"Sur une forme générale des équations de la dynamique," J. reine angew. Math. 121 (1900), 310-319.

On a general form of the equations of dynamics

(By **Paul Appell** at St. Germain-en-Laye)

Translated by D. H. Delphenich

1.

The **Lagrange** equations are not applicable when certain constraints are expressed by non-integrable differential relations or when one introduces parameters that are coupled with the coordinate non-integrable differential relations. That difficulty has been the subject of various studies, and one will find a detailed bibliography in an article that I just published in the collection *Scientia* (Carré and Naud, editors) that was entitled "Les mouvements de roulement en dynamique."

We propose to indicate a general form for the equation of motion here that is not subject to the exceptions that we just stated. In order to write the equations in that new form, it will suffice to calculate the function:

$$S = \tfrac{1}{2}\sum m\,J^2\,,$$

in which m denotes the mass of any of the points of the system, and J denotes that absolute acceleration of that point: One sees that this function S is composed of the accelerations in the same way that one-half the *vis viva* is composed of the velocities.

We have indicated the principle of the method that follows here in a note that was published in the Comptes Rendus des Séances de l'Académie des Sciences de Paris on 7 August 1899.

2.

Imagine a system that is subject to constraints such that in order to obtain the most general virtual displacement that is compatible with the constraints at the instant t, it will suffice to subject the n parameters q_1, q_2, …, q_n to arbitrary variations δq_1, δq_2, …, δq_n. If we then call the coordinates of any of the points of the system with respect to the fixed axes x, y, z then the virtual displacement of that point will have projections onto those axes that are:

$$(1)\qquad \begin{cases} \delta x = a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_n\,\delta q_n, \\ \delta y = b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_n\,\delta q_n, \\ \delta z = c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_n\,\delta q_n, \end{cases}$$

in which δq_1, δq_2, …, δq_n are arbitrary. In those formulas, the coefficients a_1, a_2, …, c_n can depend upon time t, the parameters q_1, q_2, …, q_n, and *some other parameters* q_{n+1}, q_{n+2}, …, q_{n+p} whose variations are coupled with those of the q_1, q_2, …, q_n by relations of the form:

$$\text{(2)} \qquad \left\{\begin{array}{l} \delta q_{n+1} = \alpha_1\,\delta q_1 + \alpha_2\,\delta q_2 + \cdots + \alpha_n\,\delta q_n, \\ \delta q_{n+2} = \beta_1\,\delta q_1 + \beta_2\,\delta q_2 + \cdots + \beta_n\,\delta q_n, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \delta q_{n+p} = \lambda_1\,\delta q_1 + \lambda_2\,\delta q_2 + \cdots + \lambda_n\,\delta q_n, \end{array}\right.$$

in which the coefficients α_1, α_2, …, λ_n likewise depend upon t and the set of parameters q_1, q_2, …, q_n, q_{n+1}, q_{n+2}, …, q_{n+p}. Under those conditions, the real displacement of a system during the time dt will be defined by relations of the form:

$$\text{(3)} \qquad \left\{\begin{array}{l} dx = a_1\,dq_1 + a_2\,dq_2 + \cdots + a_n\,dq_n + a\,dt, \\ dy = b_1\,dq_1 + b_2\,dq_2 + \cdots + b_n\,dq_n + b\,dt, \\ dz = c_1\,dq_1 + c_2\,dq_2 + \cdots + c_n\,dq_n + c\,dt, \end{array}\right.$$

with

$$\text{(4)} \qquad \left\{\begin{array}{l} dq_{n+1} = \alpha_1\,dq_1 + \alpha_2\,dq_2 + \cdots + \alpha_n\,dq_n + \alpha\,dt, \\ dq_{n+2} = \beta_1\,dq_1 + \beta_2\,dq_2 + \cdots + \beta_n\,dq_n + \beta\,dt, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dq_{n+p} = \lambda_1\,dq_1 + \lambda_2\,dq_2 + \cdots + \lambda_n\,dq_n + \gamma\,dt, \end{array}\right.$$

in which the coefficients $a, b, c, \alpha, \beta, \ldots, \lambda$ can depend upon t, q_1, q_2, …, q_{n+p}.

One can then obtain the equations of motion as follows:

The general equation of dynamics, which is deduced from **d'Alembert**'s principle and the principle of virtual work, is:

$$\text{(5)} \qquad \sum m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \sum\,(X\,\delta x + Y\,\delta y + Z\,\delta z),$$

in which x'', y'', z'' are the second derivatives of the coordinates with respect to time, and X, Y, Z are the projections of any of the forces.

