$ and $\mathfrak{H}$ – that form two $V_{II}^4$:

$$\mathfrak{B},\ -i\,\mathfrak{E} \qquad \text{and} \qquad \mathfrak{H},\ -i\,\mathfrak{D}.$$

One will then have the *velocity* $V_I^4$:

$$\frac{\mathfrak{q}}{\sqrt{1-\mathfrak{q}^2}}, \qquad \frac{i}{\sqrt{1-\mathfrak{q}^2}}.$$

($\mathfrak{q}$ denotes the three-dimensional velocity vector, when referred to the velocity of light.)

If one combines this $V_I^4$ with the $V_{II}^4$ $\{\mathfrak{B},\,-i\mathfrak{E}\}$ according to the schema ($6_a$) then one will obtain the $V_I^4$:

$$(7)\qquad \mathfrak{R}^e = \frac{\mathfrak{E} + [\mathfrak{q}\mathfrak{B}]}{\sqrt{1-\mathfrak{q}^2}}, \qquad U^e = \frac{i\,(\mathfrak{q}\mathfrak{E})}{\sqrt{1-\mathfrak{q}^2}},$$

which **MINKOWSKI** called the *rest electric force.* However, if one combines the $V_I^4$ of *velocity* and the $V_{II}^4$ $\{-\,i\,\mathfrak{H},\,-\,\mathfrak{D}\}$ according to the schema (6) then one will obtain the $V_I^4$ of the *rest magnetic force*:

$$(8)\qquad \mathfrak{R}^m = \frac{\mathfrak{H} - [\mathfrak{q}\mathfrak{D}]}{\sqrt{1-\mathfrak{q}^2}}, \qquad U^m = \frac{i\,(\mathfrak{q}\mathfrak{H})}{\sqrt{1-\mathfrak{q}^2}}.$$

The "rest electric and magnetic forces" are connected by $V^3$'s that determine the ponderomotive force on moving electric and magnetic poles and were written as $\mathfrak{E}'$ and $\mathfrak{H}'$ in the first paper:

(9) $$\mathfrak{E}' = \mathfrak{E} + [\mathfrak{q}\ \mathfrak{B}], \qquad \mathfrak{H}' = \mathfrak{H} - [\mathfrak{q}\ \mathfrak{D}].$$

Obviously, one will have:

($9_a$) $$\mathfrak{R}^e = k^{-1}\ \mathfrak{E}', \qquad U^e = ik^{-1}\ (\mathfrak{q}\ \mathfrak{E}'),$$

($9_b$) $$\mathfrak{R}^m = k^{-1}\ \mathfrak{H}', \qquad U^m = ik^{-1}\ (\mathfrak{q}\ \mathfrak{H}'),$$

if one takes:

($9_c$) $$k = \sqrt{1 - \mathfrak{q}^2}\ .$$

If one composes the two $V_I^4$:

$$\{\mathfrak{R}^e, U^e\} \qquad \text{and} \qquad \{\mathfrak{R}^m, U^m\}$$

according to the formula ($1_a$) then one will obtain the $V_{II}^4$:

$$\mathfrak{a} = k^{-2}\ [\mathfrak{E}'\ \mathfrak{H}'],$$
$$\mathfrak{b} = i\ k^{-2}\ \{\mathfrak{E}'\ (\mathfrak{q}\ \mathfrak{H}') - \mathfrak{H}'\ (\mathfrak{q}\ \mathfrak{E}')\} = [\mathfrak{q}\ [\mathfrak{E}'\ \mathfrak{H}']].$$

If one introduces the $V^3$:

(10) $$\mathfrak{f}' = [\mathfrak{E}'\ \mathfrak{H}']$$

then the last $V_{II}^4$ can be written:

(11) $$\mathfrak{a} = k^{-2}\ \mathfrak{f}', \qquad \mathfrak{b} = i\ k^{-2}\ [\mathfrak{q}\ \mathfrak{f}'].$$

If one multiplies the $V^3$ $\mathfrak{f}'$ by the velocity of light ($c$) then one will obtain the *relative ray* vector in my first paper [*loc. cit.* ([1]), equation IV].

Finally, one combines the $V_{II}^4$ that is represented by (11) with the $V_I^4$ of "velocity" according to the schema (6). The $V_{II}^4$ that is thus calculated is will be:

$$\mathfrak{R} = i\ k^{-3}\{\mathfrak{f}' + [\mathfrak{q}\ [\mathfrak{q}\ \mathfrak{f}']]\},$$
$$U = -\ k^{-3}\ (\mathfrak{q}\ \mathfrak{f}').$$

Multiply this by $(-\ i)$ and add **MINKOWSKI**'s $V_I^4$ of the *rest ray* to it:

(12) $$\begin{cases} \mathfrak{R} = k^{-2}\mathfrak{f}' + k^{-3}\mathfrak{q}(\mathfrak{q}\mathfrak{f}'), \\ U = ik^{-3}(\mathfrak{q}\mathfrak{f}'). \end{cases}$$

## § 3.

## Four-dimensional tensors.

A *four-dimensional tensor* ($T^4$) is a system of ten quantities that transform by **LORENTZ** orthogonal transformations as if they were the products and squares of the coordinates $x, y, z, u$:

$$x^2, y^2, z^2\,; \qquad yz, zx, xy\,; \qquad xu, yu, zu\,; \qquad u^2.$$

If one projects this onto the three-dimensional space of the $(x, y, z)$ then the first six components will form a three-dimensional tensor ($T^3$) that transform like the squares and products of the $(x, y, z)$. The following three components of the $T^4$ constitute a $V^3$, while the tenth one constitutes a scalar $S^1$.

