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FASCICLE 1.

On a general form of the equations of dynamics

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INTRODUCTION

Here, one must take the word *dynamics* in its old sense, namely, the sense of Galilei, Newton, Lagrange, d'Alembert, Carnot, Lavoisier, and Mayer.

As H. Poincaré said in his book *La valeur de la Science* (pp. 231):

> "Perhaps we must all construct a new mechanics that we can only glimpse in which inertia increases with velocity so the velocity of light will become an impassable obstacle. The simpler ordinary mechanics will remain a first approximation, since it will be true for velocities that are not very large, in such a way that we will again recover the old dynamics from the new one. We should not regret that we believed in those principles, and since very large velocities will never be anything but exceptions to the old formulas, we can even be most certain in practice that we can continue to work as if we still believed them. They are useful enough that they will still have their place. To wish to exclude them completely would be to deprive ourselves of a valuable weapon. In conclusion, I hasten to say that we have not reached that point and that nothing suggests that we will not leave it victorious and intact."

The equations that we have in mind then refer to the classical mechanics of today. As we will see, they apply regardless of the nature of the constraints, provided that the constraints are realized in such a fashion that the general equation of dynamics is exact.

One will see that in order to obtain those equations, one will be obliged to calculate the energy of acceleration of the system $S = \frac{1}{2}\sum m J^2$; i.e., to go to the second order of derivation with respect to time. If one would like to take the first order of derivation, like Lagrange, then one would be led to some very complicated equations that generalize those of Lagrange [**37**], and that one can call the Euler-Lagrange equations. That method was first studied by Volterra in 1898 ([**38**] and [**39**]). One can also consult the papers of Tzenoff [**46**] and Hamel [**47**]. We shall give some applications of it to some questions of rational mechanics. However, we hope that those equations can also be useful to physicists in the cases where the Lagrange equation and Hamilton's canonical equations, which are deduced from them, are no longer applicable.

For H. Poincaré: "The mathematician must not be simply a provider of formulas to the physicist. There must be a closer collaboration between them."

Along those lines, it is important to recall that Édouard Guillaume in Bern has applied the general equations that we shall develop to various physical theories ([**23**] and [**24**]).

I agree with Mach (Paris, Librarie Hermann, 1904, translation by Émile Betrand, with a preface by Émile Picard) when he said (pp. 465) that there exist no purely-mechanical phenomena and that all phenomena belong to all branches of physics:

> "The opinion that puts mechanics at the fundamental basis for all other branches of physics today, and according to which physical phenomena must have a *mechanical* explanation is, to me, a prejudice."

However, one must seek to explain the most possible physical phenomena *mechanically*, and then, as one has done up to now, abandon those phenomena in order to return to rational mechanics, and in that regard, to the general form that one gives to the equations that embraces more cases than the form that is due to Lagrange, which supposes that the constraints can be expressed in finite terms; i.e., in Hertz's terminology, that the constraints considered are *holonomic*. Now, one knows nothing about the constraints that are realized in the universe. H. Poincaré said: "It is a machine that is more complicated than all of those of industry, and almost all of its parts are hidden deeply from us." From the English mathematician Larmor, it is the principle of *least action* that seems to have persisted for the longest time. On the contrary, the general form that I shall present is attached to Gauss's principle of *least constraint* ([**1**], [**2**], [**3**], [**4**], [**5**], [**45**]), which Mach discussed on pages 343, *et seq.*, in the cited work. Notably, he said:

> "The examples that we just treated show that this theorem *does not represent an essentially-new concept*… The equations of motion will be the same (as they are from a direct application of the general equation of dynamics that results from the combination of d'Alembert's principle with the equation of virtual work), as one will see, moreover, by treating the same problems by d'Alembert's theorem and then by that of Gauss."

I think that the value of Gauss's principle is found in precisely that identity.

Mach's opinion is, moreover, that of Gauss himself, who said in presenting his theorem in volume IV of Crelle's Journal:

> "As one knows, the principle of virtual velocities transforms any problem in statics into a question of pure mathematics, and dynamics is, in turn, reduced to statics by d'Alembert's principle. It results from this that no fundamental principle of equilibrium and motion can be essentially distinct from the ones that we just cited, and that, be that as it may, one can always regard that principle as a more or less immediate consequence of the former ones.
>
> One must not conclude from this that any new theorem will be without merit. On the contrary, it will always be interesting and instructive to study the laws of nature from a new viewpoint that might then allow us to treat this or that particular question more simply or only obtain a much greater precision to the statements.
>
> The great geometer, whose has so brilliantly made the science of motion rest upon the principle of virtual velocities did not despair to perfect and generalize Maupertuis's principle, which relates to *least action*, and one knows that this principle is often employed by geometers in a very advantageous manner."

The great geometer that Gauss spoke of is Lagrange. One will find the works of Lagrange *on the principle of least action* on page 281 of the first volume of the third edition of *Mécanique analytique,* which was edited, corrected, and annotated by J. Bertrand (Mallet-Bachelier, 1853).

Among the applications of the general equations, I must cite the ones that Henri Beghin just made to the Anschütz and Sperry gyrostatic compass in a thesis that he presented to the science faculty in Paris [**29**] in November 1922.

I. – NATURE OF THE CONSTRAINTS.

1. Essentially holonomic or essentially non-holonomic systems. Order of a non-holonomic system. – Imagine a material system with k degrees of freedom that is composed of n points with masses m_μ ($\mu = 1, 2, \dots, n$) that have rectangular coordinates x_μ, y_μ, z_μ in an oriented trihedron of axes, and are animated with a motion of uniform, rectilinear translations with respect to axes that are considered to be fixed in classical mechanics. The displacements, velocities, and accelerations that we consider are displacements, velocities, and accelerations with respect to that trihedron.

In order to obtain the most general displacement of the system that is compatible with the constraints that exist at the instant t, it will suffice to vary k conveniently-chosen parameters $q_1, q_2, \dots, q_k$ by arbitrary infinitely-small quantities $\delta q_1, \delta q_2, \dots, \delta q_k$. One will then have that the virtual displacement of the point m_μ is:

$$(1)\qquad \left\{\begin{aligned} \delta x_\mu &= a_{\mu,1}\,\delta q_1 + a_{\mu,2}\,\delta q_2 + \cdots + a_{\mu,k}\,\delta q_k, \\ \delta y_\mu &= b_{\mu,1}\,\delta q_1 + b_{\mu,2}\,\delta q_2 + \cdots + b_{\mu,k}\,\delta q_k, \\ \delta z_\mu &= c_{\mu,1}\,\delta q_1 + c_{\mu,2}\,\delta q_2 + \cdots + c_{\mu,k}\,\delta q_k, \end{aligned}\right.$$

and for the actual displacement of the same point during the time interval dt, one has:

$$(2)\qquad \left\{\begin{aligned} dx_\mu &= a_{\mu,1}\,dq_1 + a_{\mu,2}\,dq_2 + \cdots + a_{\mu,k}\,dq_k + a_\mu\,dt, \\ dy_\mu &= b_{\mu,1}\,dq_1 + b_{\mu,2}\,dq_2 + \cdots + b_{\mu,k}\,dq_k + b_\mu\,dt, \\ dz_\mu &= c_{\mu,1}\,dq_1 + c_{\mu,2}\,dq_2 + \cdots + c_{\mu,k}\,dq_k + c_\mu\,dt. \end{aligned}\right.$$

In those equations, the coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$, a_μ, b_μ, c_μ ($\mu = 1, 2, \dots, n$; $\nu = 1, 2, \dots, k$) are arbitrary. They depend upon only the position of the system at the instant t and time t. The nature of the coefficients plays no role in the general case.

In Hertz's terminology, a system is called *holonomic* when the constraints that are imposed upon it are expressed by relations with finite terms between the coordinates that determine the positions of the various bodies that it is composed of. One can choose $q_1, q_2, \dots, q_k$ to be variables whose numerical values at the instant t determine the position of the system. The quantities $q_1, q_2, \dots, q_k$ are then the coordinates of the holonomic system whose position is determined by the figurative point whose rectangular coordinates in k-dimensional space are $q_1, q_2, \dots, q_k$. The coordinates x_μ, y_μ, z_μ are functions of the $q_1, q_2, \dots, q_k$ and time t that are expressible in finite terms, and the right-hand sides of equations (2) are total differentials of functions of a_μ, b_μ, c_μ and time t. The equations of motion will then take the form that was given by Lagrange. On the contrary, it can happen that the constraints between certain bodies of the system are expressed by *non-integrable* differential relations between the coordinates that the positions of those bodies depend upon. For example, that is what happens when a solid in the system is bounded by a surface or line that is subject to rolling without slipping on a fixed surface or on the surface of another solid of the system. Indeed, that constraint is expressed, in the former case, by writing that the velocity of the material point is *zero* at the point of contact, and in the latter, by writing that the velocities of two material points are the same at the point of contact. According to Hertz, one says that the system is not holonomic in those cases. Even if one supposes that the $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ can be expressed with the aid of only the variables $q_1, q_2, \ldots, q_k, t$, the right-hand sides of formulas (2) are not supposed to be exact differentials.

In the preceding, we, with Hertz, have considered only the systems themselves. In order to distinguish them we say that they are *essentially holonomic* or *essentially non-holonomic*. One can also define the nature of a system for a certain choice of parameters. In that regard, one can define the *order of a non-holonomic system for a choice of parameters.* There will then be two elements that one must address, namely, the material system and the choice of parameters. One says that a system is holonomic for a certain choice $q_1, q_2, \ldots, q_k$ if the Lagrange equations apply to all the parameters. One says the *order* of a non-holonomic system for a certain choice of parameters $q_1, q_2, \ldots, q_k$ to mean the number of parameters to which the Lagrange equations do not apply [**33**]. In nos. **15** and **16**, we shall see how that order can be determined when one defines the energy of the velocities $T = \frac{1}{2}\sum m\,V^2$ and the energy of the accelerations $S = \frac{1}{2}\sum m\,J^2$ for a system.

From that, a system that is *non-holonomic of order zero* for a certain choice of parameters will be *holonomic*.

The order can remain the same or change when one replaces the system of parameters $q_1, q_2, \ldots, q_k$ with another one.

Example. – Here is an example in which the order passes from 0 to 2. Take a system that is composed of just one point in the xOy plane with coordinates $x, y, 0$. It is a system with two degrees of freedom, so it is essentially holonomic. That system will be holonomic when one chooses the parameters for the coordinates of the point in an arbitrary system. For example, if one takes polar coordinates r, θ in the plane then one will have:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = 0,$$

$$T = \frac{m}{2}(x'^2 + y'^2 + z'^2) = \frac{m}{2}(r'^2 + r^2\,\theta'^2)\,.$$

Upon calling the component of the force $(X, Y, 0)$ along the perpendicular to the radius vector P and its component along the prolongation of the radius vector Q, one will have:

$$X\,\delta x + Y\,\delta y = Pr\,\delta\theta + Q\,\delta r\,.$$

The Lagrange equations apply to the parameters r and θ, but in place of θ, they take the area σ that is described by the radius vector as their parameter:

$$\delta\sigma = \tfrac{1}{2} r^2\,\delta\theta, \qquad d\sigma = \tfrac{1}{2} r^2\,d\theta,$$

$$(5) \qquad \left\{ \begin{aligned} & T = \frac{m}{2}\left(r'^2 + \frac{4\sigma'^2}{r^2}\right), \\ & X\,\delta x + Y\,\delta y = \frac{2P}{r}\delta\sigma + Q\,\delta r. \end{aligned} \right.$$

Neither of the two Lagrange equations apply, as one verifies immediately.

For the new choice of variables r and σ, the system is then non-holonomic of order 2.

One sees that the order of a non-holonomic system is defined with respect to a certain choice of parameters and that when one varies that choice, one can vary the *order*. However, there nonetheless exists *an essential order* that is attached to that system, which is the *minimum* ω of the orders that are obtained by varying the choice of parameters in an arbitrary way. For example an essentially-holonomic system is a non-holonomic system of essential order zero.