That equation can be true for all displacements (1) that are compatible with the constraints: They will then decompose into the following n equations:

$$\text{(6)} \qquad \left\{\begin{array}{l} \sum m\,(x'' a_1 + y'' b_1 + z'' c_1) = \sum (X\,a_1 + Y\,b_1 + Z\,c_1), \\ \sum m\,(x'' a_2 + y'' b_2 + z'' c_2) = \sum (X\,a_2 + Y\,b_2 + Z\,c_2), \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum m\,(x'' a_n + y'' b_n + z'' c_n) = \sum (X\,a_n + Y\,b_n + Z\,c_n). \end{array}\right.$$

The right-hand sides of those equations are calculated as they are for the **Lagrange** equations. Upon replacing δx, δy, δz with their values (1), one will have:

$$\sum (X\ \delta x + Y\ \delta y + Z\ \delta z) = Q_1\ \delta q_1 + Q_2\ \delta q_2 + \ldots + Q_n\ \delta q_n$$

for the sum of the virtual works done by applied forces. The quantities $Q_1, Q_2, \ldots, Q_n$ are the right-hand sides of equations (6):

$$Q_1 = \sum (X\ a_1 + Y\ b_1 + Z\ c_1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In order to calculate the left-hand sides, divide the relations (3) that define the real displacement by dt and let $x', y', z', q_1', q_2', \ldots, q_n'$ denote the total derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}$. We have:

$$\begin{aligned} x' &= a_1 q_1' + a_2 q_2' + \cdots + a_n q_n' + a\,, \\ y' &= b_1 q_1' + b_2 q_2' + \cdots + b_n q_n' + b\,, \\ z' &= c_1 q_1' + c_2 q_2' + \cdots + c_n q_n' + c\,, \end{aligned}$$

in which the unwritten terms do not contain $q_1'', q_2'', \ldots, q_n''$. However, one will then obviously have:

$$a_1 = \frac{\partial x''}{\partial q_1''}, \qquad b_1 = \frac{\partial y''}{\partial q_1''}, \qquad c_1 = \frac{\partial z''}{\partial q_1''},$$

$$a_2 = \frac{\partial x''}{\partial q_2''}, \qquad b_2 = \frac{\partial y''}{\partial q_2''}, \qquad c_2 = \frac{\partial z''}{\partial q_2''},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The equations of motion are then written:

$$(8) \qquad \left\{ \begin{aligned} &\sum m \left(x'' \frac{\partial x''}{\partial q_1''} + y'' \frac{\partial x''}{\partial q_1''} + z'' \frac{\partial x''}{\partial q_1''} \right) = Q_1, \\ &\sum m \left(x'' \frac{\partial x''}{\partial q_2''} + y'' \frac{\partial x''}{\partial q_2''} + z'' \frac{\partial x''}{\partial q_2''} \right) = Q_2, \end{aligned} \right.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now consider the function:

$$S = \tfrac{1}{2} \sum m\,(x''^2 + y''^2 + z''^2) = \tfrac{1}{2} \sum m\, J^2,$$

in which J is the absolute acceleration of the point m. The equations of motion (8) take the form:

$$\frac{\partial S}{\partial q_1''} = Q_1\,, \quad \frac{\partial S}{\partial q_2''} = Q_2\,, \;\ldots,\; \frac{\partial S}{\partial q_n''} = Q_n\,. \tag{9}$$

One sees that in order to write them out, it will suffice to calculate just the function S and to express it in such a manner that it no longer contains any other second derivatives than those of the parameters $q_1, q_2, \ldots, q_n$, whose variations are regarded as arbitrary. It can happen that when this function S is calculated as a function of the $q_1, q_2, \ldots, q_{n+p}$, it will contain their first derivatives $q_1', q_2', \ldots, q_{n+p}'$ and the second derivatives $q_1'', q_2'', \ldots, q_{n+p}''$. When the relations (4) are divided by dt, that will give $q_{n+1}', q_{n+2}', \ldots, q_{n+p}'$ as linear functions of $q_1', q_2', \ldots, q_n'$, and when one differentiates them with respect to time, one will likewise obtain $q_{n+1}'', q_{n+2}'', \ldots, q_{n+p}''$ as linear functions of $q_1'', q_2'', \ldots, q_n''$. One can always do that in such a way that the function S will no longer contain any other second derivatives than the $q_1'', q_2'', \ldots, q_n''$. Furthermore, it will contain those quantities in the second degree. Once the function S has been prepared in that way, one can write out equations (9). Those equations, when combined with the conditions (4), form a system of $n + p$ equations that define $q_1, q_2, \ldots, q_n$ as functions of time.

