Excerpted from P. Appell, *Traité de mécanique rationnelle,* 6th ed., v. II, Gauthier-Villars, Paris, 1953. pp. 402-416. Translated by D. H. Delphenich.

VII. – CONSTRAINTS INVOLVING SERVOS.

470. Servos (†). – In a remarkable treatise that was submitted in November 1922 to the Paris Science Faculty and which was entitled *Étude théorique des compas gyrostatiques* ANSCHÜTZ *et* SPERRY, Henri BEGHIN introduced the new notion of a "servo."

There exists an important category of mechanisms that realize their constraints by a method that is entirely different from the one that was just examined. For those mechanisms, *one cannot abstract from the way that the constraints are realized.*

The constraints that are realized by these mechanisms can be arbitrary; most often, they are holonomic. However, instead of those realizations being – so to speak – passive, such as ones that are obtained by simple contact, they use arbitrary forces (e.g., electromagnetic forces, compressed air pressure, etc.) – in a word, *auxiliary energy sources that come into play automatically and are automatically measured out in such a way as to realize this or that constraint at each instant.* One can even imagine an animate being that acts by contact and regulates its action in such a manner as to realize this or that constraint.

Let Σ be a solid body (a disc, for example) that moves around a diameter Δ under the influence of certain given forces. A solid body Σ_1 (a concentric ring, for example) of diameter Δ_1 moves around Δ without having any contact with Σ. The ring Σ_1 carries a toothed wheel *a* whose axis is Δ that meshes with a pinion *b* that is attached to the shaft of a motor *M*. It is easy to image an arrangement (1) that would make the motor turn in one sense or the other without acting directly on either Σ or Σ_1, while Σ and Σ_1 are never in the same plane. If α and α_1 are the azimuths of Σ and Σ_1, respectively, then the constraint:

$$\alpha = \alpha_1$$

will then be found to be realized in such a way that *the ring* Σ_1 *follows the disc* Σ *in all of its motions around* Δ *without being driven by it.* It is obvious that the manner in which this system behaves has nothing in common with the manner in which would behave if Σ were driven by Σ_1 by direct contact: For example, if a small spring that is fixed to Σ_1 pushes on Σ then the system will take on a uniformly accelerated motion in the case of a servo, while it will obviously remain immobile under the second hypothesis.

What are the forces of constraint in the system in the previous example? If I consider the system $\Sigma\Sigma_1$ then those forces will be, on the one hand, the reactions along the axis Δ, which are ordinary forces of constraint, and the reactions of the pinion *b* on the gear *a*.

(†) The French "liaisons par asservissement" literally means "constraints by servitude (or slavery)." However, since the standard modern term is "servo constraints," I will consistently translate "asservissement" as "servo."

(1) *See* the description of the Sperry Compass (*The Sperry Gyrocompass*, 7).

Those reactions, which play a major role in the problem, have an entirely special character, because the pinion b (viz., a foreign obstacle) that exerts them is not fixed, nor is it in a state of motion that is known in advance as a function of t : *It is an obstacle whose position is known in advance as a function of the parameters (α, α_1, here) upon which the system considered* $\Sigma\Sigma_1$ *depends.*

If I include the rotor R of the motor M in the system considered then the constraint forces will be the electromagnetic actions to which the rotor is subject on the part of the stator, in addition to the actions of contact between the fixed obstacles and the actions of contact R Σ_1, which are ordinary constraint forces. Indeed, those forces have the character of constraint forces: *They are unknown, but one knows that they have the value that is necessary in order to insure the constraint considered.*

For any elementary constraint that is compatible with the constraint $\alpha = \alpha_1$, the ordinary constraint forces will do zero work. On the contrary, the other constraint forces (whether one means the reactions of the foreign obstacles whose position depends upon parameters α, α_1 or those electromagnetic actions that are exerted at a distance on the rotor) will do non-zero work. That is how the mechanisms that include a servo are distinguished from the other ones.

General study of the mechanisms that include a servo. D'Alembert's principle. – Let Σ be a material system that presents no source of energy dissipation. In addition, suppose that no part of that system can contract or dilate, with the exception that will be assumed below.

Upon taking into account the contacts that are imposed upon it, that system will be supposed to depend upon a limited number h of parameters $q_1, q_2, \ldots, q_h$ in such a manner that the coordinates x, y, z of each element of Σ are functions of those parameters that are known in advance, and might also be:

$$(1) \qquad x = f(q_1, q_2, \ldots, q_h, t) , \qquad y = \ldots, \qquad z = \ldots$$

at time t.

Some of the foreign obstacles that Σ is in contact with are fixed or depend upon t. Others, as a result of the contacts imposed, are supposed to depend upon a certain number k of the preceding parameters – namely, $q_1, q_2, \ldots, q_k$, also possibly t.