2. Examples: Top and hoop. – The two favorite toys of children – viz., the top and the hoop – provide examples of systems that are essentially holonomic or essentially non-holonomic. In order to show that, we first define the six coordinates of an entirely-free solid body (which is an essentially-holonomic system). Let $O\xi\eta\zeta$ be three rectangular fixed axes. Call the coordinates of the center of gravity G of the solid body with respect to those axes ξ, η, ζ. Let θ, φ, ψ be the Euler angles that a system of rectangular axes $Gxyz$ that are coupled with the body makes with the axes with fixed directions $Gx_1\,y_1\,z_1$ that are parallel to the fixed axes. Those six coordinates ξ, η, ζ, θ, φ, ψ, define the position of a free solid body. The coordinates of an arbitrary point of the body are functions of those six coordinates. If one imposes constraints on the solid then, depending upon the case, that will amount to establishing certain relations in finite terms between the six coordinates or also establishing certain non-integrable first-order differential relations. The number of degrees of freedom will then be diminished.

1. *Top. Essentially-holonomic system with five degrees of freedom.* – In the absence of friction or slipping, the top is a ponderous body of revolution whose axis is terminated by a point P that slides on a perfectly-polished fixed plane Π. If one takes the axis Gz to be the axis of revolution (when counted to be positive in the sense that goes from P to G), and one lets a denote the distance PG then one will have:

$$\zeta = a \cos\theta,$$

which is a constraint equation in finite terms. The position of the top is then defined by the five coordinates:

$$\xi,\ \eta,\ \theta,\ \varphi,\ \psi.$$

The coordinates of an arbitrary point on the top with respect to the fixed axes are expressed as functions of those five coordinates. The top is then an *essentially-holonomic* system; that system is holonomic for the choice of parameters $\xi, \eta, \theta, \varphi, \psi$.

2. *Hoop. Non-holonomic system with three degrees of freedom and essential order two.* – A hoop is a solid body of revolution that is bounded by a circular edge C that is subject to *rolling without slipping* on a fixed horizontal plane Π (one neglects friction while it rolls). The center of gravity G of the hoop is supposed to be in the plane of the edge C. The axes $Gxyz$ that are coupled to the body here will be the axis of the circle Gz, which is perpendicular to the plane of the edge, and two rectangular axes Gx and Gy that are situated in the plane of the edge; the radius of the edge C is a.

As one will see in *Traité de Mécanique* by P. Appell (t. II, no. 462), one will have:

$$\begin{cases} d\xi - a\sin\psi\sin\theta\,d\theta + a\cos\psi\cos\theta\,d\psi + a\cos\psi\,d\varphi = 0, \\ d\eta + a\cos\psi\sin\theta\,d\theta + a\sin\psi\cos\theta\,d\psi + a\sin\psi\,d\varphi = 0, \\ d\zeta - a\cos\theta\,d\theta = 0 \end{cases}$$

for the actual displacement and:

$$\begin{cases} \delta\xi - a\sin\psi\sin\theta\,\delta\theta + a\cos\psi\cos\theta\,\delta\psi + a\cos\psi\,\delta\varphi = 0, \\ \delta\eta + a\cos\psi\sin\theta\,\delta\theta + a\sin\psi\cos\theta\,\delta\psi + a\sin\psi\,\delta\varphi = 0, \\ \delta\zeta - a\cos\theta\,\delta\theta = 0 \end{cases} \tag{8}$$

for the virtual displacement that is compatible with the constraints.

The last of the preceding relations is equivalent to the relation in finite terms:

$$\zeta = a \sin \theta, \tag{9}$$

which is obvious geometrically. However, neither the first two relations in (8) nor any linear combination of the relations (8) in which at least one of the first two appears, can be integrated and written in a finite form. The system considered will then be non-holonomic. It has *three degrees of freedom* (k = 3), because the most general virtual displacement that is compatible with the constraints is obtained by giving arbitrary values to $\delta\theta, \delta\varphi, \delta\psi$; $\delta\xi, \delta\eta, \delta\zeta$ are then determined by the relations (8). It remains to see that the system is *holonomic of order two.* Indeed, since the position of the hoop around its center of gravity is defined by the Euler angles θ, φ, ψ, Ferrer already showed [**6**] that the Lagrange equation can apply to the inclination θ, but it does not apply to φ and ψ. The *order* of the non-holonomic system will then be $\omega = 2$.

II. – REALIZING THE CONSTRAINTS. SUBORDINATION.

3. Realizing constraints. – In the foregoing, the constraints were considered from a purely-analytical viewpoint that was independent of the particular manner by which they were realized. (BEGHIN [**29**], *Thesis*, pp. 8). Can one now abstract the manner by which a constraint is realized? That question has been the object of numerous studies. Here are some general considerations that are borrowed from Beghin (*loc. cit.*) and Delassus ([**26**] and [**27**]). A constraint L on a system Σ can be realized with or without the help of an auxiliary system Σ_1 . In the former case, the realization of the constraint is called *perfect*; in the latter case, the realization of the constraint is still called *perfect* if the introduction of the auxiliary system Σ_1 does not imply any restriction on the virtual displacements of the system Σ, which will all remain compatible with the constraint L then. However, it is *imperfect* if the introduction of the system Σ_1 does imply restrictions on the virtual displacements of the system Σ.

Delassus [**27**] then gave the following example of the *imperfect* constraint $z = a$ that is imposed upon a material point with coordinates x, y, z. Imagine a hoop of radius a that rolls without slipping on the plane xOy. Suppose that the plane of the hoop (i.e., the plane of the circular edge C) is kept vertical by means of a tripod that carries the axis of the hoop and slides without friction on the horizontal plane xOy. The material point x, y, z is attached to the center G of the hoop C. That constitutes the system Σ ; the hoop with the tripod and the accessories constitutes the system Σ_1 . The apparatus obviously realizes the constraint $z = a$. It permits the material point to occupy all of the possible positions in the plane $z = a$. However, if one imposes a virtual displacement on the system that is compatible with the constraints then the displacement of the material point in the plane of the edge of the hoop and not an arbitrary direction in the plane $z = a$. The constraint is then realized *imperfectly*.

If, on the contrary, the material point is attached to the center of a sphere of radius a that is subject to rolling without slipping on the plane xOy then that point will be subject to the same constraint $z = a$, but it will then be realized perfectly.

4. Work done by constraint forces. – When one proves the theorem of virtual work for a system, one appeals to the hypothesis that for any virtual displacement of the system that is compatible with the constraints, *the sum of the works done by the constraint forces is zero*. Here, we take that hypothesis to be something that defines the constraints that we consider. It is that hypothesis that one then utilizes in order to apply d'Alembert's principle by writing that by virtue of the constraints that exist at the instant t there will be equilibrium between the forces of inertia and the applied forces.

5. Case of subordination. – However, one must point out that even if one confines oneself to perfect constraints, there will exist an important category of mechanisms in which the constraints are found to be realized by methods that are different from the ones that permit the pure and simple application of the general equation of dynamics. For those special constraints, one cannot abstract from the mode of realization, and one must be content with their analytical expression. Those constraints are the ones that one obtains by *subordination*. We say that there is *subordination* when the corresponding constraints, rather than being realized in a fashion that is, in some way, passive (such as the contact between two solids that slide or roll on each other, by way of example), they are realized by the appropriate use of arbitrary forces (e.g., electromagnetic forces, fluid, pressure, forces produced by a living entity, etc.). Those subordinate forces imply the constraint forces that Beghin [**29**] called ones of the *second type* and whose virtual work is generally non-zero, even when the displacement is compatible with the constraint. That means that we shall pass over that type of constraint and refer the study of that case to Beghin's thesis, which used the general form of the equations that we shall indicate. We shall confine ourselves to the classical constraints that were defined above (no. **4**).

III. – EQUATIONS.

6. General equations of motion. – We write the general equation of dynamics in such a way that it will result from d'Alembert's principle, combined with the theory of virtual work. In all of what follows, we shall employ Lagrange's notation of primes to denote the derivatives with respect to time. The general equation of dynamics is then:

$$\sum_\mu m_\mu (x''_\mu\,\delta x_\mu + y''_\mu\,\delta y_\mu + z''_\mu\,\delta z_\mu) - \sum_\mu (X_\mu\,\delta x_\mu + Y_\mu\,\delta y_\mu + Z_\mu\,\delta z_\mu) = 0, \tag{10}$$

in which the first summation is extended over all material points of the system, but the second one comprises only the material points to which the forces are applied. Upon replacing δx_μ, δy_μ, δz_μ with their values in (1), one will have an equation of the form:

$$P_1\,\delta q_1 + P_2\,\delta q_2 + \ldots + P_k\,\delta q_k - (Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k) = 0. \tag{11}$$

Since $\delta q_1, \delta q_2, \ldots, \delta q_k$ are arbitrary, equation (11) will reduce to k equations:

$$P_1 = Q_1, \qquad P_2 = Q_2, \qquad \ldots, \qquad P_k = Q_k, \tag{12}$$

which define the k parameters $q_1, q_2, \ldots, q_k$ as functions of t.

In order to write those equations, we remark that:

$$P_\nu = \sum_\mu m_\mu (x''_\mu\,a_{\mu,\nu} + y''_\mu\,b_{\mu,\nu} + z''_\mu\,c_{\mu,\nu}).$$

Now, from the relations (2), one will have:

$$\left\{\begin{aligned} x'_\mu &= a_{\mu,1}\,q'_1 + a_{\mu,2}\,q'_2 + \cdots + a_{\mu,\nu}\,q'_\nu + \cdots + a_{\mu,k}\,q'_k + a_\mu, \\ y'_\mu &= b_{\mu,1}\,q'_1 + b_{\mu,2}\,q'_2 + \cdots + b_{\mu,\nu}\,q'_\nu + \cdots + b_{\mu,k}\,q'_k + b_\mu, \\ z'_\mu &= c_{\mu,1}\,q'_1 + c_{\mu,2}\,q'_2 + \cdots + c_{\mu,\nu}\,q'_\nu + \cdots + c_{\mu,k}\,q'_k + c_\mu, \end{aligned}\right. \tag{13}$$

so, upon differentiating once with respect to time, one will get:

$$x''_\mu = \sum_{\nu=1}^{k} \left[a_{\mu,\nu}\,q''_\nu + \frac{da_{\mu,\nu}}{dt}\,q'_\nu \right] + \frac{da_\mu}{dt},$$

$$y''_\mu = \sum_{\nu=1}^{k}\left[b_{\mu,\nu}\, q''_\nu + \frac{db_{\mu,\nu}}{dt} q'_\nu\right] + \frac{db_\mu}{dt},$$

$$z''_\mu = \sum_{\nu=1}^{k}\left[c_{\mu,\nu}\, q''_\nu + \frac{dc_{\mu,\nu}}{dt} q'_\nu\right] + \frac{dc_\mu}{dt}.$$

One concludes that the only term on the right-hand side that contains q''_ν is $a_{\mu,\nu}\, q''_\nu$ in the first expression, $b_{\mu,\nu}\, q''_\nu$ in the second, and $c_{\mu,\nu}\, q''_\nu$ in the third. One will then have:

$$a_{\mu,\,\nu} = \frac{\partial x''_\mu}{\partial q''_\nu}, \qquad b_{\mu,\,\nu} = \frac{\partial y''_\mu}{\partial q''_\nu}, \qquad c_{\mu,\,\nu} = \frac{\partial z''_\mu}{\partial q''_\nu},$$

and the expression for P_ν will be written:

$$P_\nu = \sum_\mu m_\mu \left(x''_\mu \frac{\partial x''_\mu}{\partial q''_\nu} + y''_\mu \frac{\partial y''_\mu}{\partial q''_\nu} + z''_\mu \frac{\partial z''_\mu}{\partial q''_\nu} \right).$$

If one finally sets:

$$S = \tfrac{1}{2}\sum_{\mu=1}^{n} m_\mu \left(x''^2_\mu + y''^2_\mu + z''^2_\mu \right)$$

then one will have:

$$P_\nu = \frac{\partial S}{\partial q''_\nu}.$$

On the other hand, the term Q_ν has a known value. If one imposes upon the system the special virtual displacement in which all of the δq are *zero*, except δq_ν, then the sum $\mathcal{T}_\nu$ of the virtual works of the applied forces will be precisely:

$$\mathcal{T}_\nu = Q_\nu\, \delta q_\nu,$$

which gives a simple meaning to the Q_ν. One will then have the k equations of motion:

$$\text{(14)} \qquad \frac{\partial S}{\partial q''_1} = Q_1, \qquad \frac{\partial S}{\partial q''_2} = Q_2, \qquad \ldots, \qquad \frac{\partial S}{\partial q''_k} = Q_k,$$

which are the desired general equations, which are applicable to all systems – holonomic or not and for all choices of parameters – under the indicated restrictions that relate to subordination. In order to write those equations, one must form the function S [**19**].