3.

For example, take a solid body that moves around a fixed point O and calculate the function S by referring the motion to a system of axes O, x, y, z that move along with the body in space. Let Ω denote the instantaneous rotation of the trihedron $Oxyz$ and let P, Q, R be its components along the axes. Let ω be the rotation of the body, and let p, q, r be its components. A molecule m of the body with coordinates x, y, z possesses an absolute velocity v whose projections are:

$$v_x = q\,z - r\,y, \quad \ldots$$

That molecule possesses an absolute acceleration J whose projections are:

$$J_x = \frac{d}{dt}v_x + Q\,v_x - R\,v_x\,, \;\ldots, \tag{10}$$

which would result from the fact that J is the absolute velocity of the point whose coordinates are v_x, v_y, v_z. Now, upon calling the derivatives of p, q, r with respect to time p', q', r', one will have:

$$\frac{dv_x}{dt} = q\frac{dz}{dt} - r\frac{dy}{dt} + z\,q' - y\,r', \;\ldots,$$

in which $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, which are projections of the relative velocity of the molecule with respect to the axes O, x, y, z, are:

$$\frac{dx}{dt} = qz - ry - (Q\,z - R\,y), \ldots \tag{11}$$

Indeed, the relative velocity is the geometric difference between the absolute velocity and the velocity of the frame. From that, one will have the following expression for J, which we arrange with respect to x, y, z :

$$J_x = -\,x\,(q^2 + r^2) + y\,[q\,(p - P) + pQ - r'] + z\,[r\,(p - P) + pR + q']. \tag{12}$$

One get J_y and J_z similarly, and finally:

$$2S = \sum m\,(J_x^2 + J_y^2 + J_z^2)\,.$$

In order to simplify this, we write out that sum by supposing that the axes O, x, y, z are the principal axes of inertia at the point O and calling the moments of inertia with respect to those axes A, B, C. Upon confining ourselves to the terms in p', q', r', we will have:

$$\tag{13} \left\{ \begin{array}{l} 2S = A\,p'^2 + B\,q'^2 + C\,r'^2 + 2[(C-B)\,q\,r + A(r\,Q - q\,R)]\,p' \\ \quad + 2\,[(A-C)\,r\,p + B\,(p\,R - r\,P)]\,q' + 2\,[(B-A)\,p\,q + C\,(q\,P - p\,Q)]\,r' + \cdots \end{array} \right.$$

Euler equations: Take the moving axes to be three axes that are *invariably linked* to the body and coincide with the three principal axes of inertia. We will then have:

$$P = p, \quad Q = q, \quad R = r,$$

$$2S = A\,p'^2 + B\,q'^2 + C\,r'^2 + 2\,(C - B)\,q\,r\,p' + 2\,(A - C)\,r\,p\,q' + 2\,(B - A)\,p\,q\,r' + \ldots$$

Call the sums of the moments of the applied forces with respect to the axes L, M, N, and let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

be the elementary angles through which the body must turn around the axes in order to go from one position to an infinitely-close one. We shall make λ, μ, ν play the role of the parameters q_1, q_2, …, q_n . One has, on the one hand:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = L\,\delta\lambda + M\,\delta\mu + N\,\delta\nu\,,$$

and on the other hand, the components p, q, r of the instantaneous rotation of the body are:

$$p = \frac{d\lambda}{dt} = \lambda', \qquad q = \frac{d\mu}{dt} = \mu', \qquad r = \frac{d\nu}{dt} = \nu'.$$

The function S is then:

$$S = \tfrac{1}{2}(A\lambda''^2 + B\mu''^2 + C\nu''^2) + (C - B)\,\mu'\nu'\lambda'' + (A - C)\,\nu'\lambda'\mu'' + (B - A)\,\lambda'\mu'\nu'' + \ldots,$$

in which the unwritten terms do not contain λ'', μ'', ν''. The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N.$$

For example, the first of them is written:

$$A\,\lambda'' + (C - B)\,\mu''\nu'' = L\,.$$

From the values of p, q, r, that is precisely one of **Euler**'s equations.

5.

