“Sul problema delle coazioni elastiche,” Atti R. Accad. sci. di Torino **54** (1918-19), 414-418

On the problem of elastic coactions

Note by the corresponding member GIUSEPPE ALBENGA

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In a series of recent notes on elastic coactions that were published in the Rendiconti della R. Accademia dei Lincei ([1]) and in Atti della R. Accademia delle Scienze di Torino ([2]), Gustavo Colonetti discussed some interesting properties of elastic solids that were subject to only the stresses that engineers usually call “initial” ones. Given the importance of the argument, I believe that it would not be pointless to look at some of the results that can be added to that work in a simple way and by means of processes that are more familiar to engineers than those of the mathematical theory of elasticity.

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In the note “Una proprietà carrateristica delle coazioni elastiche nei solidi elasticamente omogenei,” it was show that in the absence of external forces and active constraints, the mean cubic dilatation of a solid in a state of elastic constriction will be zero. That is then a generalization of a theorem that August Föppl stated for the particular case of an isotropic sphere with internal stresses that vary according to a well-defined law as a function of the radius ([3]).

The proof is immediate for isotropic solids. Indeed, with the Grashof symbols ([4]):

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{m-2}{mE}(\sigma_x + \sigma_y + \sigma_z),$$

and upon multiplying the two sides of the preceding equation by dv and integrating over the entire volume:

([1]) GUSTAVO COLONETTI:

“Su certi stati di coazione elastica,” Rendiconti Accademia dei Lincei,” (5) **26** (1917), pp. 43.

“Una proprietà caraterreristica delle coazioni elastiche nei solidi elasticamente omogenei,” *ibidem*, (5) **27** (1918), pp. 155.

“Sul problema delle coazioni elastiche, Nota I and II,” *ibidem*, pp. 267 and 331.

([2]) GUSTAVO COLONETTI, “Applicazioni a problemi technici di un nuovo teorema sulle coazioni elastiche,” Atti R. Accad. Torino **54** (1918-19), pp. 69.

([3]) A. FÖPPL, *Vorlesungen über technische Mechanik*. V. *Die wichtigsten Lehren der höheren Elastizitätstheorien.*

([4]) See, for example, C. GUIDI, *Lezioni sulla Scienza delle Construzioni*, 7th ed., Part II, page 316.

(1) $$\int_v e\,dv = \frac{m-2}{m}\int_v (\sigma_x + \sigma_y + \sigma_z)\,dv\,.$$

Now, one can prove directly that under the conditions that were posed, the integral on the right-hand side will be zero. Indeed, $\int_v \sigma_x\,dv$ is nothing but the integral over all of the solid of $dx\int_\Sigma \sigma_x\,dy\,dz$, in which Σ denotes the total surface of a section of the solid that is made by a plane Π that is normal to the x-axis. However, for equilibrium under a translation along that axis of the truncation of the solid that is situated on one side of the plane Π, minus the surface of the solid, one must have:

(2) $$\int_\Sigma \sigma_x\,dy\,dz = 0,$$

for an arbitrary section, which will then imply that:

$$\int_v \sigma_x\,dv = 0.$$

When one operates analogously on σ_y and σ_z, one will get from (1) that:

$$\int_v e\,dv = 0,$$

as one wished to prove.

(2) and its analogues express a characteristic property of the initial stresses that present themselves when one would like to construct a possible distribution for them. It is enough to rapidly review the fundamental properties of the fusion stresses in the sphere that were studied by Föppl ([1]).

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The proof of (3) for a homogeneous elastic solid is equally simple. The existence of a homogeneous quadratic elastic potential has, as a consequence, the intercession of a homogeneous linear relationship between the dilatations and internal stresses; one will then have:

(4) $$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = k_1\,\sigma_x + k_2\,\sigma_y + k_3\,\sigma_z + k_4\,\tau_{xy} + k_5\,\tau_{yz} + k_6\,\tau_{zx}\,,$$

in which k does not change when one passes from one point to the other in the solid, thanks to its supposed elastic homogeneity. If one repeats the preceding proof then one will get:

$$\int_v \sigma_x\,dv = \int_v \sigma_y\,dv = \int_v \sigma_z\,dv = \int_v \tau_{xy}\,dv = \int_v \tau_{yz}\,dv = \int_v \tau_{zx}\,dv = 0,$$

and therefore, if one recalls (4), one will also have:

([1]) *Loco citato.*

$$\int_v e\, dv = 0.$$

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The results that were obtained in our note can be attached to ones in the beautiful and little-known paper by Luigi Donati ([1]). Some theorems were proved in it that, if one recalls the usual symbolism from the theory of the resistance of materials and denotes the internal work done by deformation (viz., the elastic potential) by L_i and the Clapeyron work that is due to the external forces by L_e then one can state: Compatible with the conditions that were imposed, the quantity:

$$L_i - 2\, L_e = \text{minimum}, \tag{5}$$

whether one considers variations of the deformations and the displacements (and one will then have Lagrange's theorem) or one examines variations of forces and internal stresses (and in that case, one will have a new form for the principle of variations of work from which one will immediately derive Castigliano's theorems).

One can annul the internal stresses in any solid by means of a series of suitable cuts. The faces, thus-liberated, are displaced with respect to each other, and the solid will assume a new configuration that is congruent and equilibrated. Let **s** be the displacement of the generic element. One can reestablish the original configuration, which is also equilibrated and congruent, by applying the initial stresses $\boldsymbol{\rho}$ to each face of the cut, which are exerted such that: The points of application of the stresses $\boldsymbol{\rho}\, dF$ will submit to a displacement – **s**. Let F denote the total surface of all the cuts that were made in the solid: From the state of affairs that is now being considered, one will have:

$$L_e = -\tfrac{1}{2}\int_F \boldsymbol{\rho}\times\mathbf{s}\, dF\ ,$$

if one lets $\boldsymbol{\rho}$ be the only external force that is applied to the solid, and then, from the second of the theorem that is expressed by (5), the state of stress considered will be the one that makes the function:

$$L_i + \tfrac{1}{2}\int_F \boldsymbol{\rho}\times\mathbf{s}\, dF \tag{6}$$

a minimum, compatible with the given **s**, which is an expression that differs only by its symbols from the one that Colonetti gave in the cited Note II "Sul problema delle coazioni elastiche."

In the case of reticular beams, one will have:

([1]) LUIGI DONATI:

"Sul lavoro di deformazione dei sistemi elastici," Memorie della R. Accademia delle Scienze dell'Istituto di Bologna (1888),

"Illustrazione dei teorema di Menabrea," *ibidem*, (1889),

"Ulteriori osservazioni sul teorema di Menabrea," *ibidem*, (1894).

$$L_e = -\tfrac{1}{2}\sum S\,\Delta l\,,$$

in which one lets Δl denote the displacements that follow from the surplus cuts in the beam that are subject to the initial stresses. In addition, it is well-known that:

$$L_i = \tfrac{1}{2}\sum \frac{S^2 s}{EF}\,.$$

It will then follow from (5) that the state of stress is the one that makes the expression:

$$\sum\left(\frac{S^2 s}{2EF} + S\,\Delta l\right) \tag{7}$$

a minimum, compatible with the given variations Δl, which is a relation that Colonetti pointed out in his Note "Applicazioni a problemi tecnici di un nuovo teorema sulle coazioni elastiche."

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