The four components of the "lor" operator transform like the components of a $V_I^4$, so when one is given a four-dimensional scalar $\varphi$ that is a function of the $x, y, z, u$, one can deduce a $T^4$ by differentiating it twice with respect to $x, y, z, u$:

$$\frac{\partial^2\varphi}{\partial x^2}, \frac{\partial^2\varphi}{\partial y^2}, \frac{\partial^2\varphi}{\partial z^2}; \frac{\partial^2\varphi}{\partial y\,\partial z}, \frac{\partial^2\varphi}{\partial z\,\partial x}, \frac{\partial^2\varphi}{\partial x\,\partial y}; \frac{\partial^2\varphi}{\partial x\,\partial u}, \frac{\partial^2\varphi}{\partial y\,\partial u}, \frac{\partial^2\varphi}{\partial z\,\partial u}; \frac{\partial^2\varphi}{\partial u^2}.$$

If one is given an $S^4$ that is a homogeneous quadratic function of the $x, y, z, u$:

$$(13) \qquad \left\{ \begin{aligned} \varphi(x, y, z, u) = {} & \tfrac{1}{2}c_{11}x^2 + \tfrac{1}{2}c_{22}y^2 + \tfrac{1}{2}c_{33}z^2 \\ & + c_{23}yz + c_{31}zx + c_{12}xy \\ & + c_{14}xu + c_{24}yu + c_{34}zu + \tfrac{1}{2}c_{44}u^2 \end{aligned} \right.$$

then the ten coefficients:

$$c_{11}, c_{22}, c_{33}, c_{23}, c_{31}, c_{12}\,; \qquad c_{14}, c_{24}, c_{34}\,; \qquad c_{44}$$

will constitute a four-dimensional tensor.

In the electrodynamics of bodies in motion, one has the following equations of impulse and energy ([8]):

([8]) **M. ABRAHAM**, *loc. cit.* ([1]), equations (6) and (7).

(14)
$$\begin{cases} \mathfrak{K}_x = \dfrac{\partial X_x}{\partial x} + \dfrac{\partial X_y}{\partial y} + \dfrac{\partial X_z}{\partial z} - \dfrac{\partial \mathfrak{g}_x}{\partial t}, \\[2ex] \mathfrak{K}_y = \dfrac{\partial Y_x}{\partial x} + \dfrac{\partial Y_y}{\partial y} + \dfrac{\partial Y_z}{\partial z} - \dfrac{\partial \mathfrak{g}_y}{\partial t}, \\[2ex] \mathfrak{K}_z = \dfrac{\partial Z_x}{\partial x} + \dfrac{\partial Z_y}{\partial y} + \dfrac{\partial Z_z}{\partial z} - \dfrac{\partial \mathfrak{g}_z}{\partial t}, \\[2ex] c\,\mathfrak{q}\mathfrak{K} + Q = -\dfrac{\partial \mathfrak{S}_x}{\partial x} - \dfrac{\partial \mathfrak{S}_y}{\partial y} - \dfrac{\partial \mathfrak{S}_z}{\partial z} - \dfrac{\partial \psi}{\partial t}. \end{cases}$$

In order to give these four equations a more symmetric form, set:

(15) $$u = ict, \qquad \mathfrak{K}_u = i\,\mathfrak{q}\mathfrak{K} + i\,\frac{Q}{c}, \qquad U_u = \psi,$$

($15_a$) $$X_n = -\,ic\,\mathfrak{g}_x, \qquad Y_n = -\,ic\,\mathfrak{g}_y, \qquad Z_n = -\,ic\,\mathfrak{g}_z,$$

($15_b$) $$U_x = -\frac{i}{c}\,\mathfrak{S}_x, \qquad U_y = -\frac{i}{c}\,\mathfrak{S}_y, \qquad U_z = -\frac{i}{c}\,\mathfrak{S}_z.$$

That will then imply that:

(16)
$$\begin{cases} \mathfrak{K}_x = \dfrac{\partial X_x}{\partial x} + \dfrac{\partial X_y}{\partial y} + \dfrac{\partial X_z}{\partial z} + \dfrac{\partial X_u}{\partial u}, \\[2ex] \mathfrak{K}_y = \dfrac{\partial Y_x}{\partial x} + \dfrac{\partial Y_y}{\partial y} + \dfrac{\partial Y_z}{\partial z} + \dfrac{\partial Y_u}{\partial u}, \\[2ex] \mathfrak{K}_z = \dfrac{\partial Z_x}{\partial x} + \dfrac{\partial Z_y}{\partial y} + \dfrac{\partial Z_z}{\partial z} + \dfrac{\partial Z_u}{\partial u}, \\[2ex] \mathfrak{K}_u = \dfrac{\partial U_x}{\partial x} + \dfrac{\partial U_y}{\partial y} + \dfrac{\partial U_z}{\partial z} + \dfrac{\partial U_u}{\partial u}. \end{cases}$$

Now, in **MINKOWSKI**' theory, the system of four quantities:

$$\mathfrak{K}_x, \mathfrak{K}_y, \mathfrak{K}_z, \mathfrak{K}_u,$$

the first three of which are the components of a $V^3$ that determines the ponderomotive force per unit spatial volume, constitute a $V_I^4$. The sixteen quantities $X_x$, $X_y$, …, $U_z$, $U_u$, from which that system is derived by means of equations (16), must transform in such a way that the latter condition is satisfied.