Body of revolution suspended by a point *O* on its axis. – Draw a fixed axis $O\alpha$ through O and take the axis Oz to be the axis of revolution, the axis Oy to be the perpendicular to the plane αOz, and the axis Ox to be the perpendicular to the plane yOz. When the position of the trihedron $Oxyz$ is known, in order to get the position of the body, it will suffice to know the angle φ that Oy makes with a ray that issues from O and is invariably coupled with the body in the xy-plane. The derivative φ' of that angle with respect to time represents the proper rotation of the body around Oz. The rotation ω of the body is then the resultant of the rotation Ω of the trihedron and the rotation φ'. One will then have:

$$p = P, \qquad q = Q, \qquad r = R + \varphi'.$$

Since $A = B$, the function S that is defined by the expression (13) will then become:

$$2S = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(p\,q' - q\,p') + \ldots \tag{14}$$

Once more, let $\delta\lambda$, $\delta\mu$, $\delta\nu$ be the elementary angles through which one must turn the body around the axes Ox, Oy, Oz in order to take it from one position to a neighboring one, and let L, M, N be the moments of the forces with respect to the axes, so one will have:

$$p = \lambda'', \qquad q = \mu'', \qquad r = \nu'',$$

as above, and the equations of motion will be:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N \,;$$

i.e., since the component R of the rotation Ω does not depend upon λ'', μ'', ν'':

$$\begin{aligned} A\,p' - (AR - Cr)\,q &= L, \\ A\,q' + (AR - Cr)\,p &= M, \\ C\,r' &= N. \end{aligned}$$

6.

Hoop. – In order to calculate the function S relative to an arbitrary system, one can employ a theorem that is analogous to one by **Koenig** for the calculation of *vis viva*. For example, take a hoop or a homogeneous disc of negligible thickness that is subject to roll in a horizontal plane. Call the radius of the hoop a and call its center G. Let $G\alpha$ be the ascending vertical that is drawn through G, and let Gz be the normal to the plane of the hoop; i.e., the axis of revolution of the body. We let Θ denote the angle $\alpha\,G\,z$.

As in the preceding example, take the axis Gy to be the perpendicular to the plane αGz and the Gx to be the perpendicular to the plane yGz. In that way, Gy is a horizontal in the plane of the hoop, and Gx is the line of greatest slope in that plane that starts from the point H where the hoop touches the fixed plane.

Take the mass of the hoop to be unity. Let J_0 denote the acceleration of the point G, and let J denote the relative acceleration of a point m on the hoop with respect to some axes with fixed directions that pass through G. Upon applying a theorem and is analogous to **Koenig**'s theorem, one will have:

$$\tfrac{1}{2}\sum m\,J^2 = \tfrac{1}{2}J_0^2 + \tfrac{1}{2}\sum m\,J^2,$$

which is a formula that we write:

$$S = \tfrac{1}{2}J_0^2 + S'.$$

The relative motion of the hoop around the point G is the motion of a body of revolution that is suspended by a point on its axis. Upon applying the notations of the preceding number to that motion, from (14), one will have:

$$2S' = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(p\,q' - q\,p') + \ldots$$

It then remains to calculate J_0^2. In order to do that, let u, v, w denote the projections of the absolute velocity of the point G onto the axes Gx, Gy, Gz. In order to express the idea that the hoop rolls, on must write out that the material point on the hoop that is found to be in contact with the base has a zero velocity at the point H. Since the velocity of that point is the resultant of its relative velocity around G and the velocity of the frame of G, one will then have:

(15) $$u = 0, \qquad v + r\,a = 0, \qquad w - qa = 0.$$

The coordinates of the point H with respect to the axes $Gxyz$ are indeed a, 0, 0.

Since the instantaneous rotation of the trihedron $Gxyz$ is Ω, the absolute acceleration of the point G will have the following projections onto the axes Gx, Gy, Gz :

$$\frac{du}{dt} + Qw - Rv,$$
$$\frac{dv}{dt} + Ru - Pw,$$
$$\frac{dw}{dt} + Pv - Qu\ ;$$

i.e., form (15):

$$q\,(Qq + Rr), \quad -ar' - aPq, \quad aq' - aPr,$$

and upon forming the sum of the squares and remarking that $P = p$, $Q = q$, one will have:

$$J_0^2 = a\,(q'^2 + r'^2) + 2a^2\,p\,(p\,r' - r\,q') + \ldots,$$

in which the terms that do not contain p', q', r' are not written out. Finally, one will then have:

$$2S = A\,p'^2 + (A + a^2)\,q'^2 + (C + a^2)\,r'^2 + 2\,(AR - Cr)\,(pq' - qp') + 2a^2\,p\,(qr' - rq') + \ldots$$