7. Energy of acceleration of a system. – The semi-*vis viva*, or kinetic energy:

(15) $$T = \tfrac{1}{2}\sum_{\mu=1}^{n} m_\mu \left(x_\mu'^2 + y_\mu'^2 + z_\mu'^2 \right) = \tfrac{1}{2}\sum m\, V^2 \, ,$$

in which V denotes the velocity of the point with mass m, can be called the *energy of velocity* of the system. The function S :

(16) $$S = \tfrac{1}{2}\sum_{\mu=1}^{n} m_\mu \left(x_\mu''^2 + y_\mu''^2 + z_\mu''^2 \right) = \tfrac{1}{2}\sum m\; J^2 \, ,$$

in which J denotes the acceleration of the point with mass m, will be called the *energy of acceleration* of the system. That terminology was introduced by A de Saint-Germain [**20**]. In order to write the equations of an arbitrary system with k degrees of freedom, with an arbitrary choice of the k parameters q_1 , q_2 , …, q_k , it will then suffice to form the energy of the accelerations S of that system. In each case, the quantity S will be a function of second degree in q_1'' , q_2'' , …, q_k'' . One can then, in turn, write the equations of motion by a simple differentiation.

One knows that if a system is essentially holonomic and if its position at the instant t depends upon k geometrically-independent coordinates then the equations of motion of the system can be written in the form that was given by Lagrange:

(17) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu \qquad (\nu = 1, 2, \ldots, k).$$

However, that form is not applicable to non-holonomic systems. It is not even adapted to an arbitrary choice of parameters for holonomic systems. In order to obtain an absolutely-general form, one agrees to calculate S as was said; i.e., to go to the second order of differentiation with respect to t.

8. Case in which the Lagrange equations apply to certain parameters. – The coefficient P_ν of δq_ν in the general equation of dynamics (10) is:

$$P_\nu = \sum_\mu m_\mu \,(x_\mu''\, a_{\mu,\nu} + y_\mu''\, b_{\mu,\nu} + z_\mu''\, c_{\mu,\nu}) \, .$$

In the case of a holonomic system (which is the only one that Lagrange considered), that coefficient can be written:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} \, .$$

In any case, one can obviously write:

$$P_\nu = \frac{d}{dt}\sum_\mu m_\mu \,(x_\mu'\, a_{\mu,\nu} + y_\mu'\, b_{\mu,\nu} + z_\mu'\, c_{\mu,\nu}) - \sum_\mu m_\mu \left(x_\mu' \frac{da_{\mu,\nu}}{dt} + y_\mu' \frac{db_{\mu,\nu}}{dt} + z_\mu' \frac{dc_{\mu,\nu}}{dt} \right).$$

Now since, from (13), $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ are equal to $\frac{\partial x'_\mu}{\partial q'_\nu}$, $\frac{\partial y'_\mu}{\partial q'_\nu}$, $\frac{\partial z'_\mu}{\partial q'_\nu}$, one will have:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \sum_\mu m_\mu \left(x'_\mu \frac{da_{\mu,\nu}}{dt} + y'_\mu \frac{db_{\mu,\nu}}{dt} + z'_\mu \frac{dc_{\mu,\nu}}{dt} \right).$$

If one sets:

$$R_\nu = \sum_\mu m_\mu \left(x'_\mu \frac{da_{\mu,\nu}}{dt} + y'_\mu \frac{db_{\mu,\nu}}{dt} + z'_\mu \frac{dc_{\mu,\nu}}{dt} \right)$$

then one will see that:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} - \left(R_\nu - \frac{\partial T}{\partial q_\nu} \right).$$

The Lagrange equation will then be applicable to the parameter q_ν if one has:

$$\Delta_\nu \equiv R_\nu - \frac{\partial T}{\partial q_\nu} = 0.$$

Now, one has:

$$(18) \qquad \Delta_\nu \equiv R_\nu - \frac{\partial T}{\partial q_\nu} = \sum_\mu m_\mu \left[x'_\mu \left(\frac{da_{\mu,\nu}}{dt} - \frac{\partial x'_\mu}{\partial q_\nu} \right) + y'_\mu \left(\frac{db_{\mu,\nu}}{dt} - \frac{\partial y'_\mu}{\partial q_\nu} \right) + z'_\mu \left(\frac{dc_{\mu,\nu}}{dt} - \frac{\partial z'_\mu}{\partial q_\nu} \right) \right].$$

If one replaces x'_μ, y'_μ, z'_μ with their expressions in terms of q'_1, q'_1, …, q'_k [eq. (13)] then one will see that $R_\nu - \frac{\partial T}{\partial q_\nu}$ is a function of degree two in q'_1, q'_1, …, q'_k. In order to apply the Lagrange equation to the parameter q_ν, it is necessary and sufficient that that function must be zero for any positions and velocities of the points of the system that are compatible with the constraints, since at each instant (which is considered to be initial), those quantities can be taken arbitrarily.

Particular case. – Suppose that the coefficients $a_{\mu,\nu}$ are functions of the q_1, q_2, …, q_k, and t, so:

$$\frac{da_{\mu,\nu}}{dt} = \frac{\partial a_{\mu,\nu}}{\partial q_1} q'_1 + \frac{\partial a_{\mu,\nu}}{\partial q_2} q'_2 + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_\nu} q'_\nu + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_k} q'_k + \frac{\partial a_{\mu,\nu}}{\partial t},$$

$$\frac{\partial x'_\mu}{\partial q_\nu} = \frac{\partial a_{\mu,1}}{\partial q_\nu} q'_1 + \frac{\partial a_{\mu,2}}{\partial q_\nu} q'_2 + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_\nu} q'_\nu + \cdots + \frac{\partial a_{\mu,k}}{\partial q_k} q'_k + \frac{\partial a_\mu}{\partial q_\nu},$$

……………………………………………………………….

The coefficient of x_μ in the difference (18) is:

$$\left(\frac{\partial a_{\mu,\nu}}{\partial q_1}-\frac{\partial a_{\mu,1}}{\partial q_\nu}\right)q_1'+\left(\frac{\partial a_{\mu,\nu}}{\partial q_2}-\frac{\partial a_{\mu,2}}{\partial q_\nu}\right)q_2'+\cdots+\left(\frac{\partial a_{\mu,\nu}}{\partial q_k}-\frac{\partial a_{\mu,k}}{\partial q_\nu}\right)q_k'+\frac{\partial a_{\mu,\nu}}{\partial t}-\frac{\partial a_\mu}{\partial q_\nu}.$$

If that coefficient is zero for any μ, as well as the analogous coefficients of y_μ', z_μ', then the quantity R_ν will be zero. The Lagrange equation will then apply to the parameter q_ν if one has:

$$(19)\qquad \left\{\begin{array}{llll}
\dfrac{\partial a_{\mu,\nu}}{\partial q_1}=\dfrac{\partial a_{\mu,1}}{\partial q_\nu}, & \dfrac{\partial a_{\mu,\nu}}{\partial q_2}=\dfrac{\partial a_{\mu,2}}{\partial q_\nu}, \quad \cdots & \dfrac{\partial a_{\mu,\nu}}{\partial q_k}=\dfrac{\partial a_{\mu,k}}{\partial q_\nu}, & \dfrac{\partial a_{\mu,\nu}}{\partial t}=\dfrac{\partial a_\mu}{\partial q_\nu}, \\[2ex]
\dfrac{\partial b_{\mu,\nu}}{\partial q_1}=\dfrac{\partial b_{\mu,1}}{\partial q_\nu}, & \dfrac{\partial b_{\mu,\nu}}{\partial q_2}=\dfrac{\partial b_{\mu,2}}{\partial q_\nu}, \quad \cdots & \dfrac{\partial b_{\mu,\nu}}{\partial q_k}=\dfrac{\partial b_{\mu,k}}{\partial q_\nu}, & \dfrac{\partial b_{\mu,\nu}}{\partial t}=\dfrac{\partial b_\mu}{\partial q_\nu}, \\[2ex]
\dfrac{\partial c_{\mu,\nu}}{\partial q_1}=\dfrac{\partial c_{\mu,1}}{\partial q_\nu}, & \dfrac{\partial c_{\mu,\nu}}{\partial q_2}=\dfrac{\partial c_{\mu,2}}{\partial q_\nu}, \quad \cdots & \dfrac{\partial c_{\mu,\nu}}{\partial q_k}=\dfrac{\partial c_{\mu,k}}{\partial q_\nu}, & \dfrac{\partial c_{\mu,\nu}}{\partial t}=\dfrac{\partial c_\mu}{\partial q_\nu}
\end{array}\right.$$

for any μ.

One can characterize this case in a different way. If the conditions (19) are assumed to have been fulfilled then determine the functions U_μ, V_μ, W_μ of and t by the formulas:

$$U_\mu=\int_{q_\nu^0}^{q_\nu} a_{\mu,\nu}\,dq_\nu\,,\qquad V_\mu=\int_{q_\nu^0}^{q_\nu} b_{\mu,\nu}\,dq_\nu\,,\qquad W_\mu=\int_{q_\nu^0}^{q_\nu} c_{\mu,\nu}\,dq_\nu\,,$$

in which q_ν^0 denotes a *constant*. From (19), one has immediately:

$$\frac{\partial U_\mu}{\partial q_1}=\int_{q_\nu^0}^{q_\nu}\frac{\partial a_{\mu,\nu}}{\partial q_1}dq_\nu=\int_{q_\nu^0}^{q_\nu}\frac{\partial a_{\mu,1}}{\partial q_\nu}dq_\nu=a_{\mu,1}-a_{\mu,1}^0\,,$$

in which $a_{\mu,1}^0$ is what $a_{\mu,1}$ will become when one replaces q_ν with the constant q_ν^0. Similarly:

$$\frac{\partial U_\mu}{\partial q_2}=a_{\mu,2}-a_{\mu,2}^0\,,\qquad \ldots,\qquad \frac{\partial U_\mu}{\partial q_k}=a_{\mu,k}-a_{\mu,k}^0\,,$$

$$\frac{\partial U_\mu}{\partial t}=\int_{q_\nu^0}^{q_\nu}\frac{\partial a_{\mu,\nu}}{\partial t}dq_\nu=\int_{q_\nu^0}^{q_\nu}\frac{\partial a_\mu}{\partial q_\nu}dq_\nu=a_\mu-a_\mu^0\,,$$

$$\frac{\partial V_\mu}{\partial q_\rho}=b_{\mu,\rho}-b_{\mu,\rho}^0\,,\qquad \frac{\partial V_\mu}{\partial t}=b_{\mu,}-b_\mu^0\,,$$

$$\frac{\partial W_\mu}{\partial q_\rho}=c_{\mu,\rho}-c_{\mu,\rho}^0\,,\qquad \frac{\partial W_\mu}{\partial t}=c_{\mu,}-c_\mu^0\,.$$

If one replaces $a_{\mu,\rho}$, $b_{\mu,\rho}$, $c_{\mu,\rho}$, a_μ, b_μ, c_μ, with their expressions that one infers from the preceding formulas then formulas (1) will become:

$$(20)\qquad \left\{\begin{aligned} \delta x_\mu &= \delta U_\mu + a^0_{\mu,1}\,\delta q_1 + a^0_{\mu,2}\,\delta q_2 + \cdots + a^0_{\mu,k}\,\delta q_k, \\ \delta y_\mu &= \delta V_\mu + b^0_{\mu,1}\,\delta q_1 + b^0_{\mu,2}\,\delta q_2 + \cdots + b^0_{\mu,k}\,\delta q_k, \\ \delta z_\mu &= \delta W_\mu + c^0_{\mu,1}\,\delta q_1 + c^0_{\mu,2}\,\delta q_2 + \cdots + c^0_{\mu,k}\,\delta q_k, \end{aligned}\right.$$

in which δU_μ, δV_μ, δW_μ are total differentials that are taken while regarding t as constant, and in which $a^0_{\mu,\nu}$, $b^0_{\mu,\nu}$, $c^0_{\mu,\nu}$ which are the coefficients of the δq_ν, are zero.

