"Sur des transformations de mouvements," J. reine angew. Math. **110** (1892), 37-41.

On some transformations of motions.

(By *Paul Appell* in Paris)

Translated by D. H. Delphenich

1. – The goal of this note is to summarize some research that is associated with the interesting article by **Stäckel** "Ueber die Differentialgliechungen der Dynamik und den Begriff der analytischen Aequivalenz dynamischer Probleme," that appeared in volume 107 of this journal (1891).

Suppose that a material system has constraints that are independent of time, and its position is defined by n parameters $p_1, p_2, \ldots, p_n$. If that system is subject to forces that depend upon the positions and velocities of the points of application then, according to **Lagrange**, the equations of motion are:

$$(1) \qquad \frac{d}{dt}\left(\frac{\partial S}{\partial p'_\alpha}\right) - \frac{\partial S}{\partial p_\alpha} = P_\alpha , \qquad p'_\alpha = \frac{dp_\alpha}{dt} ,$$

in which S denotes the semi-*vis viva* of the system, and:

$$P_1\,\delta p_1 + P_2\,\delta p_2 + \ldots + P_n\,\delta p_n$$

denotes the sum of the virtual works done by the forces that are applied directly under an arbitrary displacement that is compatible with the constraints. The quantities P_α are functions of $p_1, p_2, \ldots, p_n, p'_1, p'_2, \ldots, p'_n$. In the particular case in which the forces depend upon only the position of the system, the P_α will not contain $p'_1, p'_2, \ldots, p'_n$. The equations of motion of that system are:

$$(2) \qquad \frac{d}{dt_1}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha , \qquad q'_\alpha = \frac{dq_\alpha}{dt_1} ,$$

in which the quantities Q_α depend upon $q_1, q_2, \ldots, q_n$, and their derivatives q'_α . One must consider the two problems in mechanics to be *equivalent* if there exists a transformation of the form:

$$(3) \qquad \begin{cases} q_\alpha = \varphi_\alpha(p_1, p_2, \ldots, p_n), \\ dt = \lambda(p_1, p_2, \ldots, p_n)\,dt_1 \end{cases}$$

that transforms the system of equations (2) into the system (1).

In order to account for the possibility of such a transformation, we set:

$$S = \tfrac{1}{2}\sum_{i,j} a_{i,j}\, p_i'\, p_j', \qquad a_{i,j} = a_{j,i},$$
$$T = \tfrac{1}{2}\sum_{i,j} b_{i,j}\, q_i'\, q_j', \qquad b_{i,j} = b_{j,i}.$$

We then direct our attention to the terms in (1) and (2) that contain the second derivatives with respect to time. We have:

$$\frac{d}{dt_1}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} = b_{1\alpha}\frac{d^2 q_1}{dt_1^2} + b_{2\alpha}\frac{d^2 q_2}{dt_1^2} + \cdots + b_{n\alpha}\frac{d^2 q_n}{dt_1^2} + \cdots,$$

in which the following terms contain only first derivatives and constitute a quadratic form with respect to q_1', q_2', …, q_n'. Similarly:

$$\frac{d}{dt}\left(\frac{\partial S}{\partial p_\alpha'}\right) - \frac{\partial S}{\partial p_\alpha} = a_{1\alpha}\frac{d^2 p_1}{dt^2} + a_{2\alpha}\frac{d^2 p_2}{dt^2} + \cdots + a_{n\alpha}\frac{d^2 p_n}{dt^2} + \cdots$$

One has, from the relations (3):

$$\begin{aligned}
\frac{dq_\alpha}{dt_1} &= \frac{dq_\alpha}{dt}\lambda, \\
\frac{d^2 q_\alpha}{dt_1^2} &= \frac{d^2 q_\alpha}{dt^2}\lambda^2 + \frac{dq_\alpha}{dt}\left(\frac{\partial\lambda}{\partial p_1}p_1' + \cdots + \frac{\partial\lambda}{\partial p_n}p_n'\right)\lambda, \\
\frac{dq_\alpha}{dt} &= \frac{\partial\varphi_\alpha}{\partial p_1}p_1' + \frac{\partial\varphi_\alpha}{\partial p_2}p_2' + \cdots + \frac{\partial\varphi_\alpha}{\partial p_n}p_n', \\
\frac{d^2 q_\alpha}{dt^2} &= \frac{\partial\varphi_\alpha}{\partial p_1}\frac{d^2 p_1}{dt^2} + \frac{\partial\varphi_\alpha}{\partial p_2}\frac{d^2 p_2}{dt^2} + \cdots + \frac{\partial\varphi_\alpha}{\partial p_n}\frac{d^2 p_n}{dt^2} + \cdots,
\end{aligned}$$

in which the unwritten terms in the last equation constitute a quadratic form in p_1', p_2', …, p_n'.

