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ROLLING MOTIONS

IN

DYNAMICS

BY

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WITH TWO NOTES BY HADAMARD

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INTRODUCTION

Rolling motions occupy a special place in dynamics, as has been known for some time, and mainly by the research of Neumann [Math. Ann. **27** (1886)]. That comes down to the fact that the constraint that two solid bodies should roll on each other cannot be expressed by equations whose left-hand sides are linear, homogeneous functions of the coordinate differentials, *because those functions are not exact total differentials.* That will imply special difficulties when one wishes to apply the general methods of analytical dynamics to those problems. On the other hand, those motions are encountered constantly in applied mechanics: viz., the hoop, the unicycle, the bicycle, the rolling of balls are the simplest examples.

This booklet has the goal of making known the main methods that are employed to treat that class of problems in such a fashion that the reader himself can undertake new research.

MAIN BOOKS AND PAPERS TO CONSULT

SLESSER. – Quart. J. Math., 1861.

NEUMANN. – Math. Ann. **27** (1886), Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig, 1888.

ROUTH. – *Advanced Rigid Dynamics*, MacMillan and Co., 1884.

VIERKANDT. – "Ueber gleitende und rollende Bewegung," Mon. Math. Phys. **3** (1892), pp. 47.

HADAMARD. – "Sur les mouvements de roulement," C. R. Acad. Sci. (1894), Mémoires de la Société des Sciences physiques et naturelles de Bordeaux **5** (1895). That paper is reproduced at the end of this volume.

BOURLET. – *Bicycles et Bicyclettes*, v. 1, Équilibre et direction, v. 2, Travail, Gauthier-Villars. "Étude théorique sur la bicyclette," Bull. Soc. math. France (1899). "Sur les roulements à billes," Génie civil (1898). "Pistes de vélodrome," Rend. Circ. Mat. di Palermo (1899).

CARVALLO. – Papers submitted in competition for the Fourneyron Prize (public meeting of l'Académie des Sciences, 17 December 1898). That paper will appear next in the J. Ec. poly.

BOUSSINESQ. – Various notes in Comptes rendus, second semester 1898, first semester 1899, and J. de Math., first fascicle, 1899.

KORTEWEG ([1]). – "Ueber eine ziemlich verbreitete unrichtige Behandlungsweise eines Problemes der rollenden Bewegung, über die Theorie dieser Bewegung, und insbesondere über kleine rollende Schwingungen um eine Gleichgewichtslage," Nieuw Achief voor Wisskunde (1899). That paper will be followed by a "Note sur le mouvement de roulement d'un corps pesant de révolution sur le plan horizontal," which will appear in the same journal at the end of 1899.

APPELL. – "Sur l'intégration des équations du mouvement d'un corps pesant de révolution roulant par une aréte circulaire sur un plan horizontal; cas du cerceau." To appear in Rend. Circ. Mat. Palermo, first fascicle of 1900.

([1]) I had no knowledge of this paper when the present volume was published.

FIRST CHAPTER

GENERAL FORMULAS RELATING TO THE MOTION OF A SOLID

BOOKS TO CONSULT:

DARBOUX. – *Leçons sur la théorie générale des surfaces*, t. I, Chap. I and II.

ROUTH. – *The Advanced part of a Treatise on the Dynamics of a system of Rigid-Bodies*, Chap. I.

1. Some theorems of kinematics. – First imagine a system of invariable form in motion around a fixed point *O*. In kinematics, one shows that at each instant *t*, the velocities of the various points of the system are the same as if it turned with a certain angular velocity ω around an axis that passes through the fixed point. One represents that rotation, which is called an *instantaneous rotation*, by a vector $O\omega$ of length ω that is carried by the rotational axis with a sense such that an observer that has his feet at *O* and his head at ω will see the system turn from his left to his right.

Now imagine an invariable system that is animated with an arbitrary motion. Take a point *O* that is invariably linked with the system, and let *V* be the velocity of that point. One shows that the velocities of the various points of the system are the same as if the system were animated by both a translation whose velocity is equal to *V* and rotation whose angular velocity ω around an axis $O\omega$ passes through *O*. That is to say that the velocity of an arbitrary point of the system is the resultant of a vector that is equal to *V* and a vector that is equal to the velocity that the point would have if the system were animated by only the rotation ω. In that representation of the state of the velocities, the choice of the point *O* that is invariably linked with the system is *arbitrary*. If, at the same instant *t*, one has made a choice of another point O' that is invariably linked with the system then one will have another velocity V' of translation, but the rotation will be the same.

2. Formulas. – Consider a tri-rectangular trihedron *Oxyz* in motion. That trihedron constitutes an invariable system. We suppose that the axes are oriented in such a fashion that a rotation of 90° in the positive sense around *Oz* will take *Ox* to *Oy*.

At the instant *t*, let V' be the velocity of the point *O*, and let ω' be the instantaneous rotation of the trihedron, and denote the projections of the vectors V' and ω' onto the moving axes *Oxyz* by u', v', w' and p', q', r', respectively.

When the motion of the trihedron is given, V' and ω' will be known at each instant, so u', v', w' and p', q', r' will be known functions of time. Conversely, if those quantities are given as functions of time then one can find the motion of the trihedron, as one can see in DARBOUX's *Leçons*, vol. I, Chapter II.

Velocity of a point. – Let m be a point that is *invariably linked* with the trihedron: The coordinates x, y, z of that point with respect to the trihedron will then be constant. Let V_e denote the velocity of the point m at the instant t, and let V_{ex}, V_{ey}, V_{ez} be its projections onto the moving axes. From the known formulas for rotations:

$$(1) \qquad \begin{aligned} V_{ex} &= u' + q'z - r'y , \\ V_{ey} &= v' + r'x - p'z , \\ V_{ez} &= w' + p'y - q'x . \end{aligned}$$

If the point m is in motion with respect to the axes $Oxyz$ then its coordinates x, y, z will vary with t. The absolute velocity of the point m will then be the resultant of its relative velocity, which has $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ for its projections, and its guiding velocity, which has the quantities (1) for its projections. Upon denoting the absolute velocity of the point m by V_a and denoting its projections by V_{ax}, V_{ay}, V_{az}, one will then have:

$$(2) \qquad \begin{aligned} V_{ax} &= \frac{dx}{dt} + u' + q'z - r'y , \\ V_{ay} &= \frac{dy}{dt} + v' + r'x - p'z , \\ V_{az} &= \frac{dz}{dt} + w' + p'y - q'x . \end{aligned}$$

3. Applications:

1. *Expressing the idea that the point m is immobile in space.* – It will suffice to write down that its absolute velocity is zero:

$$(3) \qquad \frac{dx}{dt} + u' + q'z - r'y = 0, \text{ etc.}$$

2. *Expressing the idea that a line whose direction cosines with respect to the axes Oxyz are the quantities α, β, γ has a fixed direction in space.* – Upon drawing a segment OA of length 1 that is parallel to the given line through the origin, one will get a point A whose coordinates are:

$$x = \alpha, \quad y = \beta, \quad z = \gamma .$$

In order for the segment OA to displace parallel to itself, it is necessary and sufficient that the points A and O should have the same absolute velocity at each instant; one expresses that by writing that $V_{ax} = u'$, $V_{ay} = v'$, $V_{az} = z'$. One will then have the conditions:

$$\frac{d\alpha}{dt} + \gamma q' - \beta r' = 0, \quad \text{etc.} \tag{4}$$

4. Acceleration of a point. – Let m be a point in motion with respect to the moving axes: Let V_a denote its absolute velocity, and let J_a denote its absolute acceleration.

Take a point O_1 that is absolutely fixed and has coordinates a, b, c with respect to the moving axes, and draw a segment $O_1\ m_1$ through that point that is equal and parallel to V_a. From the definition of acceleration, J_a will be equal to the absolute velocity of the point m_1 . Now, the coordinates x_1, y_1, z_1 of m_1 with respect to the axes $Oxyz$ are:

$$x_1 = a + V_{ax}\,, \quad y_1 = b + V_{ay}\,, \quad z_1 = c + V_{az}\,.$$

Upon letting J_{ax} , J_{ay} , J_{az} denote the projections of the desired acceleration J_a , one will then have:

$$J_{ax} = \frac{dx_1}{dt} + u' + q' z_1 - r' y_1\,, \ldots,$$

or

$$J_{ax} = \frac{da}{dt} + \frac{dV_{ax}}{dt} + u' + (c + V_{ez})\, q' - (b + V_{ey})\ r', \ldots$$

However, if the point O_1 is fixed in space then one will have:

$$\frac{da}{dt} + u' + c\, q' - b\, r' = 0, \ldots$$

Thus, one finally has:

$$J_{ax} = \frac{dV_{ax}}{dt} + q' V_{az} - r' V_{ay}\,,$$

$$J_{ay} = \frac{dV_{ay}}{dt} + r' V_{ax} - p' V_{az}\,, \tag{5}$$

$$J_{az} = \frac{dV_{az}}{dt} + p' V_{ay} - q' V_{ax}\,.$$

5. Motion of a solid body around a fixed point. – Imagine a solid body that moves around a fixed point O under the action of given forces F_1 , F_2 , F_3 , … Let ω denote the instantaneous rotation of the body at the instant t, which is a rotation that is represented by a certain vector $O\omega$. Refer the motion of the body to a trihedron $Oxyz$ whose summit O is animated with a known motion. As above, let ω' denote the instantaneous rotation of the trihedron at time t. Let p', q', r' denote the projections of ω' onto the three axes, and let p, q, r denote those of ω. If the point O is fixed then the quantities u', v', w' will be zero.

Resultant moment of the quantities of motion. – A molecule m of the body whose coordinates are x, y, z will possess an absolute velocity at the instant t whose projections onto the moving axes will be:

$$V_{ax} = q\,z - r\,y,$$

(6) $$V_{ay} = r\,x - p\,z,$$

$$V_{az} = p\,y - q\,x\,,$$

from known formulas for rotations.

Let us construct the resultant moment $O\sigma$ of the quantities of motion of the various points of the body with respect to the point O. The moment of the quantity of motion of the point m with respect to O will have the following projections onto the axes:

$$m\,(y\,V_{az} - z\,V_{ay}),\ \ldots,$$

i.e.:

$$m\,[(y^2 + z^2)\,p - xy\,q - xz\,r],\ \ldots$$

Upon letting σ_x , σ_y , σ_z denote the projections of $O\sigma$ onto the axes Ox, Oy, Oz, one will then have:

$$\sigma_x = \sum m\,[(y^2 + z^2)\,p - xy\,q - xz\,r],\ \ldots$$

Set:

$$A = \sum m\,(y^2 + z^2), \qquad B = \sum m\,(z^2 + x^2), \qquad C = \sum m\,(x^2 + y^2),$$

(7)

$$D = \sum m\,yz, \qquad E = \sum m\,xz, \qquad F = \sum m\,xy.$$

One will get:

(8) $$\sigma_x = A\,p - F\,q - E\,r\,,$$
$$\sigma_y = B\,q - D\,r - F\,p\,,$$
$$\sigma_z = C\,r - E\,p - D\,q\,.$$

In those formulas, A, B, C are the moments of inertia of the body with respect to the axes $Oxyz$, and D, E, F are the products of inertia with respect to those axes. Since the trihedron $Oxyz$ is supposed to be animated with an arbitrary motion in space and in the body, those six quantities will vary with time.

Vis viva of the body. – The semi-*vis viva* $T = \frac{1}{2}\sum mV_a^2$ is given by the formula:

$$2T = \sum m\,(V_{ax}^2 + V_{ay}^2 + V_{az}^2)\,,$$

so, upon developing:

(9) $$2T = A\,p^2 + B\,q^2 + C\,r^2 - 2D\,qr - 2E\,rp - 2F\,pq,$$

one can verify that one has:

(10) $$\sigma_x = \frac{\partial T}{\partial p}, \qquad \sigma_y = \frac{\partial T}{\partial q}, \qquad \sigma_z = \frac{\partial T}{\partial r}.$$

Resultant moment of the forces. – Let *OS* be the resultant moment of the forces that are applied to the body with respect to the point *O*, and let S_x, S_y, S_z be its projections onto the axes. Those quantities are the sums of the moments of the forces with respect to the axes *Ox*, *Oy*, *Oz*, respectively.

Equations of motion. – From a geometric interpretation of the theorem of moments that was given by Resal (see my *Traité de mécanique*, t. II, Chapter XVIII), the absolute velocity of the point σ is equal and parallel to *S* at each instant *t*. We shall write that the projections of the absolute velocity of σ onto the axes *Oxyz* are equal to those of *S*. Now, the point σ has the coordinates σ_x, σ_y, σ_z, so the projections of its absolute velocity will be given by formulas (2), in which one replaces *x*, *y*, *z* with σ_x, σ_y, σ_z and u', v', w' with zero. One will then have the equations of motion:

(11) $$\begin{aligned} \frac{d\sigma_x}{dt} + q'\sigma_z - r'\sigma_y &= S_x, \\ \frac{d\sigma_y}{dt} + r'\sigma_x - p'\sigma_z &= S_y, \\ \frac{d\sigma_z}{dt} + p'\sigma_y - q'\sigma_x &= S_z. \end{aligned}$$

In those equations, σ_x, σ_y, σ_z have the values (8), and one must remark that in the calculation of $d\sigma_x / dt$, …, one must take into account the fact that the coefficients *A*, *B*, … vary with *t*, in general.

6. Special cases:

1. *The reference trihedron Oxyz is attached to the body.* – If the trihedron is invariably linked with the body then the instantaneous rotation of the trihedron ω' is identical to that of the body. One will then have:

$$p' = p, \qquad q' = q, \qquad r' = r.$$

In addition, *A*, *B*, *C*, *D*, *E*, *F* are *constants*. If one supposes that the reference trihedron is composed of the principal axes of inertia of the body relative to *O* then one will see that *D*, *E*, *F* are zero, and one will recover Euler's equations.

2. *The axis Oz is fixed in the body. The axes Oy and Oz move in the body.* – A point *m* that is taken on *Oz* must have the same absolute velocity whether one regards it as moving with the trihedron *Oxyz* or as moving with the body. One must then have:

$$q'z - r'y = q\,z - r\,y, \quad \text{etc.}$$

when one supposes that x and y are zero. One will then have:

$$p' = p, \qquad q' = q,$$

but r' is different from r.

For example, imagine that the ellipsoid of inertia that relates to O is one of revolution. Take the axis Oz to be the axis of revolution and the axes Ox, Oy to be two rectangular axes in the plane of the equator that move inside the body. One will then have:

$$A = B, \qquad D = E = F = 0.$$

A, B, C are constants, moreover. In this case, one will have:

$$\sigma_x = A\,p, \qquad \sigma_y = A\,q, \qquad \sigma_z = C\,r,$$

$$p' = p, \qquad q' = q, \qquad r' \neq r.$$

From (11), the equations of motion are then:

$$(12) \qquad \begin{aligned} A\frac{dp}{dt} + (C\,r - A\,r')\,q &= S_x\,, \\ A\frac{dq}{dt} + (C\,r - A\,r')\,p &= S_y\,, \\ A\frac{dr}{dt} &= S_z\,. \end{aligned}$$

7. Motion of a free solid body. – Let a free solid body be subjected to forces F_1, F_2, …, F_n . Refer the motion of the body to a reference trihedron $Oxyz$ that is animated with a known motion. As in no. **2**, we then let V' denote the velocity of O and let ω' denote the instantaneous rotation of the trihedron.