Formulas (2) likewise become:

$$(20')\qquad \left\{\begin{aligned} dx_\mu &= dU_\mu + a^0_{\mu,1}\,dq_1 + a^0_{\mu,2}\,dq_2 + \cdots + a^0_{\mu,k}\,dq_k, \\ dy_\mu &= dV_\mu + b^0_{\mu,1}\,dq_1 + b^0_{\mu,2}\,dq_2 + \cdots + b^0_{\mu,k}\,dq_k, \\ dz_\mu &= dW_\mu + c^0_{\mu,1}\,dq_1 + c^0_{\mu,2}\,dq_2 + \cdots + c^0_{\mu,k}\,dq_k. \end{aligned}\right.$$

One sees that the Lagrange equation will apply to the q_ν when δx_μ, δy_μ, δz_μ and dx_μ, dy_μ, dz_μ can be put *into the form of a total differential, followed by an expression that contains neither q_ν not δq_ν nor dq_ν* for any point of the system. One can also say that the Lagrange equation will apply to the parameter q_ν when the other parameters q_1, q_2, …, $q_{\nu-1}$, $q_{\nu+1}$, …, q_k are known as functions of t, so q_ν will become a true coordinate, in such a fashion that x_μ, y_μ, z_μ can be expressed in finite form as functions of q_ν and t.

In order for the Lagrange equations to be applicable to the parameters q_1, q_2, …, q_k it is sufficient for that condition to be true for $\nu = 1, 2, \ldots, s$; i.e., that the virtual displacements δx_μ, δy_μ, δz_μ and the actual displacements dx_μ, dy_μ, dz_μ can be put into the form:

$$\delta x_\mu = \delta U_\mu + \alpha_{\mu,s+1}\,\delta q_{s+1} + \ldots + \alpha_{\mu,k}\,\delta q_k,$$

$$\delta y_\mu = \delta V_\mu + \beta_{\mu,s+1}\,\delta q_{s+1} + \ldots + \beta_{\mu,k}\,\delta q_k,$$

$$\delta z_\mu = \delta W_\mu + \gamma_{\mu,s+1}\,\delta q_{s+1} + \ldots + \gamma_{\mu,k}\,\delta q_k,$$

$$dx_\mu = dU_\mu + \alpha_{\mu,s+1}\,dq_{s+1} + \ldots + \alpha_{\mu,k}\,dq_k + \alpha_\mu\,dt,$$

$$dy_\mu = dV_\mu + \beta_{\mu,s+1}\,dq_{s+1} + \ldots + \beta_{\mu,k}\,dq_k + \beta_\mu\,dt,$$

$$dz_\mu = dW_\mu + \gamma_{\mu,s+1}\,dq_{s+1} + \ldots + \gamma_{\mu,k}\,dq_k + \gamma_\mu\,dt,$$

in which the coefficients $\alpha_{\mu,s+1}$, …, $\alpha_{\mu,k}$, α_μ, $\beta_{\mu,s+1}$, …, $\beta_{\mu,k}$, β_μ, $\gamma_{\mu,s+1}$, …, $\gamma_{\mu,k}$, γ_μ, no longer contain the q_1, q_2, …, q_s. The system is then non-holonomic of order $k - s$ for the choice of parameters q_1, q_2, …, q_k.

IV. – APPLICATIONS.

9. Motion of a point in polar coordinates for the plane. – The equations:

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = 0$$

give

$$S = \frac{m}{2}(x''^2 + y''^2 + z''^2) = \frac{m}{2}[(r'' - r\,\theta'')^2 + (r\,\theta'' + 2r'\,\theta')^2] .$$

Upon adopting the notations of the example at the end of no. **1**, one will see that the equations of motion are:

$$\frac{\partial S}{\partial r''} = Q, \qquad \frac{\partial S}{\partial \theta''} = P\,r,$$

or

$$m\,(r'' - r\,\theta'') = Q, \qquad m\,(r\,\theta'' + 2r'\,\theta') = P.$$

Those equations are identical to those of Lagrange. With those parameters r and θ, the system is holonomic. However, if one takes the area σ that is swept out by the radius vector to be the parameter then one will have:

$$\delta\sigma = \tfrac{1}{2}\, r^2\, \delta\theta, \qquad d\sigma = \tfrac{1}{2}\, r^2\, d\theta,$$

$$\delta x = \cos\theta\,\delta r - \frac{2\sin\theta}{r}\delta\sigma ,$$

$$\delta y = \sin\theta\,\delta r + \frac{2\cos\theta}{r}\delta\sigma ,$$

$$x' = \cos\theta\, r' - \frac{2\sin\theta}{r}\sigma' , \qquad y' = \sin\theta\, r' + \frac{2\cos\theta}{r}\sigma' ,$$

$$T \equiv \frac{m}{2}\left(r'^2 + \frac{4\sigma'^2}{r}\right),$$

$$x' = \cos\theta\left(r'' - \frac{4}{r^3}\sigma'^2\right) + 2\sin\theta\frac{\sigma''}{r},$$

$$y' = \sin\theta\left(r'' - \frac{4}{r^3}\sigma'^2\right) + 2\cos\theta\frac{\sigma''}{r},$$

$$S = \frac{m}{2}\left[\left(r'' - \frac{4}{r^3}\sigma'^2\right)^2 + \frac{4}{r^2}\sigma''^2\right],$$

$$X\,\delta x + Y\,\delta y = \frac{2P}{r}\delta\sigma + Q\,\delta r.$$

The equations are then:

$$\frac{\partial S}{\partial r''} = Q, \qquad \frac{\partial S}{\partial \sigma''} = \frac{2P}{r},$$

or upon making things explicit:

$$m\left(r'' - \frac{4}{r^3}\sigma'^2\right) = Q, \qquad m\,\sigma'' = \frac{2P}{r}.$$

If the force is central then $P = 0$, and the second equation will give:

$$\sigma'' = 0, \qquad \sigma' = C,$$

which expresses the theorem of areas.

Neither of the quantities:

$$\Delta_1 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r}\right] - \frac{\partial S}{\partial r''},$$

$$\Delta_2 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \sigma'}\right) - \frac{\partial T}{\partial \sigma}\right] - \frac{\partial S}{\partial \sigma''}$$

is zero. With the choice of parameters r and σ, the system has become non-holonomic of order 2.

10. Motion of a solid body around a fixed point. – We calculate the energy of acceleration S of a solid body that moves around a point O while placing ourselves in the most general case. For each particular example, it will then suffice to employ that function S when it has been calculated once and for all. Refer the motion of the body to a tri-rectangular trihedron $Oxyz$ with its origin at O and which is animated with a known motion. Let Ω be the instantaneous rotation of that trihedron, let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ be the components of that rotation along the edges Ox, Oy, Oz. Similarly, let ω be the absolute instantaneous rotation of the solid body, and let p, q, r be its components along the axes $Oxyz$. A molecule m of the corps with coordinates x, y, z possesses an absolute velocity v with projections:

$$\text{(21)} \qquad \begin{cases} v_x = qz - ry, \\ v_y = rx - pz, \\ v_z = py - qx. \end{cases}$$

That molecule possesses an absolute acceleration J that has projections:

$$(22) \qquad \begin{cases} J_x = \dfrac{dv_x}{dt} + \mathcal{Q}\, v_z - \mathcal{R}\, v_y, \\ J_y = \dfrac{dv_y}{dt} + \mathcal{R}\, v_x - \mathcal{P}\, v_z, \\ J_z = \dfrac{dv_z}{dt} + \mathcal{R}\, v_y - \mathcal{Q}\, v_z. \end{cases}$$

Those formulas can be written down immediately when one remarks that the acceleration J is the absolute velocity of the geometric point that has v_x , v_y , v_z with respect to the moving axes $Oxyz$.

Having said that, one will have:

$$\frac{dv_x}{dt} = q\frac{dz}{dt} - r\frac{dy}{dt} + z\,q' - y\,r', \ldots,$$

in which p', q', r' denote the derivatives of p, q, r with respect to time. The quantities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the projections onto Ox, Oy, Oz of the relative velocity v_r of the molecule m with respect to those axes. If one calls the guiding velocity of that same molecule v_e then one will have:

$$(v_r)_x = v_x - (v_e)_x ;$$

i.e.:

$$\frac{dx}{dt} = q\,z - r\,y - (\mathcal{Q}\,z - \mathcal{R}\,y).$$

One will get $\frac{dy}{dt}$ and $\frac{dz}{dt}$ similarly, by permutation. From that, the expressions (22) for J_x , J_y , J_z will take the following form, in which we write only J_x :

$$J_x = q\,[(p - \mathcal{P})\,y - (q - \mathcal{Q})\,x] - r\,[(r - \mathcal{R})\,x - (p - \mathcal{P})\,z]$$
$$+ z\,q' - y\,r' + \mathcal{Q}\,(p\,y - q\,x) - \mathcal{R}\,(r\,x - p\,z),$$

or, upon rearranging:

$$J_x = -\,x\,(q^2 + r^2) + y\,[q\,(p - \mathcal{P})\,y + p\,\mathcal{Q} - r'] - z\,[r\,(p - \mathcal{P}) + p\,\mathcal{R} + q'] .$$

One will get J_y and J_z upon permuting. When one takes the sum of the squares, one will get J^2, and then the function:

$$S = \tfrac{1}{2}\sum m\,(J_x^2 + J_y^2 + J_z^2) .$$

The coefficients of the moment of inertia:

$$A = \sum m(y^2+z^2), \quad B = \sum m(z^2+x^2), \quad C = \sum m(x^2+y^2)$$

enter into that sum, along with the products of inertia:

$$D = \sum m\,y\,z, \quad E = \sum m\,z\,x, \quad F = \sum m\,x\,y$$

with respect to the axes *Oxyz*. In general, those six quantities will vary in time, since the axes *Oxyz* displace in the body.

At present, the parameters are the angles that fix the orientation of the body around the point O. The quantities p, q, r contain the *first derivatives* of those parameters with respect to time. If the trihedron *Oxyz* is animated with a known motion then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ must be regarded as known functions of time. If the motion of the trihedron is coupled to that of the body in some fashion then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ will depend upon only the first derivatives of the parameters. The second derivatives of the parameters then enter into only the p', q', r'. From a preceding remark, it will then suffice to calculate the terms in S that depend upon the accelerations; i.e., the p', q', r', because only those terms depend upon second derivatives of the parameters.

Set:

$$q\,\mathcal{R} - r\,\mathcal{Q} = \mathcal{P}_1, \qquad r\,\mathcal{P} - p\,\mathcal{R} = \mathcal{Q}_1, \qquad p\,\mathcal{Q} - q\,\mathcal{P} = \mathcal{R}_1, \tag{23}$$

to abbreviate, and let a, b, c denote the sums $\sum m\,x^2$, $\sum m\,y^2$, $\sum m\,z^2$, for the moment. One can write:

$$\begin{aligned} 2S &= a\,[(q' - \mathcal{Q}_1 - p\,r)^2 + (r' - \mathcal{R}_1 + p\,q)^2] \\ &\quad + b\,[(r' - \mathcal{R}_1 - q\,p)^2 + (p' - \mathcal{P}_1 + q\,r)^2] \\ &\quad + c\,[(p' - \mathcal{P}_1 - r\,q)^2 + (q' - \mathcal{Q}_1 + r\,p)^2] \\ &\quad - 2D\,[(q^2 - r^2)\,p' + (q' - \mathcal{Q}_1 + p\,r)\,(r' - \mathcal{R}_1 - p\,q)] \\ &\quad - 2E\,[(r^2 - p^2)\,q' + (r' - \mathcal{R}_1 + q\,p)\,(p' - \mathcal{P}_1 - q\,r)] \\ &\quad - 2F\,[(p^2 - q^2)\,r' + (p' - \mathcal{P}_1 + r\,q)\,(q' - \mathcal{Q}_1 - r\,p)] + \ldots \end{aligned} \tag{24}$$

We develop this and rearrange the result with respect to $p' - \mathcal{P}_1$, $q' - \mathcal{Q}_1$, $r' - \mathcal{R}_1$, while noting that:

$$\begin{array}{lll} b + c = A, & c + a = B, & a + b = C, \\ b - c = C - B, & c - a = A - C, & a - b = B - A. \end{array}$$

Upon dropping the terms that are independent of p', q', r', we can write:

$$\begin{aligned} 2S &= A\,(p' - \mathcal{P}_1)^2 + B\,(q' - \mathcal{Q}_1)^2 + C\,(r' - \mathcal{R}_1)^2 \\ &\quad - 2D\,(q' - \mathcal{Q}_1)\,(r' - \mathcal{R}_1) - 2E\,(r' - \mathcal{R}_1)\,(p' - \mathcal{P}_1) - 2F\,(p' - \mathcal{P}_1)\,(q' - \mathcal{Q}_1) \end{aligned} \tag{25}$$

$$+ 2 [(C - B)\, q\, r - D\, (q^2 - r^2) - E\, p\, q + F\, p\, r]\, (p' - \mathcal{P}_1)$$
$$+ 2 [(A - C)\, r\, p - E\, (r^2 - p^2) - F\, q\, r + D\, q\, p]\, (q' - \mathcal{Q}_1)$$
$$+ 2 [(B - A)\, p\, q - F\, (p^2 - q^2) - D\, r\, p + E\, r\, q\,]\, (r' - \mathcal{R}_1) + \ldots$$

Remark. – If the axes *Oxyz* are *fixed in space* then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ will be zero, and one will have:

$$\mathcal{P}_1 = \mathcal{Q}_1 = \mathcal{R}_1 = 0. \tag{26}$$

The same fact will be true if the axes are *fixed in the body*, because in that case:

$$\mathcal{P} = p, \qquad \mathcal{Q} = q, \qquad \mathcal{R} = r. \tag{27}$$

Upon making this explicit, one will get the Euler equations, which one establishes easily.