From that, one can find coefficients $R_\alpha^{(i)}$, which are n^2 in number ($i = 1, 2, \ldots, n$; $\alpha = 1, 2, \ldots, n$), depend upon only $p_1, p_2, \ldots, p_n$ in such a way that the expressions:

$$\frac{d}{dt_1}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} - \sum_{i=1}^{n} R_\alpha^{(i)}\left[\frac{d}{dt}\left(\frac{\partial S}{\partial p_i'}\right) - \frac{\partial S}{\partial p_i}\right] \tag{4}$$

will no longer contain second derivatives, and as a result, will reduce to quadratic forms Φ_α in p_1', p_2', …, p_n'. Indeed, let us annul the coefficient of $d^2 p_k / dt^2$ in the expression (4). We will then have:

$$\lambda^2 \left[b_{1\alpha} \frac{\partial \varphi_1}{\partial p_k} + b_{2\alpha} \frac{\partial \varphi_2}{\partial p_k} + \cdots + b_{n\alpha} \frac{\partial \varphi_n}{\partial p_k} \right] - \sum_{i=1}^{n} R_\alpha^{(i)} a_{ki} = 0 .$$

If one sets $k = 1, 2, \ldots, n$ then one will have n linear equations that define:

$$R_\alpha^{(1)}, \quad R_\alpha^{(2)}, \quad \ldots, \qquad R_\alpha^{(n)} .$$

The determinant of the coefficients of the unknowns is the discriminant of S is non-zero, as one knows. When the coefficients $R_\alpha^{(i)}$ have been calculated, the difference (4) will become a quadratic form Φ_α in p_1', p_2', …, p_n', and one will have the identities:

(5) $$\frac{d}{dt_1}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha + \sum_{i=1}^{n} R_\alpha^{(i)} \left[\frac{d}{dt}\left(\frac{\partial S}{\partial p_i'}\right) - \frac{\partial S}{\partial p_i} \right],$$

by virtue of which the equations of motion (1) will imply the equations of motion (2) when one sets:

(6) $$Q_\alpha = \Phi_\alpha + \sum_{i=1}^{n} R_\alpha^{(i)} P_i .$$

Since the choice of functions λ and φ_α is arbitrary, one can (and in an infinitude of ways) make any motion in the one system under the action of forces that depend upon positions and velocities correspond to an analogous motion in the other.

2. – Let us now specialize the problem and see whether we can make any motion of the first system under the action of forces that *depend upon only the position* of the system correspond to an analogous motion in the second. In order to do that, if the quantities P_α are arbitrary functions of the $p_1, p_2, \ldots, p_n$ then it will be necessary and sufficient that the quantities Q_α that are defined by equations (6) are functions of $q_1, q_2, \ldots, q_n$, and not the derivatives of those parameters. Therefore, it is necessary and sufficient that the transformation (3) must be specialized by the conditions:

(7) $$\Phi_1 = 0 , \qquad \Phi_2 = 0 , \qquad \ldots, \qquad \Phi_n = 0 .$$

Since we desire that those conditions must be fulfilled for all possible motions of the system under the actions of forces that depend upon only the position, it is necessary that equations (7) must be true for any $p'_1, p'_2, \ldots, p'_n$, since the initial conditions permit one to take those derivatives arbitrarily. The conditions (7), whose left-hand sides are quadratic forms in $p'_1, p'_2, \ldots, p'_n$, are then divided into a greater number of equations that define the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$, and λ as functions of $p_1, p_2, \ldots, p_n$. In general, those equations will be too numerous, in such a way that one can realize the transformation only if certain condition relations are true between the coefficients $a_{i,j}$ and $b_{i,j}$ of the two forms S and T.

When that particular transformation exists, while supposing that $P_1, P_2, \ldots, P_n$ are zero, one will find from (6), and by virtue of (7), that $Q_1, Q_2, \ldots, Q_n$ will also be zero. *Therefore, if the transformation exists then it must make a motion in the first system when no forces act it upon correspond to an analogous motion in the second system.* In a word, the transformation must preserve *geodesic* motions.

One then comes down to a question that was studied by **Beltrami**, **Lipschitz**, and **Dini** in their work on quadratic differential forms and by **S. Lie**.

That theorem, which was said to be probable as a result of a result by **Goursat** in an article "Sur l'homographie en mécanique" that was published in volume XII of the *American Journal of Mathematics*, was found to be proved, very briefly, in a note that we recently inserted in the *Bulletin de la Société mathématique de France* (t. XX, session on 16 March 1892). **Painlevé** proved the same theorem in some notes that are even more recent (*Comptes rendus des Séances de l'Académie des Sciences de Paris*, 11 April and 23 May 1892), along with several propositions that seem more worthy of attention to us, and which must be compared to several notes by **R. Liouville** (*Comptes rendus des Séances de l'Académie des Sciences de Paris*, 6 April 1891 and 23 May 1892).

It can happen that for a well-defined system, the only transformation that realizes the indicated conditions is the **Stäckel** transformation, accompanied by a transformation $dt = C\, dt_1$, in which C is a constant that is real or imaginary. However, others will exist for some special systems.

For example, for free material points, one can employ a homographic transformation [C. R. Acad. Sci. Paris **108** (1889)]. For a point that moves on a sphere, one can employ a transformation by central projection onto a plane (Am. J. Math., v. XIII). Finally, as **Dautheville** showed [C. R. Acad. Sci. Paris **11** (1890) and Annales de l'École Normale Supérieure **7** (1890)], one can transform the motion of a point on a surface of constant total curvature into a planar motion (which corresponds to a theorem of **Beltrami**), and more generally, one can transform the motion of a point on one surface into a motion of a point on another surface (that is not mappable) if the first surface satisfies the conditions that **Dini** found for their geodesics to correspond.

To conclude our summary of the research that has been done on the methods of transformation in mechanics, we cite the article of **Bertrand** on similitude in mechanics (J. de l' École Polytechnique, 1848), a note by **Goursat** (C. R. Acad. Sci. Paris **108**, pp. 446), a note by **Darboux** (*ibid.*, pp. 449), and a note on the interpretation of imaginary values of time that we published in volume 87 of *Comptes rendus*.

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