"Les équations du mouvement d'un fluide parfait déduites de la considération de l'énergie d'accélération," Ann. mat. pura appl. (3) **20** (1913), 37-42.

The equations of motion for a perfect fluid, deduced from the consideration of the energy of acceleration.

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I. – J. L. Lagrange deduced the equations of hydrostatics from the principle of virtual work in his *Mécanique analytique* (3rd edition, I, pp. 173-206; *Oeuvres* **11**, pp. 197-236). As for the equations of hydrodynamics, BASSET, in the work entitled *A Treatise on hydrodynamics*, Cambridge, 1888, said this on page 32: "As Larmor showed, the equations of motion can be deduced by the use of the principle of least action, combined with Lagrange's method."

I propose to deduce those equations from the following principle:

In a material system with arbitrary constraints without friction (whether holonomic or not) that is subject to forces X, Y, Z that depend upon time, the positions, and the velocities, the components x, y, z of the accelerations of the various points have values at an arbitrary instant that render the function:

$$R = \sum \left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'') \right]$$

a minimum.

One will find the proof of that principle in a note that was entitled "Sur les mouvements de roulement; équations du mouvement analogues à celles de Lagrange" [Comptes Rendus de l'Académie des Sciences de Paris **129** (7 August 1899), pp. 317-320].

The quantity:

$$S = \sum \frac{m}{2}(x''^2 + y''^2 + z''^2),$$

which is analogous to the LAGRANGE function:

$$T = \sum \frac{m}{2}(x'^2 + y'^2 + z'^2),$$

has received the name of *energy of acceleration.*

II. – Having said that, imagine a perfect fluid in motion. Let ρ denote the density of a particle whose coordinates are x, y, z at the instant t, and let X, Y, Z denote the components of the force per unit mass. Use the variables that one calls the *Lagrange variables*, and let a, b, c denote the initial coordinate of the particle x, y, z at the instant $t = t_0$, and let ρ_0 denote the initial density of that particle. The coordinates x, y, z, and the density ρ are functions of a, b, c, t.

The continuity equation is:

$$\rho D = \rho_0 , \tag{1}$$

in which D denotes the functional determinant:

$$D = \frac{d(x,y,z)}{d(a,b,c)},$$

and we shall denote minors such as $\frac{\partial y}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial z}{\partial b}\frac{\partial y}{\partial c}$ by $\frac{d(y,z)}{d(b,c)}$. Finally, the derivatives of x, y, z with respect t will be denoted by primes, following LAGRANGE's notation. The function that is analogous to R is presently expressed by a triple integral that is extended over the volume V of the fluid at the instant t :

$$R = \iiint \left[\tfrac{1}{2}\rho (x''^2 + y''^2 + z''^2) - \rho (X\, x'' + Y\, y'' + Z\, z'') \right] dx\, dy\, dz ,$$

or, upon taking the integration variables to be a, b, c :

$$R = \iiint \left[\tfrac{1}{2}\rho_0 (x''^2 + y''^2 + z''^2) - \rho_0 (X\, x'' + Y\, y'' + Z\, z'') \right] da\, db\, dc, \tag{2}$$

because one must replace $dx\, dy\, dz$ by $D\, da\, db\, dc$ or $(\rho_0 / \rho)\, da\, db\, dc$. In that integral, x'', y'', z'' are, at the instant t, functions of the a, b, c that are subject to the following relation, which is deduced from the continuity equation (1). Differentiate that equation twice with respect to t ; it will become:

$$D'' + 2\frac{\rho'}{\rho} D' + \frac{\rho''}{\rho} D = 0. \tag{3}$$

In that equation, the terms in x'', y'', z'' are provided by only D''. Now, one has:

$$\left.\begin{aligned}
D'' = {} & \frac{\partial x''}{\partial a}\frac{d(y,z)}{d(b,c)} + \frac{\partial y''}{\partial a}\frac{d(z,x)}{d(b,c)} + \frac{\partial z''}{\partial a}\frac{d(x,y)}{d(b,c)} \\
& + \frac{\partial x''}{\partial b}\frac{d(y,z)}{d(c,a)} + \frac{\partial y''}{\partial b}\frac{d(z,x)}{d(c,a)} + \frac{\partial z''}{\partial b}\frac{d(x,y)}{d(c,a)} \\
& + \frac{\partial x''}{\partial c}\frac{d(y,z)}{d(a,b)} + \frac{\partial y''}{\partial c}\frac{d(z,x)}{d(a,b)} + \frac{\partial z''}{\partial c}\frac{d(x,y)}{d(a,b)} \\
& + \dots\dots\dots\dots\dots\dots ,
\end{aligned}\right\} \tag{4}$$

in which the unwritten terms do not contain x'', y'', z''.