Similarly, upon suitably specializing the formulas, one will get the equations of motion for the classical case in which the ellipsoid of inertia relative to the fixed point *O* is one of revolution. One takes the axis *Oz* to be the axis of revolution and the axes *Ox* and *Oy* to be two axes that move in both the body and space that are defined as follows: Let Ox_1, Oy_1, Oz_1 be three fixed axes: The axis *Oy* is perpendicular to the plane zOz_1 and the axis *Ox* is perpendicular to the plane *yOz*. The angle θ is then the angle z_1Oz, and ψ is the angle x_1Oy. The instantaneous rotation Ω of the trihedron *Oxyz* is the resultant of two rotations, one of which $d\theta / dt = \theta'$ is around *Oy*, while the other $d\psi / dt = \psi'$ is around Oz_1 . The components of that rotation around *Ox*, *Oy*, *Oz* are then:

$$\mathcal{P} = -\, \psi' \sin\theta, \qquad \mathcal{Q} = \theta', \qquad \mathcal{R} = \psi' \sin\theta. \tag{28}$$

Once the trihedron *Oxyz* has been located, one must define the position of the solid with respect to that trihedron. In order to do that, it will suffice to know the angle φ that a line that is coupled to the body in the *xOy* plane makes with the axis *Oy*. The derivative $d\varphi / dt = \varphi'$ of that angle measures the proper rotation of the body around *Oz*. The instantaneous rotation ω of the body is then the resultant of the rotation Ω of the trihedron and the proper rotation φ around *Oz*. One will then have that the projections *p*, *q*, *r* of ω onto the axes *Oxyz* are:

$$p = \mathcal{P} = -\, \psi' \sin\theta, \quad q = \mathcal{Q} = \theta', \quad r = \mathcal{R} + \varphi' = \psi' \cos\theta + \varphi'. \tag{29}$$

Upon differentiating with respect to *t*, one will conclude that:

$$p' = -\, \psi'' \sin\theta + \ldots, \quad q' = \theta'', \qquad r' = \psi'' \cos\theta + \varphi'' + \ldots$$

In addition, since the ellipsoid of inertia is one of revolution around *Oz* :

$$B = A.$$

When one replaces P and Q with p and q, resp., and remarks that $D = E = F = 0$, since the moving axes are the principal axes of inertia, the general expression (25) for S will be:

(30) $$2S = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(A\,\mathcal{R} - C\,r)\,(p\,q' - q\,p') + \ldots$$

For a variation $\delta\theta$, $\delta\varphi$, $\delta\psi$ of the three angles, the sum of the works done by the applied forces will take the form:

$$\Theta\ \delta\theta + \Phi\ \delta\varphi + \Psi\ \delta\psi\,.$$

Since the virtual displacement that is obtained by setting $\delta\varphi = \delta\psi = 0$ is a rotation around Oy, Θ is the sum $\mathcal{M}_y$ of the moments of the forces with respect to Oy. Similarly, Φ is the sum $\mathcal{M}_z$ of the moments of the forces with respect to Oz, and Ψ is the sum $\mathcal{M}_{z_1}$ of the moments of the forces with respect to Oz_1. The equations are then easy to write out.

One will get them in the definitive form more quickly by introducing (as one can do in the general case) the three quantities λ, μ, ν that are defined by the relations:

(31) $$\delta\lambda = -\sin\theta\,\delta\psi, \qquad \delta\mu = \delta\theta, \qquad \delta\nu = \cos\theta\,\delta\psi + \delta\varphi$$

as the parameters, so the actual displacement will be:

(32) $$\begin{cases} p = \lambda' = -\sin\theta\,\psi', \quad q = \mu' = \theta', \quad r = \nu' = \cos\theta\,\psi' + \varphi', \\ \qquad p' = \lambda'', \quad q' = \mu'', \quad r' = \nu''. \end{cases}$$

The quantities $\delta\lambda$, $\delta\mu$, $\delta\nu$ are then the elementary rotations around Ox, Oy, Oz, and one will have:

$$\sum\ (X\,\delta x + Y\,\delta y + Z\,\delta z) = \mathcal{M}_x\,\delta\lambda + \mathcal{M}_y\,\delta\mu + \mathcal{M}_z\,\delta\nu\,.$$

The function $2S$ that is given by the expression (30) is expressed immediately as a function of λ'', μ'', ν'', and the equations of motion are:

$$\frac{\partial S}{\partial\lambda''} = \mathcal{M}_x\,, \qquad \frac{\partial S}{\partial\mu''} = \mathcal{M}_y\,, \qquad \frac{\partial S}{\partial\nu''} = \mathcal{M}_z\,,$$

or, since $\lambda'' = p'$, $\mu'' = q'$, $\nu'' = r'$:

(33) $$\frac{\partial S}{\partial p'} = \mathcal{M}_x\,, \qquad \frac{\partial S}{\partial q'} = \mathcal{M}_y\,, \qquad \frac{\partial S}{\partial r'} = \mathcal{M}_z\,.$$

These are the three equations in their simplest form. With the parameters λ, μ, ν, the system will be non-holonomic of order 3 [**31**-2].

11. Theorem analogous to that of Koenig. – With the applications that follow in mind, it will be useful to establish a theorem that is analogous to that of Koenig, in order to abbreviate the calculations. Let x, y, z be the absolute coordinates of a point in a certain system with respect to some fixed axes. Let m be the mass of that point, let ξ, η, ζ be the coordinates of the center of gravity of the system, let $M = \sum m$ be the total mass, and let x_1, y_1, z_1 be the coordinates of the point m with respect to the axes $Gx_1\,y_1\,z_1$, which are drawn through G parallel to the fixed axes. Let J_0 denote the absolute value of the acceleration of the point G :

$$J_0^2 = \xi''^2 + \eta''^2 + \zeta''^2,$$

let J denote the absolute value of the acceleration of the point m :

$$J^2 = x''^2 + y''^2 + z''^2,$$

and let J_1 be its acceleration relative to the axes $Gx_1\,y_1\,z_1$:

$$J_0^2 = x_1''^2 + y_1''^2 + z_1''^2 .$$

One has:

$$x = \xi + x_1 , \qquad y = \eta + y_1 , \qquad z = \zeta + z_1 .$$

The expression:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2) = \tfrac{1}{2}\sum m\,[(\xi'' + x_1'')^2 + (\eta'' + y_1'')^2 + (\zeta'' + z_1'')^2],$$

when one takes into account the fact that:

$$\sum m\,x_1 = 0, \qquad \sum m\,x_1'' = 0, \qquad \ldots,$$

will become:

$$S = \tfrac{1}{2}M\,J_0^2 + \tfrac{1}{2}\sum m\,J_1^2 ,$$

which one can write:

$$S = \tfrac{1}{2}M\,J_0^2 + S_1 ,$$

in which S_1 is the energy of acceleration that is calculated for the relative motion around G; one will then have the theorem:

The energy of acceleration S of a system is equal to the energy of acceleration that one would have if the total mass were concentrated at its center of gravity, plus the energy of acceleration of the system that is calculated for the relative motion of the system around its center of gravity.

12. Totally-free solid body. – One can obtain the function S for a free solid body by applying the theorem in no. **11** that is analogous to Koenig's theorem. The term $S_1 = \frac{1}{2}\sum m\,J_1^2$, which relates to the motion of the body around its center of gravity, will be given by formula (25) , which relates to the motion of a solid around a fixed point. One will then have:

$$2S' = M\,J_0^2 + 2S_1 .$$

That formula is easily applied to the motion of a ponderous homogeneous body of revolution that is subject to sliding without friction on a fixed plane [**22**].

It will likewise permit one write out the equations of motion of a ponderous homogeneous body of revolution that is subject to rolling without slipping on a fixed horizontal plane.

13. Application to a solid body that moves parallel to a fixed plane. – In the study of the motion of a solid around a fixed point, one essentially supposes that the point is at a finite distance. If it is at infinity then the solid can move parallel to a fixed plane. Take the plane of the figure to be the plane of the curve that is described by the center of gravity. Let two axes Ox and Oy be fixed in the plane, and let ξ and η be the coordinates of G. It will obviously suffice to know the motion of the plane figure (P), which is a section of the body by the plane xOy. Let θ denote the angle that Ox makes with a radius GA that is invariably coupled to that planar figure (P), while $M\,k^2$ is the moment of inertia of the body with respect to the axis that is drawn through G perpendicular to the plane xOy.

The motion of the body around the center of gravity G is a rotation around an axis that is fixed in the body, while the angular velocity of rotation is θ'. One will then have:

$$S_1 = \frac{M\,k^2}{2}(\theta''^2 + \theta'^2)$$

for the function S_1 that is calculated for the motion of the body around G.

Therefore:

$$S = \frac{M}{2}[\xi''^2 + \eta''^2 + k^2\theta''^2 + \dots],$$

in which it is pointless to write out the terms that do not contain the second derivatives.

On the other hand, if one calls the projections of the general resultant of the applied forces X_0, Y_0, and lets N_0 denote the sum of the moments of those forces with respect to the axis that is drawn through G perpendicular to the plane xOy then one will have:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = X_0\,\delta\xi + Y_0\,\delta\eta + N_0\,\delta\theta .$$

If the body is not supposed to be subject to any constraint then the parameters ξ, η, ζ will be independent, and the equations of motion will be:

$$\frac{\partial S}{\partial \xi''} = X_0\,, \qquad \frac{\partial S}{\partial \eta''} = Y_0\,, \qquad \frac{\partial S}{\partial \theta''} = Z_0\,,$$

$$M\,\xi'' = X_0\,, \qquad M\,\eta'' = Y_0\,, \qquad M\,k^2\,\theta'' = N_0\,.$$

One will then recover the equations that give the general theorems immediately.

Suppose that the body is subject to a new constraint, which can be expressed by a relation in finite terms:

$$f(\xi, \eta, \theta, t) = 0,$$

or by a differential relation:

$$A\,d\xi + B\,d\eta + C\,d\theta + D\,dt = 0,$$
$$A\,\delta\xi + B\,\delta\eta + C\,\delta\theta + \qquad = 0,$$

in which A, B, C, D are functions of ξ, η, θ, t. One can then express η'' as a function of ξ'' and θ'', for example, and $\delta\eta$ as a function of $\delta\xi$ and $\delta\theta$. As a result, one can calculate S as a function of ξ'' and θ'' and make $\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$ linear and homogeneous in $\delta\xi$ and $\delta\theta$, and then equate $\frac{\partial S}{\partial \xi''}$ to the coefficients of $\delta\xi$ and $\frac{\partial S}{\partial \theta''}$ to that of $\delta\theta$.

V. – REMARKS OF AN ANALYTICAL ORDER.

14. Some properties of the function *S*. – In this number, we suppose that the constraints do not depend upon time:

$$a_\mu = b_\mu = c_\mu = 0,$$

and that the coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ depend solely upon the $q_1, q_2, \ldots, q_k$, and not on t. The same thing will then be true of the coefficients of S.

From the expression for S that was given above:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2),$$

when that function is confined to only the useful terms, it will have the following form:

(32) $$S = \varphi(q_1'', q_2'', \ldots, q_n'') + \psi_1\,q_1'' + \psi_2\,q_2'' + \cdots + \psi_n\,q_n''\,,$$

in which φ is a quadratic form in the q'':

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \sum \alpha_{ij}\,q_i''\,q_j'' \qquad (\alpha_{ij} = \alpha_{ji}),$$

whose coefficients α_{ij} are supposed to depend upon solely the q_1, q_2, …, q_k, and in which the ψ_1, ψ_2, …, ψ_k are quadratic forms in the q'_1, q'_2, …, q'_k whose coefficients also depend upon the q_1, q_2, …, q_k.