Those contact conditions are *holonomic contact constraints.*

Suppose, in addition that the system is subject to certain non-holonomic constraints; i.e., that the parameters $q_1, q_2, \ldots, q_h$ are coupled by a certain number p of differential relations that express the conditions of rolling without slipping or pivoting at certain contacts. Those relations will permit one to express the p elementary variations:

$$dq_{n+1} , dq_{n+2} , \ldots, dq_{n+p} \qquad (n + p = h)$$

as functions of $q_1, q_2, \ldots, q_n$, and dt ; they have the form:

(2) (p relations) $$\begin{cases} A_1\, dq_1 + \cdots + A_h dq_h + A\,dt = 0, \\ B_1\, dq_1 + \cdots + B_h dq_h + B\,dt = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Those conditions are *non-holonomic contact constraints.* Those are the only two types of constraints that one encounters in modern problems.

For any elementary displacement that is compatible with the constraints that might exist at the instant t (i.e., one for which δt is zero, and $\delta q_1, \ldots, \delta q_n$ are arbitrary), the mutual reactions between the bodies of the system do zero work, as well as the reactions of the fixed obstacles or the ones that depend upon t. I will say that these reactions are *constraint forces of the first kind.*

In addition, the system Σ is supposed to be subject to other constraints that I will call *servo constraints,* which are also expressed by finite equations or linear differential equations, but are realized by means of forces that are completely different: Those forces, which will call *generalized constraint forces,* or *ones of the second kind,* are applied to the bodies in the system: They can be *external* or *internal.*

In the first case, they are either actions at a distance, such as electromagnetic ones or other kinds, which are regulated *automatically* in such a manner as to insure the finite or differential constraint that they are supposed to realize, or the contact actions with the foreign obstacles whose position is supposed to depend upon $q_1, q_2, \ldots, q_k$, t whose motion must regulated *automatically* in such a manner that certain finite or differential equations must be verified at each instant by the parameters q.

In the second case – i.e., if those constraint forces of the second kind are internal – they will be either actions at a distance, such as electromagnetic ones, or internal stresses in the bodies that can contract or dilate (e.g., compressed air, muscles in a living being), which are stresses that are regulated *automatically* – for example, the will of the living being – in such a manner as to realize this or that constraint. Except for that exception, the system will not be supposed to be compressible.

The system Σ can be composed of an electric motor whose velocity ω is independent of the load, which might be, for example a derivative motor (*moteur-dérivation*), within certain limits. The servo constraint will then be realized in the form:

$$d\theta = \omega\, dt .$$

The system can be composed of a cyclist and his machine. The cyclist can contract his muscles, not by a given quantity but by a quantity that is measured out in such a way that certain constraints are found to be realized: He will regulate the action of his legs in such a manner as to realize a constant angular velocity, or perhaps he will contract the muscles of his body in such a way to realize an inclination of the frame as a function of t, etc. The methods that will be described below will permit one to study the variation of the unknown parameters.

As an application, one can also imagine a ship Σ, with one part σ of the cargo that is put into motion automatically by a motor in such a manner as to realize certain constraints: For example, as a servo constraint, one might have that the ship must remain constantly vertical, which is realized by a roll stabilizer. A small gyrostatic mechanism that is based upon the principle of the Schlick stabilizer will indicate the true vertical onboard the ship. The servomotor will come into action when that vertical is not in the plane of symmetry of the ship. One can also regulate the motion of σ in such a manner as to realize the motion of σ in such a manner as to realize some relation between its position and the inclinations of the ship. One can then change the period of oscillation of the ship at will and avoid the synchronism of the hull, when appropriate. One can regulate the motion of σ in such a manner as to realize some condition between its position and the angular velocity of the ship that permits one to damp out the oscillations, etc. The forces of constraint of the second kind here will be the mutual actions between Σ and σ.

A material system that presents constraint forces of the second kind will be said to *include a servo.* It is obvious that *the virtual work that is done by constraint forces of the second kind is generally non-zero.*

Having posed those definitions, imagine that there are r servo relations, one of which is finite, while the others are differentials, and have the form:

$$(3) \qquad (r \text{ relations}) \qquad \begin{cases} g(q_1, \ldots, q_h, t) = 0, \quad \ldots, \\ \varepsilon_1\, dq_1 + \varepsilon_2\, dq_2 + \cdots + \varepsilon_h\, dq_h + \varepsilon\, dt = 0, \quad \ldots \end{cases}$$