Let G be the center of gravity of the body, let ξ, η, ζ be its coordinates with respect to $Oxyz$, and let V be its absolute velocity with projections u, v, w onto $Oxyz$. One will have:

$$u = \frac{d\xi}{dt} + u' + q'\zeta - r'\eta,$$

$$v = \frac{d\eta}{dt} + v' + r'\xi - p'\zeta,$$

$$w = \frac{d\zeta}{dt} + w' + p'\eta - q'\xi.$$

Similarly, let J be the absolute acceleration of the center of gravity G, and let J_x, J_y, J_z be its projections onto $Oxyz$. From (5), one has:

$$J_x = \frac{du}{dt} + q'w - r'v, \ \dots$$

Motion of the center of gravity. – The applied forces have a general resultant that has X, Y, Z for its projections onto the axes $Oxyz$. From the theorem on the motion of the center of gravity, that resultant will be equal to MJ, where M denotes the total mass of the body. One will then have the equations $M J_x = X$, …, or:

$$(13) \qquad \begin{aligned} M\left(\frac{du}{dt} + q'w - r'v\right) &= X, \\ M\left(\frac{dv}{dt} + r'u - p'w\right) &= Y, \\ M\left(\frac{dw}{dt} + p'v - q'u\right) &= Z. \end{aligned}$$

In the particular case where O coincides with G, one will have:

$$\xi = \eta = \zeta = 0, \qquad u = u', \qquad v = v', \qquad w = w'.$$

Motion around the center of gravity. – Draw axes $G\,x_1\,y_1\,z_1$ through G that are parallel to the axes $Oxyz$. The instantaneous rotation of the new trihedron is obviously the same as that of the first one, and its projections will again by p', q', r'. The theorem of moments applies to the motion around the center of gravity G as if that point were fixed. We can then apply the equations of the preceding number to it. Let p, q, r be the components of the instantaneous rotation ω of the body. Let $G\sigma'$ be the resultant moment with respect to G of the quantities of motion of the body in its motion around G. Let σ'_x, σ'_y, σ'_z denote the projections of the vector $G\sigma'$ onto the axes $G\,x_1\,y_1\,z_1$ or axes parallel to $Oxyz$. Similarly, let GS' be the resultant moment of the external forces with respect to G, and let S'_x, S'_y, S'_z be the projections of the vector GS' onto the axes. Finally, let A_1, B_1, C_1, D_1, E_1, F_1 be the moments and products of inertia of the body with respect to the axes $G\,x_1\,y_1\,z_1$:

$$A_1 = \sum m\,(y_1^2 + z_1^2), \ \dots$$

so we will have:

$$(14) \qquad \begin{aligned} \sigma'_x &= A_1\,p - F_1\,q - E_1\,r, \\ \sigma'_y &= B_1\,q - D_1\,r - F_1\,p, \\ \sigma'_z &= C_1\,r - E_1\,p - D_1\,q, \end{aligned}$$

$$\frac{d\sigma'_x}{dt} + q'\sigma'_z - r'\sigma'_y = S'_x ,$$

(15)
$$\frac{d\sigma'_y}{dt} + r'\sigma'_x - p'\sigma'_z = S'_y ,$$

$$\frac{d\sigma'_z}{dt} + p'\sigma'_y - q'\sigma'_x = S'_z .$$

Equations (13) and (15) will be the six equations of motion of the body.

Relative vis viva. – The *vis viva* $2T_1$ of the relative motion around G is given by:

$$2T_1 = A_1\, p^2 + B_1\, q^2 + C_1\, r^2 - 2D_1\, qr - 2E_1\, rp - 2F_1\, pq .$$

In those formulas, the quantities A_1 , B_1 , … generally vary with t. They remain constant when the trihedron $G\, x_1\, y_1\, z_1$ is invariably linked with the body. It can remain constant in other cases; for example, if the ellipsoid of inertia relative to G is a sphere. D_1, E_1, F_1 are then zero, while A_1, B_1, C_1 are equal to the same constant.

In particular, one can apply the remarks of no. **6** to equations (15).

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CHAPTER II

ROLLING

8. Rolling and pivoting of a surface that moves on a fixed surface. – Imagine a moving solid body that is bounded by a rigid surface S that is constrained to remain in contact with a fixed surface S_1 . At each instant t, a certain point A of the moving surface S is found to be in contact with a point A_1 of the fixed surface S_1 . If the velocity V_0 of the contact point A of the moving surface S is not zero at the instant t then that velocity will be located in the common tangent plane to the two surfaces at the point of contact: Indeed, let B be the point of contact at the instant $t + dt$, and let A' be the new position of A. The vectors BA_1 and BA' are in the common tangent place to the two surfaces at B, so the same thing will be true of the vector AA', which is the absolute displacement of A.

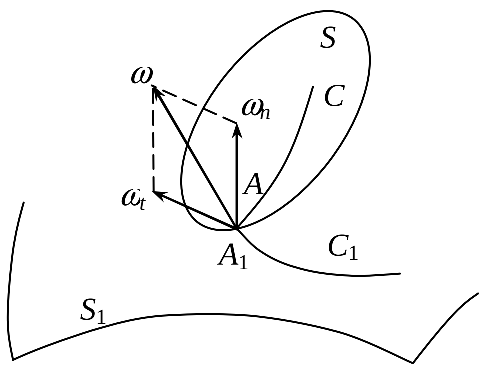

Figure 1.

The velocities of the various points of the moving solid body are the same as if the body were animated with a velocity of translation V_0 and a rotation $A\omega$ around an axis that passes through A. The velocity V_0 is the velocity of S slipping on S_1 . One says that the surface S *rolls* and *pivots* on S_1 when the *velocity of the contact point A is zero* at each instant. In that case, V_0 will be zero, so the velocities of the points of the moving solid are the same at each instant as if the body were animated with only a *rotation* $A\omega$ around an axis that passes through A. The slipping of S on S_1 will then be zero.

The locus of $A\omega$ in the body S is a ruled surface Σ, and a ruled surface Σ_1 in absolute space. The motion is obtained by rolling Σ on Σ_1 . The locus of the point A on S is a curve C that is the intersection of Σ with S. The locus of the point A_1 on S_1 is a curve C of intersection of Σ_1 with S_1. Those two curves also roll on each other. The corresponding arcs of those curves are equal.

The instantaneous rotation $A\omega$ can be decomposed into two more: one of them ω_n is normal to the two surfaces, which one calls the *angular velocity of pivoting*, while the other $A\omega_t$ is situated in the tangent plane, which is the *angular velocity of rolling, properly speaking.*

In what follows, we shall address rolling and pivoting motions without slipping.

It should be remarked that if one considers a molecule m in the body that is *located along the normal* to the two surfaces at A then the velocity of the molecule will be parallel to the common tangent plane at A. Indeed, that velocity is the geometric sum of the velocities that are due to two rotations ω_n and ω_t. Since the molecule m is located on the normal the velocity due to ω_n will be zero. All that will remain is the velocity due to ω_t, which is a velocity that is parallel to the tangent plane at A.

9. Physical conditions that determine the rolling and pivoting of a surface that moves on a fixed surface. – Imagine a moving solid body S that is subject to remain in contact with a fixed solid body S_1. If the surfaces S and S_1 are *perfectly polished* then the reaction of S_1 on S will be normal to the surfaces S and S_1 at the contact point A. In that case, no force will oppose the *slipping* of S on S_1, and that slipping will be produced, in general.

In order for the body S to not slip on S_1 – i.e., in order for it to roll and pivot on S_1 – it is necessary that the surfaces of the two bodies should be rough enough for there to be friction between them.

Let f denote the coefficient of friction of S on S_1. The reaction of S on S_1 is composed of a normal component N that is applied to A and a component F that is situated in the common tangent plane to the two surfaces at A. In order for *there to be no slipping*, it is necessary and sufficient that one must have:

$$F < f N.$$

That is then the condition for the surface S to roll and pivot without slipping on S_1. If F becomes greater than $f N$ at a certain moment then there will be slipping.

In summary, in order to study the rolling and pivoting of an unpolished body S on another body S_1 that is likewise unpolished under the action of given forces, one writes down the equations of motion of the solid S by assuming that it rolls and pivots on S_1 and introducing the normal reaction N and the tangential reaction F of the contacting surfaces as auxiliary unknowns. The motion that is provided by those equations will take place effectively as long as the values of F and N that are inferred from the equations verify the inequality:

$$F < f N.$$

On the contrary, if F becomes equal to $f N$ at an instant t_1 and then greater than it then the body S will slide on S_1 starting at that instant. The motion will enter into a new phase in which the preceding equations no longer apply. In order to get the equations of motion for that new phase, one must assume that the body S slides over S_1 and introduce auxiliary unknowns in the form of a normal component N to the reaction and a tangential component that is equal to $f N$ and directed in the opposite sense to the velocity of the material point A of the body S that is in contact with S_1. That will result from the known laws of sliding friction.

We shall confine ourselves to rolling and pivoting motions here. We shall neglect the friction of rolling and pivoting.

10. *Vis viva* of a solid body that is animated with a rolling and pivoting motion. – Take the reference trihedron to be a trihedron *Oxyz* whose origin coincides at each instant with the geometric contact point of the moving body *S* and the fixed body S_1. The origin *O* then displaces in both the body and in space. As for the directions of the axes *Oxyz*, they can vary according to an arbitrary law.

As above, let $O\omega$ denote the instantaneous rotation of the body, and let *p*, *q*, *r* be its components along the axes *Oxyz*. Let *A*, *B*, *C*, *D*, *E*, *F* be the moments and products of inertia of the body with respect to those axes. Since the velocities of the various points of the body are the same as if the body were animated with only the rotation ω, the semi-*vis viva T* of the body will be the same as that of a solid body that moves around a fixed point *O* and is animated with an instantaneous rotation ω. One will then have:

$$2T = A\,p^2 + B\,q^2 + C\,r^2 - 2D\,qr - 2E\,rp - 2\,F\,pq\,.$$

Vis viva theorem. – In the motion of the body, the work done by tangential and normal reactions *F* and *N* will be *zero*, because those forces are applied at each instant to a material point whose velocity is zero. Upon applying the *vis viva* theorem, one will then have:

$$dT = \sum \mathcal{T}_e\,,$$

in which $\sum \mathcal{T}_e$ denotes the sum of the elementary works done by applied forces.

11. Equations of motion of a body. – One writes the equations of motion by applying formulas (13) and (15), with the simplification that if one takes the reference trihedron to be the trihedron of the preceding number then the velocity of the material point that is placed at *O* will be zero. However, the velocity of the origin will be non-zero.

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CHAPTER III

APPLICATIONS

12. – The following applications are borrowed from the book by Routh: *Advanced part of a Treatise on the Dynamics of Rigid Bodies* (London, MacMillan and Co., 1884).

13. Rolling of a sphere on a surface (Routh, pp. 123). – Let a homogeneous sphere of radius a and mass 1 be constrained to roll and pivot on a given surface, and let it be acted upon by forces that admit a unique resultant that passes through the center.

Let G be the center of the sphere. Take the axis Gz to be the line that joins the contact point of the sphere and the surface to the point G and the axes Gx and Gy to be two arbitrary perpendicular axes. The plane xGy will then be parallel to the tangent plane to the surface at the contact point.

Let V denote the absolute velocity of the point G and let u, v, w be its projections onto the moving axes. Since the velocity V is parallel to the common tangent plane to the sphere and the surface on which it rolls, one will have $w = 0$. As above, let ω' be the instantaneous rotation of the trihedron $Gxyz$, and let p', q', r' be its components, while ω is that of the sphere, and its components are p, q, r.

Let X, Y, Z be the components of the resultant of the applied forces along Gx, Gy, Gz, respectively. The reaction of the surface is composed of a normal force R that is directed in the sense of Gz and a tangential force whose components along Gx and Gy we shall call F and F', resp. Furthermore, let k denote the radius of gyration of the sphere around a diameter $k = \frac{1}{5} a\sqrt{10}$. The moments of inertia with respect to the axes Gx, Gy, Gz are:

$$A = B = C = k^2 ;$$

in addition, D, E, F are zero. We apply the general equations to this case.

Motion of the center of gravity. – Since G coincides with the origin of the axes, one will have $u' = u$, $v' = v$, $w' = w$. When one sets $M = 1$, $w = 0$, equations (13) will then give:

$$\frac{du}{dt} - r'v = X + F,$$

$$(16) \qquad \frac{dv}{dt} + r'u = Y + F,$$

$$p'v - q'u = Z + R .$$

Motion around G. – For the motion around G, the resultant moment $G\sigma'$ of the quantities of relative motion will have projections:

$$\sigma'_x = k^2\, p, \qquad \sigma'_y = k^2\, q, \qquad \sigma'_z = k^2\, r\,.$$

When one divides formulas (15) by k^2 and notes that the force X, Y, Z is applied to the point G, they will give terms S'_x, S'_y, S'_z that are provided by moments of the reaction R, F, F' that is applied to the point $x = 0$, $y = 0$, $z = -a$:

(17)
$$\frac{dp}{dt} + q'r - r'q = \frac{a\,F'}{k^2},$$
$$\frac{dq}{dt} + r'p - p'r = -\frac{a\,F}{k^2},$$
$$\frac{dr}{dt} + p'q - q'p = 0\,.$$

Conditions for rolling. – The contact point $x = 0$, $y = 0$, $z = -a$ has an absolute velocity of *zero*:

(18)
$$u - a\,q = 0, \qquad v + a\,p = 0.$$

Consequences of those equations. – If we infer p and q from the equations (18) and substitute them in (17) , and then eliminate F and F' from equations (16) and (17) then we will have:

(19)
$$\frac{du}{dt} - r'v = \frac{a^2}{a^2+k^2}\,X + \frac{k^2}{a^2+k^2}\,a\,p'r\,,$$
$$\frac{dv}{dt} + r'u = \frac{a^2}{a^2+k^2}\,Y + \frac{k^2}{a^2+k^2}\,a\,q'r\,.$$

Those equations show that the center of gravity moves like the center of gravity of an identical sphere that is subject to sliding without friction on the same surface and acted upon by:

1. An applied force at G that has components $\frac{k^2}{a^2+k^2}\,a\,p'r$ and $\frac{k^2}{a^2+k^2}\,a\,q'r$ along Gx and Gy, resp.

