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ON THE

DEFORMATION OF THE ELASTIC SPHERE

BY DOCTOR

EMILIO ALMANSI

Translated by D. H. Delphenich

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I.

1. – In this article, I propose to study the deformation of a homogeneous elastic sphere whose surface is acted upon in some way, but is not subject to gravity nor any other volume forces, when one knows the components of the displacement or tension at any point of the surface.

The problem of the deformation of an elastic sphere for given forces that act upon the surface was solved for the first time by LAMÉ, who obtained the components of the displacement of any point of the solid expressed in terms of a series. The first solution to the problem by means of definite integrals was due to BORCHARDT.

Prof. BETTI ["Teoria dell'elasticità," Nouvo Cimento (2), vol. VII, *et seq.*] started from the known theorem that is called the "reciprocity theorem" and gave, for the first time, a general method for the integration of the equations of equilibrium of an isotropic elastic body that led to the determination of the dilatation and the components of the rotation.

CERRUTI applied that method, although simplified somewhat, and solved the problem for the solid that was bounded by an indefinite plane and for the sphere ["Ricerche intorno all'equilibrio dei corpi elastici isotropi," R. Accademia dei Lincei (3), vol. XIII; "Sulla deformazione di una sfera omogenea," Nuovo Cimento (3), vol. XXXII].

SOMIGLIANA ["Sopra gl'integrali delle equazioni dell'isotropia elastica," Nuovo Cimento (3), vol. XXXIV] gave another method for integrating the equations of equilibrium that permitted one to determine the components of the displacement directly. He represented those components by means of three new functions that he called "generating functions" and found that those functions could be expressed by integral formulas that were analogous to the one that is deduced from Green's Lemma and had the same significance in the problem of elasticity that the latter formula has in the

Dirichlet problem. He then discussed the nature of the generating functions in the cases of solids that were bounded by an indefinite plane and a sphere.

The problem of the deformation of a spherical solid was also treated by MARCOLONGO [“Deformazione di una sfera isotropa,” Ann. di. Mat. (2), vol. XXIII), and he studied the case in which one knows some components of the displacement and some of the external tensions on the surface.

LAURICELLA solved the problem of the sphere by a procedure that was different from the ones that Cerruti and Somigliana employed [“Equilibrio dei corpi elastici isotropic,” Ann. della R. Scuola Norm. Sup. di Pisa, vol. VII]. In that same article, the author presented a method that was analogous to Neumann’s method for integrating the differential equation $\Delta^2 = 0$ and by which, after posing some restrictions on the form and nature of the solid, he succeeded in representing the components of the displacement by means of series when one is given their value on the surface.

The method that follows in my article here is based mainly upon the property that an arbitrary function Φ presents when it satisfies the equation $\Delta^2\,\Delta^2 = 0$ that it can be represented by two functions φ, χ that satisfy the equation $\Delta^2 = 0$ by means of the formula:

$$\Phi = (x^2 + y^2 + z^2 - R^2)\ \varphi + \chi,$$

in which R is a constant. That formula makes it very simple to study the various questions that refer to the sphere, and in the special way that I propose to treat.

2. – If the function φ of the variables x, y, z satisfies the equation $\Delta^2 = 0$ and one sets:

$$c\ \varphi + x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z} = \psi,$$

in which c denotes a constant, then the function ψ, thus-expressed, will also satisfy that equation as is easy to verify.

Introduce the variables r, s, t in place of the variables x, y, z by setting:

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}\,, \\ s &= \arcsin\frac{y}{\sqrt{x^2+y^2}}\,, \\ t &= \arcsin\frac{x}{\sqrt{x^2+y^2+z^2}}\,. \end{aligned} \tag{1}$$

The preceding equation can be written:

$$c\ \varphi + r\frac{\partial \varphi}{\partial r} = \psi.$$

Now, supposing that the function Ψ, which satisfies the equation $\Delta^2 = 0$, is given, we would like to determine the function φ in such a way that it satisfies the two equations:

$$
\begin{aligned}
c\,\varphi + r\frac{\partial \varphi}{\partial r} &= \Psi, \\
\Delta^2 \Psi &= 0.
\end{aligned}
\tag{2}
$$

Suppose that the function Ψ is uniform in all of the space to which our considerations are confined, or in the sphere of radius R whose center is the coordinate origin, and that is also what is required by the function φ.

One proves that if the constant c is positive then just one function φ that satisfies those conditions will exist.

In order to prove that more than one cannot exist, it is enough to prove that if a function φ_1 is uniform inside the sphere and satisfies the two equations:

$$
c\,\varphi_1 + r\frac{\partial \varphi_1}{\partial r} = 0,
$$
$$
\Delta^2 \varphi_1 = 0
$$

then it will be zero at all points of the sphere.

Let S denote the space that is enclosed by the spherical surface of radius R, let σ denote that surface, and let n denote the inward-pointing normal, so one has the known formula:

$$
\int_S \left\{ \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 \right\} dS = -\int_\sigma \varphi_1 \frac{\partial \varphi_1}{\partial n}\, d\sigma\,. \tag{3}
$$

However, formula (2) gives:

$$
c\,\varphi_1 - R\,\frac{\partial \varphi_1}{\partial n} = 0
$$

for the points of σ, or:

$$
\frac{\partial \varphi_1}{\partial n} = \frac{c}{R}\,\varphi_1\,.
$$

If one substitutes that in formula (3) then one will have:

$$
\int_S \left\{ \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 \right\} dS = -\frac{c}{R}\int_\sigma \varphi_1^2\, d\sigma\,.
$$

However, if c is positive then that equality cannot persist unless both sides of it are zero identically. One must then have:

$$\frac{\partial \varphi_1}{\partial x} = 0, \qquad \frac{\partial \varphi_1}{\partial y} = 0, \qquad \frac{\partial \varphi_1}{\partial z} = 0$$

in all of the sphere or:

$$\varphi_1 = \text{const.}$$

and

$$\varphi_1 = 0$$

on the surface. Therefore, φ_1 must be zero at all points of the sphere.

One will arrive at the result by constructing the general integral of equation (2). It is, in fact:

$$\varphi = \frac{\kappa(s,t)}{r^c} + \frac{1}{r^c}\int_0^r r^{c-1}\,\Psi\,dr\,,$$

in which the function κ of the variables s, t is arbitrary. However, the function must not become infinite at any point of the sphere. Now, if, as one suppose, the function Ψ is uniform in S then the term:

$$\frac{1}{r^c}\int_0^r r^{c-1}\,\Psi\,dr$$

will remain finite at all points of S around the center, where it assumes the value Ψ_0 / c, where Ψ_0 denotes the value that the function Ψ takes at that point. However, since the constant c is positive, the term κ / r^c will become infinite at the center of the sphere. In order to not have that, it is necessary for κ to vanish for $r = 0$. However, κ is independent of r. It must then be zero at all points of S. Consequently, what will remain is:

$$\varphi = \frac{1}{r^c}\int_0^r r^{c-1}\,\Psi\,dr\,, \tag{4}$$

and that is the only function that is uniform in S and satisfies equation (2).