The virtual displacements of the system that are compatible with the *contact* constraints that might exist at the instant t ($\delta t = 0$) are obtained by taking $h - p$ of the elementary variations $\delta q_1, \ldots, \delta q_h$ arbitrarily; the other p are defined by the relations (1), which will reduce to:

$$(2') \qquad (r \text{ relations}) \qquad \begin{cases} A_1\, \delta q_1 + \cdots + A_h\, \delta q_h = 0, \\ B_1\, \delta q_1 + \cdots + B_h\, \delta q_h = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Among those displacements, there exist ones for which one can confirm *a priori* that the work done by the constraint forces of the second kind is zero, without knowing anything but the way that they act. We shall suppose that they are the ones that simultaneously verify the j relations:

$$(4) \qquad (j \text{ relations}) \qquad \begin{cases} a_1\, \delta q_1 + \cdots + a_h\, \delta q_h = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1\, \delta q_1 + \cdots + l_h\, \delta q_h = 0. \end{cases}$$

D'Alembert's principle, when it is applied to any of those displacements, is expressed by the equation:

$$(5) \qquad \sum m\,(x''\, \delta x + y''\, \delta z + z''\, \delta z) = \sum (X\, \delta x + Y\, \delta y + Z\, \delta z)$$

here, in which the Σ sign on the left-hand side extends over all elements of the system, while m denotes the mass of one of its elements, and x'', y'', z'' denote the projections of its acceleration, while the Σ on the right-hand side extends over all given forces X, Y, Z. Indeed, it is obvious that for those displacements the constraint forces, which are either of the first or second kind, will do zero work.

That condition decomposes into $h - p - j$, since the h elementary variations $\delta q_1, \dots, \delta q_h$ are subject to the p relations (2′) and the j relations (4), so only $h - p - j$ of those variations will be arbitrary.

In order to write those equations effectively, we employ the method of Lagrange multipliers: If x, y, z are expressed as functions of $q_1, \dots, q_h$, t by equations (1) then the left-hand side of equation (5) will be the sum of h terms of the form:

$$\delta q \sum m\left(x''\frac{\partial x}{\partial q}+y''\frac{\partial y}{\partial q}+z''\frac{\partial z}{\partial q}\right)=P\ \delta q\,, \tag{6}$$

in which q denotes any of the h parameters. The right-hand side is the sum of h terms of the form:

$$\delta q \sum\left(X\frac{\partial x}{\partial q}+Y\frac{\partial y}{\partial q}+Z\frac{\partial z}{\partial q}\right)=Q\ \delta q\,. \tag{7}$$

D'Alembert's equation is written:

$$(P_1 - Q_1)\ \delta q_1 + (P_2 - Q_2)\ \delta q_2 + \dots + (P_h - Q_h)\ \delta q_h = 0\,. \tag{8}$$

That equation must be combined with the p relations (2′), when multiplied by the coefficients A, M, …, respectively, and the j relations (4), when multiplied by λ, μ, …, respectively, where those coefficients A, M, …, λ, μ, … constitute $p + j$ auxiliary unknowns. We will get the equation:

$$\sum\ (P_i - Q_i + AA_i + MB_i + \dots + \lambda\, a_i + \mu\, b_i + \dots)\ \delta q_i = 0, \tag{9}$$

in which i represents the indices 1, 2, …, h. The multipliers A, M, …, λ, μ, … can be chosen in such a manner that the coefficients of $p + j$ of the variations δq_i will be zero, because the relations (2′) and (4) are meant to be independent in the preceding. Equation (9) must be verified for any of the other $h - q - j$ variations δq_i in such a way that the coefficients of those $h - p - j$ variations in equation (9) must also be themselves zero.

In summary, the problem comes down to solving the h equations:

$$\left\{\begin{array}{l} P_1 - Q_1 + AA_1 + MB_1 + \cdots + \lambda\, a_1 + \mu\, b_1 + \cdots = 0, \\ P_2 - Q_2 + AA_2 + MB_2 + \cdots + \lambda\, a_2 + \mu\, b_2 + \cdots = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{array}\right. \tag{10}$$

to which one must append the p equations (2) that express the non-holonomic contact constraints and the r servo constraints (3), namely, $h + p + r$ equations in $h + p + j$ unknowns (viz., $q_1, \dots, q_h$, A, M, …, λ, μ, …).

If it happens that r is greater than j then the problem will generally be impossible to solve; i.e., it will not be possible to realize a number of servo constraints that is greater than the number of restrictive conditions that one must impose upon the parameters q in order to annul the virtual work done by forces of the same kind.

If r is equal to j then the problem will be solved by equations (2), (3), and (10).

If r is less than j then the motion will be indeterminate: One can imagine, moreover, that if the function that must replace the forces of the second kind is not sufficiently well-defined then their elimination will become impossible, and that the motion cannot be studied unless one is given some of them.

