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CHAPTER XXIV

GENERAL EQUATIONS OF ANALYTICAL DYNAMICS

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440. Objective of the chapter. – In order to find the motion of a system without friction that has k degrees of freedom and is subject to given forces, one must integrate a system of k differential equations whose general form we indicated in the preceding chapter (nos. **433** and **434**).

In this chapter, we shall give some methods for writing the equations of motion that are more concise. These methods will differ according to whether the system is holonomic or not.

We shall first study holonomic systems, since they are simplest. We shall point out a form for the equations of motion of those systems that was first given by Lagrange. Let $q_1, q_2, \ldots, q_k$ be the coordinates of the holonomic system, and let $q_1', q_2', \ldots, q_k'$ be their derivatives with respect to time under the motion of the system. Following Lagrange, we show that we can write the equations of motion as soon as we know the expression for the *kinetic energy* or *energy of velocity:*

$$T = \tfrac{1}{2}\sum mv^2$$

as a function of $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k'$, and t.

We will then see that for a *non-holonomic* system, knowing the kinetic energy does not suffice to determine the equations of motion. Let $q_1, q_2, \ldots, q_k$ be the parameters whose arbitrary variations $\delta q_1, \delta q_2, \ldots, \delta q_k$ define the most general virtual displacement of the system, let $q_1', q_2', \ldots, q_k', q_1'', q_2'', \ldots, q_k''$ be their first and second derivatives with respect to time under the motion of the system, and let $\mathbf{J}$ be the acceleration of a point-mass m. We shall show that we can write the equations of motion as soon as we know the expression for the function:

$$S = \tfrac{1}{2}\sum m\mathbf{J}^2$$

as a function of $q_1, q_2, \ldots, q_k, q_1', q_2', \ldots, q_k', q_1'', q_2'', \ldots, q_k''$ and t. That function S, which is constructed from the accelerations in the same way that T is constructed from the velocities, can be called the *energy of acceleration of the system.*

I. – Holonomic systems. Lagrange equations.

441. Reduction of the equations of motion to the minimum in a system without friction. – As before, imagine that a system of n points is subject to constraints such that, from a geometric standpoint, the position of the system depends upon k geometrically-independent parameters $q_1, q_2, \ldots, q_k$. One can then express the coordinate of each point of the system as functions of those parameters. In the general case where the functions contain time, the expressions for the coordinates of the various points as functions of q_1, q_2, …, q_k will contain time t :

(1)
$$\begin{cases} x_\nu = \varphi_\nu(q_1, q_2, \ldots, q_k, t), \\ y_\nu = \psi_\nu(q_1, q_2, \ldots, q_k, t), \\ z_\nu = \varpi_\nu(q_1, q_2, \ldots, q_k, t). \end{cases}$$

When the constraints are expressed by equations such as equations (6) of no. **434**, those coordinate expressions will be, by hypothesis, such that when one substitutes them in the constraint equations, those equations will be satisfied identically for any q_1, q_2, …, q_k, t .

One will then obtain the most general virtual displacement of the system that is compatible with the constraints at the instant t by giving q_1, q_2, …, q_k arbitrarily infinitely-small increments δq_1, δq_2, …, δq_k , which will give:

$$\delta x_\nu = \frac{\partial x_\nu}{\partial q_1}\delta q_1 + \frac{\partial x_\nu}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_\nu}{\partial q_k}\delta q_k \,,$$

and two analogous formulas for δy_ν and δz_ν . When one substitutes these values in the general equation of dynamics (no. **431**), one will get an equation of the form:

(2)
$$(P_1 - Q_1)\,\delta q_1 + (P_2 - Q_2)\,\delta q_2 + \ldots + (P_k - Q_k)\,\delta q_k = 0,$$

in which one sets:

$$P_\alpha = \sum_\nu m_\nu \left(\frac{d^2 x_\nu}{dt^2}\frac{\partial x_\nu}{\partial q_\alpha} + \frac{d^2 y_\nu}{dt^2}\frac{\partial y_\nu}{\partial q_\alpha} + \frac{d^2 z_\nu}{dt^2}\frac{\partial z_\nu}{\partial q_\alpha} \right),$$

$$Q_\alpha = \sum_\nu \left(X_\nu \frac{\partial x_\nu}{\partial q_\alpha} + Y_\nu \frac{\partial y_\nu}{\partial q_\alpha} + Z_\nu \frac{\partial z_\nu}{\partial q_\alpha} \right).$$

Equation (2) must be true for any δq_1, δq_2, …, δq_k , since it must be true for any virtual displacement that is compatible with the constraints, so it will decompose into k equations:

(3)
$$P_1 - Q_1 = 0, \quad P_2 - Q_2 = 0, \quad \ldots, \quad P_k - Q_k = 0 .$$

The expressions for the quantities P_α transform as we did in the case of a material point (no. **282**).

Upon suppressing the indices ν for simplicity of notation, we can write:

$$P_\alpha = \frac{d}{dt}\sum m \left(\frac{dx}{dt}\frac{\partial x}{\partial q_\alpha} + \frac{dy}{dt}\frac{\partial y}{\partial q_\alpha} + \frac{dz}{dt}\frac{\partial z}{\partial q_\alpha} \right) - \sum m \left(\frac{dx}{dt}\frac{d\,\dfrac{\partial x}{\partial q_\alpha}}{dt} + \frac{dy}{dt}\frac{d\,\dfrac{\partial y}{\partial q_\alpha}}{dt} + \frac{dz}{dt}\frac{d\,\dfrac{\partial z}{\partial q_\alpha}}{dt} \right).$$

If we denote the derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ by x', y', z', resp., then the expression for P_α will become:

$$P_\alpha = \frac{d}{dt}\sum m\left(x'\frac{\partial x}{\partial q_\alpha} + y'\frac{\partial y}{\partial q_\alpha} + z'\frac{\partial z}{\partial q_\alpha}\right) - \sum m\left(x'\frac{d\frac{\partial x}{\partial q_\alpha}}{dt} + y'\frac{d\frac{\partial y}{\partial q_\alpha}}{dt} + z'\frac{d\frac{\partial z}{\partial q_\alpha}}{dt}\right).$$

If one likewise lets q_1', q_2', … , q_k' denote the derivatives of q_1, q_2, …, q_k, when they are considered to be functions of time, then:

$$x' = \frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2' + \cdots + \frac{\partial x}{\partial q_\alpha}q_\alpha' + \cdots + \frac{\partial x}{\partial q_k}q_k' + \frac{\partial x}{\partial t}.$$

Upon considering x' to be a function of the q, the q', and t, one will immediately see that:

$$\frac{\partial x'}{\partial q_\alpha'} = \frac{\partial x}{\partial q_\alpha};$$

one will likewise have:

$$\frac{\partial y'}{\partial q_\alpha'} = \frac{\partial y}{\partial q_\alpha}, \quad \frac{\partial z'}{\partial q_\alpha'} = \frac{\partial z}{\partial q_\alpha}.$$

The expression P_α will then become:

$$P_\alpha = \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q_\alpha'} + y'\frac{\partial y'}{\partial q_\alpha'} + z'\frac{\partial z'}{\partial q_\alpha'}\right) - \sum m\left(x'\frac{d\frac{\partial x}{\partial q_\alpha}}{dt} + y'\frac{d\frac{\partial y}{\partial q_\alpha}}{dt} + z'\frac{d\frac{\partial z}{\partial q_\alpha}}{dt}\right).$$