That proof is obviously valid for any bounding surface on the space S, as long as one finds the coordinate origin inside of it.

Let us now see whether the function that we found satisfies the equation $\Delta^2 = 0$.

When one differentiates formula (2) with respect to x, one will get:

$$c\frac{\partial \varphi}{\partial x} + \frac{x}{r}\frac{\partial \varphi}{\partial r} + r\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial r} = \frac{\partial \Psi}{\partial x}.$$

If one compares that formula with the identity:

$$\frac{\partial}{\partial r}\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial r} - \frac{1}{r}\frac{\partial \varphi}{\partial x} + \frac{x}{r^2}\frac{\partial \varphi}{\partial r}$$

then one will get:

$$(c+1)\frac{\partial \varphi}{\partial x} + r\frac{\partial}{\partial r}\frac{\partial \varphi}{\partial x} = \frac{\partial \Psi}{\partial x}.$$

From a formula that is analogous to (4), one will then have:

$$\frac{\partial \varphi}{\partial x} = \frac{1}{r^{c+1}}\int_0^r r^c \frac{\partial \Psi}{\partial x} dr \,. \tag{5}$$

If one repeats the argument then one will find that:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{r^{c+2}}\int_0^r r^{c+1} \frac{\partial^2 \Psi}{\partial x^2} dr \,, \tag{6}$$

as well as analogous expressions for $\frac{\partial^2 \varphi}{\partial y^2}, \frac{\partial^2 \varphi}{\partial z^2}$. Consequently, if one sums then one will get:

$$\Delta^2 \varphi = \frac{1}{r^{c+2}}\int_0^r r^{c+1} \frac{\partial^2 \Psi}{\partial x^2} dr \,, \tag{7}$$

and finally since $\Delta^2 \Psi = 0$, by hypothesis, one will also have:

$$\Delta^2 \varphi = 0. \tag{8}$$

We have then shown that when the constant c is positive, the two equations:

$$c\,\varphi + r\frac{\partial \varphi}{\partial r} = \Psi,$$
$$\Delta^2 \varphi = 0,$$

in which Ψ represents a function that satisfies the equation $\Delta^2 = 0$ and is uniform in the space that is bounded by an arbitrary surface, which will therefore contain the coordinate origin, will be satisfied by a unique function that is uniform in that space. That function is given by the formula:

$$\varphi = \frac{1}{r^c}\int_0^r r^{c-1}\, \Psi \, dr \,.$$

3. – Now consider another differential equation. Suppose that Φ represents a function that is uniform in a space S and satisfies the equation $\Delta^2 = 0$. One would like to determine the function φ that satisfies the equation:

$$A\,\varphi + Br\frac{\partial \varphi}{\partial r} + r^2\frac{\partial^2 \varphi}{\partial r^2} = \frac{1}{2}\left(\Phi - r\frac{\partial \Phi}{\partial r}\right), \tag{9}$$

in which *A*, *B* are constants, and in addition:

$$\Delta^2 \varphi = 0, \tag{10}$$

and it is uniform in all space.

In order to do that, set:

$$A = ab, \qquad B = a + b + 1.$$

Equation (9) can be written:

$$ab\ \varphi + (a + b + 1)\ r\frac{\partial\varphi}{\partial r} + r^2\frac{\partial^2\varphi}{\partial r^2} = \frac{1}{2}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right)$$

or:

$$a\left(b\varphi + r\frac{\partial\varphi}{\partial r}\right) + r\frac{\partial}{\partial r}\left(b\varphi + r\frac{\partial\varphi}{\partial r}\right) = \frac{1}{2}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right).$$

The constants *a*, *b* will be given by the formulas:

$$a = \tfrac{1}{2}(B-1) + \tfrac{1}{2}\sqrt{(B-1)^2 - 4A},$$

$$b = \tfrac{1}{2}(B-1) - \tfrac{1}{2}\sqrt{(B-1)^2 - 4A}.$$

If those constants prove to be real and positive then equation (11) can be integrated by applying formula (4). In fact, the function $\frac{1}{2}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right)$, like Φ, will satisfy the equation $\Delta^2 = 0$. One will then have:

$$b\ \varphi + r\frac{\partial\varphi}{\partial r} = \frac{1}{2r^c}\int_0^r r^{c-1}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right) dr,$$

and in that way, if one applies the same formula once more then:

$$\varphi = \frac{1}{2r^c}\int_0^r dr \cdot r^{b-a-1}\int_0^r r^{c-1}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right) dr,$$

and it will follow from an integration by parts that:

$$\varphi = \frac{1}{2(b-a)}\left[\frac{1}{r^a}\int_0^r r^{a-1}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right) dr - \frac{1}{r^b}\int_0^r r^{b-1}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right) dr\right].$$

If one observes that one has:

$$\int_0^r r^c\frac{\partial\Phi}{\partial r} dr = r^c\,\Phi - \int_0^r r^{c-1}\Phi\, dr$$

identically then that formula can also be written:

$$\varphi = \frac{1}{2(b-a)}\left[\frac{a+1}{r^a}\int_0^r r^{a-1}\Phi\,dr - \frac{b+1}{r^b}\int_0^r r^{b-1}\Phi\,dr\right]. \tag{12}$$

That function also satisfies the equation $\Delta^2 = 0$, since it is the sum of two terms that each satisfy it, as was proved before.

Now, on the other hand, consider the case in which the constants a, b prove to be imaginary. Set:

$$\tfrac{1}{2}(B-1) = p,$$
$$A - \tfrac{1}{4}(B-1)^2 = q^2,$$

in which p, q are real quantities. Let:

$$a = p + iq, \qquad b = p - iq.$$

One can say that in this case, as well, the function φ that is given by formula (12) is the only real function that is uniform in the space S that satisfies the two equations (9) and (10).