Special cases:

1. Suppose that equations (2′), which express the idea that the virtual displacements are compatible with the non-holonomic contact constraints, and equations (4), which one is led to introduce in order to annul the virtual work done by constraint forces of the second kind, are solved for the $p + j = m$ variations $\delta q_1, \ldots, \delta q_m$:

$$(11)\qquad \left\{\begin{array}{l} \delta q_1 = \mathcal{A}_{m+1}\,\delta q_{m+1} + \cdots + \mathcal{A}_h\,\delta q_h , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_m = \mathcal{L}_{m+1}\,\delta q_{m+1} + \cdots + \mathcal{L}_h\,\delta q_h ; \end{array}\right.$$

the Lagrange multipliers become superfluous. If one replaces the $\delta q_1 , \ldots, \delta q_m$ with these expressions in (8) then one will get an equation that is linear in $\delta q_{m+1} , \ldots, \delta q_h$, which must be verified for any variations, and therefore $h - m$ equations of the form:

$$(12)\qquad P_{m+i} - Q_{m+i} + \mathcal{A}_{m+i}\,(P_1 - Q_1) + \ldots + \mathcal{L}_{m+i}\,(P_m - Q_m) = 0,$$

in which i denotes one of the numbers 1, 2, …, $h - m$.

One must combine these equations with the p equations (2) and the r servo equations (3).

2. If the equations (11) reduce to:

$$(13)\qquad \delta q_1 = 0, \qquad \ldots, \qquad \delta q_m = 0$$

then the equations of motion will reduce to the simple form:

$$(14)\qquad P_{m+1} = Q_{m+1} , \quad \ldots, \qquad P_h = Q_h .$$

3. Suppose that the constraint forces of the second kind are uniquely the contact actions of an auxiliary system Σ_1 of moving obstacles whose positions depend upon a certain subset $q_1, \ldots, q_k$ of the parameters $q_1, \ldots, q_h$. In that case, the relations (4) will be:

$$(15)\qquad \delta q_1 = 0, \qquad \ldots, \qquad \delta q_k = 0,$$

because it is by leaving those obstacles fixed that one will annul the work done by their actions on the given system Σ. The multipliers λ, μ, … will become superfluous, because equation (8) will no longer contain δq_{k+1}, …, δq_h. Equations (10) will reduce to the following $h - k$ equations:

$$
(16) \qquad \left\{ \begin{array}{l} P_{k+1} - Q_{k+1} + AA_{k+1} + MB_{k+1} + \cdots = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ P_h - Q_h + AA_h + MB_h + \cdots = 0, \end{array} \right.
$$

and as in the general case, one must combine these with the p equations (2) and the r relations (3), so one will have $h - k + p + r$ relations in $h + p$ unknowns. The problem will be determinate when the number of servo-equations is equal to the number k of parameters that the auxiliary system Σ_1 depends upon.

4. With the same hypotheses as in the preceding paragraph (3.), we suppose, in addition, that the contact constraints on the system are all holonomic ($p = 0$). The multipliers A, M, … will also become superfluous, and equations (10) will reduce to the following $h - k$ equations:

$$
(17) \qquad P_{k+1} = Q_{k+1}, \quad \ldots, \quad P_h = Q_h,
$$

and one must append the r equations (3), which express the servo. The unknowns are uniquely q_1, …, q_h.

Remarks:

1. In systems without servos, the virtual displacements to which one applies d'Alembert's equation will be the ones that are compatible with all of the constraints. In the systems that include servos, things will be different for some displacements: There will then exist analytical reasons for the difference that exists between those two categories of systems, and one can understand all of the interest that is attached to the mechanisms that include servos from the industrial standpoint.

2. In the case where the constraint forces of the second kind are uniquely the reactions of the moving obstacles whose positions are functions of some of the parameters q (cases 3. and 4.), the solution of the problem will be independent of the inertia of those bodies and the given forces that are applied to them.

Thus, if one can define two parts Σ, Σ_1 of a system that is subject to r servo constraints such that the partial system Σ is not subject to any constraint force of the second kind, outside of the reactions of the system Σ_1, and if, on the other hand, the number of parameters that the system Σ_1 depends upon is equal to the number of servo conditions then the inertial forces and the given forces that are applied to Σ_1 will not influence the motion of Σ. The method that was indicated in the special cases 3. and 4. will permit one to put the problem into the form of equations without introducing either inertial forces or given forces. The partial system will then play an auxiliary role. That special case frequently presents itself in the applications.

Equilibrium in systems that include a servo. – D'Alembert's principle will give the equilibrium conditions when one suppresses the P, which are the terms that are provided by the inertial forces in the system considered. Equations (10), which relate to the general case, and equations (12), (14), (16) or (17), which relate to the special cases that were studied, will then give the equilibrium equations if one replaces the P with zero. One must combine those equations with the servo equations, which are finite. The differential equations that express non-holonomic constraints, which are either contact constraints or servo constraints, must obviously not be appended; they are verified identically.