2. A force that is equal to the real applied force (X, Y, Z) that has been reduced by the ratio $\frac{a^2}{a^2+k^2}$.

Geometric relations. – The center G of the sphere describes a surface S_1 that is parallel to S and is obtained by extending the normals to S by a length a. Suppose that the axes Gx, Gy are taken to be tangent to the lines of curvature of that surface S_1 . In addition, let ρ_1 and ρ_2 be the radii of principal curvature of S_1 that correspond to the principal directions Gx and Gy, resp. We shall calculate p', q', r'. In order to take the reference trihedron from its present position to an infinitely-close one, one can first make it turn around a parallel to Gy that is drawn through the center of curvature C_1 of the normal section that is tangent to Gx, then around a parallel to Gx that is drawn through the center of curvature Gz of the normal section that is tangent to Gy, and finally around Gz. Under the first rotation, the arc that is described by G will be $u\,dt$, and it is also $\rho_1 q'\,dt$; similarly, $v\,dt = -\rho_2 p'\,dt$. Therefore:

(20) $$u = \rho_1 p', \qquad v = -\rho_2 p'.$$

Finally, if G and G' are the positions of G at the instants t and $t + dt$ then $r\,dt$ will be the angle of the two successive positions of Gx. Let χ_1 and χ_2 denote the angles that principal normals to the two lines of curvature make with the normal to the surface. From Meusnier's theorem, the curvatures of the lines of curvature will be $1 / (\rho_1 \cos \chi_1)$ and $1 / (\rho_1 \cos \chi_1)$, and their geodesic curvatures will be:

$$\frac{1}{\rho_1}\tan \chi_1 \qquad \text{and} \qquad \frac{1}{\rho_2}\tan \chi_2 ,$$

resp.

In order to take G to G', first take G to H along one of the lines of curvature and then take H to G along the other. Under the first displacement, Gx will turn through the angle $(u / \rho_1)\,dt \tan \chi_1$, and under the second, it will turn through $(u / \rho_2)\,dt \tan \chi_2$. One will then have:

(20 cont.) $$r' = \frac{u}{\rho_1}\tan \chi_1 + \frac{v}{\rho_2}\tan \chi_2 .$$

If one deduces p and q from equations (18) and p' and q' from equations (20) and substitutes them in the third of the relations (17) then one will have:

(21) $$a\frac{dr}{dt} = uv\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right).$$

Those are the equations of the problem.

One can write down an integral by applying the *vis viva* theorem.

The moments of inertia relative to three axes parallel to $Gxyz$ that are drawn through the contact point of the sphere with the surface will be $(a^2 + k^2)$, $(a^2 + k^2)$, and k^2. Since the velocities are the same as if the sphere turned around that point, the *vis viva* will be:

$$(a^2 + k^2)\,(p^2 + q^2) + k^2\,r^2.$$

If one then lets Φ denote the force function for (X, Y, Z) then one will have:

$$(a^2 + k^2)(p^2 + q^2) + k^2 r^2 = 2\Phi + h\,.$$

h is an arbitrary constant.

Remark. – The geometric relations (20) and (20, cont.) are very special cases of the formulas of Codazzi and Bonnet, which one will find in the *Leçons sur la théorie générale des surfaces* by Darboux, Part Two, Book V, Chaps. II and III. We refer to that book for the rigorous proofs of the formulas above.

14. Examples. – If the fixed surface on which the sphere rolls is a plane then ρ_1 and ρ_2 will be infinite, so p' and q' will be zero. Therefore: *If a homogeneous sphere rolls and pivots on a fixed plane under the action of forces that admit a unique resultant that passes through its center then the motion of the center will be the same as if the plane were perfectly polished and the applied forces reduced to* 5 / 7 *of their values.* (Routh, *loc. cit.*, pp. 126)

For other examples, we refer to Routh's treatise, which contains a large number of elegant exercises, notably the rolling of a sphere on a sphere, a cylinder, and a cone, and small oscillations about a stable equilibrium position or a stable motion.

15. Equations of motion of a heavy solid constraint to roll and pivot on a horizontal plane (Routh, *loc. cit.*, pp. 143). – Take the principal axes of inertia relative to the center of gravity $Gxyz$ to be the reference trihedron. Let $\varphi(x, y, z) = 0$ be the equations of the surface that bounds the body with respect to those axes. Call the coordinates of the contact point P of the surface with the horizontal plane x, y, z, and let α, β, γ be the direction cosines of the normal to the surface at P :

$$(N)\qquad \frac{\alpha}{\dfrac{\partial\varphi}{\partial x}} = \frac{\beta}{\dfrac{\partial\varphi}{\partial y}} = \frac{\gamma}{\dfrac{\partial\varphi}{\partial z}} = \pm\frac{1}{\sqrt{\left(\dfrac{\partial\varphi}{\partial x}\right)^2 + \left(\dfrac{\partial\varphi}{\partial y}\right)^2 + \left(\dfrac{\partial\varphi}{\partial z}\right)^2}}\,,$$

in which the sign is chosen in such a fashion that the sense of α, β, γ is the normal that is directed along the descending vertical. Suppose that the mass of the body is equal to 1, and let X, Y, Z denote the components along Gx, Gy, Gz, resp., of the total reaction of the plane (normal and tangential reaction) that is applied to the point P. Finally, note that the projections of the weight along the axes $Gxyz$ are $g\alpha$, $g\beta$, $g\gamma$.

Motion of the center of gravity. – Let u, v, w denote the velocity of the center of gravity on the moving axes $Gxyz$, and let p, q, r be the components of the instantaneous rotation of the body along those axes. Upon noting that the quantities that are called u', v', w', p', q', r' in the general case are presently equal to u, v, w, p, q, r, we will have:

$$\frac{du}{dt} + q\,w - r\,v = g\,\alpha + X,$$

(22)
$$\frac{dv}{dt} + q\,u - p\,w = g\,\beta + Y,$$

$$\frac{dw}{dt} + p\,v - q\,u = g\,\gamma + Z\,.$$

Motion around the center of gravity. – Here, the equations are Euler's equations:

$$A\frac{dp}{dt} + (C - B)\,q\,r = y\,Z - z\,Y,$$

(23)
$$A\frac{dq}{dt} + (A - C)\,r\,p = z\,X - x\,Z\,,$$

$$A\frac{dr}{dt} + (B - A)\,p\,q = x\,Y - y\,X\,.$$

Geometric conditions. – First of all, if the line (α, β, γ) remains vertical (no. **3**) then one will have:

$$\frac{d\alpha}{dt} + q\,\gamma - r\,\beta = 0,$$

(24)
$$\frac{d\beta}{dt} + r\,\alpha - p\,\alpha = 0,$$

$$\frac{d\gamma}{dt} + p\,\beta - q\,\alpha = 0\,.$$

Now, in order to express the rolling, one must write that the absolute velocity of the contact point (x, y, z) is *zero* :

(25)
$$\begin{aligned} u + q\,z - r\,y &= 0, \\ v + r\,x - p\,z &= 0, \\ w + p\,y - q\,x &= 0, \end{aligned}$$

One will then have twelve equations in twelve unknowns u, v, w, p, q, r, x, y, z, X, Y, Z. The quantities α, β, γ are known as functions of x, y, z by equations (N).

The integral of the *vis viva* is presently:

$$u^2 + v^2 + w^2 + A\,p^2 + B\,q^2 + C\,r^2 = 2g\,(\alpha\,x + \beta\,y + \gamma z) + h\,,$$

because the height of the center of gravity is the projection of GP onto the vertical.

That is easy to verify by appealing to the relation:

$$\alpha\, dx + \beta\, dy + \gamma\, dz = 0,$$

which results from equations (*N*).

16. Rolling and pivoting of a heavy body of revolution on a horizontal plane. – This problem is likewise treated in Routh. We shall give a solution that is deduced from the preceding general equations and compare it to that of Routh.

Imagine a heavy solid body that is subject to the following conditions:

1. The ellipsoid of inertia relative to the center of gravity *G* is one of revolution around and axis *Gz*.

2. The body touches a fixed horizontal plane for a surface of revolution around the same axis.

Those conditions are fulfilled for a heavy homogeneous solid of revolution, in particular.

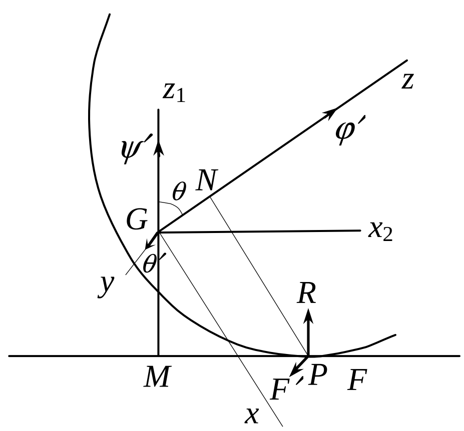

Figure 2.

Represent the meridian of the surface of revolution along which the body touches the fixed plane in (Fig. 2). The tangent plane at a point *P* of the meridian is perpendicular to the meridian plane *zGP*, and the trace of the meridian on the tangent plane is *PM*. Let ζ be the distance *GM* from the center of gravity to the tangent plane, and let θ be the angle between that perpendicular *GM* and *Gz* : ζ is a function of θ:

$$\zeta = f(\theta),$$

which will be defined once the meridian is given. Conversely, one can give the function $f(\theta)$ *a priori* : The corresponding surface will have a curve for its meridian that is the envelope of lines *PM* that verify that condition. In addition, it is obvious that once the meridian is determined, the distance *PM* is also a known function of θ. In order to determine that function, we remark that the tangent *PM* has the equation:

$$x \sin \theta - z \cos \theta = f(\theta)$$

with respect to the axes *Gx* an *Gz* that are situated in the meridian plane.

Since the meridian is the envelope of that line when θ varies, one will get the coordinates of the contact point *P*, while associating the preceding equation with its derivative with respect to θ:

$$x \cos \theta + z \sin \theta = f'(\theta).$$

The latter equation represents a line that passes through *P*, which is the normal *PR*; its distance from the point *G* is equal to *MP*. One will then have:

$$MP = \pm f'(\theta).$$

In addition, upon solving the two equations above for *x* and *z*, one will have the coordinates of *P*:

$$(P) \qquad \begin{aligned} PN &= x = f'(\theta) \cos \theta + f(\theta) \sin \theta, \\ GN &= z = f'(\theta) \sin \theta - f(\theta) \cos \theta. \end{aligned}$$

Having said that, place the solid on a fixed horizontal plane on which it can roll and pivot. Let *P* be the contact point, and let *GM* be the distance from the center of gravity to the plane. The vertical $M\,Gz_1$ makes an angle of θ with *Gz*, and from the preceding, one will have:

$$GM = \zeta = f(\theta).$$

Take the reference trihedron to be the trihedron that is composed of the axis of revolution *Gz*, the axis *Gx*, which is the perpendicular to *Gz* in the meridian plane *PGz* to the contact point, and finally the axis *Gy*, which is perpendicular to the preceding ones. The plane *zGx* is vertical, while the axis *Gy* is horizontal. Under those conditions, the instantaneous rotation ω' of the moving trihedron *Gxyz* is the resultant of two rotations, one of which $d\theta / dt = \theta'$ is around *Gy*, while the other one $d\psi / dt = \psi'$ is around Gz_1. The components p', q', r' of that rotation along *Gx*, *Gy*, *Gz* are then:

$$(\omega') \qquad \begin{aligned} p' &= -\psi' \sin \theta, \\ q' &= \theta', \\ r' &= \psi' \cos \theta. \end{aligned}$$

In order to fix the orientation of the solid around the point *G*, one must know the position of the solid with respect to the axes *Gxyz*. In order to do that, it is sufficient to know the angle φ that a line that is fixed in the body in the plane *xGy* makes with the line *Gy*. The derivative $d\varphi / dt = \varphi'$ of that angle measures the proper rotation of the body around *Gz*.

The instantaneous rotation ω of the body is the resultant of the rotation ω' of the trihedron *Gxyz* and the proper rotation φ' around *Gz*. One will then have sums of the projections of ω' and φ' for the projections *p*, *q*, *r* of ω:

$$(\omega)\qquad \begin{aligned} p &= p' &&= -\psi'\sin\theta, \\ q &= q' &&= \theta', \\ r &= r' + \varphi' &&= \psi'\cos\theta + \varphi'. \end{aligned}$$

Equations of motions of the center of gravity. – Let u, v, w be the projection onto $Gxyz$ of the velocity of the point G, and let A, B, C be the moments of inertia with respect to Gx, Gy, Gz ($A = B$). Upon taking the mass of the body to be unity, the applied forces will be the weight g, whose projections along the axes $Gxyz$ are:

$$+ g\sin\theta, \qquad 0, \qquad - g\cos\theta,$$

and the total reaction of the plane that is applied to the point P (viz., the normal reaction R and the tangential reaction), which will have the projections:

$$X, \quad Y, \quad Z.$$

One will then have equations (13):

$$(26)\qquad \begin{aligned} \frac{du}{dt} + q\,w - r'v &= g\sin\theta + X, \\ \frac{dv}{dt} + r'u - p\,w &= Y, \\ \frac{dw}{dt} + p\,v - q\,u &= - g\cos\theta + Z. \end{aligned}$$

In the general equations of no. **7**, we replaced q with q' and p with p', by using the preceding equations that gave ω.

Motion around G. – Here, we can apply equations (12) of no. **6**, since the body is one of revolution around Gz. Upon noting that the coordinates of P are x, 0, z, we will then have:

$$(27)\qquad \begin{aligned} A\frac{dp}{dt} + (C\,r - A\,r')\,q &= - z\,X, \\ B\frac{dq}{dt} - (C\,r - A\,r')\,p &= z\,X - x\,Z, \\ C\frac{dr}{dt} &= x\,Y. \end{aligned}$$

Geometric condition. – The absolute velocity of the molecule at the contact point P is *zero:*

$$(28)\qquad \begin{aligned} u + q\,z &= 0, \\ v + r\,x - q\,z &= 0, \end{aligned}$$

$$w - q\,x = 0.$$

When one eliminates the auxiliary unknowns X, Y, Z, u, v, w from equations (26), (27), and (28), one will have three second-order equations that define θ, φ, ψ.

Another form of the equations. – In order to compare those equations with Routh's, take three moving axes in the following fashion: One axis Gz_1 is vertically ascending, one axis Gx_1 is horizontal in the plane zGz_1, and finally a perpendicular axis Gy coincides with the axis that was employed before. That system of axes is animated with an instantaneous rotation ω_1' that takes place effectively around Gz_1 with the angular velocity:

$$\frac{d\psi}{dt} = \psi'.$$

One will then have:

$$p_1' = 0, \qquad q_1' = 0, \qquad r_1' = \psi'$$

for the components of that rotation along $G\,x_1\,y_1\,z_1$.