The semi-*vis viva* of the system:

$$T = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2)$$

is a quadratic form in the q'_1, q'_2, …, q'_k whose coefficients are the same as those of the form φ, in such a way that:

$$T = \varphi(q'_1, q'_2, \ldots, q'_k) = \sum \alpha_{ij}\, q'_i\, q'_j\,; \tag{34}$$

that fact results from calculating the two functions S and T. In order to simplify the writing, we make:

$$\varphi(q''_1, q''_2, \ldots, q''_n) = \varphi_2\,,$$
$$\varphi(q'_1, q'_2, \ldots, q'_k) = \varphi_2\,,$$

so

$$\begin{cases} S = \varphi_2 + \psi_1\, q''_1 + \psi_2\, q''_2 + \cdots + \psi_k\, q''_k, \\ T = \varphi_1. \end{cases} \tag{35}$$

It is easy to verify that one has:

$$\frac{dT}{dt} = \frac{\partial S}{\partial q''_1}\, q'_1 + \frac{\partial S}{\partial q''_2}\, q'_2 + \cdots + \frac{\partial S}{\partial q''_k}\, q'_k \tag{36}$$

identically.

Let us see what that identity gives from the forms (35) of S and T. It becomes:

$$\begin{aligned} &q'_1 \frac{\partial \varphi_2}{\partial q''_1} + q'_2 \frac{\partial \varphi_2}{\partial q''_1} + \cdots + q'_k \frac{\partial \varphi_2}{\partial q''_1} + \psi_1\, q'_1 + \psi_2\, q'_2 + \cdots + \psi_k\, q'_k \\ &= q''_1 \frac{\partial \varphi_1}{\partial q'_1} + q''_2 \frac{\partial \varphi_1}{\partial q'_2} + \cdots + q''_k \frac{\partial \varphi_1}{\partial q'_k} + q'_1 \frac{\partial \varphi_1}{\partial q_1} + q'_2 \frac{\partial \varphi_1}{\partial q_2} + \cdots + q'_k \frac{\partial \varphi_1}{\partial q_k}. \end{aligned} \tag{37}$$

The right-hand side of this is the developed expression for dT / dt that would result from the fact that T depends upon t by the intermediary of q'_1, q'_2, …, q'_k, q_1, q_2, …, q_k. Now, the first part of the left-hand side of (37) is identical to the first part on the right-hand side, from an elementary property of quadratic forms. The identity (37) will then reduce to:

$$\psi_1\, q'_1 + \psi_2\, q'_2 + \ldots + \psi_k\, q'_k = \frac{\partial \varphi_1}{\partial q_1}\, q'_1 + \frac{\partial \varphi_1}{\partial q_2}\, q'_2 + \cdots + \frac{\partial \varphi_1}{\partial q_k}\, q'_k\,. \tag{38}$$

That relation must be true for any $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'$. It will then establish the necessary relations between the coefficients of the forms $\psi_1, \psi_2, \ldots, \psi_k$ and the coefficients α_{ij} of φ_1. To abbreviate the writing, we denote both sides of the identity (38) by a single letter and set:

$$E = \frac{\partial \varphi_1}{\partial q_1} q_1' + \frac{\partial \varphi_1}{\partial q_2} q_2' + \cdots + \frac{\partial \varphi_1}{\partial q_k} q_k' = \psi_1 \, q_1' + \psi_2 \, q_2' + \ldots + \psi_k \, q_k' , \tag{39}$$

so the function E will be a cubic form in the $q_1', q_2', \ldots, q_k'$.

15. Correction terms in the Lagrange equations. – If the identity (38) is supposed to be fulfilled, then look for an expression for the difference:

$$\Delta_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} - \frac{\partial S}{\partial q_1''} . \tag{40}$$

From the notations of no. **8**, one will have:

$$\Delta_1 = R_1 - \frac{\partial T}{\partial q_1} .$$

Since we have set $T = \varphi_1$, we will have:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = {} & \frac{\partial^2 \varphi_1}{\partial q_1'^2} q_1'' + \frac{\partial^2 \varphi_1}{\partial q_1' \partial q_2'} q_2'' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1' \partial q_k'} q_k'' \\ & + \frac{\partial^2 \varphi_1}{\partial q_1' \partial q_1} q_1' + \frac{\partial^2 \varphi_1}{\partial q_1' \partial q_2} q_2' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1' \partial q_k} q_k' , \end{aligned}$$

because $\dfrac{\partial T}{\partial q_1'}$ or $\dfrac{\partial \varphi_1}{\partial q'}$ depend upon t by the intermediary of $q_1', q_2', \ldots, q_k', q_1, q_2, \ldots, q_k$.

Upon specifying the first row and taking into account the expression for E, one can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = 2(\alpha_{11} \, q_1'' + \alpha_{11} \, q_2'' + \cdots + \alpha_{11} \, q_k'') + \frac{\partial E}{\partial q_1'} - \frac{\partial \varphi_1}{\partial q_1} .$$

On the other hand:

$$\frac{\partial T}{\partial q_1} = \frac{\partial \varphi_1}{\partial q_1} ,$$

$$\frac{\partial S}{\partial q_1''} = 2(\alpha_{11}\, q_1'' + \alpha_{11}\, q_2'' + \cdots + \alpha_{11}\, q_k'') + \psi_1 .$$

After reduction, the difference (40), which is called Δ_1, will then become:

$$\Delta_1 = \frac{\partial E}{\partial q_1'} - 2\frac{\partial \varphi_1}{\partial q_1} - \psi_1 .$$

Upon setting:

$$\Delta_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} - \frac{\partial S}{\partial q_\nu''} = R_\nu - \frac{\partial T}{\partial q_\nu},$$

one will likewise have:

(41) $$\Delta_\nu = \frac{\partial E}{\partial q_\nu'} - 2\frac{\partial \varphi_1}{\partial q_\nu} - \psi_\nu .$$

Having said that, the equations of motion can be written:

(42) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu + \Delta_\nu \qquad (\nu = 1, 2, \ldots, k),$$

in which the term Δ_ν is expressed by the quantity (41) . Those quantities Δ_ν form what one call the *correction terms* in the Lagrange equations. One sees that the Lagrange equations can apply to the system if those terms Δ_ν are all identically zero. That will be the case when the system considered is holonomic and the parameters are true coordinates.

If the system is not holonomic then the motion of the system is the same as that of a holonomic system that admits the same *vis viva* $2T$ as the first one and is acted upon by "generalized forces":

$$Q_1 + \Delta_1, \qquad Q_2 + \Delta_2, \qquad \ldots, \qquad Q_k + \Delta_k .$$

The proof of the fact that a non-holonomic system and a holonomic system can have the same T identically can be found in a simple example that we gave in the Journal für die reine und angewandte Mathematik (Crelle's Journal), v. **122**, pp. 205.

The order of a non-holonomic system for the choice of parameters $q_1, q_2, \ldots, q_k$ is the number of Δ_ν that are non-zero.

Equation of *vis viva*. Verification. – If the constraints are independent of time then the equation of the *vis viva* will be:

(43) $$\frac{dT}{dt} = Q_1\, q_1' + Q_2\, q_2' + \cdots + Q_k\, q_k' .$$

In order to deduce that equation from equations (42), one must multiply the first of those equations by q_1', the second one by q_2', etc., the last one by q_k', and add them.

One will then get equation (43), because one has:

$$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_k q_k' = 0 \tag{44}$$

identically; in other words, the apparent forces that are characterized by Δ_1, Δ_2, …, Δ_k are *gyroscopic*, in the terminology of Sir. W. Thomson [**44**].

Indeed, from the expressions (41) for the quantities Δ_ν and the definition of E, one will have:

$$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_k q_k' = q_1' \frac{\partial E}{\partial q_1} + \cdots + q_k' \frac{\partial E}{\partial q_k} - 3E .$$

However, since E is homogeneous of degree three in q_1', q_2', …, q_k', the right-hand side will be zero identically, from the theorem on homogeneous functions.

16. General case. – If the constraints depend upon time, one can once more set:

$$\Delta_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} - \frac{\partial S}{\partial q_\nu''} .$$

The order of the non-holonomic system for the choice of parameters q_1, q_2, …, q_k will again be the number of Δ_ν (n = 1, 2, …, k) that are not zero [**33**].

VI. – FORMULATING THE EQUATIONS OF A PROBLEM IN DYNAMICS REDUCES TO THE SEARCH FOR THE MINIMUM OF A SECOND-DEGREE FUNCTION. GAUSS'S PRINCIPLE OF LEAST CONSTRAINT.

17. Problem of the minimum of a second-degree function. – If one considers the function R :

$$R = S - Q_1 q_1'' - Q_2 q_2'' - \cdots - Q_k q_k'' ,$$

which one can call the analytical expression for the constraint, then R will be a function of degree two in the q_1'', q_2'', …, q_k''. The equations of motion are written:

$$\frac{\partial R}{\partial q_1''} = 0, \qquad \frac{\partial R}{\partial q_2''} = 0, \qquad \ldots, \qquad \frac{\partial R}{\partial q_k''} = 0 .$$

The values of q_1'', q_2'', …, q_k'' that are inferred from those equations will then make R a maximum or minimum. Since R is a function of degree two in the q_1'', q_2'', …, q_k'' whose second-degree terms constitute a positive-definite form, the function R will be a *minimum* for the values of q_ν'' that correspond to the motion. It is obvious that one can make any function that differs from R by terms that are independent of the q''_ν play the same role as R. From the expressions for the x''_μ, y''_ν, z''_μ, δx_μ, δy_μ, δz_μ, that function R will have the same terms in q''_1, q''_2, …, q''_k as:

$$S - \sum_\mu [X\, x''_\mu + Y\, y''_\mu + Z\, z''_\mu]$$

or

$$\tfrac{1}{2}\sum m\, J^2 - \sum F\, J \cos FJ$$

or

$$R_0 = \sum \frac{1}{m}[(m\, x'' - X)^2 + (m\, y'' - Y)^2 + (m\, z'' - Z)^2].$$

One can then say that the accelerations that take the system from each instant, which are characterized by the values of q''_1, q''_2, …, q''_k, will make R_0 a *minimum*. If the system is free then that minimum will obviously be *zero*. If there are no external forces then R_0 will reduce to S.

18. Gauss's principle of least constraint. – From the translation of Gauss's paper, the principle of least constraint can be stated as follows:

> "The new principle is the following one:
>
> The motion of a system of material points that are coupled to each other in an arbitrary manner and subject to arbitrary influences will, at each instant, happen with the most perfect agreement possible with the motion that it would have if its were entirely free; i.e., with the smallest constraint possible, by taking the measure of the constraint that it experiences during an infinitely-small time interval to be the sum of the products of the mass of each point with the square of the quantity by which it deviates from the position that it would have taken had it been free.
>
> Let m, m', m'' be the masses of the points, let a, a', a'' be their respective positions, and let b, b', b'', resp., be the positions that they will occupy after an infinitely-small time dt, by virtue of the forces that act upon them and the velocities that had had acquired at the beginning of that instant. The preceding statement amounts to saying that the positions c, c', c'', resp., that they will take will be, among all of the ones that are allowed by the constraints, the ones for which the sum:
>
> $$m\,\overline{bc}^2 + m'\,\overline{b'c'}^2 + m''\,\overline{b''c''}^2 + \dots$$
>
> will be a minimum.
>
> Equilibrium is a special case of the general law that will be true when the sum:

$$m\,\overline{ab}^{\,2} + m'\,\overline{a'b'}^{\,2} + \dots$$

> is a minimum (since the points have no velocity), or in other words, when the conservation of the system of points in the rest state is *closer* to the free motion that each would tend to take than any possible displacement that one can imagine."

Here is the proof of that principle:

The preceding equation proves that principle. One can say that the equation is its analytical expression. On page 343 of his book [**11**], Mach, speaking of Gauss's principle, considered the expression:

$$N = \sum m\left[\left(\frac{X}{m} - \xi\right)^2 + \left(\frac{Y}{m} - \eta\right)^2 + \left(\frac{Z}{m} - \zeta\right)^2\right],$$

(in which ξ, η, ζ denote the projections of the acceleration of the point m), and sought the conditions that ξ, η, ζ must fulfill in order for N to be a *minimum*; he then came back to the general equation of dynamics.