In order to transform the second parenthesis, we remark that one will have:

$$\frac{d\frac{\partial x}{\partial q_\alpha}}{dt} = \frac{\partial^2 x}{\partial q_\alpha\,\partial q_1}q_1' + \frac{\partial^2 x}{\partial q_\alpha\,\partial q_2}q_2' + \cdots + \frac{\partial^2 x}{\partial q_\alpha\,\partial q_k}q_k' + \frac{\partial^2 x}{\partial q_\alpha\,\partial t},$$

since $\partial x / \partial q_\alpha$ is a function of the variables q_1, q_2, …, q_k, t. One immediately verifies that this expression is identical to the derivative of x with respect to q_α:

$$\frac{d\frac{\partial x}{\partial q_\alpha}}{dt} = \frac{\partial x'}{\partial q_\alpha},$$

and one will similarly have:

$$\frac{d\dfrac{\partial y}{\partial q_\alpha}}{dt} = \frac{\partial y'}{\partial q_\alpha}, \qquad \frac{d\dfrac{\partial z}{\partial q_\alpha}}{dt} = \frac{\partial z'}{\partial q_\alpha}.$$

One will then have:

$$P_\alpha = \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q'_\alpha} + y'\frac{\partial y'}{\partial q'_\alpha} + z'\frac{\partial z'}{\partial q'_\alpha}\right) - \sum m\left(x'\frac{\partial x'}{\partial q_\alpha} + y'\frac{\partial y'}{\partial q_\alpha} + z'\frac{\partial z'}{\partial q_\alpha}\right).$$

Now let T be the total semi-*vis viva* of the system:

$$T = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2).$$

Upon considering T to be a function of $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k, t$, one will see that the sums that enter into the expression for P_α are $\dfrac{\partial T}{\partial q'_\alpha}$ and $\dfrac{\partial T}{\partial q_\alpha}$, respectively; one will then have:

$$P_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha},$$

in such a way that the equations of motion will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, 3, \ldots, k).$$

These are the Lagrange equations.

The function T has degree two with respect to $q'_1, q'_2, \ldots, q'_k$. Consequently, the preceding equations have order two; they give $q_1, q_2, \ldots, q_k$ as functions of time and $2k$ arbitrary constants. We remark that *in the case where the constraints are independent of time*, one can arrange that the expressions φ, ψ, ϖ that are obtained for the coordinates do not contain t explicitly. The function T will then be *homogeneous* and have degree two with respect to $q'_1, q'_2, \ldots, q'_k$. From its very definition, it is an essentially positive quantity, moreover. T will then be a positive-definite quadratic form in $q'_1, q'_2, \ldots, q'_k$.

In general, in order to calculate the Q_α, one only has to form the expression for the sum of the virtual works done by the given forces for the most general displacement that is compatible with the constraints at the instant t. As we just saw, that sum is $Q_1\,\delta q_1 + \ldots + Q_k\,\delta q_k$. If one wishes that Q_α should be determinate then it will suffice to consider the virtual displacement that is obtained by keeping t and all the q constant, except for q_α, which one varies by δq_α; the sum of the given forces will then be $Q_\alpha\,\delta q_\alpha$.

The quantities Q_α take a remarkable form when there is a force function for the given forces. That function $U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ can be expressed as a function of $q_1, q_2, \ldots, q_k, t$, and one will then have:

$$\frac{\partial U}{\partial q_\alpha} = \sum_\nu \left(\frac{\partial U}{\partial x_\nu}\frac{\partial x_\nu}{\partial q_\alpha} + \frac{\partial U}{\partial y_\nu}\frac{\partial y_\nu}{\partial q_\alpha} + \frac{\partial U}{\partial z_\nu}\frac{\partial z_\nu}{\partial q_\alpha} \right).$$

By hypothesis, the quantities X_ν, Y_ν, Z_ν are equal to $\frac{\partial U}{\partial x_\nu}$, $\frac{\partial U}{\partial y_\nu}$, $\frac{\partial U}{\partial z_\nu}$. One will then have:

$$\frac{\partial U}{\partial q_\alpha} = \sum_\nu \left(X_\nu \frac{\partial x_\nu}{\partial q_\alpha} + Y_\nu \frac{\partial y_\nu}{\partial q_\alpha} + Z_\nu \frac{\partial z_\nu}{\partial q_\alpha} \right) = Q_\alpha ,$$

and the Lagrange equations will take the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} = \frac{\partial U}{\partial q_\alpha} .$$

That same form will persist when X_ν, Y_ν, Z_ν are the partial derivatives with respect to x_ν, y_ν, z_ν of a function $U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t)$ that contains time explicitly. One will see that from the same calculation.