Extending the Lagrange equations. – With the same general conditions that were defined at the outset of this discussion, the coordinates x, y, z of the various elements of the system considered Σ can be expressed by finite expressions [eq. (1)] as functions of time t and the parameters $q_1, \ldots, q_h$ that depend upon the system when one takes into account only the holonomic contact constraints; now, the expression:

$$P = \sum m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right)$$

will have the value:

$$P = \frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q}.$$

One will then extend the Lagrange equations to the systems that include a servo by replacing $P_1, \ldots, P_h$ with their expression in equations (10).

It is essential to remark that the *vis viva* must be calculated as functions of the $q_1, \ldots, q_h, q_1', \ldots, q_h', t$ *without taking into account the servo constraints.* The same thing will be true for the elementary work:

$$Q_1\,\delta q_1 + \ldots + Q_h\,\delta q_h$$

done by the given forces. If those forces admit a force functions – i.e., if $Q_1, \ldots, Q_h$ are the derivatives $\frac{\partial U}{\partial q_1}, \ldots, \frac{\partial U}{\partial q_h}$ of a function U of $q_1, \ldots, q_h, t$ – then that function U will be calculated without addressing the servo. It is only in the equations themselves – i.e., in the expressions Q, $\frac{\partial T}{\partial q}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right)$ – that one can take them into account. Meanwhile, when the derivative of $\partial T / \partial q'$ with respect to t is taken for the real motion, which is compatible with the servo constraints, one can carry out all of the simplifications on $\partial T / \partial q'$ that result from those constraints before differentiating with respect to t. In summary:

One can take the servo into account after concluding the calculation of the three categories of expression Q, $\frac{\partial T}{\partial q}, \frac{\partial T}{\partial q'}$.

Vis viva equation. – Since the contact constraints are not supposed to depend upon t, in particular, equations (2), which represent the non-holonomic constraints, have no terms in dt ($A = B = \ldots = 0$), because the given forces are supposed to admit the force function $U(q_1, \ldots, q_h)$, we multiply equations (10), which give the motion in the general case, by $dq_1, \ldots, dq_h$, resp., which are elementary variations of the parameters under the real displacement, and the expression:

$$P_1\, dq_1 + \ldots + P_h\, dq_h$$

will give the work done by the inertial forces, with the sign changed:

$$\sum m\,(x''\,dx + y''\,dy + z''\,dz)\ ;$$

i.e., the differential dT of one-half the *vis viva*.

The expression:

$$Q_1\, dq_1 + \ldots + Q_h\, dq_h$$

is equal to dU. The multiplier A has the coefficient:

$$A_1\, dq_1 + \ldots + A_h\, dq_h\,,$$

which is zero, since the displacement verifies equations (2). The same thing will be true for the analogous coefficients $M, \ldots$

One will then have the equation:

$$d\,(T - U) + \lambda\,(a_1\, dq_1 + \ldots + a_h\, dq_h) + \mu\,(b_1\, dq_1 + \ldots + b_h\, dq_h) + \ldots = 0\,.$$

One sees that $T - U$ is not constant. Since the terms in $\lambda, \mu, \ldots$ represent the elementary work done by the constraint forces of the second kind, which is not zero, in general, the conditions (4) will not be imposed upon the real displacement. According to its sign, that work will correspond to a *gain* or a *loss of mechanical energy* for the system Σ considered.

The same thing will be true in each of the special cases that were defined before: The combination of the *vis vivas* will not be given by the expression $d\,(T - U)$, because only some of the expressions $P_1, \ldots, P_h, Q_1, \ldots, Q_h$ will enter into the equations of motion.

It is interesting to conclude that *the servo can permit one to increase or decrease the desired mechanical energy of a system, and in particular, it can damp out the oscillations of a system that presents no source of dissipation of energy.*

Application. – Let a plate Σ in a fixed plane articulate with a circular base plate Σ_1 that moves around its center O at a point C. A force that is parallel to a fixed line Ox and has a constant magnitude F is exerted on the plate Σ at a point A that is located along the line that joins C to the center of gravity G. A servomotor M acts on the base plate Σ_1 by way of gears, in such a manner as to constantly realize the constraint:

$$\alpha - \beta = \frac{\pi}{2} \tag{1}$$

$$[\alpha = (Ox, OC), \beta = (Ox, CA), OC = R, CA = a, CG = b].$$

Since there is just one servo constraint, and on the other hand, the base plate Σ_1 depends upon just one parameter α, the system Σ, taken by itself, will belong to the special case 4 (pp. 7). One can then apply the Lagrange equations to just the plate Σ. One sees that the mass of the base plate Σ_1 will have no influence on the motion. The *vis viva* of Σ is:

$$2T = M\,(R^2\,\alpha'^2 + b^2\,\beta'^2 + 2R\,b\,\alpha'\beta'\cos(\alpha - \beta) + k\,\beta'^2),$$

where Mk^2 denotes the moment of inertia of Σ about G.