We determine those sixteen quantities in the following way: They reduce to ten:

(17) $$\begin{cases} X_x, X_y, X_z & Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x, \\ X_u = U_x, & Y_u = U_y, \quad Z_u = U_z, \quad U_u, \end{cases}$$

*which are the components of a $T^4$.*

It will then indeed follow from the properties of the transformation of the components of a $T^4$ and the components of the "lor" operator that the four quantities that are derived from (16) transform like the coordinates *x*, *y*, *z*, *u* of a point in four-dimensional space; i.e., like the components of a $V_I^4$, conforming to the principle of relativity. Hence, the determination that I chose is not the only one that corresponds to that principle. **MINKOWSKI** himself preferred another determination that does not satisfy the conditions of symmetry that are contained in (17). However, the determination that I postulated in my system of electrodynamics of moving bodies is the one that is indicated at the moment.

Our goal is to form a $T^4$ whose components correspond to the expressions that were pointed out in my first paper for the special case of **MINKOWSKI**'s theory. Having done that, it would be obvious that these expressions would satisfy the principle of relativity.

In order to obtain such a $T^4$, one calculates an $S^4$ of the form (13); i.e., a homogeneous quadratic function of the *x*, *y*, *z*, *u* that is invariant under the **LORENTZ** transformations:

(18) $$\varphi(x, y, z, u) = \Phi(x, y, z) - i\,u\,(\mathfrak{r}\,\mathfrak{f}) + \tfrac{1}{2}u^2\psi.$$

One will obtain the six **MAXWELL** pressures from the $S^3$ that is a second-order homogeneous function of the *x*, *y*, *z*:

($18_a$) $$\Phi(x, y, z) = \tfrac{1}{2}X_x\,x^2 + \tfrac{1}{2}Y_y\,y^2 + \tfrac{1}{2}Z_z\,z^2 + Y_z\,yz + Z_x\,zx + X_y\,xy\,.$$

From ($15_{a,b}$), the $V^3$ that is presently denoted by $\mathfrak{f}$ gives both the energy current $\mathfrak{S}$ and the density of electromagnetic impulse $\mathfrak{g}$:

($18_b$) $$\mathfrak{f} = c\,\mathfrak{g} = \frac{1}{c}\mathfrak{S}.$$

Finally, the tenth component $\psi$ of the $T^4$ that is derived from (18) will determine the density of the electromagnetic energy.

In order to add to a suitable four-dimensional scalar that is a homogeneous function of second order in the coordinates *x*, *y*, *z*¸ *u* with coefficients that are bilinear in the components of the electromagnetic vectors, one first forms the radius vector $\{\mathfrak{r}, u\}$ in four-dimensional space according to the scheme (6), and from the $V_{II}^4\,\{\mathfrak{a}, \mathfrak{b}\}$, the $V_I^4$:

$$\mathfrak{R} = u\,\mathfrak{a} + [\mathfrak{r}\,\mathfrak{b}], \qquad U = -(\mathfrak{r}\,\mathfrak{a}).$$

In an analogous way, if one is given another $V_{II}^4\{\mathfrak{a}, \mathfrak{b}\}$ and the $V_I^4\{\mathfrak{r}, \mathfrak{a}\}$ then one can compose the $V_I^4$:

$$\mathfrak{R}' = u\,\mathfrak{a}' + [\mathfrak{r}\,\mathfrak{b}'], \qquad U' = -(\mathfrak{r}\,\mathfrak{a}').$$

Now, according to the schema (2), one will obtain the $S^4$:

$$S = \mathfrak{R}\mathfrak{R}' + UU' = u^2\,\mathfrak{a}\mathfrak{a}' + u\,\mathfrak{a}\,[\mathfrak{r}\mathfrak{b}'] + u\,\mathfrak{a}'\,[\mathfrak{r}\mathfrak{b}] + [\mathfrak{r}\mathfrak{b}][\mathfrak{r}\mathfrak{b}'] + (\mathfrak{r}\mathfrak{a})\,(\mathfrak{r}\mathfrak{a}'),$$

which can be written:

$$S = (\mathfrak{r}\mathfrak{a})\,(\mathfrak{r}\mathfrak{a}') - (\mathfrak{r}\mathfrak{b})(\mathfrak{r}\mathfrak{b}') + \mathfrak{r}^2\,(\mathfrak{a}\mathfrak{a}') + u\,\mathfrak{r}\,[\mathfrak{a}\mathfrak{b}'] + u\,\mathfrak{r}\,[\mathfrak{a}'\,\mathfrak{b}] + u^2\,(\mathfrak{a}\mathfrak{a}'). \tag{19}$$