Remark. – The calculations that we did in order to find the expression for P_α in terms of the function T did not suppose that the parameters q were independent. They will persist when we introduce some new constraints that are expressed by the relations:

(4)
$$\begin{cases} g_1(q_1, q_2, \ldots, q_k, t) = 0, \\ g_2(q_1, q_2, \ldots, q_k, t) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_\mu(q_1, q_2, \ldots, q_k, t) = 0, \end{cases}$$

and the hypothesis of the independence of the parameters is, in fact, introduced only in order to deduce the formulas:

$$Q_1 - P_1 = 0, \quad Q_2 - P_2 = 0, \quad \ldots, \quad Q_k - P_k = 0,$$

from equation (2) .

The number m of the new conditions must obviously be less than k . The variations of the parameters are then coupled by the relations:

$$\frac{\partial g_1}{\partial q_1}\delta q_1 + \frac{\partial g_1}{\partial q_2}\delta q_2 + \cdots + \frac{\partial g_1}{\partial q_k}\delta q_k = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\partial g_\mu}{\partial q_1}\delta q_1 + \frac{\partial g_\mu}{\partial q_2}\delta q_2 + \cdots + \frac{\partial g_\mu}{\partial q_k}\delta q_k = 0,$$

which shows that $k - \mu$ of them are arbitrary. In order to express the idea that the equation:

$$(Q_1 - P_1)\,\delta q_1 + \ldots + (Q_k - P_k)\,\delta q_k = 0$$

is then satisfied for all of those $k - \mu$ arbitrary variations, we shall employ the method of undetermined multipliers, which will give us the equations of motion in the form:

$$P_\alpha = Q_\alpha + \lambda_1 \frac{\partial g_1}{\partial q_\alpha} + \lambda_2 \frac{\partial g_2}{\partial q_\alpha} + \cdots + \lambda_\mu \frac{\partial g_\mu}{\partial q_\alpha} \qquad (\alpha = 1, 2, \ldots, k),$$

or, upon replacing P_α with its value:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha + \lambda_1 \frac{\partial g_1}{\partial q_\alpha} + \lambda_2 \frac{\partial g_2}{\partial q_\alpha} + \cdots + \lambda_\mu \frac{\partial g_\mu}{\partial q_\alpha} \qquad (\alpha = 1, 2, \ldots, k)\,.$$

When those k equations are combined with the constraint equations, that will permit one to determine the $k + \mu$ unknowns:

$$q_1, q_2, \ldots, q_k, \lambda_1, \lambda_2, \ldots, \lambda_k$$

as functions of time.

442. First example:

Problem. – Find the motion of a system that consists of two heavy homogeneous bars AB, $A'B'$ that are linked by a massless string and have the same length, where the line AB is subject to rotating around its center O, and the entire system must remain in a fixed vertical plane.

That problem was treated in no. **366**, example V. Upon employing the notations that were employed before, one will have:

$$T = \frac{M}{2}(2k^2\,\varphi'^2 + l^2\theta'^2)\,.$$

Here, there is a force function that is given by:

$$U = Mg\,\xi = M\,gl\cos\theta,$$

when one lets ξ denote the height of the center of gravity O' of the bar $A'B'$. Indeed, the sum of the works done by the given forces will reduce to the work $Mg\,\delta\xi$ that is done by gravity.

The Lagrange equations that relate to the parameters φ and θ are then:

$$\frac{d}{dt}(2M\,k^2\,\varphi'^2) = 0, \qquad \frac{d}{dt}(2M\,l^2\theta') = -M\,gl\sin\theta\,.$$

Those are the same equations that were found directly.