The fact that equations (9) and (10) are satisfied is obvious and does not depend upon the nature of the constants a, b. In addition, whereas the function φ contains the imaginary quantities a, b, is it real, since switching those two quantities with each other, while changing i into $-i$, will not change that function. Its expression without imaginaries is easily constructed by replacing the quantities a, b with their values $p + iq$, $p - iq$, and applying the formula:

$$r^{\pm iq} = \cos(q\log r) \pm i\sin(q\log r).$$

One then obtains:

$$\varphi = -\frac{1}{2qr^p}\Big[\{(p+1)\cos(q\log r) + q\sin(q\log r)\}\int_0^r r^{p-1}\sin(q\log r)\,\Phi\,dr$$
$$+\{q\cos(q\log r) - (p+1)\sin(q\log r)\}\int_0^r r^{p-1}\cos(q\log r)\,\Phi\,dr\Big],$$

and if one sets:

\

$$-\frac{\sqrt{(p+1)+q^2}}{2q} = \kappa; \qquad \frac{p+1}{\sqrt{(p+1)^2+q^2}} = \sin\varepsilon, \qquad \frac{q}{\sqrt{(p+1)^2+q^2}} = \cos\varepsilon$$

then the formula that was found can be written more simply as:

$$\varphi =$$

$$\frac{\kappa}{r^p}\left\{\sin(q\log r+\varepsilon)\int_0^r r^{p-1}\sin(q\log r)\,\Phi\,dr + \cos(q\log r+\varepsilon)\int_0^r r^{p-1}\cos(q\log r)\,\Phi\,dr\right\}. \tag{13}$$

One can then say that the function φ is uniform in all of the space S when Φ is also like that. Indeed, the unique point for which any doubt would remain is the coordinate origin. However, if one observes formula (12) then one will easily see that for $r = 0$, the function φ will assume the value $\frac{\Phi_0}{2ab}$, or $\frac{\Phi_0}{2(p^2+q^2)}$, where Φ_0 is the value that the function Φ takes at that point, and that its derivative with respect to r will vanish.

Therefore, the function φ, which is given by formula (13), will satisfy all of the required conditions.

We now address the proof that it is the only function that satisfies them. In order to do that, it is enough to show that no real and uniform function will satisfy the differential equation:

$$a\left(b\varphi + r\frac{\partial\varphi}{\partial r}\right) + r\frac{\partial}{\partial r}\left(b\varphi + r\frac{\partial\varphi}{\partial r}\right) = 0.$$

The general integral of that equation is:

$$\varphi = \frac{U}{r^a} + \frac{V}{r^b},$$

in which U, V are independent of the variable r.

One can also write:

$$\varphi = \frac{U}{r^{p+iq}} + \frac{V}{r^{p-iq}}$$

or

$$\varphi = \frac{1}{r^p}\left(U\, r^{-iq} + V\, r^{iq}\right).$$

In order for that expression to be real, it must happen that the quantities U, V are conjugate imaginaries. Set:

$$U = u + iv, \qquad V = u - iv.$$

One will have:

$$\varphi = \frac{1}{r^p}\left\{u\,(r^{-iq} + r^{iq}) + v\,(r^{-iq} - r^{iq})\right\}$$

or

$$\varphi = \frac{2}{r^p}\left\{u\cos(q\log r) + v\sin(q\log r)\right\}.$$

However, that function will become indeterminate when r tends to 0: It will not satisfy the required conditions unless one does not have:

$$u = 0, \quad v = 0,$$

and therefore:

$$\varphi = 0,$$

which is precisely what we wished to prove.

In summary, if Φ is a function that is uniform in a certain space and satisfies the equation $\Delta^2 = 0$ then the only function φ that is real and uniform in that same space and satisfies the equations:

$$A\varphi + Br\,\frac{\partial\varphi}{\partial r} + r^2\,\frac{\partial^2\varphi}{\partial r^2} = \frac{1}{2}\left(\Phi - r\,\frac{\partial\Phi}{\partial r}\right),$$

$$\Delta^2\varphi = 0$$

is the function that is given by the formula:

$$\varphi =$$

$$\frac{\kappa}{r^p}\left\{\sin(q\log r+\varepsilon)\int_0^r r^{p-1}\sin(q\log r)\,\Phi\,dr + \cos(q\log r+\varepsilon)\int_0^r r^{p-1}\cos(q\log r)\,\Phi\,dr\right\}, \quad (14)$$

or

$$p = \tfrac{1}{2}(B-1), \qquad q = \sqrt{A - \tfrac{1}{4}(B-1)^2},$$

$$\kappa = -\frac{\sqrt{(p+1)^2+q^2}}{2q}, \qquad \varepsilon = \arctan\frac{p+1}{q}. \quad (15)$$

4. – We shall now prove the theorem that was stated to begin with, viz., that a function Φ that is uniform in a space S and satisfies the equation $\Delta^2\Delta^2 = 0$ can always be put into the form:

$$\Phi = (x^2 + y^2 + z^2 - R^2)\,\varphi + \chi,$$

in which φ, χ are functions that are uniform in all space and satisfy the equation $\Delta^2 = 0$.

In order to do that, it is enough to prove that there always exists a function φ that is uniform in all of space and satisfies the equation $\Delta^2 = 0$, and is such that the difference:

$$\Phi - (x^2 + y^2 + z^2 - R^2)\,\varphi$$

will also satisfy that equation; i.e., that one will have:

$$\Delta^2\,\Phi - \sigma\varphi - 4\left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right) = 0,$$

or, if 4Ψ denotes the function $\Delta^2\,\Phi$, which satisfies the equation $\Delta^2 = 0$, then:

$$\frac{3}{2}\varphi + r\frac{\partial\varphi}{\partial r} = \Psi.$$

However, that is not equation (2) when the positive constant c equals 3 / 2. One must then have $\Delta^2 = 0$, so one can then apply formula (4) and then obtain the unique function φ that satisfies all of the required conditions.

The theorem is then proved.

It then results from this that in a problem that involves n functions that satisfy the equation $\Delta^2\,\Delta^2 = 0$, one can introduce $2n$ functions that satisfy the equation $\Delta^2 = 0$ in their place. Now, one can show that there are three functions u, v, w such that the functions $\Delta^2 u$, $\Delta^2 v$, $\Delta^2 w$ are the derivatives with respect to x, y, z of the same function κ that satisfies the equation $\Delta^2 = 0$; i.e.:

$$\Delta^2 u = \frac{\partial \kappa}{\partial x}, \qquad \Delta^2 v = \frac{\partial \kappa}{\partial y}, \qquad \Delta^2 w = \frac{\partial \kappa}{\partial z}. \tag{16}$$