As would follow from ($6_a$), one can permute $\mathfrak{a}$ with $\mathfrak{b}$ and $\mathfrak{a}'$ with $\mathfrak{b}'$ and obtain another $S^4$ in a corresponding way:

$$S^* = (\mathfrak{r}\mathfrak{a})\,(\mathfrak{r}\mathfrak{b}') - (\mathfrak{r}\mathfrak{a})(\mathfrak{r}\mathfrak{a}') + \mathfrak{r}^2\,(\mathfrak{a}\mathfrak{a}') + u\,\mathfrak{r}\,[\mathfrak{a}\mathfrak{b}'] + u\,\mathfrak{r}\,[\mathfrak{a}'\,\mathfrak{b}] + u^2\,(\mathfrak{b}\mathfrak{b}'). \tag{$19_a$}$$

Set:

$$4\varphi = S - S^*;$$

it will result that:

$$\begin{aligned} 2\varphi = {} & (\mathfrak{r}\mathfrak{a})(\mathfrak{r}\mathfrak{a}') - \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{a}\mathfrak{a}') - (\mathfrak{r}\mathfrak{b})(\mathfrak{r}\mathfrak{b}') + \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{b}\mathfrak{b}') \\ & + u\,\mathfrak{r}\,[\mathfrak{b}\mathfrak{a}'] + u\,\mathfrak{r}\,[\mathfrak{b}'\,\mathfrak{a}] + \tfrac{1}{2}u^2\{(\mathfrak{a}\mathfrak{a}') - (\mathfrak{b}\mathfrak{b}')\}. \end{aligned} \tag{20}$$

Now, identify the function $\varphi$ that is homogeneous of second order in the $x$, $y$, $z$, $u$ and invariant under **LORENTZ** transformation with the $S^4$ that is given in (18), and one will find the expressions:

$$2\Phi = (\mathfrak{r}\mathfrak{a})(\mathfrak{r}\mathfrak{a}') - \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{a}\mathfrak{a}') - (\mathfrak{r}\mathfrak{b})(\mathfrak{r}\mathfrak{b}') + \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{b}\mathfrak{b}'), \tag{$20_a$}$$

$$2\mathfrak{f} = i\,[\mathfrak{b}\mathfrak{a}'] + i\,[\mathfrak{b}'\,\mathfrak{a}], \tag{$20_b$}$$

$$2\psi = (\mathfrak{a}\mathfrak{a}') - (\mathfrak{b}\mathfrak{b}'). \tag{$20_c$}$$

Introduce the $V_{II}^4$ of **MINKOWSKI**'s electrodynamics, and set:

$$\left\{ \begin{array}{ll} \mathfrak{a} = \mathfrak{H}, & \mathfrak{b} = -i\,\mathfrak{D}, \\ \mathfrak{a}' = \mathfrak{B}, & \mathfrak{b}' = -i\,\mathfrak{E}. \end{array} \right. \tag{21}$$

If one takes ($18_a$) into account then the following expressions will result:

$$\left\{ \begin{array}{l} X_x x^2 + Y_y y^2 + Z_z z^2 + 2Y_z yz + 2Z_x zx + 2X_y xy \\ = (\mathfrak{r}\mathfrak{E})(\mathfrak{r}\mathfrak{D}) - \tfrac{1}{2}\mathfrak{r}^2(\mathfrak{E}\mathfrak{D}) + (\mathfrak{r}\mathfrak{H})(\mathfrak{r}\mathfrak{B}) - \tfrac{1}{2}\mathfrak{r}^2(\mathfrak{H}\mathfrak{B}), \end{array} \right. \tag{$21_a$}$$

$$2\mathfrak{f} = [\mathfrak{E}\mathfrak{H}] + [\mathfrak{D}\mathfrak{B}], \tag{$21_b$}$$

(21$_c$) $$2\psi = \mathfrak{E}\mathfrak{D} + \mathfrak{H}\mathfrak{B}.$$

They give the values of the **MAXWELL** pressures, and the energy current and density for empty space, in which $\mathfrak{D}$ is identified with $\mathfrak{E}$, and $\mathfrak{H}$ is identified with $\mathfrak{B}$. For ponderable bodies in a state of rest, the values (21$_a$) and (21$_c$) of the pressures and the energy density are acceptable, but the value (21$_b$) will be:

$$\mathfrak{D} = \varepsilon\,\mathfrak{E}, \qquad \mathfrak{B} = \mu\,\mathfrak{H},$$

while the energy current will be:

$$\mathfrak{S} = c\,\mathfrak{f} = \left(\frac{\varepsilon\mu + 1}{2}\right) c\,[\mathfrak{E}\,\mathfrak{H}],$$

which will differ from the current that is given by the **POYNTING** vector:

$$\mathfrak{S} = c\,[\mathfrak{E}\,\mathfrak{H}]$$

by:

$$\left(\frac{\varepsilon\mu - 1}{2}\right) c\,[\mathfrak{E}\,\mathfrak{H}].$$

Therefore, we must subtract another $S^4$ that contains the factor $(\varepsilon\mu - 1)$, which will be zero for empty space, from the invariant $\varphi$ that is given by equation (20).

