"Sur une forme générales des équations de la dynamique et sur le principe de Gauss," J. reine angew. Math. **122** (1900), 205-208.

# On a general form of the equations of dynamics and Gauss's principle

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We published a paper "Sur une forme générales des équations de la dynamique" on pp. 310 in v. 121 of this journal. We now ask permission to present two complementary remarks in regard to that subject about **Gauss**'s principle of least constraint, one of which is of a mathematical order, while the other is of a bibliographic order.

## 1.

The **Lagrange** equations are applicable when the constraints on a system without friction can be expressed in finite terms, and when one employs parameters that are true coordinates. Suppose, to simplify, that there exists a force function $U$. One can then write the equations of motion once one knows the expression for one-half the *vis viva* $T$ and $U$ as functions of the independent parameters.

On the contrary, if the constraints cannot all be expressed by relations in finite terms then one can no longer apply **Lagrange**'s equations. In order to write out the equations of motion, it suffices to know $U$ and the function $S = \frac{1}{2}\sum m J^2$, which is composed from the accelerations in the same way that $T$ is composed from the velocities. But is that necessary?

Might there not exist equations of motion that are more general than **Lagrange**'s that are applicable to all cases and require only that one must know the two functions $T$ and $U$ in order to write them down? We shall show that such equations do not exist. In order to do that, we shall indicate two different systems in which the functions $T$ and $U$ are identically the same, although the equations of motion are not the same.

**First system:** Imagine a ponderous solid that fulfills the following conditions:

1. The solid is bounded by a sharp edge that has the form of a circle $K$ of radius $a$.

2. The center of gravity $G$ of the body is situated at the center of the circle $K$.

3. The ellipsoid of inertia that relates to the center of gravity $G$ is an ellipsoid of revolution around the perpendicular $Gz$ to the plane of the circle.

Now suppose that the solid body, thus-constructed, is subject to rolling without slipping on a fixed horizontal plane and that it touches the circular edge $K$.

Let $G\alpha$ be the ascending vertical that is drawn through $G$, take the $Gy$ axis to be the perpendicular to the plane $\alpha Gz$, and the $Gx$ to be the perpendicular to the plane $yGz$. $Gy$ is then a horizontal to the plane of the circle $K$, and $Gx$ is a line of greatest slope to that plane that ends at the point where the circle touches the fixed plane. Let $\Theta$ denote the angle between $Gz$ and the ascending vertical $G\alpha$, and let $\psi$ be the angle between $Gy$ and a fixed horizontal. Those two angles determine the orientation of the trihedron $Gxyz$. In order for fix the position of the solid body with respect to the trihedron $Gayz$, it will suffice to know the angle $\varphi$ that a radius of the circle $K$, which invariable coupled with the body, makes with the axis $Gy$. The instantaneous rotation $\omega$ of the body is then the resultant of the rotation of the trihedron and a rotation $d\varphi\,/dt = \varphi'$ around $Gz$. The components $p$, $q$, $r$ of $\omega$ are then:

$$p = -\psi' \sin\Theta, \qquad q = \Theta', \qquad r = \psi' \cos\Theta + \varphi'.$$

On the other hand, the condition that the circle $K$ is rolling shows that the square of the velocity of the center of gravity $G$ will be $a^2\,(q^2 + r^2)$. By definition, if one takes the mass of the body to be unity and lets $A$ and $C$ denote the moments of inertia about $Gx$ and $Gy$, respectively, then one will have:

$$2T = a^2\,(q^2 + r^2) + A\,(p^2 + q^2) + C\,r^2,$$

so, one has:

$$(1)\qquad \begin{cases} 2T = A\psi'^2 \sin^2\Theta + (A + a^2)\,\Theta'^2 + (C + a^2)(\psi' \cos\Theta + \varphi')^2 \\ U = -ga \sin\Theta \end{cases}$$

for the defining expression for the functions $T$ and $U$.

**Second system:** Let a second ponderable body have the same form, the same radius $a$, and the same mass as before. Imagine that the distribution of the mass is different, in such a way that if one lets $A_1$ and $C_1$ denote the moments of inertia that are analogous to $A$ and $C$, resp., then one will have:

$$A_1 = A, \qquad C_1 = C + a^2.$$

Subject the body to the following two constraints: The body touches a fixed horizontal plane $P_1$ on which it slides without friction at the circular edge $K$. The center of gravity $G$ of the body slides without friction on a fixed vertical circumference whose radius is $a$ and whose center $O$ is in the fixed plane $P_1$.

In order to express those constraints, we take the same moving axes $Gxyz$ and the same notations as above. Let $x_1$, $y_1$, $z_1$ denote the absolute coordinates of the point $G$ with respect to the two axes $Ox_1$ and $Oy_1$ in the plane $P_1$ and an ascending vertical $Oz_1$. One can suppose that the fixed vertical circumference that is described by $G$ is in the plane $x_1Oz_1$ . One will then have:

$$\textit{First constraint:} \quad z_1 = a \sin \Theta,$$

$$\textit{Second one:} \quad y_1 = 0, \qquad x_1^2 + y_1^2 = a^2,$$

so one obviously has:

$$x_1 = a \cos \Theta.$$

Under those conditions, one has:

$$2T_1 = x_1'^2 + y_1'^2 + z_1'^2 + A_1 (p^2 + q^2) + C_1 r^2,$$

or, from the values of $x_1$ , $y_1$ , $z_1$ , $A_1$ , and $C_1$ :

$$(2) \qquad \begin{cases} 2T_1 = A\psi'^2 \sin^2\Theta + (A + a^2)\,\Theta'^2 + (C + a^2)(\psi' \cos\Theta + \varphi')^2, \\ U_1 = -\,ga \sin\Theta. \end{cases}$$

One sees that the functions $T$ and $T_1$, $U$ and $U_1$ are identical. Meanwhile, the equations of motion are different, since **Lagrange**'s equations apply to the second system and not to the first. That is what we would like to show.

One can point out that of the three equations of motion, two of them can be put into the same form in the two systems. Indeed, the integral of the *vis viva* is obviously the same for both of them. Moreover, as **Slesser** has already shown in an article in the Quarterly Journal of Mathematics (1873), one has the right to write down the **Lagrange** equation that relates to $\Theta$ for the first system, which one can obviously do for the second one. However, the third equations are different for the two motions: For the second system, one has the integral $r = r_0$ , which does not exist for the first one.

It is obvious that the difference between the two motions will appear immediately when one forms the two functions $S$ and $S_1$ by applying the formulas in our preceding paper. (See also Journal de Mathématiques pures et appliqués, first fascicle, 1900.)

## 2.

**Bibliographic notes.** At the end of the preceding paper, we gave some very quick and very incomplete indications in regard to the analytical statement of **Gauss**'s principle. **A. Mayer** of Leipzig has been most helpful in providing the following historical and bibliographic information: The analytical statement of **Gauss**'s principle was indicated already by **Jacobi** in a lecture that is no longer in print. It was given, independently of **Jacobi**, by **Scheffler** (Volume III of Schlömilch's Zeitschrift, pp. 197). It was found to be reproduced in **Mach** (*Die Mechanik in ihrer Entstehung historisch-kritisch dargestellt*, Laipzig, 1883), in **Hertz**, which we have cited, and in **Boltzmann** (*Vorlesungen über die Principe der Mechanik*, Leipzig, 1897). Finally, **J. Willard Gibbs**, in a beautiful paper "On the fundamental formulae of Dynamics" (American Journal of Mathematics, vol. II, 1879), gave the analytical statement of **Gauss**'s principle and some applications to various problems, and notably to the question of the rotation of solid bodies.