In the German edition of the *Enzyklopädie der mathematischen Wissenschaften* and on page 84 of his article "Die Prinzipien der rationelle Mechanik," A. Voss [**28**] proceeded as follows in order to establish Gauss's principle: The position c of the point m has the following abscissa at the instant $t + dt$:

$$x + x'dt + \frac{x''}{1\cdot 2}dt^2 .$$

The sum that Gauss considered to be the measure of the constraint, namely:

$$m\,\overline{bc}^{\,2} + m'\,\overline{b'c'}^{\,2} + \dots$$

will then be:

$$\tfrac{1}{4}dt^4 \sum m\left[\left(x'' - \frac{X}{m}\right)^2 + \left(y'' - \frac{Y}{m}\right)^2 + \left(z'' - \frac{Z}{m}\right)^2\right].$$

Now, that sum is precisely:

$$\tfrac{1}{4}dt^4 R_0 .$$

It will be a minimum among all of the possible motions because the accelerations will make R_0 a minimum.

VII. – APPLICATIONS TO MATHEMATICAL PHYSICS.

19. Electrodynamics. – In a volume in the *Collection Scientia*, "L'électricité déduite de l'expérience et ramenée au principe des travaux virtuels," Carvallo studied the application of the Lagrange equations to electrodynamical phenomena according to Maxwell's theory. In regard to the Barlow wheel, he pointed out that those equations are not always applicable to electrodynamical phenomena, notably in the case of two or three-dimensional conductors. He observed that the phenomenon of the Barlow wheel depended upon three parameters θ, q_1, q_2 whose arbitrary variations define the most general displacement of the system. He indicated that those parameters are not true coordinates and that the system behaves in regard to them in the same way that a hoop behaves in regard to the three parameters θ, φ, and ψ (no. **2**). Under those conditions, the Lagrange equations will not be applicable, and if one can hope to attach the equations of electrodynamics to those of analytical mechanics then one must choose a form for the equations that will be applicable to all systems, whether holonomic or not [**19**].

For the Barlow wheel, when one employs Carvallo's notations (*loc. cit.*, pp. 76 and 80), the equations of motion will be:

$$\begin{aligned} I\,\theta'' - K\,q_1'\,q_2' &= Q, \\ L_1\,q_1'' + K\theta'\,q_2' &= E_1 - r_1\,q_1', \\ L_2\,q_2'' &= E_\nu - r_2\,q_2', \end{aligned}$$

in which the right-hand sides are the generalized forces that we have previously denoted by Q_1, Q_2, Q_3. Now, the left-hand sides of those equations are written:

$$\frac{\partial S}{\partial \theta''}, \quad \frac{\partial S}{\partial q_1''}, \quad \frac{\partial S}{\partial q_2''}$$

if one sets:

$$S = \tfrac{1}{2}[I\,\theta''^2 + L_1\,q_1''^2 + L_2\,q_2''^2 + 2K\,q_2''(\theta' q_1'' - q_1'\,\theta'') + \cdots],$$

in which the unwritten terms no longer contain the second derivatives of the parameters. The equations of motion are indeed of the general form that was studied in that volume, but it would be important to know whether the function S, thus-formed analytically, can be shown directly to be the energy of acceleration $S = \frac{1}{2}\sum m\,J^2$ by physical considerations.

20. Extension to the physics of continuous media. Application to the theory of electrons. – In this number, we shall reproduce almost *verbatim* a note by Guillame in Bern [**24**].

> "One can remark that if a system possesses a potential energy W then one will have:

$$\sum_{i=1}^{k} Q_i\, q_i'' = -\, W'' + U,$$

in which the sum is extended over the parameters that the potential energy depends upon, U denotes a term that is independent of the q_i'', and Q_i are the forces that are derived from the potential. The remaining forces will be called forces *external* to the system, and we set:

$$E = \sum_{i=1}^{k} Q_i\, q_i'',$$

in which the sum extends over those forces. If there are constraint equations $L_j = 0$ then one can introduce functions λ_j by a generalization of the method of Lagrange multipliers, as Poincaré showed in his *Leçons sur la théorie de l'élasticité*, in such a fashion that $\sum_j \lambda_i\, L_j$ can be considered to be a supplementary potential energy.

In the particular case where the kinetic energy:

$$T = \tfrac{1}{2} \sum m\,(x'^2 + y'^2 + z'^2)$$

is expressed in Cartesian coordinates, one will have:

$$T' = \sum m\,(x'\,x'' + y'\,y'' + z'\,z''),$$

$$T'' = \sum m\,(x''^2 + y''^2 + z''^2) + \sum m\,(x'\,x''' + y'\,y''' + z'\,z'''),$$

from which, one will deduce that:

$$\frac{\partial S}{\partial x''} = \frac{1}{2} \frac{\partial T''}{\partial x''}.$$

The expression:

$$R = S - \sum_{\nu=1}^{k} Q_\nu\, q_\nu''$$

wiil then be replaced by:

$$R = \tfrac{1}{2} T'' + W'' + \sum_j \lambda_j\, L_j - E. \tag{45}$$

If the coordinates are arbitrary then one must put S in place of $\tfrac{1}{2}T''$. From that, it will be easy to write R for continuous media. In that case, instead of the motion of a point m, one considers the motion of an element $d\tau$ of a certain volume V that is bounded by a surface Σ. The functions S,

T, or W become integrals that are extended over the volume V. The term that relates to the constraint equations will be obtained upon multiplying the left-hand sides of those equations by $\lambda_j\, d\tau$, adding them together, and integrating over the volume V. The term E can give both a volume integral and a surface integral. By definition, R has the form:

$$R = \iiint \varphi_0 \, d\tau + \iint \psi_0 \, d\sigma ,$$

in which φ_0 and ψ_0 can contain the accelerations and their partial derivatives. One then specifies the accelerations in such a fashion as to put R into the form:

$$R = \iiint \varphi_1 \, d\tau + \iint \psi_1 \, d\sigma ,$$

in which φ_1 and ψ_1 are polynomials of degree two or three in the accelerations. That transformation will be possible if the system is mechanical. Upon varying the accelerations, one will form the variation δR, which must be zero for any variation of the accelerations. Upon annulling the coefficients of those variations, one will obtain the desired equations.

Application to the theory of electrons. – In order to establish a mathematical link between mechanics and electrical phenomena, Maxwell appealed to Lagrange's equations; he then supposed that the corresponding systems were holonomic. H. A. Lorentz [**40**] reprised and generalized Maxwell's ideas. In particular, he showed the following: Consider the energy of the magnetic field:

$$T = \tfrac{1}{2} \iiint \mathfrak{h}^2 \, d\tau \tag{46}$$

to be a kinetic energy, and the energy of the electric field:

$$W = \tfrac{1}{2} \iiint \mathfrak{d}^2 \, d\tau \tag{47}$$

to be a potential energy, where the vectors $\mathfrak{h}$ and $\mathfrak{d}$ satisfy the two constraint equations:

$$c \operatorname{rot} \mathfrak{h} - \mathfrak{v} \operatorname{div} \mathfrak{d} - \mathfrak{d}' = 0, \tag{48}$$

$$\operatorname{div} \mathfrak{h} = 0, \tag{49}$$

in which $\mathfrak{v}$ denotes the velocity of matter, and c denotes the velocity of light. One can then establish the fundamental equation:

$$\text{rot}\,\mathfrak{d} = -\frac{1}{c}\mathfrak{h}' \tag{50}$$

by means of d'Alembert's principle.

The proof demands certain restrictions that are due to the use of quantities of electricity as coordinates and the introduction of all the virtual displacements. Lorentz was then led to define a new class of constraints that he called *quasi-holonomic*: He supposed that a system of electrons belongs to that class. Upon starting from the expression (45), and being given equations (46), (47), (48), (49), one can then establish equation (50), by supposing that the system is non-holonomic, in a general fashion.

Indeed, conforming to the meanings of T and W, the magnetic field $\mathfrak{h}$ is analogous to a velocity, so its derivative $\mathfrak{h}'$ will be analogous to an acceleration, and the electric field $\mathfrak{d}$ measures the deformation that produces the potential energy, so its first derivative $\mathfrak{d}'$ will be the rate of the variation of that deformation, and $\mathfrak{d}''$ will be its acceleration. Equation (48) permits one to immediately express $\mathfrak{d}''$ as a function of $\mathfrak{h}'$, in such a way that one will no longer have an equation of constraint (49) to consider. Let $\mathcal{F}\,d\sigma$ denote the force that acts on the element $d\sigma$, so one has:

$$R = \iiint\left[\tfrac{1}{2}\mathfrak{h}'^2 + c\,\mathfrak{d}\,\text{rot}\,\mathfrak{h}' - 2\lambda'\,\text{div}\,\mathfrak{h}'\right]d\tau - \iint \mathcal{F}\,\mathfrak{h}'\,d\sigma + \ldots$$

$$= \iiint\left[\tfrac{1}{2}\mathfrak{h}'^2 + c\,\mathfrak{h}'\,\text{rot}\,\mathfrak{d} + 2\mathfrak{h}'\,\text{grad}\,\lambda'\right]d\tau - \iint[c\,(\mathfrak{d}\,\mathfrak{h}')_n + \lambda'\,\mathfrak{h}' + \mathcal{F}\,\mathfrak{h}']\,d\sigma + \ldots$$

One infers from the volume integral that:

$$\mathfrak{h}' = -c\,\text{rot}\,\mathfrak{d} - 2\,\text{grad}\,\lambda'. \tag{51}$$

In order to determine λ', it suffices to form div $\mathfrak{h}'$, while taking equations (49) into account. One will then find that λ' must be constant. Its gradient will then be zero, and equation (51) will reduce to the desired equation (50). The surface integral permits one to determine the force $\mathcal{F}$. In order to find its significance, it will suffice to look for the work done per unit time. One finds that by taking the constant λ' to be equal to zero:

$$\mathcal{F}\,\mathfrak{h} = -c\,[\mathfrak{d}\;\mathfrak{h}]_n\,;$$

i.e., the Poynting energy flux.

If one starts with the same equations, while remaining in the ether, then the expression (45) will permit one to determine equation (48) without the term that relates to matter. One can then exhibit the duality that often observed in electricity in a striking way.

The fecundity of the method that was proposed here comes from the fact that one substitutes virtual accelerations for virtual displacements. The quantities of electricity do not come into play. There is no need to go deeper into the mechanism for the phenomenon. The possibility of establishing the expressions φ_1 and ψ_1 for the theory of electrons will imply the possibility of a mechanical interpretation for that theory. In addition to d'Alembert's principle, one has tried (above all, since Helmholtz) to extended Hamilton's principle to all of physics. Now, those principles apply poorly to the theory of electrons. One has the right to think that Appell's principle, thus-generalized, can be substituted for them advantageously; at least, in a number of cases.

One can see that the considerations above extend to the mechanics of Einstein. In it, one introduces the function (41):

$$H = -m_0 c \sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}$$

in order to form the equations of Lagrange and Hamilton in his mechanics. It is easy to see that H is the analogue of T in ordinary mechanics. Indeed, one has:

$$\frac{1}{2}\frac{\partial H''}{\partial \mathfrak{v}'} = \mathcal{F},$$

in which $\mathcal{F}$ denotes a force. That is the fundamental equation of motion in the new mechanics. The function R is obtained by replacing T'' with H'' in the expression (45)."

VIII. – CONSTRAINTS THAT ARE NONLINEAR IN THE VELOCITIES.

21. Possibility of non-linear constraints. – In his book on mechanics [**10**], Hertz showed that constraints are expressed by linear relations. However, it is possible that when certain masses or certain geometric quantities tend to zero, a set of linear constraints will produce a non-linear constraint that is imposed on a point of a system in the limit. One can then apply the preceding general equations to the corresponding motions. That is what I did in 1911 in a note in Comptes rendus [**25**] and then in two articles in the Rendiconti del Circolo Matematico di Palermo [**25**-2].

Delassus, a professor on the Science Faculty at Bordeaux, dedicated some important notes that were included in Comptes rendus de l'Académie des Sciences de Paris [**26**] in 1911 to a general study of the question and several papers "Sur les liaisons et les mouvements des systèmes matériels" that were printed in the Annales de l'École Normale supérieure [**27**]. In a letter that he wrote to me in 1911, Professor Hamel in Brünn (without knowing of the research of Delassus), likewise pointed out the difficulties that present themselves when one passes to the limit.