In order to obtain such an $S^4$, consider two $V_I^4$. The first $V_I^4$ is the *velocity:*

$$\mathfrak{r}_1 = k^{-1}\,\mathfrak{q}, \qquad u_1 = i\,k^{-1},$$

and then the *rest radius*, which is given by equations (12):

$$\mathfrak{R} = k^{-1}\,\mathfrak{f}' + k^{-3}\,\mathfrak{q}\,(\mathfrak{q}\mathfrak{f}'), \qquad U = i\,k^{-3}(\mathfrak{q}\mathfrak{f}').$$

Introduce the $V^3$:

(22) $$\mathfrak{W} = (\varepsilon\mu - 1)\,k^{-1}\,\mathfrak{R} = (\varepsilon\mu - 1)\,(k^{-2}\,\mathfrak{f}' + k^{-4}\,\mathfrak{q}\,(\mathfrak{q}\mathfrak{f}')\}.$$

$(\varepsilon\mu - 1)$ is an $S^4$, so:

$$\mathfrak{r}_2 = (\varepsilon\mu - 1)\,\mathfrak{R} = k\,\mathfrak{W},$$
$$u_2 = (\varepsilon\mu - 1)\,U = ik\,(\mathfrak{q}\,\mathfrak{W})$$

constitute a $V_I^4$.

Now, compose two $S^4$'s according to the schema (2):

$$\mathfrak{r}\mathfrak{r}_1 + uu_1 = k^{-1}\,\{(\mathfrak{r}\mathfrak{q}) + iu\},$$
$$\mathfrak{r}\mathfrak{r}_2 + uu_2 = k\{(\mathfrak{r}\mathfrak{W}) + iu\,(\mathfrak{q}\mathfrak{W})\,\},$$

which are both linear in $x$, $y$, $z$, $u$, and multiply them. An $S^4$ will then result that is a second-order homogeneous function in the $x$, $y$, $z$, $u$:

(23) $$2\chi = (\mathfrak{r}\mathfrak{q})(\mathfrak{r}\mathfrak{W}) + (iu\,\mathfrak{r}, \mathfrak{W} + \mathfrak{q}\,(\mathfrak{q}\mathfrak{W})) - u^2\,(\mathfrak{q}\,\mathfrak{W}).$$

If one sums the $S^4$'s $\varphi$ and $\chi$ that are given by (20) and (23), resp., then one will define a new $S^4$, namely:

(24) $$f = \varphi + \chi.$$

Instead of $\varphi$, take the following characteristic invariant to determine the electromagnetic pressures and the energy current and density:

$$f\,(x, y, z, u) = \Phi\,(x, y, z) - iu\,(\mathfrak{r}\;\mathfrak{f}) + \tfrac{1}{2}u^2\,\psi.$$

Instead of $(21_{a,b,c})$, one will then have the formulas:

$(24_a)$ $$\begin{cases} 2\Phi = X_x x^2 + Y_y y^2 + Z_z z^2 + 2Y_z yz + 2Z_x zx + X_y xy \\ \quad = (\mathfrak{r}\mathfrak{E})(\mathfrak{r}\mathfrak{D}) - \tfrac{1}{2}\mathfrak{r}^2(\mathfrak{E}\mathfrak{D}) + (\mathfrak{r}\mathfrak{H})(\mathfrak{r}\mathfrak{B}) - \tfrac{1}{2}\mathfrak{r}^2(\mathfrak{H}\mathfrak{B}) + (\mathfrak{r}\mathfrak{q})(\mathfrak{r}\mathfrak{W}) \end{cases}$$

$(24_b)$ $$2\mathfrak{f} = [\mathfrak{E}\mathfrak{H}] + [\mathfrak{D}\mathfrak{B}] - \mathfrak{W} - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}),$$

$(24_c)$ $$2\psi = \mathfrak{E}\mathfrak{D} + \mathfrak{H}\mathfrak{B} - 2\,(\mathfrak{q}\,\mathfrak{W}).$$

These values are identical to the ones that are derived in my system of the electrodynamics of moving bodies in the first paper for the case of **MINKOWSKI**'s theory.

In that theory, one will have the relations [9]:

(25) $$\begin{cases} \mathfrak{D} = \varepsilon\,\mathfrak{E}' - [\mathfrak{q}\mathfrak{H}], & \mathfrak{E} = \mathfrak{E}' - [\mathfrak{q}\mathfrak{B}], \\ \mathfrak{B} = \mu\,\mathfrak{H}' + [\mathfrak{q}\mathfrak{E}], & \mathfrak{H} = \mathfrak{H}' + [\mathfrak{q}\mathfrak{D}]. \end{cases}$$

A calculation (which we shall not reproduce) will give:

$$[\mathfrak{D}\mathfrak{B}] - [\mathfrak{E}\mathfrak{H}] = k^{-2}\,(\varepsilon\mu - 1)\,[\mathfrak{E}'\mathfrak{H}'] = k^{-2}\,(\varepsilon\mu - 1)\,\mathfrak{f}'.$$

On the other hand, according to (22), one will have:

(26) $$\mathfrak{W} - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}) = k^{-1}\,(\varepsilon\mu - 1)\,\mathfrak{f}'.$$

One will therefore get the relation:

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([9]) **M. ABRAHAM**, *loc. cit.* ([1]), equations (36) and (37).

$$(26_a) \qquad \mathfrak{W} - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W}) = [\mathfrak{D}\mathfrak{B}] - [\mathfrak{E}\mathfrak{H}],$$

which is a formula that was found already in the first paper ([10]).