443. Euler equations. – The Lagrange equations permit one to rapidly find the Euler equations for the motion of a solid body around a fixed point.

With the notations that were employed before (no. **383**), one will see that the position of the system depends upon three independent parameters ψ, θ, φ, and the semi-*vis viva* of the system T will have the expression:

$$T = \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2)\,,$$

where p, q, r have the following values as functions of ψ, θ, φ:

$$\begin{aligned} p &= \psi' \sin\theta \sin\varphi + \theta' \cos\varphi\,, \\ q &= \psi' \sin\theta \cos\varphi - \theta' \sin\varphi\,, \\ r &= \varphi' + \psi' \cos\theta\,. \end{aligned}$$

As for the expression for the sum of the works done by the given forces:

$$\sum (X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu),$$

it will take the form:

$$\Theta\,\delta\theta + \Phi\,\delta\varphi + \Psi\,\delta\psi\,.$$

Let us write the Lagrange equation that relates to the variable φ:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \varphi'}\right) - \frac{\partial T}{\partial \varphi} = \Phi\,.$$

However:

$$\frac{\partial T}{\partial \varphi'} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial \varphi'} = C\,r$$

and

$$\frac{\partial T}{\partial \varphi} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial \varphi} + \frac{\partial T}{\partial q}\frac{\partial q}{\partial \varphi} = Ap\frac{\partial p}{\partial \varphi} + Bq\frac{\partial q}{\partial \varphi}\,.$$

Now, from the values of p and q, one will have:

$$\frac{\partial p}{\partial \varphi} = \psi' \cos\varphi \sin\theta - \theta' \sin\varphi = q, \qquad \frac{\partial q}{\partial \varphi} = -\,p\,.$$

We will then have:

$$\frac{\partial T}{\partial \varphi} = pq\,(A - B),$$

and our equation will become:

$$C\frac{dr}{dt} + (B - A)\,pq = \Phi.$$

It remains for us to see whether Φ is the sum N of the moments of the given forces with respect to Oz. Indeed, $\Phi\ \delta\varphi$ is the sum of the virtual works done by the given forces under an elementary displacement that is obtained by leaving ψ and θ constant; i.e., under a motion that is a rotation $\delta\varphi$ around Oz. Now, we saw that if a body turns through an angle $\delta\varphi$ around Oz then the sum of the works done by the given forces (no. **181**) will be:

$$\Phi\ \delta\varphi = \sum (X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu) = \sum (x_\nu\,Y_\nu - y_\nu\,X_\nu)\ \delta\varphi,$$

so

$$\Phi = \sum (x_\nu\,Y_\nu - y_\nu\,X_\nu) = N\,.$$

We will then have one of the Euler equations:

$$C\frac{dr}{dt} + (B - A)\,pq = N\,,$$

but p, q, r play absolutely the same role in the question, and the equation above does not contain the angles ψ, θ, φ. By symmetry, it will then result that we can write down the other two equations:

$$A\frac{dp}{dt} + (C - B)\,qr = L\,,$$

$$B\frac{dr}{dt} + (A - C)\,rp = M\,.$$

One can deduce the Lagrange equations that relate to θ and ψ, moreover; however, the calculation would be more complicated than it was for the variable φ and pointless.

444. Examples of constraints that depend upon time. – In order to treat an example in which the constraints depend upon time, take the problem of no. **333**, namely, *an insect walking on a bar.*

The position of the system at time t depends upon one parameter, namely the angle θ. The constraints depend upon time, because the motion of the insect on the line is prescribed in advance. If one employs the same notations as in no. **333** then one will have:

$$2T = m\,k^2\,\theta'^2 + m\,(\rho'^2 + \rho^2\,\alpha'^2)$$

for the total *vis viva*, which is the sum of the *vis viva* of the bar and the insect.