Once more, let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

denote the infinitely-small angles through which one must turn the hoop around the axes Gx, Gy, Gz, resp., in order to move it from one position to an infinitely-close one. Those quantities are arbitrary and determined completely by the displacement of the hoop. We take λ, μ, ν to be the parameters q_1, q_2, …, q_n, and we will once more have:

$$p' = \lambda'', \qquad q' = \mu'', \qquad r' = \nu''.$$

We can then write the left-hand sides of the equations of motion such as (9). It remains for us to calculate the right-hand sides. In order to do that, one can calculate the sum of the works done by the applied forces:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

and put it into the form:

$$L_1\,\delta\lambda + M_1\,\delta\mu + N_1\,\delta\nu\,.$$

L_1, M_1, N_1 will be the right-hand sides of the equations. Those quantities have a simple meaning: Draw three axes Hx_1, Hx_2, Hx_3 through the point of contact H with the base that are parallel to the axes Gx, Gy, Gz, resp. L_1, M_1, N_1 will be the sums of the moments of the applied forces with respect to those new axes, respectively. Indeed, the velocity of the molecule that is placed at H is zero for a displacement that is compatible with the constraints, so the infinitely-small displacement of the hoop is the resultant displacement of the three elementary rotations $\delta\lambda$, $\delta\mu$, $\delta\nu$ around the axes Hx_1, Hx_2, Hx_3, resp., without displacing H. That proves the proposition.

If the only applied force is the weight g applied to G then one will obviously have:

$$L_1 = 0, \qquad N_1 = 0,$$

$$M_1 = -g\,a\cos\Theta.$$

The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = 0, \qquad \frac{\partial S}{\partial \mu''} = -g\,a\cos\Theta, \qquad \frac{\partial S}{\partial \nu''} = 0\,;$$

i.e., from the value of S :

$$Ap' - (AR - Cr)\,q = 0,$$

$$(A + a^2)\,q' + (AR - Cr)\,p - a^2\,p\,r = -g\,a\cos\Theta,$$

$$(C + a^2)\,r' + a^2\,p\,q = 0.$$

Korteweg and myself have pointed out (almost at the same time) that integrating those equations comes down to integrating **Gauss**'s hypergeometric equation that follows from one quadrature. (See an article in the Rendiconti del Circolo Matematico di Palermo, which is followed by a letter by **Korteweg**, in the first fascicle of 1900.)

7.

In the preceding, we have deduced equations (9) from **d'Alembert**'s principle, along with the principle of virtual work. One can also attach it to **Gauss**'s principle of least constraint (Crelle's Journal, t. IV). Furthermore, things could not be otherwise, since, as **Gauss** pointed out, all principles of equilibrium and motion that are equivalent to the principle of virtual velocities and **d'Alembert**'s principle must necessarily be equivalent to each other.

If one forms the function:

$$R = S - (Q_1\,q_1'' + Q_2\,q_2'' + \cdots + Q_n\,q_n''),$$

which contains the symbols q'' in degree two, then one will see that the equations of motion (9) can be written:

$$(16) \qquad \frac{\partial R}{\partial q_1''} = 0, \qquad \frac{\partial R}{\partial q_2''} = 0, \qquad \ldots, \qquad \frac{\partial R}{\partial q_n''} = 0.$$

Those are the equations that one has to write down in order to find the values of q_1'', q_2'', …, q_n'' that make R a *minimum*. Conversely, the values of q'' that one infers from those equations will make R a minimum, because the terms in R that are homogeneous of degree two will come from S and constitute a positive-definite quadratic form. Since the values of q'' determine the acceleration, one can interpret that result by saying that *the values of the acceleration at each instant will make R a minimum.*

One can replace the function R in that statement with any other function that differs from it only by terms that are independent of the accelerations; for example, by the following two functions:

$$\tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2) - \sum (X\,x'' + Y\,y'' + Z\,z''),$$

$$\tfrac{1}{2}\sum \frac{1}{m}[(m\,x'' - X)^2 + (m\,y'' - Y)^2 + (m\,z'' - Z)^2].$$

The fact that the accelerations make the latter function a minimum is an immediate consequence of the **Gauss**'s principle of least constraint, as **A. Mayer** showed in an interesting article that was entitled "Ueber die Aufstellung der Differentialgleichungen der Bewegung für reibungslose Punktsysteme, die Bedingungsgleichungen unterworfen sind," and "Zur Regulierung der Stösse in reibungslosen Punktsystemen, die dem Zwange von Bedingungsgleichungen unterliegen." Printed in the Berichten der mathematisch-physikalischen Klasse der Königl. Sächs. Gesellschaft der Wissenschaft zu Leipzig. Session in 2 July 1899. That statement of **Gauss**'s principle that I gave, for my own part, following **Mayer**, in the Comptes Rendus in 11 September 1899, is already found in volume III of the works of **Hertz**, page 224 (Leipzig, 1894).