The three functions u, v, w, which must obviously satisfy the equation $\Delta^2\,\Delta^2 = 0$, can be expressed by means of just four functions φ, λ, μ, ν, which will satisfy the equation $\Delta^2 = 0$ if one sets:

$$u = (x^2 + y^2 + z^2 - R^2)\frac{\partial \varphi}{\partial x} + \lambda,$$

$$v = (x^2 + y^2 + z^2 - R^2)\frac{\partial \varphi}{\partial y} + \mu,$$

$$w = (x^2 + y^2 + z^2 - R^2)\frac{\partial \varphi}{\partial z} + \nu.$$

In fact, one will get from those equations:

$$\Delta^2 u = 6\frac{\partial \varphi}{\partial x} + 4\left(x\frac{\partial^2 \varphi}{\partial x^2} + y\frac{\partial^2 \varphi}{\partial x\,\partial y} + z\frac{\partial^2 \varphi}{\partial x\,\partial z}\right),$$

$$\Delta^2 v = 6\frac{\partial \varphi}{\partial y} + 4\left(x\frac{\partial^2 \varphi}{\partial y\,\partial x} + y\frac{\partial^2 \varphi}{\partial y^2} + z\frac{\partial^2 \varphi}{\partial y\,\partial z}\right),$$

$$\Delta^2 w = 6\frac{\partial \varphi}{\partial z} + 4\left(x\frac{\partial^2 \varphi}{\partial z\,\partial x} + y\frac{\partial^2 \varphi}{\partial z\,\partial y} + z\frac{\partial^2 \varphi}{\partial z^2}\right),$$

or

$$\Delta^2 u = \frac{\partial}{\partial x}\left\{4\left(x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z}\right) + 2\varphi\right\},$$

$$\Delta^2 v = \frac{\partial}{\partial y}\left\{4\left(x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} + z\frac{\partial \varphi}{\partial z}\right) + 2\varphi\right\},$$

$$\Delta^2 w = \frac{\partial}{\partial z}\left\{4\left(x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}\right)+2\varphi\right\}.$$

When one compares these equations with (16), it will result that one can set:

$$4\left(x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}+z\frac{\partial\varphi}{\partial z}\right)+2\varphi=\kappa+C,$$

in which C denotes an arbitrary constant, and also:

$$\tfrac{1}{2}\varphi+r\frac{\partial\varphi}{\partial r}=\tfrac{1}{4}(\kappa+C).$$

Observe that the functions u, v, w will not change if one varies the function φ by a constant. That constant can be taken in such a way that the other constant C disappears from the preceding equation. One will then have:

$$\tfrac{1}{2}\varphi+r\frac{\partial\varphi}{\partial r}=\tfrac{1}{4}\kappa. \tag{18}$$

One must then have $\Delta^2\varphi = 0$. However, as one knows, there always exists a function that is uniform in the space in which the function κ, which satisfies those two equations, is like that. When one gets φ from formula (17), one will get the functions λ, μ, n, which must necessarily satisfy the equation $\Delta^2 = 0$.

The theorem is then proved. What is more, one finds equation (18), which governs the two functions φ, κ.

II.

1. – Consider an isotropic, elastic solid that is bounded by a surface σ. Let E its modulus of normal elasticity, and let m be its coefficient of contraction. An infinitesimal force $F\,d\sigma$ is applied to each element $d\sigma$ of its surface in such a way the solid will submit to a certain deformation.

Refer the solid to a system of orthogonal axes O (x, y, z) and say that for any of its points whose coordinates are x, y, z:

$$\xi,\quad \eta,\quad \zeta$$

are the components of the displacement, and:

$$\begin{matrix} T_{xx}\,, & T_{yy}\,, & T_{zz}\,, \\ T_{yz}\,, & T_{zx}\,, & T_{xy} \end{matrix}$$

are the normal and tangential internal stresses, resp.

Write T_{yz} or T_{zy} , etc., indifferently.

The internal stresses are coupled to the displacements by the formulas:

$$
\left.\begin{aligned}
2\,(1+m)\,T_{xx} &= E\left(\frac{2m}{1-2m}\Theta + 2\frac{\partial\xi}{\partial x}\right),\\
2\,(1+m)\,T_{yy} &= E\left(\frac{2m}{1-2m}\Theta + 2\frac{\partial\xi}{\partial y}\right),\\
2\,(1+m)\,T_{zz} &= E\left(\frac{2m}{1-2m}\Theta + 2\frac{\partial\xi}{\partial z}\right),\\
2\,(1+m)\,T_{yz} &= E\left(\frac{\partial\zeta}{\partial y} + \frac{\partial\eta}{\partial z}\right),\\
2\,(1+m)\,T_{zx} &= E\left(\frac{\partial\xi}{\partial z} + \frac{\partial\zeta}{\partial x}\right),\\
2\,(1+m)\,T_{xy} &= E\left(\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\right),
\end{aligned}\right. \tag{19}
$$

in which one sets:

$$\Theta = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z},$$

for simplicity.

If one sums the first three and sets:

$$T_{xx} + T_{yy} + T_{zz} = T$$

then one will get:

$$\frac{E}{1-2m}\Theta = T.$$

Therefore, one can also write:

$$2\,(1+m\,)\,T_{xx} = 2mT + E\left(\frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial x}\right),$$

$$2\,(1+m\,)\,T_{yy} = 2mT + E\left(\frac{\partial\eta}{\partial y} + \frac{\partial\eta}{\partial y}\right),$$

$$2\,(1+m\,)\,T_{zz} = 2mT + E\left(\frac{\partial\zeta}{\partial z} + \frac{\partial\zeta}{\partial z}\right).$$

The six stresses are coupled by the three equations:

$$\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}=0,$$

$$\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}=0,$$

$$\frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}=0,$$

which represent the equilibrium conditions inside the deformed solid, under the hypothesis that no volume forces act upon its elements. The following equations must be satisfied on the surface:

$$T_{xx}\cos\alpha+T_{xy}\cos\beta+T_{xz}\cos\gamma=-F_x\,,$$

$$T_{yx}\cos\alpha+T_{yy}\cos\beta+T_{yz}\cos\gamma=-F_y\,,$$

$$T_{zx}\cos\alpha+T_{zy}\cos\beta+T_{zz}\cos\gamma=-F_z\,,$$

in which F_x , F_y , F_z represent the components of the external force, and α, β, γ are the angles of the normal, which points inside of σ.

If we avail ourselves of formulas (19) and express the stresses in terms of the components of the displacements in equations (22) then we will get:

$$\Delta^2\xi+\frac{1}{1-2m}\frac{\partial\Theta}{\partial x}=0,$$

$$\Delta^2\eta+\frac{1}{1-2m}\frac{\partial\Theta}{\partial y}=0, \tag{23}$$

$$\Delta^2\zeta+\frac{1}{1-2m}\frac{\partial\Theta}{\partial z}=0.$$

If one differentiates the first of these with respect to x, the second with respect to y, and the third one with respect to z and sums then one will get:

$$\Delta^2\,\Theta=0.$$

Finally, if one eliminates the three functions ξ, η, ζ from equations (19) and (23) then one will get the following six equations:

$$\Delta^2 T_{xx} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial x^2}, \qquad \Delta^2 T_{yz} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial y\,\partial z},$$

$$\Delta^2 T_{yy} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial y^2}, \qquad \Delta^2 T_{zx} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial z\,\partial x}, \tag{24}$$

$$\Delta^2 T_{zz} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial z^2}, \qquad \Delta^2 T_{zy} = -\frac{1}{1+m}\frac{\partial^2 T}{\partial x\,\partial y}.$$

The function T, like Θ, which differs from it by a constant factor, satisfies the equation $\Delta^2 = 0$.