The virtual work done by the force F is:

$$d\mathcal{T} = F\,\delta(R\cos\alpha + a\cos\beta),$$

except that the equation that relates to β is written:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \beta'}\right) - \frac{\partial T}{\partial \beta} = -F\,a\sin\beta\,. \tag{4}$$

Now:

$$\frac{\partial T}{\partial \beta'} = M\,[b^2\,\beta' + 2R\,b\,\alpha'\cos(\alpha - \beta) + k\,\beta'] = M\,(b^2 + k^2)\,\beta',$$

if one takes the servo constraint into account. On the other hand:

$$\frac{\partial T}{\partial \beta} = MR\,b\,\alpha'\beta\sin(\alpha - \beta) = MR\,b\,\beta'^2.$$

The equation of motion is then:

$$M\,(b^2 + k^2)\,\beta'' - MR\,b\,\beta'^2 + F\,a\sin\beta = 0\,. \tag{5}$$

If the constraint $\alpha - \beta = \pi / 2$ is realized by direct contact between Σ and Σ_1 then the motion will be completely different: It will be regulated by the equation:

$$[M\,(b^2 + k^2 + k^2)\,\beta'' + F\,(a\sin\beta + R\cos\beta) = 0, \tag{6}$$

in which I_1 denotes the moment of inertia of the base plate about O. Equation (5) will easily give the motion: β'^2 is obtained by adding a term that is sinusoidal in β to a term that is exponential. β varies between two limits, one of which can be pushed out to infinity. On the contrary, equation (6) will give a pendulum motion.

The equilibrium positions are obtained by annulling the right-hand side of equation (4). One will then find the two positions for which CA is parallel to the force. On the contrary, equation (6) will give the positions for which OA is parallel to the force.

Extending the equations in no. 465. – The equations in no. 465 (†) present the following advantages:

1. They can be applied to systems that are subject to non-holonomic constraints without one having to introduce a system of multipliers as auxiliary unknowns.

2. They permit one use auxiliary parameters that are coupled with the true coordinates $q_1, \ldots, q_h$ by some differential relations.

Therefore, let Σ be a system that fulfills the conditions that were indicated at the beginning of this article (pp. 2). Upon taking into account the *holonomic contact constraints* that are imposed upon its position, which depends upon h parameters $q_1, \ldots, q_h$, and possibly t, in such a way that the coordinates of each element of matter are finite functions of the form:

$$(1) \qquad x = f(q_1, \ldots, q_h, t), \qquad y = \ldots, \qquad z = \ldots$$

Suppose that these parameters are combined with s auxiliary parameters $q_{h+1}, \ldots, q_{h+s}$ that are coupled with the preceding ones by some differential relations that serve as their definitions, *which are relations that do not, in turn, depend upon any constraint force.* One counts them with the relations that express the non-holonomic contact constraints, because they enter into the formulation of equations in the same way.

We then have p differential relations ($p \geq s$) of the form:

$$(2) \qquad (p \text{ relations}) \qquad \begin{cases} A_1\,dq_1 + \cdots + A_{h+s}\,dq_{h+s} + A\,dt = 0, \\ A_1\,dq_1 + \cdots + A_{h+s}\,dq_{h+s} + A\,dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Suppose that the servo constraints are represented by r finite or differential relations:

(†) Translator: No. 465 had the title "General form of the equations of motion that is suitable for all holonomic and non-holonomic motion," and was concerned with d'Alembert's equations, as well as the Appell equations.

$$(3) \qquad (r \text{ relations}) \qquad \begin{cases} g(q_1, \cdots, q_{h+s}, t) = 0, \\ \dots\dots\dots\dots\dots\dots, \\ \varepsilon_1\, dq_1 + \cdots + \varepsilon_{h+s}\, dq_{h+s} + \varepsilon\, dt = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Finally, the virtual displacements that annul the work done by constraint forces of the second kind are the ones that verify the j relations:

$$(4) \qquad (j \text{ relations}) \qquad \begin{cases} a_1\, \delta q_1 + \cdots + a_{h+s}\, \delta q_{h+s} = 0, \\ b_1\, dq_1 + \cdots + b_{h+s}\, \delta q_{h+s} = 0, \\ \dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Having said that, form the expression:

$$S = \tfrac{1}{2} \sum m\,(x''^2 + y''^2 + z''^2)\,,$$

which is called the *energy of acceleration*. If we express x'', y'', z'' by means of the parameters $q_1, \dots, q_h$, which are functions of t, and the first and second derivatives of the parameters q with respect to t then we have seen that the terms P in the d'Alembert equation will have the expressions:

$$P_1 = \frac{\partial S}{\partial q_1''}, \qquad \dots, \qquad P_h = \frac{\partial S}{\partial q_h''};$$

hence, one establishes the equations of motion.