As for the instantaneous rotation ω of the body, it will have components along those axes that equal:

$$(\omega) \qquad \begin{aligned} p_1 &= p\cos\theta + r\sin\theta &&= \varphi'\sin\theta, \\ q_1 &= q &&= \theta', \\ r_1 &= -p\sin\theta + r\cos\theta &&= \varphi'\cos\theta + \psi'. \end{aligned}$$

If one calls the projections of the velocity of the point G onto those new axes u_1, v_1, w_1 then one will have:

$$\begin{aligned} u_1 &= u\cos\theta + w\sin\theta, \\ v_1 &= v, \\ w_1 &= -u\sin\theta + w\cos\theta. \end{aligned}$$

Finally, let F, F', R denote the components of the reaction of the plane along those axes, while R is the normal component of that reaction, and the resultant of F, F' is the tangential component, so:

$$\begin{aligned} F &= X\cos\theta + Z\sin\theta, \\ F' &= Y, \\ R &= -X\sin\theta + Z\cos\theta. \end{aligned}$$

The equations of motion of the center of gravity are then:

$$\frac{du_1}{dt} + q_1'\,w_1 - r_1'\,v_1 = F,$$

$$\frac{dv_1}{dt} + r_1'\,u_1 - p_1'\,w_1 = F',$$

$$\frac{dw_1}{dt} + p_1' v_1 - q_1' u_1 = R - g\ ;$$

i.e., from the values of p_1', q_1', r_1' :

$$\frac{du_1}{dt} - v_1 \psi' = F,$$

(29)
$$\frac{dv_1}{dt} + u_1 \psi' = F',$$

$$\frac{d^2\zeta}{dt^2} = R - g,$$

in which the last equation results from the fact that if ζ is the height of the point G then one will have $w_1 = d\zeta / dt$.

As for the equations of motion around G, Routh wrote them in the form (27) by putting the right-hand sides into the following form: If the coordinates of the contact point P with respect to the axes Gx, Gy are called x and z then its coordinates with respect to the axes Gx_1 , Gz_1 will be:

$$x_1 = \quad x \cos \theta + z \sin \theta = \quad f'(\theta),$$
$$z_1 = -x \sin \theta + z \cos \theta = -f'(\theta) = -\zeta .$$

The right-hand sides of equations (27) are then:

$$-z\,T = -F' \cdot GN \quad = -F\,|\, f'(\theta) \sin \theta - f'(\theta) \cos \theta \,|,$$

$$zX - xZ = z_1 F - x_1 R \;= -F' \cdot GN - R \cdot MP,$$

$$= -F f(\theta) - R f'(\theta),$$

$$x\,Y = F' \cdot PN \qquad = F' |\, f'(\theta) \sin \theta + f'(\theta) \cos \theta \,| .$$

Finally, the geometric conditions (28) can be replaced with the following ones: The point P (x_1, 0, z_1) has a velocity of zero, so upon projecting onto the axes $Gx_1 y_1 z_1$, one will have:

$$u_1 + q_1 z_1 - r_1 y_1 = 0,$$
$$v_1 + r_1 x_1 - p_1 z_1 = 0,$$
$$w_1 + p_1 y_1 - q_1 x_1 = 0\ ;$$

i.e., from the values of p_1 , q_1 , r_1 :

$$u_1 = -q\, z_1 = q\, GM = q\, f(\theta) = \theta' f(\theta),$$

(28 cont.) $$v_1 = - r_1\, MP - p_1\, GM = - r\, PN + q\, GN,$$
$$= - r_1\, f'(\theta) - p_1\, f(\theta),$$

$$w_1 = q\, x_1 = f'(\theta)\, \theta'.$$

The last equation is obvious *a priori*, because the height ζ of the center of gravity is:

$$\zeta = GM = f(\theta)\,,$$

so the projection w_1 of its velocity onto the vertical will be:

$$w_1 = \frac{d\zeta}{dt} = f'(\theta)\, \theta'.$$

Vis viva integral. – The differential equations of motion admit the following first integral that is provided by the *vis viva* theorem:

$$u^2 + v^2 + w^2 + A\,(p^2 + q^2) + C\, r^2 = - 2g\, \zeta + h,$$

which is an equation in which $u^2 + v^2 + w^2$ is identical to $u_1^2 + v_1^2 + w_1^2$.

17. Applications. – Routh studied the case in which the solid rolls in such a fashion that its axis makes a constant angle with the vertical, and then he studied the small oscillations around that motion (pp. 141). He studied the particular case in which the solid is a disc or a hoop (pp. 142) of radius a. In that case, one will have:

$$z = a \sin \theta, \qquad f(\theta) = a \sin \theta.$$

There is not enough space for us to treat those various questions.

18. Carvallo's research on the hoop. – The motion of the hoop was recently studied by Carvallo in a paper that was submitted in competition for the Fourneyron prize and presented at a public meeting of the Paris Academy of Sciences (December 1898).

If we keep the axes $Gxyz$ that were employed in the preceding general case then we will see that Gz is the perpendicular to the plane of the hoop, Gx is the line that joins the center to the contact point P, and Gy is the horizontal to the plane of the hoop. If one calls the radius of the hoop a then the coordinates of the point P with respect to the axes $Gxyz$ will be:

(P) $$x = a, \qquad y = 0, \qquad z = 0.$$

The function $f(\theta)$ is $a \sin \theta$.

If we suppose that the hoop reduces to a material circumference of radius a then we will have (the mass of the hoop being 1):

$$A = B = \tfrac{1}{2} a^2, \quad C = a^2.$$

The expressions (28) of u, v, w become:

$$\begin{aligned} u &= 0, \\ v + ar &= 0, \\ w - aq &= 0. \end{aligned}$$

The equations of motion (26) of the center of gravity then become:

$$\begin{aligned} a\,(q^2 + r\,r') &= g \sin\theta + X, \\ -a\frac{dr}{dt} - a\,p\,q &= Y, \\ a\frac{dr}{dt} - a\,p\,r &= -g\cos\theta + Z. \end{aligned}$$

Finally, the equations of motion (27) around G will become:

$$\begin{aligned} \tfrac{1}{2} a^2 \frac{dp}{dt} + a^2\left(r - \tfrac{1}{2} r'\right) q &= 0, \\ \tfrac{1}{2} a^2 \frac{dq}{dt} - a^2\left(r - \tfrac{1}{2} r'\right) p &= -a\,Z, \\ a^2 \frac{dr}{dt} &= a\,Y. \end{aligned}$$

If we eliminate X, Y, and Z from those equations then we will have the three equations of motion:

$$(30) \qquad \begin{aligned} \tfrac{1}{2} \frac{dp}{dt} + \left(r - \tfrac{1}{2} r'\right) q &= 0, \\ \tfrac{3}{2} \frac{dq}{dt} - \left(2r - \tfrac{1}{2} r'\right) p &= -\frac{g}{a}\cos\theta, \\ 2\frac{dr}{dt} + pq &= 0. \end{aligned}$$

Upon remarking that:

$$r' = -p\cot\theta,$$

one will get equations (30) in the form that Carvallo gave from the formulas that determine p, q, r and p', q', r'.

We refer to his paper for the conclusions that he inferred from those equations in regard to the following question:

Equilibrium conditions for a regime of the hoop.
Stability of a regime of equilibrium.
Tendency to slip.
Discussion of the equilibrium states.

We shall confine ourselves to making the following remark in regard to the integration of the system (30).

Upon recalling that $q = \theta'$, one can write those equations as:

$$\frac{dp}{dt} + (2r + p\cot\theta)\frac{d\theta}{dt} = 0,$$

$$(30) \qquad 3\frac{d^2\theta}{dt^2} - (4r + p\cot\theta)\,p = -\frac{2g}{a}\cos\theta,$$

$$2\frac{dr}{dt} + p\frac{d\theta}{dt} = 0\,,$$

and one will see that one will have to integrate a system of three equations that define p, r, and θ as functions of t. The first and the last of them are of order one in p and r and will permit one to find p and r as functions of θ.

One knows one integral of those equations from the *vis viva* theorem:

$$p^2 + 4r^2 + 3\,\theta'^2 = -\frac{4g}{a}\sin\theta + h\,.$$

Eliminating p from the first and last of equations (3) will give the linear equation:

$$\frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\cot\theta - r = 0$$

for the determination of r as a function of θ, which will reduce to Gauss's hypergeometric series by taking $\cos^2\theta$ to be the variable. That equation will give r as a function of θ, so one will then have $p = -2\,dr / d\theta$. Finally, the *vis viva* integral will give t as a function of θ by a quadrature.

That method of integration can be extended to the rolling of a body of revolution ([1]).

([1]) See an article that will appear in the Rend. Circ. Mat. Palermo after a letter by Korteweg (1900).

19. The bicycle problem. – One of the more important applications of rolling motion is the bicycle problem. There is not enough space for us to treat it here, so we refer to the volumes that were published by Bourlet in the Gauthier-VIllar collection (*Equilibre et direction, Travail*), the paper that Bourlet presented to the Academy in December 1898 (Fourneyron Prize), a paper that was published in Bull. Soc. Math. France in 1899, the cited paper by Carvallo, which was likewise submitted to the Academy, in which one finds a theory of the unicycle, an English book by Sharp entitled *Bicycles and Tricycles*, and finally, to several notes by Boussinesq that were published in Comptes rendus (2nd semester, 1898 and 1st semester, 1899) and in Jordan's Jour. d. Math. in 1898.

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CHAPTER IV

ANALYTICAL MECHANICS. LAGRANGE EQUATIONS.

20. Rolling is a constraint that cannot generally be expressed by equations in finite terms. – The position of an entirely-free solid body depends upon six parameters, which are, for example, the three coordinates of the center of gravity and the three Euler angles. In order to express the idea that the body rolls and pivots on a fixed surface, one must write that the velocity of the molecule at the contact point is zero. Now, upon calling the six parameter $q_1, q_2, q_3, q_4, q_5, q_6$, that condition will be expressed by relations of the form:

$$A_1\,dq_1 + A_2\,dq_2 + \ldots + A_6\,dq_6 = 0 \tag{33}$$

whose coefficients are functions of $q, q_1, q_2, \ldots, q_6$, but *whose left-hand side is not an exact differential, in general and does not admit an integrating factor.*

The constraint that is imposed on the body can therefore not be expressed by relations *in finite terms* between the parameters. This will result in some particular difficulties in the application of the theorems of analytical mechanics, the most salient of which is that the Lagrange equations cannot be applied when one takes those exceptional constraints into account in order to modify the expression for the *vis viva T*.

The difficulties that result from that viewpoint on that type of constraint have been pointed out and studied by C. Neumann ["Grundzüge der Analytischen Mechanik," Ber. Kön. Sächs. Ges. Wiss Leipzig (1888), pp. 32], by Vierkandt ["Ueber gleitende und rollende Bewegung," Monats. Math. Phys. **3** (1892)], by Hadamard ["Sur les mouvements de roulement," Soc. Sci. Bordeaux (1895)], and finally by Carvallo in his paper that was cited above in the context of problems with the hoop, the unicycle, and bicycle, and by Korteweg.

For example, take a homogeneous sphere of radius *a* that is constrained to roll on a fixed plane. Take the fixed axes to be two axes $O\xi$, $O\eta$ in the plane and a perpendicular axes $O\zeta$ on the side where one finds the sphere. Let ξ, η, ζ be the coordinates of the center *G* of the sphere with respect to those axes ($\zeta = a$). Draw three axes $Gx_1y_1z_1$ through *G* that are parallel to the axes $O\xi\eta\zeta$ and call the components of the instantaneous rotation of the sphere around those axes p_1, q_1, r_1. Upon writing down that the point of the sphere that makes contact has a velocity of zero, one will have:

$$\frac{d\xi}{dt} - a\,q_1 = 0, \qquad \frac{d\eta}{dt} + a\,p_1 = 0, \qquad \frac{d\zeta}{dt} = 0, \tag{34}$$

Moreover, if ψ, φ, θ are the Euler angles of a system of axes *Gxyz* that is fixed in the sphere with respect to the axes $G\,x_1\,y_1\,z_1$ then from some known formulas (see my *Traité de mécanique*, t. II, pp. 257), one will have:

$$p_1 = \theta'\cos\psi + \varphi'\sin\theta\sin\psi,$$

$$q_1 = \theta' \sin\psi - \varphi' \sin\theta \cos\psi,$$
$$r_1 = \psi' + \varphi' \cos\theta.$$

The relations (34), which express the idea that the real displacement is a rolling motion, are then written:

$$(35)\qquad \begin{aligned} &d\xi - a\sin\psi\, d\theta + a\sin\theta\cos\psi\, d\varphi = 0, \\ &d\eta + a\cos\psi\, d\theta + a\sin\theta\sin\psi\, d\varphi = 0. \end{aligned}$$

Similarly, a virtual displacement that is compatible with the constraint is characterized by:

$$(36)\qquad \begin{aligned} &\delta\xi - a\sin\psi\, \delta\theta + a\sin\theta\cos\psi\, \delta\varphi = 0, \\ &\delta\eta + a\cos\psi\, \delta\theta + a\sin\theta\sin\psi\, \delta\varphi = 0. \end{aligned}$$

If the coordinate ζ is constant then the position of the system will depend upon five parameters ξ, η, θ, φ, ψ that are coupled by the relations (35) *whose left-hand sides are not exact total differentials.*

21. Applying the general equation of dynamics. – The general equation of dynamics:

$$(37)\qquad \sum\left[\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right] = 0$$

results from d'Alembert's principle, combined with the theorem of virtual work. For any displacement that is compatible with the constraints, it expresses the idea that the sum of the works done by the applied forces (X, Y, Z) and the inertial forces is *zero*. That equation applies to every constraint that fulfills the following condition: *For any virtual displacement that is compatible with the constraints, the sum of the works done by the forces of constraint is zero.* That results from the classical proof of the virtual work theorem, which is reproduced in the first volume of my *Traité de mécanique*, for example. Now, the constraints that consist of requiring a body to roll and pivot on another body fulfill that condition. The general equation of the dynamics then applies to the particular type of problems that we treat here.

22. Use of the Lagrange equations. – Imagine a general system that is first subject to some constraints *that are expressible by relations in finite terms between the coordinates of the various points.* Upon taking those constraints into account, let k be the number of independent parameters $q_1, q_2, \ldots, q_k$ that fix the system. When one supposes that the constraints are independent of time, one will have:

$$(38)\qquad \begin{aligned} &x = f(q_1, q_2, \ldots, q_k), \\ &y = \varphi(q_1, q_2, \ldots, q_k), \end{aligned}$$

$$z = \psi(q_1, q_2, \ldots, q_k)$$

for the coordinates of an arbitrary point of the system.

One obtains a virtual displacement that is compatible with those constraints by varying $q_1, q_2, \ldots, q_n$ by $\delta q_1, \delta q_2, \ldots, \delta q_n$, resp. The equation of dynamics (37) will then take the form:

$$(P_1 - Q_1)\,\delta q_1 + (P_2 - Q_2)\,\delta q_2 + \ldots + (P_k - Q_k)\,\delta q_k = 0, \tag{39}$$

in which:

$$P_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}.$$

If there are no other constraints then the $\delta q_1, \delta q_2, \ldots, \delta q_k$ will be arbitrary, and equation (39) will provide k equations that are the Lagrange equations.