The expressions for ρ and α show that:

$$\rho' = \frac{v^2 t}{\rho}, \qquad \alpha' = \theta' + \frac{v\sqrt{R^2 - a^2}}{\rho^2}\,.$$

Hence, upon substituting:

$$2T = m\,k^2\,\theta'^2 + \frac{mv^4t^2}{\rho^2} + m\left(\rho\,\theta'' + v\frac{\sqrt{R^2 - a^2}}{\rho}\right)^2,$$

where ρ is a function of only t :

$$\rho = \sqrt{R^2 - a^2 + v^2t^2}\,.$$

Since the work done by the forces other than the constraint forces (viz., weight) is zero, the right-hand sides of the Lagrange equations are *zero*. Presently, there is only one parameter θ; the single equation of motion will then be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'}\right) - \frac{\partial T}{\partial \theta} = 0,$$

or, since T does not contain θ, $\partial T / \partial \theta' =$ const.:

$$k^2\,\theta' + \rho'^2\,\theta' + v\sqrt{R^2 - a^2} = c\,.$$

The constant c must be determined by the initial conditions. From the particular initial conditions that were indicated in no. **333**, one must take $c = 0$, and one will then recover the equation that was obtained directly.

II. – APPLICATIONS OF THE LAGRANGE EQUATIONS.

445. *Vis viva* integral. – When the constraints are independent of time and realized without friction, the *vis viva* theorem is expressed by the equation:

$$dT = \sum (X\,dx + Y\,dy + Z\,dz),$$

in which only the elementary works done by the given forces are involved. In particular, if those forces are derived from a force function U then one will have the *vis viva* integral $T = U + h$. Those theorems are easy to recover by starting from the Lagrange equations.

When the constraints are independent of time, one can always choose the parameters $q_1, \ldots, q_k$ in such a fashion that the x, y, z are expressed as functions of those parameters without t entering into them explicitly. Under those conditions, one will have:

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \cdots + \frac{\partial x}{\partial q_k} dq_k\,, \quad \ldots,$$

$$\sum (X\,dx + Y\,dy + Z\,dz) = Q_1\,dq_1 + Q_2\,dq_2 + \ldots + Q_k\,dq_k\,.$$

The *vis viva* equation is then written:

$$\frac{d}{dt}T = Q_1\, q_1' + Q_2\, q_2' + \cdots + Q_k\, q_k' .$$

That equality, which is a consequence of d'Alembert's principle, must be a consequence of the Lagrange equations. One verifies that in the following manner: In the case that is presently being considered, T is a homogeneous polynomial that has degree two in the q'. Now calculate the Lagrange equations from the quantity $Q_1\, q_1' + Q_2\, q_2' + \cdots + Q_k\, q_k'$. One finds that:

$$Q_1\, q_1' + Q_2\, q_2' + \cdots + Q_k\, q_k' = q_1' \frac{d}{dt}\frac{\partial T}{\partial q_1'} + \cdots + q_k' \frac{d}{dt}\frac{\partial T}{\partial q_k'} - q_1' \frac{\partial T}{\partial q_1} - \cdots - q_k' \frac{\partial T}{\partial q_k}$$

$$= \frac{d}{dt}\left(q_1' \frac{\partial T}{\partial q_1'} + \cdots + q_k' \frac{\partial T}{\partial q_k'} \right) - q_1'' \frac{\partial T}{\partial q_1'} - \cdots - q_k'' \frac{\partial T}{\partial q_k'} - q_1' \frac{\partial T}{\partial q_1} - \cdots - q_k' \frac{\partial T}{\partial q_k} .$$

By virtue of Euler's theorem on homogeneous functions:

$$q_1' \frac{\partial T}{\partial q_1'} + q_2' \frac{\partial T}{\partial q_2'} + \cdots + q_k' \frac{\partial T}{\partial q_k'}$$

is equal to $2T$; on the other hand, since T does not contain t explicitly:

$$\frac{dT}{dt} = \frac{\partial T}{\partial q_1'} q_1'' + \cdots + \frac{\partial T}{\partial q_k'} q_k'' + \frac{\partial T}{\partial q_1} q_1' + \cdots + \frac{\partial T}{\partial q_k} q_k' .$$

From that:

$$Q_1\, q_1' + Q_2\, q_2' + \cdots + Q_k\, q_k' = \frac{d(2T)}{dt} - \frac{dT}{dt} = \frac{dT}{dt} ,$$

which is the *vis viva* equation.

