"Remarque d'ordre analytique su une nouvelle forme des équations de la Dynamique," J. Math. pures appl. (5) **7** (1901), 5-12.

Remarks of an analytical order about a new form of the equations of dynamics

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1. – As we showed in a paper that was included in the first fascicle of the year 1900 of this collection, a material system is characterized by the function:

$$S = \tfrac{1}{2}\sum m\, J^2,$$

in which J denotes the acceleration of the point of mass m. Upon calling the parameters $q_1, q_2, \ldots, q_n$, whose virtual variations are arbitrary, the function S will be a function of degree two in $q_1'', q_2'', \ldots, q_n''$ that one can suppose to be reduced to only the terms in q_1'', $q_2'', \ldots, q_n''$. The coefficients of that function can depend upon $q_1, q_2, \ldots, q_n$, and some other parameters whose virtual variations are given linear, homogeneous functions of the variations of $q_1, q_2, \ldots, q_n$. For an arbitrary virtual displacement that is imposed upon the system, the sum of the works done by the applied force will be:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_n\,\delta q_n .$$

Moreover, the equations of motion are written:

$$\frac{\partial S}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, n).$$

De Saint-Germain proposed [Comptes rendus, **130** (1900), pp. 1174] to call the function S the *energy of acceleration*, by analogy with the terms *kinetic energy* or *energy of velocity* that are given to one-half the *vis viva* T.

We now propose to show that the function S can be chosen arbitrarily as a function of the parameters under only certain conditions on the degrees of $q_1'', q_2'', \ldots, q_n''$ and q_1', $q_2', \ldots, q_n'$. If the function S is supposed to be known then we will show how one can deduce the correcting terms in the Lagrange equations. Finally, we will give some indications of how one applies transformation methods to the problems of dynamics to which the Lagrange equations do not apply.

We suppose, to simplify, that the constraints do not depend upon time and that the coefficients of S contain only $q_1, q_2, \ldots, q_n$.

2. – From the expression for S that was given in the preceding paper:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2)\,,$$

that function will have the following form:

$$(1)\qquad S = \varphi(q_1'', q_2'', \ldots, q_n'') + \psi_1\, q_1'' + \psi_2\, q_2'' + \ldots + \psi_n\, q_n''\,,$$

in which φ is a quadratic form in the q'':

$$(2)\qquad \varphi(q_1'', \ldots, q_n'') = \sum a_{ij}\, q_i'' q_j'' \qquad\qquad (a_{ij} = a_{ji})$$

whose coefficients a_{ij} are supposed to depend upon only $q_1, q_2, \ldots, q_n$, and in which ψ_1, ψ_2, …, ψ_n, are quadratic forms in q_1', q_2', …, q_n' whose coefficients also depend upon $q_1, q_2, \ldots, q_n$.

One-half the *vis viva* of the system:

$$T = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2)$$

is a quadratic form in the q_1', q_2', …, q_n' whose coefficients are the same as the one in the form φ, in such a way that:

$$(3)\qquad T = \varphi(q_1', q_2', \ldots, q_n') = \sum a_{ij}\, q_i' q_j'\,.$$

That results from calculating with the two functions S and T. In order to simplify the notation, we set:

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \varphi_2\,,$$

$$\varphi(q_1', q_2', \ldots, q_n') = \varphi_1\,.$$

One will then have:

$$(4)\qquad \begin{cases} S = \varphi_2 + \psi_1\, q_1'' + \psi_2\, q_2'' + \cdots + \psi_n\, q_n''\,, \\ T = \varphi_1. \end{cases}$$

3. Necessary conditions that *S* must fulfill. – As is easy to show, and as we showed at the end of the preceding paper, one will have:

$$(5)\qquad \frac{dT}{dt} = \frac{\partial S}{\partial q_1''}\, q_1' + \frac{\partial S}{\partial q_2''}\, q_2' + \cdots + \frac{\partial S}{\partial q_n''}\, q_n'\,.$$

Let us see what the identity gives from the forms (4) of S and T. It will become:

(6)
$$\begin{cases} q_1'\dfrac{\partial\varphi_2}{\partial q_1''}+q_2'\dfrac{\partial\varphi_2}{\partial q_2''}+\cdots+q_n'\dfrac{\partial\varphi_2}{\partial q_n''}+\psi_1\,q_1'+\psi_2\,q_2'+\cdots+\psi_n\,q_n' \\ \quad = q_1''\dfrac{\partial\varphi_1}{\partial q_1''}+q_2''\dfrac{\partial\varphi_1}{\partial q_2''}+\cdots+q_n''\dfrac{\partial\varphi_1}{\partial q_n''}+q_1'\dfrac{\partial\varphi_1}{\partial q_1''}+q_2'\dfrac{\partial\varphi_1}{\partial q_2''}+\cdots+q_n'\dfrac{\partial\varphi_1}{\partial q_n''}\end{cases}$$

identically. The right-hand side of this is the developed expression for dT / dt that would result from T by the intermediary of q_1', q_2', …, q_n', q_1, q_2, …, q_n. Now the first part of the left-hand side of (6) is identical to the first part of the right-hand side, from an elementary property of quadratic forms. The identity (6) will then reduce to:

(7)
$$\psi_1\,q_1'+\psi_2\,q_2'+\cdots+\psi_n\,q_n' \;=\; q_1'\frac{\partial\varphi_1}{\partial q_1''}+q_2'\frac{\partial\varphi_1}{\partial q_2''}+\cdots+q_n'\frac{\partial\varphi_1}{\partial q_n''}.$$

That relation must be true for any q_1, q_2, …, q_n, q_1', q_2', …, q_n'. It then establishes necessary conditions between the coefficients of the forms ψ_1, ψ_2, …, ψ_n and the coefficients a_{ij} of φ_1. To abbreviate the notation, we shall denote the two sides of the identity (7) by the same symbol. When one sets:

(8)
$$E = \frac{\partial\varphi_1}{\partial q_1}q_1'+\frac{\partial\varphi_1}{\partial q_2}q_2'+\cdots+\frac{\partial\varphi_1}{\partial q_n}q_n' \equiv \psi_1\,q_1'+\psi_2\,q_2'+\cdots+\psi_n\,q_n',$$

the function E will be a cubic form in the q_1', q_2', …, q_n'.

4. Correcting terms in the Lagrange equations. – Suppose that the identity (7) is fulfilled and look for an expression for the difference:

(9)
$$\Delta_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right)-\frac{\partial T}{\partial q_1}-\frac{\partial S}{\partial q_1''}.$$

Since we have set $T = \varphi_1$, we will have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right)=\frac{\partial^2\varphi_1}{\partial q_1'^2}\;q_1''+\frac{\partial^2\varphi_1}{\partial q_1'\partial q_2'}\;q_2''+\cdots+\frac{\partial^2\varphi_1}{\partial q_1'\partial q_n'}\;q_n''$$

$$+\frac{\partial^2\varphi_1}{\partial q_1'\partial q_1'}\;q_1+\frac{\partial^2\varphi_1}{\partial q_1'\partial q'2}\;q_2+\cdots+\frac{\partial^2\varphi_1}{\partial q_1'\partial q_n'}\;q_n,$$

because $\dfrac{\partial T}{\partial q_1'}$ or $\dfrac{\partial\varphi_1}{\partial q_1}$ depend upon t by the intermediary of q_1', q_2', …, q_n', q_1, q_2, …, q_n.

Upon specifying the first row and taking the expression for E into account, one can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = 2(a_{11}\,q_1''+a_{12}\,q_2''+\cdots+a_{1n}\,q_n'')+\frac{\partial E}{\partial q_1'}-\frac{\partial \varphi_1}{\partial q_1}.$$

On the other hand:

$$\frac{\partial T}{\partial q_1} = \frac{\partial \varphi_1}{\partial q_1},$$

$$\frac{\partial S}{\partial q_1''} = 2(a_{11}\,q_1''+a_{12}\,q_2''+\cdots+a_{1n}\,q_n'')+\psi_1.$$

After reduction, the difference (9) that was called Δ_1 will then become:

$$\Delta_1 = \frac{\partial E}{\partial q_1'}-2\frac{\partial \varphi_1}{\partial q_1}-\psi_1 .$$

Upon setting:

$$\Delta_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha'}\right)-\frac{\partial T}{\partial q_\alpha}-\frac{\partial S}{\partial q_\alpha''},$$

one will have:

$$\Delta_\alpha = \frac{\partial E}{\partial q_\alpha'}-2\frac{\partial \varphi_1}{\partial q_\alpha}-\psi_\alpha . \tag{10}$$