Having said that, from LAGRANGE's method for the calculus of variations, upon denoting an arbitrary function of a, b, c, t by λ, one must determine x'', y'', z'' in such a fashion that the variation of the integral:

$$I = \iiint \left[\tfrac{1}{2}\rho_0 (x''^2 + y''^2 + z''^2) - \rho_0 (X\, x'' + Y\, y'' + Z\, z'') + \lambda\, D'' \right] da\, db\, dc$$

will be zero when x'', y'', z'' are subjected to arbitrary infinitely-small variations $\delta x''$, $\delta y''$, $\delta z''$. Now:

$$\delta I = \iiint [\rho_0 (x'' - X)\, \delta x'' + \cdots + \lambda\, \delta D''] \; da\, db\, dc$$

in which, from (4):

$$\delta D'' = \frac{\partial \delta x''}{\partial a}\frac{d(y,z)}{d(b,c)} + \frac{\partial \delta y''}{\partial a}\frac{d(z,x)}{d(b,c)} + \frac{\partial \delta z''}{\partial a}\frac{d(x,y)}{d(b,c)} + \ldots + \ldots,$$

in which we have written only the first line. We now apply the formula for integration by parts, which would result from GREEN's theorem (see, for example, my *Traité de mécanique*, t. III, Chap. XXVIII) to terms such as:

$$\iiint \lambda \frac{d(y,z)}{d(b,c)} \frac{\partial \delta x''}{\partial a}\, da\, db\, dc. \tag{5}$$

Upon letting $d\sigma_0$ denote an element of the bounding surface S_0 of the fluid, and letting α_0, β_0, γ_0 denote the direction cosines of the exterior normal to it, one will have

$$\iiint \frac{\partial (PQ)}{\partial a}\, da\, db\, dc = \iint_{S_0} P\, Q\, \alpha_0\, d\sigma_0$$

for two any functions P and Q, so:

$$\iiint P \frac{\partial Q}{\partial a}\, da\, db\, dc = \iint_{S_0} P\, Q\, \alpha_0\, d\sigma_0 - \iiint Q \frac{\partial P}{\partial a}\, da\, db\, dc.$$

The term (5) can then be replaced with:

$$\iint_{S_0} \lambda \frac{d(y,z)}{d(b,c)}\, \delta x''\, \alpha_0\, d\sigma_0 - \iiint \frac{\partial}{\partial a}\left(\lambda \frac{d(y,z)}{d(b,c)} \right) \delta x''\, da\, db\, dc.$$

We treat each of the nine terms that are provided by $\delta D''$ similarly, and remark that:

$$\frac{\partial}{\partial a}\left(\lambda\frac{d(y,z)}{d(b,c)}\right)+\frac{\partial}{\partial b}\left(\lambda\frac{d(y,z)}{d(c,a)}\right)+\frac{\partial}{\partial c}\left(\lambda\frac{d(y,z)}{d(a,b)}\right)$$

$$= \frac{\partial\lambda}{\partial a}\frac{d(y,z)}{d(b,c)}+\frac{\partial\lambda}{\partial b}\frac{d(y,z)}{d(c,a)}+\frac{\partial\lambda}{\partial c}\frac{d(y,z)}{d(a,b)}$$

$$= D\frac{\partial\lambda}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial\lambda}{\partial x},$$

as one will see upon writing:

$$\frac{\partial\lambda}{\partial a} = \frac{\partial\lambda}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial\lambda}{\partial y}\frac{\partial y}{\partial a}+\frac{\partial\lambda}{\partial z}\frac{\partial z}{\partial a},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We finally have:

$$\left.\begin{aligned}\delta I = \iint_{S_0}\left(\alpha_0\frac{d(y,z)}{d(b,c)}+\beta_0\frac{d(y,z)}{d(c,a)}+\gamma_0\frac{d(y,z)}{d(a,b)}\right)\lambda\,\delta x''\,d\sigma_0 \\ +\cdots+\cdots \\ +\iiint\left(x''-X-\frac{1}{\rho}\frac{\partial\lambda}{\partial x}\right)\delta x''\,\rho_0\,da\,db\,dc+\cdots+\cdots,\end{aligned}\right\} \tag{6}$$

in which we have written only the terms in $\delta x''$ in both the partial integral (viz., the double integral) and the triple integral. Later, we shall see that the partial integral is *zero*. We equate the coefficients of $\delta x''$, $\delta y''$, $\delta z''$ in the triple integral to zero. We will then get the classical equations:

$$x''-X-\frac{1}{\rho}\frac{\partial\lambda}{\partial x} = 0, \quad y''-Y-\frac{1}{\rho}\frac{\partial\lambda}{\partial y} = 0, \quad z''-Z-\frac{1}{\rho}\frac{\partial\lambda}{\partial z} = 0,$$

in which the pressure is equal to $-\lambda$.

III. – It remains for us to see that the partial integral (viz., the double integral) in formula (6) is *zero*, as a result of the values of $\delta x''$, $\delta y''$, $\delta z''$ on the boundary surface.

One knows that the fluid particles that are on the boundary surface S_0 at the initial instant will remain on the boundary surface S at the instant t. The element $d\sigma_0$ of S_0 will become an element $d\sigma$ on S. One lets α, β, γ denote the direction cosines of the normal to $d\sigma$. One will then have:

$$\left(\alpha_0\frac{d(y,z)}{d(b,c)}+\beta_0\frac{d(y,z)}{d(c,a)}+\gamma_0\frac{d(y,z)}{d(a,b)}\right)d\sigma_0 = \alpha\,d\sigma. \tag{7}$$

Indeed, a, b, c are functions of two parameters p and q on S_0, and when one sets:

$$d\sigma_0 = k\, dp\, dq,$$

one will have:

$$\alpha_0 = \frac{1}{k}\frac{d(b,c)}{d(p,q)}, \qquad \beta_0 = \frac{1}{k}\frac{d(c,a)}{d(p,q)}, \qquad \gamma_0 = \frac{1}{k}\frac{d(a,b)}{d(p,q)}.$$

Since x, y, z are functions of p and q by the intermediary of a, b, c, one will have:

$$d\sigma = h\, dp\, dq,$$

$$\alpha = \frac{1}{h}\frac{d(y,z)}{d(p,q)} = \frac{1}{h}\left[\frac{d(y,z)}{d(b,c)}\frac{d(b,c)}{d(p,q)} + \frac{d(y,z)}{d(c,a)}\frac{d(c,a)}{d(p,q)} + \frac{d(y,z)}{d(a,b)}\frac{d(a,b)}{d(p,q)}\right],$$

which is precisely formula (7). The double integral that figures in δI can then be written:

$$\iint_S (\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z'')\,\lambda\, d\sigma,$$

in which the integration is extended over the surface S of the fluid at the instant t. However, the differential elements will then be *zero*. Indeed, let:

$$f(x, y, z, t) = 0$$

be the equation of the bounding surface S. Upon differentiating with respect to t and following the particle x, y, z in its motion, one will have:

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' + \frac{\partial f}{\partial t} = 0,$$

and then:

$$\frac{\partial f}{\partial x}x'' + \frac{\partial f}{\partial y}y'' + \frac{\partial f}{\partial z}z'' + \ldots = 0, \tag{8}$$

in which the unwritten terms do not contain x'', y'', z''. The relation (8) shows that the variations $\delta x''$, $\delta y''$, $\delta z''$ will verify the condition:

$$\frac{\partial f}{\partial x}\delta x'' + \frac{\partial f}{\partial y}\delta y'' + \frac{\partial f}{\partial z}\delta z'' = 0$$

on the surface, or when one calls the direction cosines of the normal α, β, γ:

$$\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z'' = 0.$$

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