III.

1. – Let an elastic sphere be given whose radius is R and whose center is supposed to be located at the coordinate origin. The sphere will submit to a certain deformation under the action of the tensions that act upon its surface. If the deformation of its surface is given, or if one knows the components:

$$\xi_\sigma, \qquad \eta_\sigma, \qquad \zeta_\sigma$$

of the displacement at any of its point, then one would like to see how one can determine the functions ξ, η, ζ for any point of the sphere.

Equations (23) must be satisfied, or the three functions $\Delta^2\xi$, $\Delta^2\eta$, $\Delta^2\zeta$ must be the derivatives with respect to x, y, z, resp., of the function $-\frac{1}{1-2m}\Theta$, which satisfies the equation $\Delta^2 = 0$. Hence, if one applies the theorem that was proved before (I.**4**) then one can set:

$$\xi = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial\varphi}{\partial x} + \lambda,$$

$$\eta = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial\varphi}{\partial y} + \mu, \tag{25}$$

$$\zeta = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial\varphi}{\partial z} + \nu,$$

in which the functions φ, λ, μ, ν satisfy the equation $\Delta^2 = 0$. The relation that is expressed by formula (18) must exist between the two functions φ, $-\frac{1}{1-2m}\Theta$, or:

$$\tfrac{1}{2}\varphi + r\frac{\partial\varphi}{\partial r} = -\frac{1}{4(1-2m)}\Theta. \tag{26}$$

However, if one recalls (20) then the preceding formulas will give:

$$\Theta = 2\left(x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right) + \frac{\partial\lambda}{\partial x} + \frac{\partial\mu}{\partial y} + \frac{\partial\nu}{\partial z},$$

or, if one takes:

$$\Phi = \frac{\partial\lambda}{\partial x} + \frac{\partial\mu}{\partial y} + \frac{\partial\nu}{\partial z},$$

for the same of simplicity:

$$\Theta = 2r\frac{\partial\varphi}{\partial r} + \Phi .$$

Therefore, if one substitutes this in equation (26) then one will have:

$$\tfrac{1}{2}\varphi + r\frac{\partial\varphi}{\partial r} = -\frac{1}{4(1-2m)}\left(2r\frac{\partial\varphi}{\partial r} + \Phi\right),$$

or also:

$$\frac{1-2m}{3-4m}\varphi + r\frac{\partial\varphi}{\partial r} = -\frac{1}{2(3-4m)}\Phi. \tag{27}$$

We shall now address the determination of the four functions φ, λ, μ, ν.

On the surface of the sphere, one will have:

$$\xi = \lambda, \qquad \eta = \mu, \qquad \zeta = \nu,$$

so one must have:

$$\lambda = \xi_\sigma, \quad \mu = \eta_\sigma, \quad \nu = \zeta_\sigma .$$

However, the functions λ, μ, ν satisfy the equation $\Delta^2 = 0$. Therefore, one can determine those functions for all points of the sphere under certain conditions that the quantities ξ_σ, ξ_σ, ζ_σ must satisfy.

Let M be any of its points, so one will have:

$$\lambda = \frac{1}{2\pi R}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,\xi_\sigma\,d\sigma,$$

$$\mu = \frac{1}{2\pi R}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,\eta_\sigma\,d\sigma, \tag{28}$$

$$\nu = \frac{1}{2\pi R}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,\zeta_\sigma\,d\sigma,$$

in which ξ_σ, ξ_σ, ζ_σ are the components of the displacement of a point *N* of σ, *r* is the distance from the center *O* of the sphere to the point *M*, and ω is the angle $\measuredangle MON$.

We now move on to the determination of the functions φ. We shall get that from equation (27). Set:

$$\frac{1-2m}{3-4m} = c, \qquad -\frac{1}{2(3-4m)} = A,$$

for simplicity.

We will have the equation:

$$c\,\varphi + r\,\frac{\partial\varphi}{\partial r} = A\,\Phi.$$

The constant *m* is found between 0 and 1 / 2. Therefore, the constant *c* is positive. In addition, the function Φ, like the functions λ, μ, ν, upon which it depends, must satisfy the equation $\Delta^2 = 0$. The function φ must also satisfy that equation. One can then apply formula (4) and have:

$$\varphi = A\frac{1}{r^c}\int_0^r r^{c-1}\Phi\,dr\,. \tag{29}$$

We have thus determined all four functions that are included in the right-hand sides of formulas (25).

2. – In order to calculate the derivatives of the function φ with respect to the variables *x*, *y*, *z*, we shall first see how to express the function Φ.

It is given by the formula:

$$\Phi = \frac{\partial\lambda}{\partial x} + \frac{\partial\mu}{\partial y} + \frac{\partial\nu}{\partial z}.$$

Take the first of formulas (28):

$$\lambda = \frac{1}{2\pi R}\int_\sigma \frac{(R^2 - r^2)\,\xi_\sigma}{(R^2 + r^2 - 2R\,r\cos\omega)^{3/2}}\,d\sigma\,.$$

If *x*, *y*, *z* denote the coordinates of the point *M*, and *X*, *Y*, *Z* denote the coordinates of the point *N* then one will have:

$$\left.\begin{aligned} r^2 &= x^2 + y^2 + z^2,\\ R\,r\cos\omega &= x\,X + y\,Y + z\,Z. \end{aligned}\right\} \tag{30}$$

Therefore, the preceding formula will imply that:

$$\frac{\partial\lambda}{\partial x} = \frac{1}{2\pi R}\int_\sigma \frac{-2x\,\xi_\sigma (R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{3/2} - 3(R^2 + r^2)(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)(x - X)\,\xi_\sigma}{(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{5/2}}\,d\sigma\,,$$

or

$$\frac{\partial \lambda}{\partial x} = -\frac{1}{2\pi R}\int_\sigma \frac{(5R^2 + r^2 - 4\,\mathrm{R}\,r\cos\omega)^{3/2}\,x\,\xi_\sigma - 3(R^2 + r^2)X\,\xi_\sigma}{(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{5/2}}\,d\sigma\,.$$

If one does the same thing with the other two formulas (28) and sums then one will get:

$$\Phi =$$

$$-\frac{1}{2\pi R}\int_\sigma \frac{(5R^2 - r^2 - 4\,\mathrm{R}\,r\cos\omega)^{3/2}(x\,\xi_\sigma + y\eta_\sigma + z\zeta_\sigma) - 3(R^2 + r^2)(X\,\xi_\sigma + Y\eta_\sigma + Z\zeta_\sigma)}{(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{5/2}}\,d\sigma\,. \quad (31)$$