Having said that, the equations of motion will be:

$$\frac{\partial S}{\partial q_\alpha''} = Q_\alpha .$$

One can then write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha'}\right)-\frac{\partial T}{\partial q_\alpha} = Q_\alpha+\Delta_\alpha \qquad (\alpha = 1, 2, \ldots, n), \tag{11}$$

in which the term Δ_α is expressed by the quantity in (10). Those quantities Δ_α constitute what one can call the *correcting terms* in the Lagrange equations. One sees that the Lagrange equations can apply to the system when those terms Δ_α are identically zero. That situation will come about when the system considered is subject to constraints that can all be expressed in finite form and the parameters are true coordinates. Following Hertz, one then calls the system *holonomic*.

If the system is not holonomic then the motion of the system is the same as that of a holonomic system that admits the same *vis viva* $2T$ as the first one and is subjected to forces:

$$Q_1 + \Delta_1 , \qquad Q_2 + \Delta_2 , \qquad \ldots, \qquad Q_n + \Delta_n .$$

The fact that a non-holonomic system and a holonomic system can have the same T identically can be proved by a simple example that we gave in the Journal für die reine und angewandte Mathematik, founded by Crelle, v. 122, pp. 205.

3. *Vis viva* equation: verification. – If the constraints are independent of time then the *vis viva* equation will be:

$$\frac{dT}{dt} = Q_1 q_1' + Q_2 q_2' + \cdots + Q_n q_n' . \tag{12}$$

In order to deduce that equation from (11), one must multiply the first of those equations by q_1', the second by q_2', …, and the last one by q_n', and then add them together.

One will then get equation (12), because one will have:

$$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_n q_n' = 0 \tag{13}$$

identically.

Indeed, from the expressions (10) for the quantities Δ_α and the definition of E [viz., equation (8)], one will have:

$$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_n q_n' = q_1' \frac{\partial E}{\partial q_1'} + \cdots + q_n' \frac{\partial E}{\partial q_n'} - 3E.$$

However, since E is homogeneous and of degree three in q_1', q_2', …, q_n', the right-hand side of that is zero identically, from the theorem on homogeneous functions.

6. Application of transformation methods. – We conclude by indicating a problem that presents itself naturally. If the components of the forces Q_1, Q_2, …, Q_n depend upon only q_1, q_2, …, q_n, and not upon the velocities then the right-hand sides of the equations of motion (11) can nonetheless contain the velocities in the terms Δ_α when the system is not holonomic: Those terms have degree two in q_1', q_2', …, q_n'.

Can one make the terms of that nature disappear by performing a change of variables that involves the parameters and time?

In particular, one can try a transformation of the form:

(14)
$$\begin{cases} p_\alpha = f_\alpha(q_1, q_2, \dots, q_n) \qquad (\alpha = 1, 2, \dots, n), \\ dt = \lambda(q_1, q_2, \dots, q_n)\, dt_1, \\ p'_\alpha = \dfrac{dp_\alpha}{dt_1}, \end{cases}$$

in which f_α and λ are functions of q_1, q_2, …, q_n, p_α are the new parameters, and t_1 is the new time. From a calculation that we made in an article "Sur des transformations de mouvement" (Crelle's Journal, v. 110, pp. 37), the equations of motion (11) will take the form:

(15)
$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial p'_2}\right) - \frac{\partial T_1}{\partial p_2} = \Phi_\alpha + \sum_{i=1}^{n} R_\alpha^i (Q_i + \Delta_i),$$

in which Φ_α is a quadratic form in q'_1, q'_2, …, q'_n, and in which the R_α^i depend upon only the q_1, q_2, …, q_n. One can make the derivatives in the right-hand side disappear, moreover, if one can specialize the transformation in such a way that one has:

(16)
$$\Phi_\alpha + \sum_{i=1}^{n} R_\alpha^i \Delta_i = 0 \qquad (\alpha = 1, 2, \dots, n)$$

identically.

Those conditions, whose left-hand sides are quadratic forms in q'_1, q'_2, …, q'_n, must be true no matter what those derivatives are, so upon equating the coefficients of the various powers of q'_1, q'_2, …, q'_n to zero, one will get a larger number of defining equations for the functions f_1, f_2, …, f_n, and λ. Those equations will generally be quite numerous, and the problem can be solved only for special systems.