Equation ($24_b$) can then be written:

$$(26_b) \qquad \mathfrak{f} = [\mathfrak{E}\mathfrak{H}] - \mathfrak{q}\,(\mathfrak{q}\,\mathfrak{W})$$

or

$$(26_c) \qquad \mathfrak{f} = [\mathfrak{D}\mathfrak{B}] - \mathfrak{W}.$$

Obviously, ($26_b$) and ($18_b$) will imply the energy current that is postulated by **POYNTING**'s theorem in the rest case. The values of the energy current and the impulse density agree with the ones that were found in the first paper ([11]). Furthermore, the expression ($24_c$) for the energy density was indicated already in ([12]).

It remains for us to prove that the electromagnetic pressures that are determined from equations ($24_a$) are the same as the ones that result in the first paper.

In order to prove this, we need to introduce the *relative pressures* that are defined by ([13]):

$$\begin{aligned}
X'_x &= X_x + \mathfrak{q}_x\,\mathfrak{f}_x, & X'_y &= X_y + \mathfrak{q}_y\,\mathfrak{f}_x, & X'_z &= X_z + \mathfrak{q}_z\,\mathfrak{f}_x, \\
Y'_x &= Y_x + \mathfrak{q}_x\,\mathfrak{f}_y, & Y'_y &= Y_y + \mathfrak{q}_y\,\mathfrak{f}_y, & Y'_z &= Y_z + \mathfrak{q}_z\,\mathfrak{f}_y, \\
Z'_x &= Z_x + \mathfrak{q}_x\,\mathfrak{f}_z, & Z'_y &= Z_y + \mathfrak{q}_y\,\mathfrak{f}_z, & Z'_z &= Z_z + \mathfrak{q}_z\,\mathfrak{f}_z.
\end{aligned}$$

If one takes into account the symmetry conditions:

$$Y_z = Z_y, \qquad Z_x = X_z, \qquad X_y = Y_x$$

then one will have:

$$\begin{aligned}
Y'_z - Z'_y &= \mathfrak{q}_z\,\mathfrak{f}_y - \mathfrak{q}_y\,\mathfrak{f}_z, \\
Z'_x - X'_z &= \mathfrak{q}_x\,\mathfrak{f}_z - \mathfrak{q}_z\,\mathfrak{f}_x, \\
X'_y - Y'_x &= \mathfrak{q}_y\,\mathfrak{f}_x - \mathfrak{q}_x\,\mathfrak{f}_y,
\end{aligned}$$

which are relations that the indicated expressions for the relative pressures will satisfy, as was proved in the first paper. It is therefore enough to prove that if the function:

$$2\Phi' = X'_x\,x^2 + Y'_y\,y^2 + Z'_z\,z^2 + (Y'_z + Z'_y)\,yz + (Z'_x + X'_z)\,zx + (X'_y + Y'_z)\,xy$$

is taken to be equal to:

$$2\Phi' = 2\Phi + x^2\mathfrak{q}_x\,\mathfrak{f}_x + y^2\mathfrak{q}_y\,\mathfrak{f}_y + z^2\mathfrak{q}_z\,\mathfrak{f}_z + (\mathfrak{q}_y\,\mathfrak{f}_z + \mathfrak{q}_z\,\mathfrak{f}_y)\,yz + (\mathfrak{q}_z\,\mathfrak{f}_x + \mathfrak{q}_x\,\mathfrak{f}_z)\,zx + (\mathfrak{q}_x\,\mathfrak{f}_y + \mathfrak{q}_y\,\mathfrak{f}_x)\,xy$$

([10]) **M. ABRAHAM**, *loc. cit.*([1]), equation ($40_c$).

([11]) **M. ABRAHAM**, *loc. cit.*([1]), equations (40), ($40_a$), and (42) .

([12]) **M. ABRAHAM**, *loc. cit.*([1]), equation ($44_a$).