However, now suppose that one adds some new constraints that are independent of time to the previous constraints, and which are expressed by *non-integral differential relations between the parameters* $q_1, q_2, \ldots, q_k$. For a virtual displacement that is compatible with those constraints, one will have:

$$(40)\qquad \begin{aligned} &A_1\,\delta q_1 + A_2\,\delta q_2 + \ldots + A_k\,\delta q_k = 0, \\ &B_1\,\delta q_1 + B_2\,\delta q_2 + \ldots + B_k\,\delta q_k = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &L_1\,\delta q_1 + L_2\,\delta q_2 + \ldots + L_k\,\delta q_k = 0, \end{aligned}$$

in which the left-hand sides *are not exact differentials and do not admit integrable combinations.*

Under those conditions, equation (39) must be true for all displacements $\delta q_1, \delta q_2, \ldots, \delta q_k$ that verify the conditions (40). From the method of Lagrange multipliers, the equations will then be:

$$(41)\qquad \begin{aligned} &P_1 = Q_1 + \lambda_1 A_1 + \lambda_2 B_1 + \ldots + \lambda_p L_1, \\ &P_2 = Q_2 + \lambda_1 A_2 + \lambda_2 B_2 + \ldots + \lambda_p L_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &P_k = Q_k + \lambda_1 A_k + \lambda_2 B_k + \ldots + \lambda_p L_k, \end{aligned}$$

in which P_α has the expression above. Those equations are then combined with the p equations:

$$A_1\,dq_1 + A_2\,dq_2 + \ldots + A_k\,dq_k = 0,$$

$$(42)\qquad \begin{aligned} &B_1\, dq_1 + B_2\, dq_2 + \ldots + B_k\, dq_k = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &L_1\, dq_1 + L_2\, dq_2 + \ldots + L_k\, dq_k = 0, \end{aligned}$$

which expresses the idea that the real displacement is compatible with the constraints that determine $q_1, q_2, \ldots, q_k$ and $\lambda_1, \lambda_2, \ldots, \lambda_p$.

That method was employed by Routh (*loc. cit.*, pp. 132) and by Vierkandt (*loc. cit.*, pp. 47-50).

23. Impossibility of directly applying the Lagrange equations with a minimum number of parameters. – We just saw how one could use the Lagrange equations by taking into account the relations (40) with the method of multipliers.

However, one can try to reduce the parameters to the least-possible number by appealing to the relations (40) in order to leave the minimum number of parameters remaining in the expression for the virtual displacement and to equations (42) in order to leave the minimum number of parameters remaining in the expression for the semi-*vis viva:*

$$T = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2)\,.$$

With those modifications, the *Lagrange equations will no longer be applicable.* We shall show that rapidly, following the authors that were cited above.

A virtual displacement that is compatible with all the constraints imposed on the system is defined for the point x, y, z by:

$$\delta x = \frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f}{\partial q_k}\delta q_k\,,$$

$$\delta y = \frac{\partial \varphi}{\partial q_1}\delta q_1 + \frac{\partial \varphi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \varphi}{\partial q_k}\delta q_k\,,$$

$$\delta z = \frac{\partial \psi}{\partial q_1}\delta q_1 + \frac{\partial \psi}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \psi}{\partial q_k}\delta q_k\,,$$

in which $\delta q_1, \delta q_2, \ldots, \delta q_k$ are coupled by the p relations (40). Infer p variations δq_k, $\delta q_{k-1}, \ldots, \delta q_{k-p+1}$ from those relations as linear, homogeneous functions of the other ones; upon substituting them in $\delta x, \delta y, \delta z$ and setting $n = k - p$:

$$(43)\qquad \begin{aligned} \delta x &= a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_n\,\delta q_n\,, \\ \delta y &= b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_n\,\delta q_n\,, \end{aligned}$$

$$\delta z = c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_n\,\delta q_n\,,$$

in which δq_1, δq_2, …, δq_n are now arbitrary. If one substitutes those values of δx, δy, δz in the general equation of dynamics (37) then one will get a relation in which the coefficients δq_1, δq_2, …, δq_n must be zero, and one will then have the equations of motion:

$$\sum m\left(a_1\frac{d^2x}{dt^2}+b_1\frac{d^2y}{dt^2}+c_1\frac{d^2z}{dt^2}\right) = \sum(a_1X+b_1Y+c_1Z) = Q_1\,,$$

(44) ………………………………………………………………….,

$$\sum m\left(a_\nu\frac{d^2x}{dt^2}+b_\nu\frac{d^2y}{dt^2}+c_\nu\frac{d^2z}{dt^2}\right) = \sum(a_\nu X+b_\nu Y+c_\nu Z) = Q_\nu\,,$$

$$(\nu = 1, 2, \ldots, n),$$

in which we have denoted the right-hand sides by Q_ν.

Furthermore, since the real displacement is currently compatible with the constraints, one will have, from (13):

$$dx = a_1\,dq_1 + a_2\,dq_2 + \ldots + a_n\,dq_n\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or, upon adopting Lagrange's notation for derivatives:

$$x' = a_1\,q_1' + a_2\,q_2' + \cdots + a_n\,q_n'\,,$$
$$y' = b_1\,q_1' + b_2\,q_2' + \cdots + b_n\,q_n'\,,$$
$$z' = c_1\,q_1' + c_2\,q_2' + \cdots + c_n\,q_n'\,.$$

Let us try to follow the method that led to the Lagrange equations with the first of equations (44). To simplify, we suppose that the coefficients a_1, b_1, c_1, …, a_2, b_2, c_2, …, a_n, b_n, c_n *depend upon only* q_1, q_2, …, q_n. We can write the first equation (44) ($\nu = 1$) as:

(45) $$\frac{d}{dt}\sum m\,(a_1\,x' + b_1\,y' + c_1\,z') - R_1 = Q_1\,,$$

in which R_1 denotes the quantity:

$$R_1 = \sum m\left(x'\frac{da_1}{dt}+y'\frac{db_1}{dt}+z'\frac{dc_1}{dt}\right).$$

Now, a_1, b_1, c_1 are obviously equal to $\frac{\partial x'}{\partial q_1'}$, $\frac{\partial y'}{\partial q_1'}$, $\frac{\partial z'}{\partial q_1'}$, resp., so the first term in equation (45) will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right),$$

as in the Lagrange equations; however, the second R_1 is not equal to $\frac{\partial T}{\partial q_1}$, in general. Indeed, one has:

$$\frac{\partial T}{\partial q_1} = \sum m\left(x'\frac{\partial x'}{\partial q_1} + y'\frac{\partial y'}{\partial q_1} + z'\frac{\partial z'}{\partial q_1}\right).$$

Therefore:

$$R_1 - \frac{\partial T}{\partial q_1} = \sum m\left[x'\left(\frac{da_1}{dt} - \frac{\partial x'}{\partial q_1}\right) + y'\left(\frac{db_1}{dt} - \frac{\partial y'}{\partial q_1}\right) + z'\left(\frac{dc_1}{dt} - \frac{\partial z'}{\partial q_1}\right)\right]. \tag{46}$$

Now, the coefficients a_1, b_1, … are supposed to be functions of q_1, q_2, …, q_n, so one will have:

$$\frac{da_1}{dt} = \frac{\partial a_1}{\partial q_1}q_1' + \frac{\partial a_1}{\partial q_2}q_2' + \cdots + \frac{\partial a_1}{\partial q_n}q_n',$$

$$\frac{\partial x'}{\partial q_1} = \frac{\partial a_1}{\partial q_1}q_1' + \frac{\partial a_2}{\partial q_1}q_2' + \cdots + \frac{\partial a_n}{\partial q_1}q_n'.$$

The coefficient of x' in the difference $R_1 - \frac{\partial T}{\partial q_1}$ is then:

$$\left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1}\right)q_2' + \left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right)q_3' + \left(\frac{\partial a_1}{\partial q_n} - \frac{\partial a_n}{\partial q_1}\right)q_n'; \tag{47}$$

it is not zero, in general. The coefficients of y' and z' have analogous forms. From the values of x', y', z' as functions of q_1', q_2', …, q_2', the difference $R_1 - \frac{\partial T}{\partial q_1}$ will then be a quadratic form of q_1', q_2', …, q_2', in general. In order for R_1 to be equal to $\frac{\partial T}{\partial q_1}$ – i.e., for the Lagrange equation to be applicable to the parameter q_1 – it is necessary and sufficient that that quadratic form should be identically zero for any q and q'. It results from that analysis that some of the equations that were employed by Lindelof in his paper in *Acta societatis Scientiarum Fennicae*, t. XXI must be modified ([1]).

Special cases:

1. If the expressions (43) for δx, δy, δz are exact total differentials then all of the quantities such as:

$$\left(\frac{\partial a_i}{\partial q_\nu}-\frac{\partial a_\nu}{\partial q_i}\right),\quad \left(\frac{\partial b_i}{\partial q_\nu}-\frac{\partial b_\nu}{\partial q_i}\right),\quad \left(\frac{\partial c_i}{\partial q_\nu}-\frac{\partial c_\nu}{\partial q_i}\right)$$

will be zero. Expressions such as (46) will be zero, and the Lagrange equations will apply to all of the parameters. In that case, one can integrate the expressions (43) and express x, y, z in finite form as functions of $q_1, q_2, \ldots, q_n$.

2. Here is a case in which the Lagrange equation applies to the parameter q_1 . Suppose that one has:

$$\begin{aligned}
&\frac{\partial a_1}{\partial q_2}=\frac{\partial a_2}{\partial q_1}, \quad \frac{\partial a_1}{\partial q_3}=\frac{\partial a_3}{\partial q_1}, \quad \ldots, \quad \frac{\partial a_1}{\partial q_n}=\frac{\partial a_n}{\partial q_1},\\
&\frac{\partial b_1}{\partial q_2}=\frac{\partial b_2}{\partial q_1}, \quad \frac{\partial b_1}{\partial q_3}=\frac{\partial b_3}{\partial q_1}, \quad \ldots, \quad \frac{\partial b_1}{\partial q_n}=\frac{\partial b_n}{\partial q_1},\\
&\frac{\partial c_1}{\partial q_2}=\frac{\partial c_2}{\partial q_1}, \quad \frac{\partial c_1}{\partial q_3}=\frac{\partial c_3}{\partial q_1}, \quad \ldots, \quad \frac{\partial c_1}{\partial q_n}=\frac{\partial c_n}{\partial q_1}.
\end{aligned} \tag{48}$$

The quantities such as (47) that define the coefficients of x', y', z' in $R_1 - \partial T / \partial q_1$ will then be *zero*, and R_1 will be equal to $\partial T / \partial q_1$. One can characterize that case differently. If the conditions (48) are assumed to be fulfilled then one determines the functions of q_1, $q_2, \ldots, q_n$ by the conditions:

$$U_1 = \int_{q_1^0}^{q_1} a_1\,dq_1, \qquad V_1 = \int_{q_1^0}^{q_1} b_1\,dq_1, \qquad W_1 = \int_{q_1^0}^{q_1} c_1\,dq_1,$$

in which q_1^0 is an arbitrary constant, and the integration is performed over q_1 . From the conditions (48), one will find immediately that:

$$\frac{\partial U_1}{\partial q_2} = \int_{q_1^0}^{q_1} \frac{\partial a_1}{\partial q_2} dq_1 = \int_{q_1^0}^{q_1} \frac{\partial a_2}{\partial q_1} dq_1 = a_2 - a_2^0,$$

([1]) Those equations were reproduced in the first examples in Tome II of my *Traité de mécanique*. They were corrected in the following examples: The end of no. **452** was modified and a no. **452** (*cont.*) was added.

in which a_2^0 is what a_2 will become when one replaces q_1 with the constant q_1^0 in it. Similarly:

$$\frac{\partial U_1}{\partial q_3} = a_3 - a_3^0, \;\ldots, \qquad \frac{\partial U_1}{\partial q_n} = a_n - a_n^0.$$

One will have analogous relations for V_1 and W_1. One can then write:

$$\delta x = \delta U_1 + a_2^0\,\delta q_2 + a_3^0\,\delta q_3 + \cdots + a_n^0\,\delta q_n,$$

$$\delta y = \delta V_1 + b_2^0\,\delta q_2 + b_3^0\,\delta q_3 + \cdots + b_n^0\,\delta q_n,$$

$$\delta z = \delta W_1 + c_2^0\,\delta q_2 + c_3^0\,\delta q_3 + \cdots + c_n^0\,\delta q_n.$$

Hence, the Lagrange equation will apply to q_1 when δx, δy, δz can be put into the form of *a total differential, followed by a differential expression that does not contain* q_1.

One can also say that if q_2, q_3, …, q_n are known as functions of t then q_1 will become a true coordinate, because one can express x, y, z as functions of q_1 in finite form.

In his cited paper, Carvallo elegantly appealed to the Lagrange equations, which he modified, as necessary, in order to treat the problems of the hoop, the unicycle, and the bicycle: He gave a simple and general method for calculating the terms R_1, R_2, …, R_n. In addition, he showed that the Lagrange equation can be applied without modification to the parameter that determines the inclination of the hoop over the plane. As we just said, that amounts to the fact that once the other parameters are known as functions of t, the inclination will become a true coordinate. However, before Carvallo, Hadamard went even deeper into those questions of analytical mechanics in his research with great generality, which is why we shall reproduce the text of that article in the following section of this book.

Use of Hamilton's principle. – One can conveniently deduce the Lagrange equations from Hamilton's principle. It is obvious that this proof breaks down in the case of the exceptional constraints that we are dealing with. I have shown that from the viewpoint of education and without pretending to add anything to the problem at its basis in a small note in the Bull. Soc. Math. France (December, 1898).

24. Equations that can replace those of Lagrange. – Upon differentiating the equations that give x', y', z' with respect to time, one will have:

$$x'' = a_1\,q_1'' + a_2\,q_2'' + \cdots + a_n\,q_n'' + \cdots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

in which the unwritten terms in x'' do not contain the q''. However, one will then obviously have:

$$a_1 = \frac{\partial x''}{\partial q_1''}, \qquad b_1 = \frac{\partial y''}{\partial q_1''}, \qquad c_1 = \frac{\partial z''}{\partial q_1''},$$

and the first of equations (44) can be written:

$$\sum m \left(x'' \frac{\partial x''}{\partial q_1''} + y'' \frac{\partial y''}{\partial q_1''} + z'' \frac{\partial z''}{\partial q_1''} \right) = Q_1 ;$$

hence, upon setting:

$$S = \tfrac{1}{2} \sum m \, (x''^2 + y''^2 + z''^2) = \tfrac{1}{2} \sum m \, J^2 ,$$

in which J is the acceleration of the point m, the equation can be written:

$$\frac{\partial S}{\partial q_1''} = Q_1 .$$

One will get other equations similarly. (See a note that was included in Comptes rendus, 7 August 1899).

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ON ROLLING MOTIONS

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[Extract from the Mémoires of the Société des sciences physiques et naturelles de Bordeaux, (4) **5** (1895)]

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1. – As C. Neumann ([1]) has remarked, the study of rolling motions occupies a special place in dynamics due to the analytical form that the constraint equations are clad in. Indeed, the condition that two bodies in a system should roll on each other without slipping does not translate into equations in finite terms between the desired parameters, but into linear equations in total differentials that are *not integrable*. C. Neumann, and later Vierkandt ([2]), have established those equations by adopting a special notation. Here, I shall employ Darboux's ([3]) notations, which lead to the same result very easily.

