"Sur les liaisons cachées et les forces gyroscopiques apparentes dans les systèmes non holonomes," C. R. Acad. Sci. Paris **162** (1916), 27-29.

On hidden constraints and apparent gyroscopic forces in non-holonomic systems

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I. – In the search for a mechanical representation of a phenomenon, one can assume *a priori* only that the hidden constraints are of a special nature: In order to embrace the most general case, one must then suppose that those constraints are not holonomic and employ, not the Lagrange equations nor the canonical equations, but the general equations that would result from considering the energy of acceleration.

That viewpoint, which seems to have interested the physicists, was pointed out in a note "Sur l'emploi possible de l'énergie d'accelération dans les équations de l'Électrodynamique" that I presented to the Academy during the session on 22 April 1922 (C. R. Acad. Sci. Paris **154**, pp. 1037). It was developed in a note by Édouard Guillaume, "Sur l'extension des équations mécanique de M. Appell à la physique des milieu continus; application à la théorie des électrons" (C. R. Acad. Sci. Paris **156**, 10 March 1913, pp. 875).

If one mistakenly employs the Lagrange equations then one will be led to introduce, along with the forces that are actually applied, some *apparent* forces, which, in the terminology of Sir William Thomson (*Treatise on Natural Philosophy*, Vol. I, Part I, new edition, Cambridge, 1879, pp. 391-415), are *gyroscopic forces*, like the ones that present themselves in certain electromagnetic phenomena.

It is that fact that I propose to illuminate in a general manner.

II. – Although the consideration of the isolated system under study will suffice, it seems preferable to me to make the comparison that I would like to employ.

Imagine two systems (A) and (B), with hidden constraints that are independent of time and frictionless, the one (A) being holonomic, while the other (B) is non-holonomic. Suppose that those two systems have the same number k of degrees of freedom and the same expressions for their kinetic energies:

$$2T = \sum a_{ij}\, q_i'\, q_j' \,,$$

in which the coefficients a_{ij} are functions of the parameters $q_1, q_2, \ldots, q_k$. Finally, suppose that the forces that are actually applied to the two systems are derived from the same force function:

$$U(q_1, q_2, \ldots, q_k),$$

or, more generally, that the sum of the works done by those forces for an arbitrary displacement $\delta q_1, \delta q_2, \ldots, \delta q_k$ will have the same expression:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k$$

in the two systems.

The fact that systems of that type exist results from an example that I gave in my *Traité de Mécanique* (2nd ed., t. 2, pp. 385, no. 469).

Under those conditions, the equations of motion of system (A) are:

$$\text{(A)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, \ldots, k) .$$

Those of the system (B) can be written:

$$\frac{\partial S}{\partial q_i''} = Q_i ,$$

in which S denotes the energy of acceleration of that system. One can also put them into the form:

$$\text{(B)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i + \Delta_i ,$$

in which the Δ_i are correction terms that are homogeneous of order two in the components q_1', q_2', …, q_k' of the velocities. The analytical composition of the terms Δ_i was indicated in an article that was entitled "Remarques d'ordre analytique sur une nouvelle forme des équations de la Dynamique" and that I published in Jordan's *Journal de Mathématiques* [(5) **7** (1901), 5-12]. One can also obtain those terms by using Hertz's calculations (*Œuvres*, t. 3).

One then sees that for the observer who believes that the constraints on the system (B) are holonomic, it would seem that the system is subject to not only the real forces that give rise to the terms $Q_1, Q_2, \ldots, Q_k$, but also to apparent forces that give rise to the correction terms $\Delta_1, \Delta_2, \ldots, \Delta_k$.

Furthermore, those apparent forces are *gyroscopic*. Indeed, an application of the *vis viva* theorem shows that the systems of equation (A) and (B) will imply the same *vis viva* equation:

$$\frac{dT}{dt} = Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k' ,$$

which is obtained by adding them, after they have been multiplied by q_1', q_2', …, q_k', respectively.

One then obtains the relation:

(C) $$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_k q_k' = 0,$$

which will be true for any velocity components q_1', q_2', …, q_k' and parameters $q_1, q_2, \ldots, q_k$, since all of those quantities can be taken arbitrarily at the initial instant. The sum of the works done by apparent forces Δ_i is therefore zero under the real displacement: Those forces are *gyroscopic*.

If one makes a change of variables *in finite form:*

$$q_i = f_i (p_1, p_2, \ldots, p_k) \qquad (i = 1, 2, \ldots, k),$$

then equations (A) and (B) will keep the same forms, in which the q and q' are replaced by p and p', resp., the Q are replaced with the P that are deduced from the identity:

$$Q_1\, \delta q_1 + Q_2\, \delta q_2 + \ldots + Q_k\, \delta q_k \equiv P_1\, \delta p_1 + P_2\, \delta p_2 + \ldots + P_k\, \delta p_k ,$$

and in which the Δ that define the apparent forces are replaced by the Γ that are deduced from the analogous identity:

$$\Delta_1\, \delta q_1 + \Delta_2\, \delta q_2 + \ldots + \Delta_k\, \delta q_k \equiv \Gamma_1\, \delta p_1 + \Gamma_2\, \delta p_2 + \ldots + \Gamma_k\, \delta p_k .$$

The apparent forces then behave like true forces under changes of variables.

The simplest example of the considerations that were just developed is provided by the theory of the hoop and the bicycle, such as was presented in Carvallo's memoir [Journal de l'École Polytechnique (11) (1900-1901), 5th and 6th Cahiers].

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