Case where the differential equations of contact constraint and the definitions (2) *are solved for the p variations dq.* – In order for the equations of motion to appear with their full simplicity, it is useful to solve those p equations (2) for p of the $h + s = n + p$ variations dq . On the one hand, one expresses the p derivatives $q'_{n+1}, \dots, q'_{n+p}$ as functions of the $q'_1, \dots, q'_n$ by means of relations of the form:

$$(5) \qquad \begin{cases} q'_{n+1} = \alpha_1\, q'_1 + \cdots + \alpha_n\, q'_n + \alpha, \\ \dots\dots\dots\dots\dots\dots\dots \\ q'_{n+p} = \gamma_1\, q'_1 + \cdots + \gamma_n\, q'_n + \alpha, \end{cases}$$

and on the other hand, the p virtual displacements $\delta q_{n+p}, \dots, \delta q_{n+1}$ as functions of the δq_1, $\dots$, δq_n :

(5)
$$\begin{cases} \delta q_{n+1} = \alpha_1\,\delta q_1 + \cdots + \alpha_n\,\delta q_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_{n+p} = \gamma_1\,\delta q_1 + \cdots + \gamma_n\,\delta q_n; \end{cases}$$

the coefficients α_i , …, α_i are functions of q_1, …, q_{n+p} , t. Of course, those parameters q_1, …, q_n can be chosen from among the true coordinates, as well as from among the auxiliary parameters q_{h+1} , …, q_{n+s} .

Having said that, instead of expressing S as a function of the parameters q_1, …, q_n and their first and second derivatives, as we supposed in the preceding paragraph, it can be interesting to use equations (5), which replace equations (2). Upon differentiating them with respect to t, we will express the second derivatives q''_{n+1}, …, q''_{n+p} as functions of the q''_1, …, q''_n, and the first and derivatives of the parameters q. We can then make the p second derivatives q''_{n+1}, …, q''_{n+p}, disappear from S. S will become a function of the q_1, …, q_{h+s} , t, q'_1, …, q'_{h+s}, and the n second derivatives q''_1, …, q''_n. We know that under those conditions the virtual work that is done by the inertial forces (with the sign changed) will be:

(7)
$$\left(\frac{\partial S}{\partial q''_1}\right)\delta q_1 + \cdots + \left(\frac{\partial S}{\partial q''_n}\right)\delta q_n .$$

On the other hand, if one expresses the virtual work done by the given forces in terms of the δq_1, …, δq_n by using only the relations (6) then one will an expression of the form:

(8)
$$Q_1\,\delta q_1 + \ldots + Q_n\,\delta q_n$$

for that work.

Those two expressions must be equal for any displacement that annuls the work done by constraint forces of the second kind; i.e., one that verifies the j relations (4). Here again, it is interesting to take the relations (6) into account, which will permit one to make the δq_{n+1} , …, δq_{n+p} disappear from equations (4); when those equations are solved for j of the remaining variations δq_1 , …, δq_n , they will be written:

(9)
$$\begin{cases} \delta q_1 = \mathcal{A}_{j+1}\,\delta q_{j+1} + \cdots + \mathcal{A}_n\,\delta q_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_j = \mathcal{L}_{j+1}\,\delta q_{j+1} + \cdots + \mathcal{L}_n\,\delta q_n. \end{cases}$$

If one replaces δq_1 , …, δq_j with these values in the expressions (7) and (8) and expresses their equality for any of the arbitrary remaining δq_{j+1} , …, δq_n then one will get the equations of motion in the form:

$$(10)\qquad \begin{cases} \left(\dfrac{\partial S}{\partial q''_{j+1}} - Q_{j+1}\right) + \mathcal{A}_{j+1}\left(\dfrac{\partial S}{\partial q''_1} - Q_1\right) + \cdots + \mathcal{L}_{j+1}\left(\dfrac{\partial S}{\partial q''_j} - Q_j\right) = 0, \\ \dotfill \\ \left(\dfrac{\partial S}{\partial q''_n} - Q_n\right) + \mathcal{A}_n\left(\dfrac{\partial S}{\partial q''_1} - Q_1\right) + \cdots + \mathcal{L}_n\left(\dfrac{\partial S}{\partial q''_j} - Q_j\right) = 0. \end{cases}$$

Those equations are simpler than the Lagrange equations that one can write for the same problem [*see* eq. (12), pp. 6], because the number of terms in each of the preceding equations is $j + 1$, instead of $m + 1 = p + j + 1$ in the case of the Lagrange equations. The complication that is thus introduced by the presence of the coefficients $\mathcal{A}$ and $\mathcal{L}$ is provided solely by the relations that express the idea that the work done by constraint forces of the second type is zero, and is not at all provided by the non-holonomic constraints.

One can append the p equations (5) to equations (10), along with the r equations (3) that express the servo constraint.