Indeed, let S, $S^{(1)}$ be two surfaces of the system that are constrained to be mutually-tangent, and on each of them, we choose a system of curvilinear coordinates, as well as a trihedron that is attached to the surface at each point. The relative position of the two bodies will be defined by the coordinates u, v ; $u^{(1)}$, $v^{(1)}$ of the contact point on S, as well as on $S^{(1)}$, and by the angle φ that the axes that are attached to $S^{(1)}$ make with the axes that are attached to S. Now, if we would like to express the idea that the two surfaces S, $S^{(1)}$ roll on each other without slipping then we must write down that the infinitely-small displacements of the contact point between the two surfaces are identical, which will give us:

$$(1) \qquad \begin{cases} \xi\,du + \xi_1\,dv = (\xi^{(1)}\,du^{(1)} + \xi_1^{(1)}\,dv^{(1)})\cos\varphi - (\eta^{(1)}\,du^{(1)} + \eta_1^{(1)}\,dv^{(1)})\sin\varphi, \\ \eta\,du + \eta_1\,dv = (\xi^{(1)}\,du^{(1)} + \xi_1^{(1)}\,dv^{(1)})\sin\varphi + (\eta^{(1)}\,du^{(1)} + \eta_1^{(1)}\,dv^{(1)})\cos\varphi, \end{cases}$$

in which ξ, ξ_1, η, η_1 have the same meaning in relation to the surface S that they have in Darboux's *Leçons*, and $\xi^{(1)}$, $\xi_1^{(1)}$, $\eta^{(1)}$, $\eta_1^{(1)}$ denote the analogous quantities that relate to the surface $S^{(1)}$, which are referred to the curvilinear coordinates $u^{(1)}$, $v^{(1)}$.

2. – Moreover, there exist some problems in which other equations of an analogous form are relevant. For example, suppose that not only the sliding friction, but the pivoting friction, take a considerable value (while the rolling friction is always zero), in such a way that pivoting is, in turn, made impossible. That condition is expressed (as always, using Darboux's notations) by the equation:

([1]) "Grundzüge der Analytischen Mechanik," Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig (1888), pp. 32.
([2]) "Ueber gleitende und rollende Bewegung," Monats. Math. Phys. **3** (1892), pp. 47.
([3]) *Leçons sur la théorie générale des surfaces,* Book V, Chap. II.

(2) $$r\,du + r_1\,dv + d\varphi - R\,du^{(1)} - R_1\,dv^{(1)} = 0,$$

in which R, R_1 denote the quantities that are analogous to r, r_1, resp., on the second surface, and the left-hand side is the value of the normal component of the elementary rotation. Furthermore, the pivoting friction translates into a couple whose axis is normal to the two surfaces, while the work that it does will be zero when equation (2) is satisfied, which will permit one to apply the principles of analytical dynamics.

3. – In the second place, let a curve and a surface of the system be constrained to remain tangent to each other. Their relative position is defined by:

1. The coordinates u, v of the contact point on the surface.

2. The arc length l of the curve that is found between the contact point and a fixed origin.

3. The angle ω that the tangent to curve (taken in the sense of increasing l) makes with the x-axis of the trihedron that is attached to the surface.

4. The angle θ that the osculating plane to the curve makes with the tangent plane to the surface.

The absence of slipping – i.e., the identity of the infinitely-small displacements of the contact point – is expressed by the equations:

(3) $$\begin{cases} \xi\,du + \xi_1\,dv = dl\cos\omega, \\ \eta\,du + \eta_1\,dv = dl\sin\omega. \end{cases}$$

If one would like to write down the absence of pivoting then one must append the equation:

(4) $$r\,du + r_1\,dv + d\omega - \frac{\cos\theta}{\rho}\,dl = 0$$

(in which ρ denotes the radius of curvature of the curve) whose left-hand side will be the normal component of the rotation.

4. – Problems of that type, in which the parameters q_1, q_2, …, q_{m+p} that define the state of the system are coupled by linear equations:

$$E_h \qquad (h = 1, 2, \ldots, p)$$

in total differentials, are treated ([1]) by a method that is entirely analogous to the one that is employed when the parameters are coupled by equations in finite terms. After writing out the expression $Q_1\,\delta q_1 + \ldots + Q_{m+p}\,\delta q_{m+p}$ in which:

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} - \frac{\partial T}{\partial q_i},$$

one writes that this expression is zero, not for all values of the δq, but only for all of the ones that that satisfy the linear equations E_h .

The calculation of the semi-*vis viva* for the case in which the two surfaces are tangent was performed in the previously-cited papers. It will be easier with the present notation, since the known formulas permit one to evaluate the infinitely-small displacements of the center of gravity and the elementary rotation.

5. – If the equations E_h result from the differential of an equation in finite terms $\mathcal{E}_h$ then one can appeal to them in order to replace p of the quantities q with their values as functions of the m other ones in the expression for T, because that would basically come down to introducing only m parameters ([2]) into the presentation of the equation, while the other p are expressed as functions of the first ones with the aide of the equations $\mathcal{E}$.

However, that will no longer be the case when the equations E do not define an integral system, and the method supposes essentially ([3]) that the semi-*vis viva* T has been calculated *as if the parameters q were independent.*

One can propose to verify that by direct calculation up to the point that caution becomes indispensible, and that study will lead to some results of an interesting form, as we shall see.

First take a special case, to fix ideas: $m = 2$, $p = 2$. Write out the equations E when they have been solved for the two of the differentials:

$$\left\{\begin{aligned} a_1^3\,dq_1 + a_2^3\,dq_2 - dq_3 &= 0, \\ a_1^4\,dq_1 + a_2^4\,dq_2 - dq_4 &= 0, \end{aligned}\right. \tag{5}$$

or, upon dividing by dt :

([1]) Vierkandt, *loc. cit.*, pp. 47-50.

([2]) Granted, the elimination of the parameters q_{m+1} , …, q_{m+p} from the expression T with the aid of the equations $\mathcal{E}$ consists of two operations:

1. Replacing those parameters with the corresponding differentials of their values that one infers from the integral equations $\mathcal{E}$.

2. Replacing the corresponding differentials with their values that one infers from the differential equations E.

However, one easily sees that the first of those operations can be performed after the partial differentiations, while the second one has been performed before those differentiations.

([3]) Cf., Vierkandt, *loc. cit.*, pp. 52-54.

(5′) $$\begin{cases} \mathcal{A}_3 = a_1^3\, q_1' + a_2^3\, q_2' - q_3' = 0, \\ \mathcal{A}_4 = a_1^4\, q_1' + a_2^4\, q_2' - q_4' = 0. \end{cases}$$

Algebraically speaking, appealing to the equations $\mathcal{A}_3 = 0$, $\mathcal{A}_4 = 0$ means combining the expression T with another expression of the form $\lambda_3\, \mathcal{A}_3 + \lambda_4\, \mathcal{A}_4$, in which λ_3 , λ_4 are arbitrary functions of the q and q'. (In general, there is good reason to take λ_3 , λ_4 to be linear in the q', so T will be quadratic in the those same quantities.) The addition of such an expression will introduce a new term into each expression Q_i , namely, the value that Q_i will take when the form of T has been replaced with $\lambda_3\, \mathcal{A}_3 + \lambda_4\, \mathcal{A}_4$, and U has been replaced with 0, and the addition will be legitimate only if the new term disappears in the final result. We must then write out that the equations of motion reduce to identities for the values $T = \lambda_3\, \mathcal{A}_3 + \lambda_4\, \mathcal{A}_4$, $U = 0$.

We observe, first of all, that:

1. If T contains terms of second degree in at least $\mathcal{A}_3$, $\mathcal{A}_4$ (in other words, if λ_3, λ_4 contain terms that are linear combinations of those quantities) then the partial differentiation will preserve $\mathcal{A}_3$, $\mathcal{A}_4$ to degree one, and as a result, those terms will have no influence, since one must take equations (5) into account after that differentiation.

2. For the same reason, all of the weight of partial differentiation must bear upon $\mathcal{A}_3$, $\mathcal{A}_4$, and not upon λ_3, λ_4 .

Under those conditions, when one forms the equations of motion, which are:

(6) $$\begin{cases} Q_1 + a_1^3\, Q_3 + a_1^4\, Q_4 = 0, \\ Q_2 + a_2^3\, Q_3 + a_2^4\, Q_4 = 0, \end{cases}$$

one will see the terms in $d\lambda_3 / dt$, $d\lambda_4 / dt$ disappear, and when one replaces q_3', q_4' with their values that one infers from equations (5′), one will find simply:

(7) $$\begin{aligned} q_2'\ (\lambda_3\, H_3 + \lambda_4\, H_4) &= 0, \\ q_1'\ (\lambda_3\, H_3 + \lambda_4\, H_4) &= 0, \end{aligned}$$

upon setting:

$$H_3 = \frac{\partial a_1^3}{\partial q_2} - \frac{\partial a_2^3}{\partial q_1} + a_2^3\,\frac{\partial a_1^3}{\partial q_3} - a_1^3\,\frac{\partial a_2^3}{\partial q_3} + a_2^4\,\frac{\partial a_1^3}{\partial q_4} - a_1^4\,\frac{\partial a_2^3}{\partial q_4}\,,$$

$$H_4 = \frac{\partial a_1^4}{\partial q_2} - \frac{\partial a_2^4}{\partial q_1} + a_2^3\,\frac{\partial a_1^4}{\partial q_3} - a_1^3\,\frac{\partial a_2^4}{\partial q_3} + a_2^4\,\frac{\partial a_1^4}{\partial q_4} - a_1^4\,\frac{\partial a_2^4}{\partial q_4}\,.$$

The relations $H_3 = 0$, $H_4 = 0$ express the integrability conditions for the system (5). Therefore:

When equations (5) *define an integrable system, and only in that case, one can immediately take those equations into account in the calculation of T.*

However, for arbitrary values of the coefficients a, one will see that only the ratios of λ_3, λ_4 are determined, and one take $\lambda_3\ \mathcal{A}_3 + \lambda_4\ \mathcal{A}_4$ to be an arbitrary multiple of the linear combination:

$$\mathcal{C} = H_3\ \mathcal{A}_3 - H_4\ \mathcal{A}_4\,.$$

One can then appeal to the equation $\mathcal{C} = 0$ *before doing any differentiation.*

6. – Things can happen quite differently when the number of parameters changes: For example, if the coefficients a depend upon a fifth parameter q_5 , because equations (5) must then be completed with terms in q_5'. When those new equations are considered to be identities in the q', they will obviously have an algebraic character that is quite unlike the first ones, and will be verified only under very exceptional circumstances that we shall not go into.

7. – Now, imagine the general case, and let the differential equations of constraint be:

$$\mathcal{A}_k = \sum_{h=1}^{m} a_h^k\, q_h' - q_k' = 0 \qquad (k = m + 1, \dots, m + p). \tag{8}$$

As before, give T the value:

$$T = \sum_{k=m+1}^{m+p} \lambda_k\, \mathcal{A}_k\,,$$

with $U = 0$. One will easily see by means of the same remarks as above that:

$$Q_i = \sum_{k=m+1}^{m+p} \frac{d}{dt}(\lambda_k\, a_i^k) - \sum_{k=m+1}^{m+p} \lambda_k \left(\sum_{k=1}^{m} q_h' \frac{\partial a_h^k}{\partial q_i} \right) \qquad (i = 1, 2, \dots, m),$$

$$Q_l = \frac{d\lambda_l}{dt} - \sum_{k=m+1}^{m+p} \lambda_k \left(\sum_{k=1}^{m} q_h' \frac{\partial a_h^k}{\partial q_i} \right) \qquad (i = m + 1, \dots, m + p),$$

and when one develops da_i / dt, replaces the q_k' with their values that one infers from equations (8), and arranges things with respect to the q_h', the equations of motion:

$$Q_i + \sum_l a_i^l Q_l = 0$$

can be written:

(9) $$\sum_{h=i}^{m} q'_h P_{i,h} = 0 \qquad (i = 1, 2, \ldots, m),$$

in which one sets:

$$P_{i,h} = \sum_{k=m+1}^{m+p} \lambda_k \left[\frac{\partial a_i^k}{\partial q_h} - \frac{\partial a_h^k}{\partial q_i} - \sum_{l=m+1}^{m+p} \left(a_i^l \frac{\partial a_h^k}{\partial q_l} - a_h^l \frac{\partial a_i^k}{\partial q_l} \right) \right],$$

so one has, in particular:

(10) $$P_{i,h} = - P_{h,i} .$$

Any system of values for λ_k that is independent of the q' and verifies equations (9) will correspond to a linear combination $\mathcal{C}$ of equations (8) that can be used before any differentiation. Such a system must satisfy the equations:

$$P_{i,h} = 0,$$

and from the relation (10), their number must reduce by $\frac{1}{2} m\,(m-1)$. There will also be combinations $\mathcal{C}$ such that p is greater than $\frac{1}{2} m\,(m-1)$ ([1]), and their number will be at least $p - \frac{1}{2} m\,(m-1)$.

In particular, one can infer $p - \frac{1}{2} m\,(m-1)$ of the differentials q' as functions of the others from those equations, and consequently:

If the number of independent parameters is m then one can always reduce the form T to something that contains only $\frac{1}{2} m\,(m-1)$ *differentials.*

8. – For special values of the coefficients a, the number of independent solutions can be greater than what we just indicated. However, *one can appeal to all of equations* (8) *only when* the coefficients of all λ are zero; i.e., *when the system* (8) *is integrable*, as one can easily assure oneself.

9. – In order to see what analytical property characterizes the combinations $\mathcal{C}$, we shall study how one forms them when one supposes that the equations E are given in their general form, but not solved for some of the parameters.

First of all, take the special case that we began with; let:

([1]) From the preceding, one sees that when differentials enter into the coefficients a of the equations E, one must be careful to count the number of parameters in the same way as when the differentials do not enter into them.

(11) $$\begin{cases} \mathcal{A} = A_1\, q_1' + A_2\, q_2' + A_3\, q_3' + A_4\, q_4' = 0, \\ \mathcal{B} = B_1\, q_1' + B_2\, q_2' + B_3\, q_3' + B_4\, q_4' = 0, \end{cases}$$

or

(11′) $$\begin{cases} A_1\, dq_1 + A_2\, dq_2 + A_3\, dq_3 + A_4\, dq_4 = 0, \\ B_1\, dq_1 + B_2\, dq_2 + B_3\, dq_3 + B_4\, dq_4 = 0 \end{cases}$$

be the differential equations, and starting with $T = \lambda\ \mathcal{A} + \mu\ \mathcal{B}$, $U = 0$, form the expressions Q_i , while always neglecting to write the partial derivatives that act on λ, μ . Upon multiplying by dt :

$$Q_i\ dt = A_i\ d\lambda + B_i\ d\mu + \lambda \sum_{h=1}^{4} dq_k \left(\frac{\partial A_i}{\partial q_h} - \frac{\partial A_h}{\partial q_i} \right) + \mu \sum_{h=1}^{4} dq_k \left(\frac{\partial B_i}{\partial q_h} - \frac{\partial B_h}{\partial q_i} \right) .$$

We must now write down that the expression:

(12) $$(Q_1\ \delta q_1 + Q_2\ \delta q_2 + Q_3\ \delta q_3 + Q_4\ \delta q_4)\ dt$$

is zero whenever equations (11′) are verified by the δq, on the one hand, and the dq, on the other. We see immediately that by virtue of those equations, the terms $d\lambda$, $d\mu$ will disappear from the expression (12), which will take the simple form:

(13) $$\mathcal{P} = \sum_{h,i} \left[\lambda \left(\frac{\partial A_i}{\partial q_h} - \frac{\partial A_h}{\partial q_i} \right) + \mu \left(\frac{\partial B_i}{\partial q_h} - \frac{\partial B_h}{\partial q_i} \right) \right] (dq_h\ \delta q_i - dq_i\ \delta q_h) .$$

10. – An initial geometric interpretation will permit us to write out the equation that relates λ and μ immediately.