([13]) **M. ABRAHAM**, *loc. cit.*([1]), equation (10), in which, one must set $\mathfrak{w} = c\,\mathfrak{q}$, $\mathfrak{f} = c\,\mathfrak{g}$.

or

(27) $$2\Phi' = 2\Phi + (\mathfrak{r}\mathfrak{q})(\mathfrak{r}\mathfrak{f})$$

and one introduces the values ($24_a$) of $2\Phi$ then one will get an expression that is identical to the one that resulted from the fundamental formulas ($V_a$) in the first paper. That will ultimately give:

($27_a$) $$2\Phi' = (\mathfrak{r}\ \mathfrak{E}')(\mathfrak{r}\ \mathfrak{D}) - \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{E}'\mathfrak{D}) + (\mathfrak{r}\ \mathfrak{H}')(\mathfrak{r}\ \mathfrak{B}) - \tfrac{1}{2}\mathfrak{r}^2\,(\mathfrak{H}'\mathfrak{B}).$$

The identity of the values (27) and ($27_a$) will be proved as long as the following relation is satisfied:

(28) $$\begin{cases} (\mathfrak{r}\mathfrak{q})(\mathfrak{r}\mathfrak{f}) + (\mathfrak{r}\mathfrak{q})(\mathfrak{r}\mathfrak{B}) \\ = (\mathfrak{r}, \mathfrak{E}' - \mathfrak{E})(\mathfrak{r}\mathfrak{D}) + (\mathfrak{r}, \mathfrak{H}' - \mathfrak{H})(\mathfrak{r}\mathfrak{B}) - \tfrac{1}{2}\mathfrak{r}^2\{\mathfrak{E}' - \mathfrak{E}, \mathfrak{D}\} + (\mathfrak{H}' - \mathfrak{H}, \mathfrak{B})\}. \end{cases}$$

If one takes ($26_c$) and (25) into account then that can be written as:

($28_a$) $$\begin{cases} (\mathfrak{r}\mathfrak{q})(\mathfrak{r}[\mathfrak{D}\mathfrak{B}]) = (\mathfrak{r}[\mathfrak{q}\mathfrak{B}])(\mathfrak{r}\mathfrak{D}) - (\mathfrak{r}[\mathfrak{q}\mathfrak{D}])(\mathfrak{r}\mathfrak{B}) - \tfrac{1}{2}\mathfrak{r}^2\{\mathfrak{D}[\mathfrak{q}\mathfrak{B}] - \mathfrak{B}[\mathfrak{q}\mathfrak{D}]\} \\ \qquad = \mathfrak{r}[\mathfrak{q}, \mathfrak{B}(\mathfrak{r}\mathfrak{D}) - \mathfrak{D}(\mathfrak{r}\mathfrak{B})] + \mathfrak{r}^2(\mathfrak{q}\,[\mathfrak{D}\mathfrak{B}]). \end{cases}$$

Now, since:

$$[\mathfrak{q}, \mathfrak{B}\,(\mathfrak{r}\ \mathfrak{D}) - \mathfrak{D}\,(\mathfrak{r}\ \mathfrak{B})] = -\,[\mathfrak{q}\,[\mathfrak{r}\,[\mathfrak{D}\ \mathfrak{B}]] = -\,\mathfrak{r}\,(\mathfrak{q}\,[\mathfrak{D}\mathfrak{B}]) + (\mathfrak{r}\mathfrak{q})\,[\mathfrak{D}\mathfrak{B}],$$

the second part of equation ($28_a$) will, in fact, give:

$$(\mathfrak{r}\mathfrak{q})(\mathfrak{r}\,[\mathfrak{D}\mathfrak{B}]),$$

so the relation (28) will be satisfied identically. It therefore follows from formula ($27_a$) that was postulated in our system of electrodynamics that the values of the **MAXWELL** pressures obey the principle of relativity in the special case of **MINKOWSKI**'s theory, in according with the relations ($24_a$).

## § 4.

## The equations of motion.

In **MINKOWSKI**'s mechanics, one encounters the so-called "proper time" of a point – i.e., a four-dimensional scalar ($\tau$) that is defined by ([14]):

([14]) **H. MINKOWSKI**, *loc. cit.* ([5]), equation (3), page 48.

(29) $$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\mathfrak{q}^2}} = k^{-1}.$$

If one divides the four-dimensional radius vector of the point with respect to $\tau$ and divides by the velocity of light ($c$) then what will result is **MINKOWSKI**'s "velocity $V^4$:

(30) $$\begin{cases} \frac{1}{c}\frac{dx}{d\tau} = \frac{1}{c}\frac{dx}{dt}\cdot\frac{dt}{d\tau} = \mathfrak{q}_x \frac{dt}{d\tau} = \mathfrak{q}_x k^{-1}, \\ \frac{1}{c}\frac{dy}{d\tau} = \frac{1}{c}\frac{dy}{dt}\cdot\frac{dt}{d\tau} = \mathfrak{q}_y \frac{dt}{d\tau} = \mathfrak{q}_y k^{-1}, \\ \frac{1}{c}\frac{dz}{d\tau} = \frac{1}{c}\frac{dz}{dt}\cdot\frac{dt}{d\tau} = \mathfrak{q}_z \frac{dt}{d\tau} = \mathfrak{q}_z k^{-1}, \\ \frac{1}{c}\frac{du}{d\tau} = \frac{1}{c}\frac{du}{dt}\cdot\frac{dt}{d\tau} = i \frac{dt}{d\tau} = i k^{-1}. \end{cases}$$

Obviously, the four components of the $V^4$ of "velocity" satisfy the equation:

($30_a$) $$\left(\frac{1}{c}\frac{dx}{d\tau}\right)^2 + \left(\frac{1}{c}\frac{dy}{d\tau}\right)^2 + \left(\frac{1}{c}\frac{dz}{d\tau}\right)^2 + \left(\frac{1}{c}\frac{du}{d\tau}\right)^2 = -1$$

identically.