That equation provides a first integral whenever $Q_1\, dq_1 + \ldots + Q_k\, dq_k$ is the exact total differential of a function U of $q_1, q_2, \ldots, q_k$. One will then have:

$$dT = dU, \qquad T = U + h .$$

As we saw, that fact presents itself when the given forces are derived from a force function:

$$U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) .$$

Since the *vis viva* integral is a consequence of the Lagrange equations, one can simplify their integration by replacing one of them, such as the most complicated one, with the *vis viva* integral.

That calculation supposes, in an essential way, that the work done by the reactions enters into the variation of the *vis viva*, because one cannot suppose then that x, y, z are expressed as functions of q_1, q_2, …, q_k by formulas that do not contain t.

446. Problem. – Two identical heavy homogeneous lines AB, $A'B'$ that articulate at the common extremity A slide without friction on a horizontal plane. One wishes to know the motion of that system. (*Licence*)

The position of the combination of the two bars depends upon four parameters; we define them to be:

1. The coordinates ξ, η of the center of gravity G, which is found at the midpoint of line CC' that joins the centers of the two bars.

2. The angle θ that the line GA makes with the x-axis.

3. The half-angle α between the two bars.

One easily convinces oneself that these four parameters suffice to define the system completely: Once one has located the center of gravity G, one draws the line GA, which is known from the angle θ, and measures out $GA = l \cos \alpha$, where the length of one of the lines is $2l$. If one constructs an angle α on either side of AG at A then one will have the positions of the two bars.

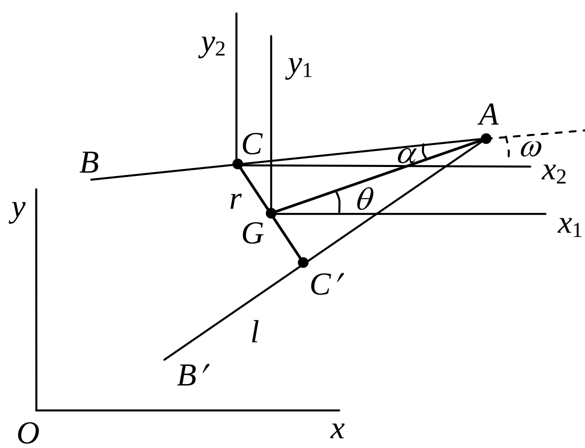

Figure 261.

We shall first look for the expression for the total *vis viva*. It is composed of the *vis viva* of the mass $2M$ that is concentrated at the center of gravity:

$$2M\,(\xi'^2 + \eta'^2),$$

in which M is the mass of one of the bars, and the *vis viva* under the motion of the system around the center of gravity. In order to get the *vis viva* of one of the bars (AB, for example) under the motion with respect to the axes $x_1\,y_1$, which are parallel to xy and pass through

G, we appeal to the same theorem as before; that *vis viva* is equal to that of the mass *M* if it were placed at *C* – namely, $M\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]$ – so it is:

$$M\,(l^2\,\alpha'^2\cos^2\alpha + l^2\,\theta'^2\sin^2\alpha),$$

since we have $GC = r = l\sin\alpha$, $x_1GC = \theta + \pi/2$, increased by the *vis viva* of *AB* under the rotation around *C* . Now, the latter *vis viva* will have the expression $Mk^2\left(\frac{d\omega}{dt}\right)^2 = Mk^2(\theta' - \alpha')^2$ when we note that the angle ω between the bar and Cx_2 is $\theta - \alpha$.