Now, let κ_σ denote the displacement of a point of σ that has the components ξ_σ, η_σ, ζ_σ, and let δ, ε denote the angles that its direction makes with the directions *OM*, *ON*, resp. One will obviously have:

$$\begin{aligned} x\,\xi_\sigma + y\,\eta_\sigma + z\,\zeta_\sigma &= r\,\kappa_\sigma\cos\delta, \\ X\,\xi_\sigma + Y\,\eta_\sigma + Z\,\zeta_\sigma &= R\,\kappa_\sigma\cos\varepsilon\,. \end{aligned} \quad (32)$$

Hence, formula (31) can be written:

$$\Phi = -\frac{1}{2\pi R}\int_\sigma \frac{(5R^2 + r^2 - 4\,\mathrm{R}\,r\cos\omega)^{3/2}\,r\cos\delta - 3(R^2 + r^2)R\cos\varepsilon}{(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{5/2}}\,\kappa_\sigma\,d\sigma\,. \quad (33)$$

Set:

$$\frac{(5R^2 + r^2 - 4\,\mathrm{R}\,r\cos\omega)^{3/2}\,r\cos\delta - 3(R^2 + r^2)R\cos\varepsilon}{(R^2 + r^2 - 2\,\mathrm{R}\,r\cos\omega)^{5/2}} = H,$$

for simplicity, and get the formula:

$$\Phi = -\frac{1}{2\pi R}\int_\sigma H\,\kappa_\sigma\,d\sigma\,. \quad (34)$$

Having said that, calculate the derivatives of the function φ with respect to the variables x, y, z. As we have seen before (I.**2**), we will have:

$$\frac{\partial \varphi}{\partial x} = A\,\frac{1}{r^{c+1}}\int_0^r r^c\,\frac{\partial \Phi}{\partial x}\,dr\,, \qquad \ldots$$

However, one gets from formula (34) that:

$$\frac{\partial \Phi}{\partial x} = -\frac{1}{2\pi R}\int_\sigma \frac{\partial H}{\partial x}\,\kappa_\sigma\,d\sigma\,, \quad \ldots,$$

and the derivatives of the function H with respect to the variables x, y, z are easily obtained by observing that they are included in r^2, $r\cos\omega$, $r\cos\delta$. Indeed, one has:

$$r^2 = x^2 + y^2 + z^2,$$

$$r \cos \omega = \frac{1}{R}(xX + yY + zZ),$$

$$r \cos \delta = \frac{1}{R}(x\,\xi_\sigma + y\,\eta_\sigma + z\,\zeta_\sigma)\,.$$

One will then have:

$$\frac{\partial \varphi}{\partial x} = -A\,\frac{1}{r^{c+1}}\int_0^r r^c dr \int_\sigma \frac{\partial H}{\partial x}\kappa_\sigma d\sigma\,,\ \ldots,$$

as well as:

$$\frac{\partial \varphi}{\partial x} = -\frac{A}{2\pi R}\int_\sigma \left[\frac{1}{r^{c+1}}\int_0^r r^c \frac{\partial H}{\partial x} dr\right]\kappa_\sigma d\sigma\,,$$

and analogously:

$$\frac{\partial \varphi}{\partial y} = -\frac{A}{2\pi R}\int_\sigma \left[\frac{1}{r^{c+1}}\int_0^r r^c \frac{\partial H}{\partial y} dr\right]\kappa_\sigma d\sigma\,, \qquad (35)$$

$$\frac{\partial \varphi}{\partial z} = -\frac{A}{2\pi R}\int_\sigma \left[\frac{1}{r^{c+1}}\int_0^r r^c \frac{\partial H}{\partial z} dr\right]\kappa_\sigma d\sigma\,.$$

If one takes:

$$\frac{1}{(R^2 + r^2 - 2R\,r\cos\omega)^{3/2}} = K,$$

for simplicity, then formulas (28) can be written:

$$\lambda = \frac{R^2 - r^2}{2\pi R}\int_\sigma K\,\xi_\sigma d\sigma\,,$$

$$\mu = \frac{R^2 - r^2}{2\pi R}\int_\sigma K\,\eta_\sigma d\sigma\,, \qquad (36)$$

$$\nu = \frac{R^2 - r^2}{2\pi R}\int_\sigma K\,\zeta_\sigma d\sigma\,.$$

Finally, if one substitutes the expressions that are found for $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \varphi}{\partial y}$, $\frac{\partial \varphi}{\partial z}$, λ, μ, ν in formulas (25) then one will get:

$$\xi = \frac{R^2 - r^2}{2\pi R}\int_\sigma \left[\frac{A}{r^{c+1}}\int_0^r r^c \frac{\partial H}{\partial x} dr \cdot \kappa_\sigma + K\xi_\sigma\right] d\sigma\,,$$

$$\eta = \frac{R^2 - r^2}{2\pi R} \int_\sigma \left[\frac{A}{r^{c+1}} \int_0^r r^c \frac{\partial H}{\partial y} dr \cdot \kappa_\sigma + K\eta_\sigma \right] d\sigma ,$$

$$\zeta = \frac{R^2 - r^2}{2\pi R} \int_\sigma \left[\frac{A}{r^{c+1}} \int_0^r r^c \frac{\partial H}{\partial z} dr \cdot \kappa_\sigma + K\zeta_\sigma \right] d\sigma .$$

The problem is then solved.

IV.

1. – We now pass on to the second of the two problems that we proposed to solve. We address the determination of the deformation of a sphere when one knows the components F_x , F_y , F_z for any point of its surface.

Consider the three functions:

$$\begin{aligned} U &= x\, T_{xx} + y\, T_{xy} + z\, T_{xz} , \\ V &= x\, T_{yx} + y\, T_{yy} + z\, T_{zz} , \\ W &= x\, T_{zx} + y\, T_{zy} + z\, T_{zz} , \end{aligned} \tag{37}$$

which, when divided by r, will give the components of the tension that acts upon the sphere of radius r that is concentric to the given sphere.

Say that the functions $\Delta^2 U$, $\Delta^2 V$, $\Delta^2 W$ are the derivatives with respect to x, y, z, resp., of a function that satisfies the equation $\Delta^2 = 0$.