Indeed, consider dq_1 , dq_2 , dq_3 , dq_4 ; δq_1 , δq_2 , δq_3 , δq_4 to represent the homogeneous coordinates of two points in ordinary space. The expressions $dq_h\ \delta q_i - dq_i\ \delta q_h$ represent the Plückerian coordinates of the line that joins those two points; i.e., (due to the conditions that were imposed on the d and the δ) of the intersection of the two planes that is represented by equations (11′). It will suffice to substitute the coordinates in the equation $\mathcal{B} = 0$, which is the equation of a linear complex that our line belongs to, in order to obtain the desired condition:

(14) $$\sum_{h,i,k,l} \left[\lambda \left(\frac{\partial A_i}{\partial q_h} - \frac{\partial A_h}{\partial q_i} \right) + \mu \left(\frac{\partial B_i}{\partial q_h} - \frac{\partial B_h}{\partial q_i} \right) \right] (A_k\ B_l - A_l\ B_k) = 0,$$

in which the indices h, i, k, l are the indices 1, 2, 3, 4, when they are displaced by any alternating permutation.

11. – In the general case, where the $m + p$ parameters are related by p equations:

$$(15)\qquad \begin{cases} \mathcal{A} = A_1\, q_1' + \cdots + A_{m+p}\, q_{m+p}' = 0, \\ \mathcal{B} = B_1\, q_1' + \cdots + B_{m+p}\, q_{m+p}' = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ \mathcal{L} = L_1\, q_1' + \cdots + L_{m+p}\, q_{m+p}' = 0, \end{cases}$$

or

$$(15')\qquad \begin{cases} A_1\, dq_1 + \cdots + A_{m+p}\, dq_{m+p} = 0, \\ B_1\, dq_1 + \cdots + B_{m+p}\, dq_{m+p} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ L_1\, dq_1 + \cdots + L_{m+p}\, dq_{m+p} = 0, \end{cases}$$

we will have to treat the expression:

$$(Q_1\ \delta q_1 + \dots + Q_{m+p}\ \delta q_{m+p})\ dt$$

in the same way, which is formed under the hypothesis that $U = 0$, $T = \lambda\,\mathcal{A} + \mu\,\mathcal{B} + \dots + \sigma\,\mathcal{L}$, and which will become:

$$(16)\qquad \mathcal{P} = \sum_{h,i}\left[\lambda\left(\frac{\partial A_i}{\partial q_h} - \frac{\partial A_h}{\partial q_i}\right) + \mu\left(\frac{\partial B_i}{\partial q_h} - \frac{\partial B_h}{\partial q_i}\right) + \cdots + \sigma\left(\frac{\partial L_i}{\partial q_h} - \frac{\partial L_h}{\partial q_i}\right)\right](dq_h\ \delta q_i - dq_i\ \delta q_h)\ .$$

If we suppose that $dq_1, \dots, dq_{m+p}$, on the one hand, and $\delta q_1, \dots, \delta q_{m+p}$, on the other, represent homogeneous coordinates in an $(m + p - 1)$-dimensional space then the expressions $dq_h\ \delta q_i - dq_i\ \delta q_h$ will be the coordinates of the line that joins the two points that are defined in that way. For $m = 2$, equations (15′) represent a line ([1]), which must, in turn, belong to the complex that is defined by the equation $\mathcal{P} = 0$, and for $m > 2$, they must represent a $(m - 1)$-fold linear multiplicity in which all lines must belong to that complex. Since such a multiplicity contains $\frac{1}{2}m\,(m - 1)$ lines with linearly-independent coordinates, we will indeed recover the number of conditions that we obtained before.

12. – Now consider the parameters q themselves to be the (absolute) coordinates of a point in an $(m + p)$-dimensional space. Equations (15′) can be considered to be the equations of an m-fold planar multiplicity to which all of the curves that pass through a well-defined point M and satisfy those equations must be tangent. An m-dimensional surface (Σ) that is tangent to that hyperplane at M will be the locus of curves that verify the aforementioned differential equations *at the point M.*

([1]) Of course, we shall not speak of the case of $m = 1$, for which equations (15′) will reduce to ordinary differential equations.

We can, to fix ideas, suppose that the differentials d and δ are taken along curves that belong to that surface, and upon denoting a system of m curvilinear coordinates on (Σ) by $t, u, v, \ldots, w$, set $d = dt\dfrac{\partial}{\partial t}$, $\delta = du\dfrac{\partial}{\partial u}$.

Now, upon denoting the first of the differentials in (15′) by dV, the expression $\mathcal{P}$ will reduce to $d\delta V - \delta dV$ for:

$$\lambda = 1, \quad \mu = \nu = \ldots = \sigma = 0.$$

The equation $\mathcal{A} = 0$ will give one of the desired combinations only when $\dfrac{d\delta V - \delta dV}{dt\,du}$ is zero at the point M.

Hence, *a combination $\mathcal{C}$ is characterized by the fact that it is an exact differential at an arbitrary point M on the corresponding surface* (Σ), where we intend that to mean the conditions that express the integrability of the differential dV are verified on (Σ), which was just confirmed by the evaluations that were just obtained, since those conditions are $\frac{1}{2}m\,(m-1)$ in number.

In other words, when the integral $\int dV$ is taken along a closed curve that is traced on the surface (Σ) in the neighborhood of the point M, it will not be identically zero, but infinitely-close of higher order in a small (two-dimensional) surface element that is bounded by that curve.

13. – The case of two surfaces that roll on each other corresponds to $m = 3$, $p = 2$. No combinations $\mathcal{C}$ will exist then, in general, and one will immediately see that they never do.

By contrast, the case of two surfaces roll on each other without pivoting [which are conditions that are represented by equations (1) and (2)], for which $m = 2$, $p = 3$, offers two such combinations. It is indeed remarkable that those two combinations are nothing but the two equations (1), *which express the absence of slipping.*

Indeed, if we employ the geometric interpretation of no. **11** then since the quantities:

$$dq_1 = du, \qquad dq_2 = dv, \qquad dq_3 = du^{(1)}, \qquad dq_4 = dv^{(1)}, \qquad dq_5 = d\varphi$$

are homogeneous coordinates in four-dimensional space, when one sets:

$$\left\{\begin{array}{ll} X = \xi^{(1)}\cos\varphi - \eta^{(1)}\sin\varphi, & X_1 = \xi_1^{(1)}\cos\varphi - \eta_1^{(1)}\sin\varphi, \\ Y = \xi^{(1)}\sin\varphi + \eta^{(1)}\cos\varphi, & Y_1 = \xi_1^{(1)}\sin\varphi + \eta_1^{(1)}\cos\varphi, \end{array}\right. \tag{17}$$

equations (1) and (2) will be written:

$$(15'')\begin{cases} A_1\, dq_1 + A_2\, dq_2 + A_3\, dq_3 + A_4\, dq_4 + A_5\, dq_5 = \xi\, du + \xi_1\, dv - X\, du^{(1)} - X_1\, dv^{(1)} = 0, \\ B_1\, dq_1 + B_2\, dq_2 + B_3\, dq_3 + B_4\, dq_4 + B_5\, dq_5 = \eta\, du + \eta_1\, dv - Y\, du^{(1)} - Y_1\, dv^{(1)} = 0, \\ C_1\, dq_1 + C_2\, dq_2 + C_3\, dq_3 + C_4\, dq_4 + C_5\, dq_5 = r\, du + r_1\, dv - R\, du^{(1)} - R_1\, dv^{(1)} + d\varphi = 0, \end{cases}$$

which define a line whose coordinates will be the various determinants that one deduces from the matrix:

$$\left\| \begin{matrix} \xi & \xi_1 & -X & -X_1 & 0 \\ \eta & \eta_1 & -Y & -Y_1 & 0 \\ r & r_1 & -R & -R_1 & 1 \end{matrix} \right\| .$$

Let $|\, h\; i\,|$ denote the determinant that is obtained by suppressing the columns of rank h, i and arranging the others in such a fashion that the order of the five indices thus-disposed will be derived them in a natural way from an even number of transpositions, so the coefficients of λ, μ in the expression will become:

$$(18) \qquad \sum_{h,i} \left(\frac{\partial A_i}{\partial q_h} - \frac{\partial A_h}{\partial q_i} \right) |\, h\, i\,|, \qquad \sum_{h,i} \left(\frac{\partial B_i}{\partial q_h} - \frac{\partial B_h}{\partial q_i} \right) |\, h\, i\,| .$$

The coefficients of dq_1, dq_2 depend upon only q_1, q_2 ; those of dq_3 dq_4 are independent of q_1, q_2, and that of dq_5 is zero or constant, so the only combinations of indices that we must consider are:

$$h,\ i = \begin{cases} 1, 2 \\ 3, 4 \\ 3, 5 \\ 4, 5 \end{cases},$$

respectively, which correspond to the determinants:

$$|\,1\;2\,| = X\, Y_1 - Y\, X_1\,, \qquad |\,3\;4\,| = \xi\, \eta_1 - \eta\, \xi_1\,,$$

$$|\,3\;5\,| = \begin{vmatrix} \xi & \xi_1 & X_1 \\ \eta & \eta_1 & Y_1 \\ r & r_1 & R_1 \end{vmatrix}, \qquad |\,4\;5\,| = - \begin{vmatrix} \xi & \xi_1 & X \\ \eta & \eta_1 & Y \\ r & r_1 & R \end{vmatrix},$$

respectively.

Moreover, the coefficients in equations (15″) satisfy the differential relations ([1]):

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = \eta\, r_1 - r\, \eta_1\,, \qquad \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = r\, \xi_1 - \xi\, r_1\,,$$

([1]) Darboux, *loc. cit.*, pp. 382.

$$\frac{\partial X}{\partial v^{(1)}} - \frac{\partial X_1}{\partial u^{(1)}} = Y R_1 - R Y_1, \qquad \frac{\partial Y}{\partial v^{(1)}} - \frac{\partial Y_1}{\partial u^{(1)}} = R X_1 - X R_1,$$

$$\frac{\partial X}{\partial \varphi} = -Y, \qquad \frac{\partial X_1}{\partial \varphi} = -Y_1, \qquad \frac{\partial Y}{\partial \varphi} = X, \qquad \frac{\partial Y_1}{\partial \varphi} = X_1,$$

which will give us the following expressions:

$$-(\eta r_1 - r\eta_1)(X Y_1 - Y X_1) + (X R_1 - R X_1)(\xi \eta_1 - \xi \eta_1)$$

$$- Y \begin{vmatrix} \xi & \xi_1 & X_1 \\ \eta & \eta_1 & Y_1 \\ r & r_1 & R_1 \end{vmatrix} + Y_1 \begin{vmatrix} \xi & \xi_1 & X \\ \eta & \eta_1 & Y \\ r & r_1 & R \end{vmatrix},$$

$$-(r \xi_1 - \xi r_1)(X Y_1 - Y X_1) + (R X_1 - X R_1)(\xi \eta_1 - \eta \xi_1)$$

$$+ X \begin{vmatrix} \xi & \xi_1 & X_1 \\ \eta & \eta_1 & Y_1 \\ r & r_1 & R_1 \end{vmatrix} - X_1 \begin{vmatrix} \xi & \xi_1 & X \\ \eta & \eta_1 & Y \\ r & r_1 & R \end{vmatrix}$$

for the coefficients (18), which vanish identically.

Our conclusion is established then: When there is rolling without pivoting, the equations of rolling can be used to calculate *T*.

14. – If we form the coefficient that corresponds to the third equation similarly then we will find simply:

$$\left(\frac{\partial r_1}{\partial u} - \frac{\partial r}{\partial v}\right)(X Y_1 - Y X_1) - \left(\frac{\partial R_1}{\partial u} - \frac{\partial R}{\partial v}\right)(\xi \eta_1 - \eta \xi_1),$$

which is annulled (since $X Y_1 - Y X_1 = \xi^{(1)} \eta_1^{(1)} - \xi_1^{(1)} \eta^{(1)}$) only if one has:

$$\frac{\dfrac{\partial r_1}{\partial u} - \dfrac{\partial r}{\partial v}}{\xi \eta_1 - \eta \xi_1} = \frac{\dfrac{\partial R_1}{\partial u} - \dfrac{\partial R}{\partial v}}{\xi^{(1)} \eta_1^{(1)} - \eta_1^{(1)} \eta^{(1)}};$$

i.e., *if the two surfaces have equal constant curvatures.*

Indeed, equations (1) and (2) will form an integrable system in that case. In order to realize it, it will suffice to remark that in the case of rolling without pivoting, the loci of contact points on the two surfaces have the same geodesic curvature. Now, the two surfaces can be mapped to the same sphere here, and our two lines on that sphere will correspond to two equal lines. There will then exist relations (that contain three arbitrary constants) between the parameters u, v ; $u^{(1)}$, $v^{(1)}$ of the contact point that define a rotation (or more precisely, a symmetry) of the sphere. As for the angle φ, which is the angle that the x-axis of the trihedron that is attached to the sphere at the point that is the transform under symmetry makes with the new position of the original x-axis, it is expressed as a function of u, v and the same constants. The three relations thus-written are the integrals of the differential system.

15. – The fact that a spherical curve is determined when one gives the radius of geodesic curvature as a function of the arc is almost obvious *a priori*. Moreover, it comes down immediately to some kinematical considerations with the aid of a tri-rectangular trihedron that has its summit at the center of the sphere, one edge terminating at a point of the curve, and one face that is tangent to the cone that has that curve for its base and the center for its summit. The spherical extremity of the normal edge to the cone describes the polar spherical curve to the first one, and the tangents to the two curves are parallel. Upon supposing, to simplify, that the radius of the sphere is equal to 1, the ratio ds / ds_1 of the arc lengths of the original curve and the polar curve will be equal to the radius of geodesic curvature ρ_s . Now, if one takes the independent variable to be the arc length s then one will see that the instantaneous rotation of the trihedron will have ds_1 / ds, 0, 1 for its projections onto the edges. The motion of that trihedron is then known when one gives the geodesic curvature as a function of the arc.

It should be remarked that the equality $\rho_g = ds / ds_1$ gives an immediate proof of the proposition:

When two figures are mutually polar on the sphere of radius 1, *the area of each of them will be equal to the perimeter of the other one (up to a hemisphere), at least if one counts the arc lengths to be positive or negative according to whether the corresponding tangents of the two curves have the same or opposite senses, respectively.*

That is because the integral $\int \frac{ds}{\rho_g} = \iint d\sigma$ will then reduce to $\int ds_1$.