Now define the four-dimensional scalar:

(31) $$\Psi = \mathfrak{K}_x \frac{dx}{c\,d\tau} + \mathfrak{K}_y \frac{dy}{c\,d\tau} + \mathfrak{K}_z \frac{dz}{c\,d\tau} + \mathfrak{K}_u \frac{du}{c\,d\tau}$$

from the $V_I^4$'s of "velocity" and "force" according to the schema (2).

If one introduces the ponderomotive force of the electromagnetic field, whose components are determined by (16) and takes equations (15) and (30) into account then one will find that:

($31_a$) $$\Psi = -\frac{Q}{c}k^{-1},$$

in which $Q$ is the **JOULE** heat that is produced in a unit of time and space.

Now, **MINKOWSKI** gave the following form to the equations of motion of an element of matter ([15]):

([15]) **H. MINKOWSKI**, *loc. cit.* ([5]), equation (20), page 54.

$$(32) \qquad \begin{cases} \nu \dfrac{d^2 x}{d\tau^2} = \mathfrak{K}_x + \Psi \dfrac{dx}{c\,d\tau}, \\ \nu \dfrac{d^2 y}{d\tau^2} = \mathfrak{K}_y + \Psi \dfrac{dy}{c\,d\tau}, \\ \nu \dfrac{d^2 z}{d\tau^2} = \mathfrak{K}_z + \Psi \dfrac{dz}{c\,d\tau}, \\ \nu \dfrac{d^2 u}{d\tau^2} = \mathfrak{K}_u + \Psi \dfrac{du}{c\,d\tau}, \end{cases}$$

in which the $S^4$ ($\nu$) determines the "rest density" of matter. The identity ($30_a$), from which it follows that:

$$\frac{dx}{d\tau}\frac{d^2 x}{d\tau^2} + \frac{dy}{d\tau}\frac{d^2 y}{d\tau^2} + \frac{dz}{d\tau}\frac{d^2 z}{d\tau^2} + \frac{du}{d\tau}\frac{d^2 u}{d\tau^2} = 0,$$

is satisfied, due to equations (32).

**MINKOWSKI** called the $V^3$ whose components the right-hand sides of the first three equations of motion (32) the "ponderomotive force" of the electromagnetic field; i.e., the $V^3$:

$$(32_a) \qquad \mathfrak{K} + \Psi\,\mathfrak{q}\,k^{-1} = \mathfrak{K} - \frac{\mathfrak{q}\cdot Q}{ck^2}.$$

That vector is not identical to the force that was determined by the impulse theorem (14), but differs from it by:

$$-\frac{\mathfrak{q}\cdot Q}{ck^2}.$$

Therefore, when the **JOULE** heat is produced in matter, **MINKOWSKI** mechanics must add that additional force to the ponderomotive force that is derived from the impulse theorem.

If one considers how important the impulse theorem is in electromagnetic mechanics then one would prefer to preserve that principle in the electrodynamics of bodies in motion. One can remove **MINKOWSKI**'s additional force, which will give the equations of motion the form that is suggested by the mechanical laws of impulse, instead of (32), namely:

(33)
$$\begin{cases} \dfrac{d}{d\tau}\left(\nu \dfrac{dx}{d\tau}\right) = \mathfrak{K}_x, \\ \dfrac{d}{d\tau}\left(\nu \dfrac{dy}{d\tau}\right) = \mathfrak{K}_y, \\ \dfrac{d}{d\tau}\left(\nu \dfrac{dz}{d\tau}\right) = \mathfrak{K}_z, \\ \dfrac{d}{d\tau}\left(\nu \dfrac{du}{d\tau}\right) = \mathfrak{K}_u. \end{cases}$$

Since $\tau$ and $\nu$ are $S^4$'s, both sides of those equations are the components of a $V_I^4$; therefore, those equations agree with the principle of relativity. The identity ($30_a$) will be satisfied if one sets:

$$\frac{d\nu}{d\tau}\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{du}{d\tau}\right)^2\right\} = \mathfrak{K}_x \frac{dx}{d\tau} + \mathfrak{K}_y \frac{dy}{d\tau} + \mathfrak{K}_z \frac{dz}{d\tau} + \mathfrak{K}_u \frac{du}{d\tau}.$$

If one takes ($30_a$), (31), and ($31_a$) into account then one will find that:

$$\frac{d\nu}{d\tau} = -\frac{\Psi}{c} = \frac{Q}{c^2 k},$$

or, according to (29):

($33_a$)
$$\frac{d\nu}{d\tau} = \frac{Q}{c^2}.$$

Therefore, $\nu$ (viz., the "rest density" of mass) must be variable and increase whenever **JOULE** heat is produced in matter. If one accepts that hypothesis, which was introduced for the first time by **EINSTEIN** and **PLANCK**, then one will avoid the additional **MINKOWSKI** force.

One can pass from the equations of motion (33), which refer to a unit volume of extended matter, to the equations of motion of a material point by the same way that **MINKOWSKI** indicated for equations (32).

Milan, 17 January 1910

MAX ABRAHAM