One has, in fact:

$$\Delta^2 U = x\, \Delta^2\, T_{xx} + y\, \Delta^2\, T_{xy} + z\, \Delta^2\, T_{xz} + 2\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right),$$

$$\Delta^2 V = x\, \Delta^2\, T_{yx} + y\, \Delta^2\, T_{yy} + z\, \Delta^2\, T_{yz} + 2\left(\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} \right),$$

$$\Delta^2 W = x\, \Delta^2\, T_{zx} + y\, \Delta^2\, T_{zy} + z\, \Delta^2\, T_{zz} + 2\left(\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \right),$$

or, by virtue of formulas (22) and (24):

$$\Delta^2 U = -\frac{1}{1+m} \left(x \frac{\partial^2 T}{\partial x^2} + y \frac{\partial^2 T}{\partial x \partial y} + z \frac{\partial^2 T}{\partial x \partial z} \right),$$

$$\Delta^2 V = -\frac{1}{1+m}\left(x\frac{\partial^2 T}{\partial y\,\partial x} + y\frac{\partial^2 T}{\partial y^2} + z\frac{\partial^2 T}{\partial y\,\partial z}\right),$$

$$\Delta^2 W = -\frac{1}{1+m}\left(x\frac{\partial^2 T}{\partial z\,\partial x} + y\frac{\partial^2 T}{\partial z\,\partial y} + z\frac{\partial^2 T}{\partial z^2}\right),$$

or also:

$$\Delta^2 U = -\frac{1}{1+m}\frac{\partial}{\partial x}\left(x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} + z\frac{\partial T}{\partial z} - T\right),$$

$$\Delta^2 V = -\frac{1}{1+m}\frac{\partial}{\partial y}\left(x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} + z\frac{\partial T}{\partial z} - T\right),$$

$$\Delta^2 W = -\frac{1}{1+m}\frac{\partial}{\partial z}\left(x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} + z\frac{\partial T}{\partial z} - T\right).$$

The expressions $\Delta^2 U$, $\Delta^2 V$, $\Delta^2 W$ are therefore the derivatives of the function:

$$-\frac{1}{1+m}\left(r\frac{\partial T}{\partial r} - T\right),$$

which satisfies the equation $\Delta^2 = 0$, like T. That was to be proved.

One can then set:

$$U = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial \varphi}{\partial x} + \lambda,$$

$$V = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial \varphi}{\partial y} + \mu, \tag{38}$$

$$W = (x^2 + y^2 + z^2 - R^2)\,\frac{\partial \varphi}{\partial z} + \nu,$$

in which φ, λ, μ, ν are functions that satisfy the equation $\Delta^2 = 0$, and from formula (18), one will have:

$$\tfrac{1}{2}\varphi + r\frac{\partial \varphi}{\partial r} = -\frac{1}{4(1+m)}\left(r\frac{\partial T}{\partial r} - T\right). \tag{39}$$

We then have a first relation between the functions φ, T.

A second relation is obtained in the following way: Differentiate the first of equations (37) with respect to x, the second one with respect to y, and the third one with respect to z, and sum them. That will give:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}$$
$$= T_{xx} + T_{yy} + T_{zz} + x\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z}\right)+y\left(\frac{\partial T_{yx}}{\partial x}+\frac{\partial T_{yy}}{\partial y}+\frac{\partial T_{yz}}{\partial z}\right)+z\left(\frac{\partial T_{zx}}{\partial x}+\frac{\partial T_{zy}}{\partial y}+\frac{\partial T_{zz}}{\partial z}\right),$$

or, from formulas (21) and (22):

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=T.$$

On the other hand, from formulas (38), one can set:

$$\frac{\partial \lambda}{\partial x}+\frac{\partial \mu}{\partial y}+\frac{\partial \nu}{\partial z}=\Phi,$$

for simplicity, then one will get:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}= 2\left(x\frac{\partial \varphi}{\partial x}+y\frac{\partial \varphi}{\partial y}+z\frac{\partial \varphi}{\partial z}\right)+\Phi,$$

or

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=2r\,\frac{\partial \varphi}{\partial r}+\Phi.$$

One will then have:

$$T = 2r\,\frac{\partial \varphi}{\partial r}+\Phi,$$

and that is the second relation between φ and T.

One gets from this:

$$\frac{\partial T}{\partial r}= 2\frac{\partial \varphi}{\partial r}+2r\frac{\partial^2 \varphi}{\partial r^2}+\frac{\partial \Phi}{\partial r}.$$

Hence, when one replaces T and $\partial T / \partial r$ with the values that were found in equation (39), the last equation will become:

$$\tfrac{1}{2}\varphi+r\frac{\partial \varphi}{\partial r}=-\frac{1}{4(1+m)}\left(2r^2\frac{\partial^2 \varphi}{\partial r^2}+r\frac{\partial \varphi}{\partial r}-\Phi\right),$$

or, when one multiplies by 2 (1 + m):

$$(1 + m)\,\varphi + 2\,(1 + m)\,r\frac{\partial \varphi}{\partial r}+r^2\frac{\partial^2 \varphi}{\partial r^2}=\tfrac{1}{2}\left(\Phi-r\frac{\partial \Phi}{\partial r}\right). \tag{41}$$

Having said that, let us see how one can determine the four functions λ, μ, ν, φ.

We have already observed that the functions U / r, V / r, W / r represent the components of the tension that acts upon the elements of the spherical surface of radius r that is concentric to σ. Therefore, at any point N of σ, one will have:

$$U = R\,F_x\,, \qquad V = R\,F_y\,, \qquad W = R\,F_z\,,$$

and therefore:

$$\lambda = R\,F_x\,, \qquad \mu = R\,F_y\,, \qquad \nu = R\,F_z\,,$$

as well, and consequently if one supposes that certain conditions are satisfied by the quantities and one is given that ω is the angle $\measuredangle MON$ then one will have:

$$\lambda = \frac{1}{2\pi}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,F_x\,d\sigma\,,$$

$$\mu = \frac{1}{2\pi}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,F_y\,d\sigma\,,$$

$$\nu = \frac{1}{2\pi}\int_\sigma \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\omega)^{3/2}}\,F_z\,d\sigma$$

at any point M of the sphere. The function Φ, or $\dfrac{\partial\lambda}{\partial x} + \dfrac{\partial\mu}{\partial y} + \dfrac{\partial\nu}{\partial z}$, can be determined by a procedure that is identical to the one that was followed in the preceding problem. If one is given F as the force that is applied to a point N of σ and one lets δ, ε denote the angles that its direction makes with the directions OM, ON, resp., then one will get:

$$\Phi = -\frac{1}{2\pi}\int_\sigma \frac{(5R^2 - r^2 - 2Rr\cos\omega)\,r\cos\delta - 3(R^2 - r^2)\cos\varepsilon}{(R^2 - r^2 - 2Rr\cos\omega)^{5/2}}\,F\,d\sigma$$

at the point M of the sphere.