16. – A very simple rolling motion is that of an indefinite plane rolling on a fixed surface in the absence of accelerating forces. By the term "indefinite plane," we mean a plane on which masses are arranged at very large distance from each other in such a manner that the principal moments of inertia will be very large. One can even suppose that the masses are external to the plane and linked with that plane by the condition that the center of gravity must be on the plane and the central ellipsoid revolves around the normal to the plane in such a way that if m denotes the mass of the system then the principal moments of inertia will be mk^2, mk^2, and $\lambda\, mk^2$, where k^2 is very large. From that, we can neglect the terms into which k^2 does not enter as a factor in comparison to the ones that contain it, and reduce the *vis viva* $2T$ to the *vis viva* of rotation around the center of gravity:

(19) $$\frac{2T}{mk^2} = (p\,u' + p_1\,v')^2 + (q\,u' + q_1\,v')^2 + \lambda\,(\varphi + r\,u' + r_1\,v')^2\,.$$

The equation that relates to the variable φ then reduces to:

(20) $$\varphi + r\,u' + r_1\,v' = c.$$

As for the equations that relate to u, v, by virtue of the constraint equations, which are written:

(21) $$\begin{cases} \xi\,du + \xi_1\,dv = dx\cos\varphi - dy\sin\varphi, \\ \eta\,du + \eta_1\,dv = dx\sin\varphi + dy\cos\varphi \end{cases}$$

(x, y denote the coordinates of the contact point in the moving plane), they will become linear combinations of the equations that relate to x, y. However, the left-hand sides of the latter, which do not contain any term in k^2, are negligible. One can then write:

(22) $$\begin{cases} \dfrac{d}{dt}[p\,(pu' + p_1v')] - \left(\dfrac{\partial p}{\partial u}u' + \dfrac{\partial p_1}{\partial u}v'\right)(pu' + p_1v') \\ \quad + \dfrac{d}{dt}[q\,(qu' + q_1v')] - \left(\dfrac{\partial q}{\partial u}u' + \dfrac{\partial q_1}{\partial u}v'\right)(qu' + q_1v') \\ \quad + \lambda\,c\left(\dfrac{dr}{dt} - \dfrac{\partial r}{\partial u}u' - \dfrac{\partial r_1}{\partial u}v'\right) = 0, \\ \\ \dfrac{d}{dt}[p_1\,(pu' + p_1v')] - \left(\dfrac{\partial p}{\partial v}u' + \dfrac{\partial p_1}{\partial v}v'\right)(pu' + p_1v') \\ \quad + \dfrac{d}{dt}[q_1\,(qu' + q_1v')] - \left(\dfrac{\partial q}{\partial v}u' + \dfrac{\partial q_1}{\partial v}v'\right)(qu' + q_1v') \\ \quad + \lambda\,c\left(\dfrac{dr_1}{dt} - \dfrac{\partial r}{\partial v}u' - \dfrac{\partial r_1}{\partial v}v'\right) = 0. \end{cases}$$

Equations (22), to which one must append the *vis viva* integral, by virtue of equation (20), determine u, v. They contain only rotations as characteristic element of the surface; i.e., elements that depend upon the spherical representation.

As usual, set:

(23) $$q\,u' + q_1\,v' = \frac{d\sigma}{dt}\cos\theta, \qquad p\,u' + p_1\,v' = -\frac{d\sigma}{dt}\sin\theta,$$

in which σ is the arc length of the spherical representation of the trace of the rolling of the surface, and θ is the angle that the spherical representation makes with the x-axis of the trihedron of the surface. When the *vis viva* integral is simplified with the help of equation (20), it will reduce to:

(24) $$\frac{d\sigma}{dt} = h.$$

On the other hand, the known differential relations between the rotations permit one to transform the two equations (22). In the first of them, one replaces $\frac{\partial p_1}{\partial u}$, $\frac{\partial q_1}{\partial u}$, $\frac{\partial r_1}{\partial u}$ with their values $\frac{\partial p}{\partial v} + r\,q_1 - q\,r_1$, $\frac{\partial q}{\partial v} + p\,r_1 - r\,p_1$, $\frac{\partial r}{\partial v} + q\,p_1 - p\,q_1$, and upon operating analogously on the second one, those equations will be represented by:

$$p\frac{d}{dt}(p\,u' + p_1\,v') + q\frac{d}{dt}(q\,u' + q_1\,v') = (p\,q_1 - q\,p_1)\,v'\,(r\,u' + r_1\,v' - \lambda\,c),$$

$$p_1\frac{d}{dt}(p\,u' + p_1\,v') + q_1\frac{d}{dt}(q\,u' + q_1\,v') = -(p\,q_1 - q\,p_1)\,u'\,(r\,u' + r_1\,v' - \lambda\,c),$$

respectively, or, by an immediate linear combination:

$$\frac{d}{dt}(p\,u' + p_1\,v') = (q\,u' + q_1\,v')\,(r\,u' + r_1\,v' - \lambda\,c),$$

$$\frac{d}{dt}(q\,u' + q_1\,v') = -(p\,u' + p_1\,v')\,(r\,u' + r_1\,v' - \lambda\,c),$$

which reduce to just:

(25) $$\theta' + r\,u' + r_1\,v' = \lambda\,c,$$

by virtue of equations (23) and (24).

The left-hand side of this expresses the geodesic curvature of the spherical representation, *which is a circle* that describes a uniform motion, from equation (24).

As for the angle φ, it is given by the condition that is deduced from a combination of equations (20), (25) that $\theta - \varphi$ – i.e., the angle between the x-axis of the moving plane and spherical representation of the trace of the rolling motion – increases in proportion to time, or rather, that the component of the pivoting motion $\varphi' + r\,u' + r_1\,v'$ is constant.

Finally, equations (21) give x and y by quadratures. The locus of the contact point on the moving plane can be considered to be defined by its arc length, moreover, which is the same as that of the surface, and the angle between the tangent and the x-axis of the moving plane, which is $\omega - \theta$; i.e., the sum of $\omega - \theta$ and a quantity that is proportional to time.

17. – When the given surface is a sphere, the traces of the rolling motion on the moving plane will likewise be circles, since $\omega - \theta$ is zero and the arc length of the curve and the angle $\omega - \varphi$ are both proportional to time. One easily sees that this cannot happen for other forms of the given surface.

Can two different surfaces give the same rolling traces on the moving plane, in general? We can see that this is impossible, at least when one demands that the correspondence between homologous points on the two surfaces must be the same in any case. Indeed, the two must then be mappable to each other. As a result, a point-like correspondence must exist between the two spherical representations that preserves areas and transforms circles into circles. It is clear that such a correspondence can be realized only by a simple rotation of the sphere, and our two surfaces can be considered to be mappable to each other with parallelism of their tangent planes. There will then be two associated minimal surfaces, which is an inadmissible solution in our problem, in which only convex surfaces can occur.

18. – The case of a line that rolls on a surface is the case of $m = 2$, $p = 2$, since the parameter θ does not enter into the equations of constraint. For:

$$\mathcal{A} = \xi\, u' + \xi_1\, v' - l' \cos \omega,$$

$$\mathcal{B} = \eta\, u' + \eta_1\, v' - l' \sin \omega,$$

equation will reduce to:

$$(\xi\, \eta_1 - \eta\, \xi_1)\,(\lambda \sin \omega - \mu \cos \omega) = 0.$$

The factor $\xi\, \eta_1 - \eta\, \xi_1$ is essentially non-zero, so the equation that one is permitted to use is:

$$(\xi\, u' + \xi_1\, v') \cos \omega + (\eta\, u' + \eta_1\, v') \sin \omega - l' = 0 \; ; \tag{26}$$

i.e., the one that *expresses the absence of longitudinal slipping.*

19. – Take the example of a line that is not acted upon by any force and rolls on a surface in such a way that the contact point describes a certain line L. Let masses be distributed in arbitrary fashion along that line whose sum is m, but we can always assume that the center of gravity corresponds to $l = 0$ and the principal moment of inertia is mk^2.

The *vis viva* of rotation around the center of gravity is:

$$mk^2\left(\frac{d\varepsilon}{dt}\right)^2 = mk^2\,\{[(p\, u' + p_1\, v') \sin \omega - (q\, u' + q_1\, v') \cos \omega]^2 + (\omega' + r\, u' + r_1\, v')^2\},$$

in which ε denotes the infinitely-small angle through which the line turns. As for the velocity of the center of gravity, since we have removed all longitudinal slipping from the line, we can consider it to consist of:

1. A transversal velocity:

$$l\,(\omega' + r\,u' + r_1\,v') + (\xi\,u' + \xi_1\,v')\sin\omega - (\eta\,u' + \eta_1\,v')\cos\omega.$$

2. A normal velocity:

$$l\,[(p\,u' + p_1\,v')\sin\omega - (q\,u' + q_1\,v')\cos\omega].$$

Its square will then be:

$$l^2\left(\frac{d\varepsilon}{dt}\right)^2 + 2l\,(\omega' + r\,u' + r_1\,v')\,[(\xi\,u' + \xi_1\,v')\sin\omega - (\eta\,u' + \eta_1\,v')\cos\omega]$$
$$+ [(\xi\,u' + \xi_1\,v')\sin\omega - (\eta\,u' + \eta_1\,v')\cos\omega]^2,$$

the last term of which can be neglected, since it is homogeneous of degree two in the left-hand sides of the equations of constraint. One will then have:

$$(27)\quad \begin{cases} \dfrac{2T}{m} = (k^2+l^2)\{[(p\,u'+p_1\,v')\sin\omega-(q\,u'+q_1v')\cos\omega]^2+(\omega'+r\,u'+r_1\,v')^2\} \\ \qquad +2l\,(\omega'+r\,u'+r_1\,v')[(\xi\,u'+\xi_1\,v')\sin\omega-(\eta\,u'+\eta_1\,v')\cos\omega]. \end{cases}$$

There are two differential equations that one must write, one of which can be replaced with the *vis viva* equation:

$$(28)\qquad (k^2+l^2)\{[(p\,u'+p_1\,v')\sin\omega-(q\,u'+q_1v')\cos\omega]^2+(\omega'+r\,u'+r_1\,v')^2\} = \alpha^2.$$

We can take the second equation to be the one that relates to ω, for which one only has one expression to calculate (namely Q), since the differential $d\omega$ does not enter into the constraint equations. Upon taking the equations $\mathcal{A} = 0$, $\mathcal{B} = 0$ into account, we will then have:

$$(29)\begin{cases} \dfrac{d}{dt}[(k^2+l^2)(\omega'+r\,u'+r_1\,v')] \\ -(k^2+l^2)[(p\,u'+p_1\,v')\sin\omega-(q\,u'+q_1\,v')\cos\omega][(p\,u'+p_1\,v')\cos\omega+(q\,u'+q_1\,v')\sin\omega] \\ -l\,l'(\omega'+r\,u'+r_1\,v')=0. \end{cases}$$

One must append the equations:

$$(30)\qquad \begin{cases} \xi\,du+\xi_1\,dv = dl\cos\omega, \\ \eta\,du+\eta_1\,dv = dl\sin\omega, \end{cases}$$

which express the idea that l is nothing but the arc length of the line L, and ω is the angle between that line and the x-axis.

The *vis viva* equation gives us:

$$(p\,u' + p_1\,v')\sin\omega - (q\,u' + q_1\,v')\cos\omega = \frac{\alpha\cos\varpi}{\sqrt{k^2+l^2}},$$

$$\omega + r\,u' + r_1\,v' = \frac{\alpha\sin\varpi}{\sqrt{k^2+l^2}},$$

in which ϖ is the angle between the osculating plane to L and the normal to the surface. If we note that the quantity:

$$(p\,u' + p_1\,v')\cos\omega + (q\,u' + q_1\,v')\sin\omega$$

represents $\frac{dl}{dt}\left(\frac{d\varpi}{dl} - \frac{1}{\tau}\right)$ then we will see that equation (29) reduces to:

$$2\cos\varpi\sqrt{k^2+l^2} = 0.$$

The line then rolls on a planar section of the surface.

20. – If the rolling is constrained to take place without pivoting then the number of combinations $\mathcal{C}$ will be equal to two. It is almost obvious *a priori* (when one considers the line to be the limit of a surface) that those combinations are the precisely the equations of rolling. One verifies that immediately from the form of equations (3) and (4). Indeed, if one once more lets q_1, q_2, q_3, q_4, q_5 denote the variables u, v, l, ω, θ, respectively, to fix ideas, then one will see that all of the terms in the expressions (18) contain zero factors.

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ON CERTAIN SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS

By HADAMARD

(Oral presentation to a session of the Société des sciences physiques et naturelles de Bordeaux, 1895-1895)

A system of p linear total differential equations:

$$(S)\qquad \begin{aligned} &A_1\,dq_1 + A_2\,dq_2 + \ldots + A_{m+p}\,dq_{m+p} = 0,\\ &B_1\,dq_1 + B_2\,dq_2 + \ldots + B_{m+p}\,dq_{m+p} = 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &L_1\,dq_1 + L_2\,dq_2 + \ldots + L_{m+p}\,dq_{m+p} = 0, \end{aligned}$$

in which one considers dq_1, dq_2, …, dq_{m+p} to be homogeneous coordinates in $(m + p)$-dimensional space, represents an $(m - 1)$-dimensional planar multiplicity. However, that multiplicity can also be represented by a system (Σ) of n tangential equations, namely, the conditions that the coefficients (viz., tangential coordinates) of an $(m + p - 2)$-dimensional hyperplane must fulfill in order for that hyperplane to contain our multiplicity. Two linear systems such as (S) and (Σ) can be called *reciprocal systems*, for brevity. If one denotes the tangential coordinates by $\frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_{m+p}}$ then the system (Σ) will be nothing but the system of linear partial differential equations that the integrals of the system (S).

Having said that, when the system (S) represents the equations of constraint in a dynamical problem, as was explained in the paper above, one will be led to consider certain special combinations $\mathcal{C}$ within the system (S) whose number is equal to at least $p - \frac{1}{2}m\,(m - 1)$ and which play a special role in the formation of the Lagrange equations.

Now, if we are dealing with the system (Σ) then the calculations of the combinations $\mathcal{C}$ will come down to a well-known theory. Indeed, if one would like to apply Lie's method to the system (Σ) then one must append $\frac{1}{2}m\,(m - 1)$ brackets (E_i, E_k) to the equations $E_1, E_2, \ldots, E_m$ of that system. One will then obtain a new system (Σ′) of $m + \lambda$ equations $[0 \leq \lambda \leq \frac{1}{2}m\,(m - 1)]$ on which one must recommence with that same operation. Here, on the contrary, we shall stop with the system (Σ) and take the reciprocal (S'); *the latter is composed of precisely the desired combinations* $\mathcal{C}$.

One will then indeed see why the number of those combinations is at least $p - \frac{1}{2}m(m-1)$ and can be p only if the system is integrable.