Case where the displacements that annul the virtual work done by the constraint forces of the second kind are defined by j relations of the form:

$$(11)\qquad \delta q_1 = 0, \qquad \ldots, \qquad \delta q_j = 0\,.$$

With the same hypotheses as before, suppose that the conditions that a displacement must fulfill in order to annul the virtual work done by constraint forces of the second kind have the simple form (11). One can, moreover, always place oneself in that case by introducing some conveniently-chosen auxiliary parameters, if necessary.

In that case – which is, in summary, the general case – if one performs the calculations as was just said then equations (10) will be simplified and will take the same form as in the case of a system without servos:

$$(12)\qquad \frac{\partial S}{\partial q''_{j+1}} = Q_{j+1}\,, \quad \ldots, \qquad \frac{\partial S}{\partial q''_n} = Q_n\,.$$

One sees that the equations of no. 465 will give a general solution of the question in a form that is simpler than the Lagrange equations. One must combine those $n - j$ equations with the p equations (5) and the r servo equations (3). If $r = j$ then the number of equations will be equal to the number of unknowns.

Application. – A material plane P can slide by translation on a fixed horizontal plane xOy. A sphere Σ of radius R can roll without slipping on that plane. The motion of the plane P is regulated automatically in such a manner that the center of the sphere turns uniformly around Oz with the velocity ω with respect to the fixed axes Ox, Oy, Oz. Let us study the motion by means of the equations of no. 465.

Let u, v be the coordinates of a distinguished point A on the plane P with respect to the axes Ox, Oy, Oz. The position of that plane is defined by only those two parameters. The position of the sphere is defined by the coordinates ξ, η of its center, and for example, the Euler angles φ, θ, ψ, which define its orientation.

If p, q, r are the projections onto the axes of the instantaneous rotation of the sphere then the conditions that express the rolling without slipping will be obtained by writing that the material element of the sphere and the material element of the plane, which coincide at the instant t, have the same velocity:

$$\xi' - q\,R = u', \quad \eta' + p\,R = v' . \tag{1}$$

There are two servo constraints:

$$d\xi + \omega\eta\,dt = 0, \qquad dh - \omega\xi\,dt = 0 . \tag{2}$$

Since the number of these relations is equal to the number of parameters that the position in the plane P depend upon, one can answer the question by applying the equations of no. 465 to just the sphere Σ.

Upon taking just the holonomic contact constraints into account, the sphere will be considered to depend upon the seven parameters u, v, ξ, η, φ, θ, ψ ($h = 7$). It is interesting to combine them with three auxiliary parameters ($s = 3$) that are coupled with the preceding one by the relations:

$$d\lambda = p\,dt, \qquad d\mu = q\,dt, \qquad d\nu = r\,dt . \tag{3}$$

These $h + s = 10$ parameters are coupled with those three relations and by the two relations (1) that express the non-holonomic contact constraints. Those relations (1) can be written:

$$d\xi - R\,d\mu = du, \qquad d\eta + R\,d\lambda = d\nu . \tag{1′}$$

The relations (3) and (1′) are the p differential relations [eq. (2), pp. 11] of the general theory ($p = 5$).

We keep $h + s - p = n = 5$ parameters from the $h + s = 10$ parameters; we choose u, v, ξ, η, ν. We express the energy of acceleration S of the sphere as a function of the second derivatives of those n parameters by using the $p = 5$ relations (3) and (1′). Now, the value of S is defined by:

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5} M\,R^2\,(p'^2 + q'^2 + r'^2),$$

or, from (3) and (1′):

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5} M\,R^2\,[(v'' - \eta'')^2 + (\xi'' - u'')^2 + R^2\nu''^2] .$$

The virtual displacements annul the work done by constraint forces of the second kind are defined by the $j = 2$ conditions:

(5) $$\delta u = 0, \qquad \delta v = 0,$$

since those forces are the reactions of the plane on the sphere. Those conditions have the form indicated in the preceding paragraph [eq. (11)], in such a way that the equations of motion have the form [eq. (12)]:

(6) $$\frac{\partial S}{\partial \xi''} = \Xi\,, \qquad \frac{\partial S}{\partial \eta''} = \mathrm{H}\,, \qquad \frac{\partial S}{\partial \nu''} = \mathrm{N}\,.$$

The right-hand sides are zero, since the given forces (viz., weight of the sphere) do zero work, and we get the equations:

(7) $$7\xi'' = 2u'', \qquad 7\eta'' = 2v'', \qquad \nu'' = 0,$$

which will answer the question when they are combined with the servo equations (2). Those five equations can be integrated immediately and will show that the point A describes a cycloid. The formulas (1′) show that the instantaneous rotation vector will remain parallel to the generators of an oblique cone whose base is a horizontal circle that describes the angular velocity ω.

____