We must now find the function φ. It must satisfy the equation:

$$(1 + m)\,\varphi + 2\,(1 + m)\;r\frac{\partial\varphi}{\partial r} + r^2\frac{\partial^2\varphi}{\partial r^2} = \tfrac{1}{2}\left(\Phi - r\frac{\partial\Phi}{\partial r}\right),$$

along with:

$$\Delta^2\varphi = 0,$$

and it must be uniform in all of the sphere, in addition. However, we have seen (I.**3**) that there exists just one function that satisfies those conditions, and it was given by formula (14), or:

$$\varphi = \frac{\kappa}{r^p}\left\{\sin(q\log r + \varepsilon)\int_0^r r^{p-1}\sin(q\log r)\,\Phi\,dr + \cos(q\log r + \varepsilon)\int_0^r r^{p-1}\cos(q\log r)\,\Phi\,dr\right\}. \qquad (42)$$

In our case, we will have:

$$A = 1 + m, \qquad B = 2\,(1 + m).$$

Therefore, from formulas (15), we will have:

$$p = \tfrac{1}{2} + m, \qquad q = \tfrac{1}{2}\sqrt{3-4m^2}\,,$$
$$\kappa = -\sqrt{\frac{3(1+m)}{3-4m^2}}\,, \qquad \varepsilon = \arctan\frac{3+2m}{\sqrt{3-4m^2}}\,, \tag{43}$$

and if m is less than 1 / 2 then the constant q will be real.

We thus have to calculate the four functions λ, μ, ν, φ, and therefore know the three functions U, V, W from formulas (38). If we then replace φ in formula (40) with the expression that is found for it then we will have the function T. When the latter is multiplied by the constant factor $(1 - 2m) / E$, that will give the dilatation Θ at any point of the sphere.

2. – Now, recall the formulas:

$$U = x\,T_{xx} + y\,T_{xy} + z\,T_{xz}\,,$$

$$V = x\,T_{yx} + y\,T_{yy} + z\,T_{yz}\,,$$

$$W = x\,T_{zx} + y\,T_{zy} + z\,T_{zz}\,.$$

Replace the stresses with their expressions that were given by formula (19) and (19′) and set:

$$\frac{1}{E}\{2\,(1 + m)\,U - 2m\,x\,T\} = P,$$
$$\frac{1}{E}\{2\,(1 + m)\,V - 2m\,y\,T\} = Q, \tag{44}$$
$$\frac{1}{E}\{2\,(1 + m)\,W - 2m\,z\,T\} = R,$$

for simplicity. One will get:

$$x\left(\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial x}\right)+y\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)+z\left(\frac{\partial\xi}{\partial z}+\frac{\partial\zeta}{\partial x}\right)=P,$$

$$x\left(\frac{\partial\xi}{\partial x}+\frac{\partial\xi}{\partial x}\right)+y\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)+z\left(\frac{\partial\xi}{\partial z}+\frac{\partial\zeta}{\partial x}\right)=Q, \tag{45}$$

$$x\left(\frac{\partial\zeta}{\partial x}+\frac{\partial\xi}{\partial z}\right)+y\left(\frac{\partial\zeta}{\partial y}+\frac{\partial\eta}{\partial z}\right)+z\left(\frac{\partial\zeta}{\partial z}+\frac{\partial\zeta}{\partial z}\right)=R.$$

If we already know the functions U, V, W, T then the functions P, Q, R will be known. We shall now address the determination of the functions ξ, η, ζ. In order for them to prove to be determined completely, we shall pose the condition that the translation and rotation of the material particle that is found at the center of the sphere is zero, or that for $r = 0$, we have:

$$\xi = 0, \qquad \eta = 0, \qquad \zeta = 0,$$

$$\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}=0, \qquad \frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}=0, \qquad \frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}=0.$$

If one differentiates the second of equations (45) with respect to z and the third one with respect to y and subtracts the former from the latter then one will get:

$$x\frac{\partial}{\partial x}\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)+z\frac{\partial}{\partial z}\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right) = \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},$$

or

$$\frac{\partial}{\partial r}\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right) = \frac{1}{r}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right).$$

If one supposes that the component $\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}$ of the rotation is zero at the center of the sphere then when one integrates the last equation, one will get:

$$\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}=\int_0^r\frac{1}{r}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dr,$$

and analogously:

$$\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}=\int_0^r\frac{1}{r}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dr,$$

$$\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}=\int_0^r\frac{1}{r}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dr.$$

We will then know the components of the rotation at all points of the sphere.

Now observe that the first of equations (45) can be written:

$$2\left(x\frac{\partial\xi}{\partial x}+y\frac{\partial\xi}{\partial y}+z\frac{\partial\xi}{\partial z}\right)+y\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right)-x\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)=P,$$

and one will get the other ones in an analogous manner. If one takes the preceding formulas into account then:

$$2r\frac{\partial\xi}{\partial r}=P+\ z\int_0^r\frac{1}{r}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dr-y\int_0^r\frac{1}{r}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dr\,,\qquad\ldots \tag{46}$$

One finds the functions P, Q, R, which have been determined already, in the right-hand side of these equations. Therefore, if one recalls that the functions ξ, η, ζ must vanish for $r = 0$ then one will easily obtain their values at all points of the sphere.

Equation (46) and the analogous ones can be transformed, as before, by setting:

$$\frac{x}{r}=\cos\alpha,\qquad\frac{y}{r}=\cos\beta,\qquad\frac{z}{r}=\cos\gamma$$

for any point of the sphere and observing that the angles α, β, γ will remain constant along the same radius. If one divides by $2r$ then one will have:

$$\frac{\partial\xi}{\partial r}=\frac{P}{2r}+\frac{1}{2}\int_0^r\frac{1}{r}\left\{\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\gamma-\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\beta\right\}dr\,,\ \ldots$$

We now go back to letting x / r, y / r, z / r denote the cosines of the angles α, β, γ, and set:

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=H_1\,,\qquad\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=H_2\,,\qquad\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=H_3\,,$$

in addition.

One has:

$$\frac{\partial\xi}{\partial r}=\frac{P}{2r}+\frac{1}{2}\int_0^r\frac{1}{r^2}(zH_2-yH_3)\,dr\,,\qquad\ldots$$

If one recalls that the displacement of the center of the sphere is assumed to be zero then when one integrates, one will finally get the formulas:

$$\xi=\int_0^r\left[\frac{P}{2r}+\frac{1}{2}\int_0^r\frac{1}{r^2}(zH_2-yH_3)\,dr\right]dr\,,$$

$$\eta=\int_0^r\left[\frac{Q}{2r}+\frac{1}{2}\int_0^r\frac{1}{r^2}(xH_3-zH_1)\,dr\right]dr\,,$$

$$\zeta = \int_0^r \left[\frac{R}{2r} + \frac{1}{2} \int_0^r \frac{1}{r^2} (yH_1 - xH_2)\, dr \right] dr \, ,$$

and with that, the problem is solved.