Delassus called "the motions studied by Appell" or "abstract motions" the motions that are obtained by the extending the principle of the minimum of the function:

$$R = \sum\left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')\right]$$

to nonlinear constraints. In his note to the Comptes rendus on 16 October 1911 [**26**], he realized those motions as motions that were limited by means of *realizations with perfect tendency*. For example, is L is the constraint:

$$(L) \qquad x'^2 + y'^2 = z'^2.$$

Delassus considered a linear constraint L' that contained arbitrary constants that gave the single relation:

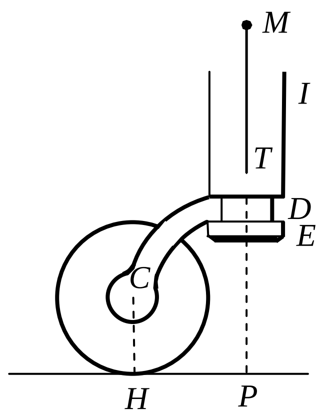
$$x'^2 + y'^2 = z'^2$$

between x', y', z', but produced some supplementary relations between x'', y'', z'' that disappeared in the limit.

My viewpoint will be different in what follows: In order to arrive at the limiting realization of the constraint L, I will consider a linear constraint L that contains an arbitrary constant ρ *that does not give any relation between* x', y', z', *but which produces the relation* (L) *in the limit* $\rho = 0$.

From the mechanical viewpoint, those two concepts are quite distinct.

I shall present that passage to the limit in an example. One will find some examples of the other viewpoint in the publications [**26** and **27**].



22. Example. – Imagine a caster that rolls without slipping on the horizontal plane xOy. The caster swivels about EI. Its wheel turns around a horizontal axis C that is carried by a fork CD that surrounds the swivel axis with a collar D. That collar can turn freely about the swivel axis in such a fashion that when one wishes to push the caster in a certain direction, the wheel will turn around the axle and will be located in a vertical plane through the direction of the swivel axis. The system poses no resistance to displacement in any direction.

In order to now complete our mechanism, we must suppose that there is just one wheel and just one swivel *EI* that is constrained to remain vertical by lateral shafts that rest on the floor without friction or sliding. A vertical shaft *TM* slides without friction inside the swivel. It is activated by the wheel with the aid of a mechanism that is easy to imagine, in such a way that it will be raised or lowered by a distance that is proportional to the angle φ through which the wheel has turned, in one sense or the other. The extremity of that shaft carries a point *M* of mass *m* and rectangular coordinates *x*, *y*, *z* on which an arbitrary force *F* acts. That system will give a quadratic constraint of the form:

$$\delta z^2 = k^2\,(\delta x^2 + \delta y^2)$$

in the limit, in which *k* denotes a constant when one supposes that:

1. All of the masses, except for *M*, first become zero.

2. The distance *HP* from the center *C* of the wheel to the shaft *TM* then tends to zero.

Indeed, in that limit, if the wheel turns through $\delta\varphi$ then its center *C* will experience a displacement whose projection onto the horizontal plane *xOy* will have components δx, δy such that:

$$\sqrt{\delta x^2 + \delta y^2} = a\ \delta\varphi ,$$

in which *a* denotes the radius of the wheel. On the other hand, the point *M* will experience a vertical displacement that is proportional to $\delta\varphi$:

$$\delta z = b\ \delta\varphi ;$$

one will then have:

$$\delta z^2 = k^2\,(\delta x^2 + \delta y^2), \qquad k^2 = \frac{b^2}{a^2} .$$

Before passing to the limit, one will have a system with linear constraints with two parameters *x* and *y*, to which one can apply the general equations that consist of writing out that under the motion, the values of the acceleration are the ones that make the function:

$$R = \sum\left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')\right]$$

a minimum. If one writes out those equations and passes to the aforementioned limit on the indicated order then one will find equations for the motion of *M* that express the idea that the function:

$$\tfrac{1}{2} m\,(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')$$

is a minimum when *x*, *y*, *z* are coupled by the relation:

$$k^2 (x'x'' + y'y'') - z'z'' = 0$$

that is obtained by differentiating the constraint equation:

$$k^2 (x'^2 + y'^2) - z'^2 = 0.$$

When one employs the method of Lagrange multipliers in order find the minimum, those equations will be:

$$m\,x'' = X + \lambda\,k^2\,x',$$
$$m\,y'' = Y + \lambda\,k^2\,y',$$
$$m\,z'' = Z - \lambda\,z'.$$

The force of constraint, whose projections are $\lambda\,k^2\,x'$, $\lambda\,k^2\,y'$, $-\lambda\,z'$, is perpendicular at M to the tangent plane to the cone whose summit is M and is defined by the set of all virtual displacements such that:

$$\delta z^2 = k^2 (\delta x^2 + \delta y^2),$$

while the plane is tangent along the actual displacement dx, dy, dz.

The work done by that constraint force is zero for the actual displacement: *It will not be zero for a virtual displacement that is compatible with the constraint.*

We perform the calculation that we just indicated. In the system of the figure: Let x, y, z denote the coordinates of M, let ξ, η denote those of the center C of the wheel, and z will remain constant for that point. ρ is the distance HP, and θ is the angle between HP and Ox. One will then have:

$$x = \xi + \rho \cos\theta, \qquad y = \eta + \rho \sin\theta.$$

The virtual displacements are defined by the relations:

$$\delta\xi = a \cos\theta\,\delta\varphi, \qquad \delta\eta = a \sin\theta\,\delta\varphi,$$

$$\delta x = a \cos\theta\,\delta\varphi - \rho \sin\theta\,\delta\theta,$$
$$\delta y = a \sin\theta\,\delta\varphi + \rho \cos\theta\,\delta\theta,$$
$$\delta z = b\,\delta\varphi\,.$$

The actual displacement is subject to the following conditions:

$$\xi' = a \cos\theta\,\varphi', \qquad \eta' = a \sin\theta\,\varphi',$$

$$x' = a \cos\theta\,\varphi' - \rho \sin\theta\,\theta',$$
$$y' = a \sin\theta\,\varphi' + \rho \cos\theta\,\theta',$$
$$z' = b\,\varphi',$$

so

$$(52) \qquad \begin{cases} \xi'' = a\cos\theta\,\varphi'' - a\sin\theta\,\varphi'\,\theta', \\ \eta'' = a\sin\theta\,\varphi'' + a\cos\theta\,\varphi'\,\theta', \end{cases}$$

$$x'' = (a\,\varphi'' - \rho\,\theta'^2)\cos\theta - (\rho\,\theta'' + a\,\varphi'\,\theta')\sin\theta,$$
$$y'' = (a\,\varphi'' - \rho\,\theta'^2)\sin\theta + (\rho\,\theta'' + a\,\varphi'\,\theta')\cos\theta,$$
$$z'' = b\,\varphi'',$$

from which, one infers that:

(53) $$\begin{cases} a\varphi'' - \rho\theta'^2 = \quad x''\cos\theta + y''\sin\theta, \\ \rho\theta'' + a\varphi'\theta' = -x''\sin\theta + y''\cos\theta. \end{cases}$$

The energy of acceleration S of the system is composed of the energy S_1 of the wheel and the energy S_2 of the point M upon neglecting the mass of the shaft and that of the piece CD :

$$S = S_1 + S_2 .$$

Now, from the preceding, one will have:

$$2S_1 = m\,(\xi''^2 + \eta''^2) + A\,\theta''^2 + B\,\varphi''^2 + \ldots,$$

when one calls the total mass of the wheel μ, while A and B are its principle moments of inertia relative to its center; hence:

$$2S_2 = m\,(x''^2 + y''^2 + z''^2) .$$

Since $z'' = b\,\varphi''$, from (52), one will have:

$$2S = (\mu\,a^2 + B + m\,b^2)\,\varphi''^2 + A\,\theta''^2 + m\,(x''^2 + y''^2) + \ldots,$$

or, upon replacing φ'' and θ'' with their expressions that are inferred from (53):

(54) $$\begin{aligned} 2S = {} & \frac{\mu\,a^2 + B + mb^2}{a^2}(x''\cos\theta + y''\sin\theta + \rho\,\theta'^2)^2 \\ & + \frac{A}{\rho^2}(x''\sin\theta - y''\cos\theta + a\,\varphi'\,\theta')^2 + m\,(x''^2 + y''^2) + \ldots \end{aligned}$$

in which the unwritten terms no longer contain second derivatives.

Now, make a force act upon the point M whose projections are X, Y, Z. The elementary work done by that force under a virtual displacement is:

$$X\,\delta x + Y\,\delta y + Z\,\delta z,$$

in which:

$$\delta z = b\,\delta\varphi = (\delta x\cos\theta + \delta y\sin\theta),$$

$$\delta z = k\,(\delta x\cos\theta + \delta y\sin\theta) .$$

The virtual work is then:

$$(X + k\,Z \cos\theta)\,\delta x + (Y + k\,Z \sin\theta)\,\delta y .$$

The equations of motion are then:

$$(55) \qquad \begin{cases} \dfrac{\partial S}{\partial x''} = X + k\,Z\cos\theta, \\ \dfrac{\partial S}{\partial y''} = Y + k\,Z\sin\theta. \end{cases}$$

Now, pass to the limit, while making the mass μ of the wheel and ρ go to zero. The coefficients B and A will also tend to zero. However, here we see the indeterminacy that Delassus pointed out in the general case. If A and ρ tend to zero at the same time then the limiting value of S will depend upon the behavior of K / ρ^2. We shall arrange that in such a way that A / ρ^2 tends to *zero*. S will then tend to the limit:

$$S = \tfrac{1}{2} m\,k^2\,(x'' \cos\theta + y'' \sin\theta)^2 + m\,(x''^2 + y''^2) + \ldots$$

The equations of motion keep the form (55). They are then:

$$\begin{aligned} m\,k^2\,(x'' \cos\theta + y'' \sin\theta) \cos\theta + m\,x'' &= X + k\,Z \cos\theta, \\ m\,k^2\,(x'' \cos\theta + y'' \sin\theta) \sin\theta + m\,y'' &= Y + k\,Z \sin\theta . \end{aligned}$$

On the other hand, one will then have:

$$x' = a \cos\theta\,\varphi', \qquad y' = a \sin\theta\,\varphi', \qquad z' = b\,\varphi',$$

$$\cos\theta = k\frac{x'}{z'}, \quad \sin\theta = k\frac{y'}{z'},$$

when one sets:

$$\frac{Z - m\,z''}{z'} = \lambda .$$

IX. – REMARKS ON NON-HOLONOMIC SYSTEMS THAT ARE SUBJECT TO PERCUSSIONS OR ANIMATED WITH VERY SLOW MOTIONS.

23. Application of Lagrange's equations in the case of percussions. – Beghin and Rousseau showed in a paper in the Journal de Mathématiques [**30**] that the form of the equations of the theory of percussions, which I have deduced from the Lagrange equations for the holonomic systems, further applies to non-holonomic systems, even though the Lagrange equations will then break down. One can establish that result by a method that is analogous to the one that I pointed out in no. **15**. Take the equations of an arbitrary system in the form of no. **15**:

(56) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu + \Delta_\nu \qquad (\nu = 1, 2, \ldots, k) .$$

The Δ_ν are correction terms that depend upon only the $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$ and time. Those correction terms are *zero* if the system is holonomic. However, if the percussions take place during the very short interval $t_1 - t_0$ then multiply the two terms in equation (56) by dt and integrate from t_0 to t_1. The integrals of $\frac{\partial T}{\partial q_\nu} dt$ and $\Delta_\nu\, dt$ will be negligible, because the q_ν and q'_ν will remain finite, and the equations will give:

$$\left(\frac{\partial T}{\partial q'_\nu}\right)_1 - \left(\frac{\partial T}{\partial q'_\nu}\right)_0 = \int_{t_0}^{t_1} Q_\nu \, dt .$$

Up to the difference in notations, these are precisely the equations that one can deduce from those of Beghin and Rousseau [**48**].

24. Case of very slow motions. – One can make a remark of the same type for the application of the Lagrange equations to the very slow motions of a non-holonomic system with constraints that are independent of time.

If the motion is very slow then the velocities will be very small. Consequently, the quantities $q'_1, q'_2, \ldots, q'_k$ will remain very small. Suppose that one neglects the squares and products of those quantities then. The terms that enter into equations (56) are quadratic forms in the $q'_1, q'_2, \ldots, q'_k$, so they are *negligible*, and the approximate equations take the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu ,$$

in which it will remain for one to suppress the terms of degree two in $q'_1, q'_2, \ldots, q'_k$.

In that case, the Lagrange equations will then provide some approximate equations of motion, although that form of equations is not rigorously applicable.

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