One calculates the *vis viva* of *AB′* in its motion around *G* by changing the sign of α, and upon adding that, one will get the total semi-*vis viva*:

$$T = M\,[\xi'^2 + \eta'^2 + (l^2\cos^2\alpha + k^2)\,\alpha'^2 + (l^2\sin^2\alpha + k^2)\,\theta'^2]\,.$$

In the present case, the only given forces will be the weights of the bars, and the works that they do will be zero. The force function will then be $U = 0$, and the right-hand sides of the Lagrange equations will be zero. If we write out the equation that relates to ξ then we will have $d\xi'/dt = 0$, so $\xi' = \xi'_0$. Similarly, the equation that relates to η will give $\eta' = \eta'_0$. The motion of the center of gravity is then uniform and rectilinear, which will immediately give the theorem of the motion of the center of gravity. The equation in θ:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial U}{\partial \theta}$$

will reduce to $\frac{d}{dt}\left(\frac{\partial T}{\partial \theta'}\right) = 0$, because neither *T* nor *U* contains θ. It is integrated immediately and gives $\frac{\partial T}{\partial \theta'}$ = const., or:

(I) $$(l^2\sin^2\alpha + k^2) = C\,.$$

We can likewise write the equation for α, but it would be too complicated. We replace it with the *vis viva* integral, which reduces to *T* = const. here; i.e.:

(II) $$(l^2\cos^2\alpha + k^2)\,\alpha'^2 + (l^2\sin^2\alpha + k^2)\,\theta'^2 = A^2,$$

since ξ' and η' are constant. We can equate that expression to an essentially-positive constant, since the left-hand side is a sum of two squares. Equation (I) shows that θ' has a variable sign: The line *GA* always turns in the same sense around *G*, since the angular velocity of that motion is, moreover, necessarily found between C/k^2 and $C/(l^2 + k^2)$. Upon substituting the preceding the value for θ' in equation (II), it will become:

$$\alpha'^2 (l^2 \cos^2 \alpha + k^2)(l^2 \sin^2 \alpha + k^2) = A^2 l^2 \sin^2\alpha + A^2 k^2 - C^2 ;$$

since the left-hand side is always positive, the same thing must always be true for the right-hand side.

If $C^2 - A^2 k^2$ were negative then α could obviously take any value that one desired, and the bars would move apart or approach each other according to whether α were positive or negative, resp., up to the moment that they collide ($\alpha = 0$ or $\alpha = \pi$). If $C^2 - A^2 k^2$ were positive then one could set it equal to $A^2 l^2 \sin^2\beta$, where β denotes a real constant. Indeed, $C^2 - A^2 k^2$ is always less than $A^2 l^2$, since the fact that α is real at the initial instant when $\alpha = \alpha_0$ will imply that one has $A^2 l^2 \sin^2\alpha_0 > C^2 - A^2 k^2$. The condition that α must satisfy will then reduce to:

$$\sin^2\alpha > \sin^2\beta ,$$

in such a way that α varies between β and $\pi - \beta$; the motion of each rod with respect to GA will then be oscillatory.

Finally, if one has $C^2 - A^2 k^2 = 0$ then α can take on all values, but when α tends to π or zero, t will increase indefinitely; the two lines will then tend to superpose, but never attain that state.

447. Heavy bodies of revolution rolling without slipping on a horizontal plane. – While using the notations of no. **407**, one will have the following expression for the *vis viva* $2T$:

$$2T = M(\xi'^2 + \eta'^2) + [M f'^2(\theta) + A]\,\theta'^2 + A\,\psi'^2 \sin^2\theta + C(\varphi' + \psi' \cos\theta)^2,$$

and for the force function:

$$U = -Mg\,\zeta = -Mg\,f(\theta) .$$

The five parameters upon which the position of the body depends are $\xi, \eta, \theta, \varphi, \psi$. One will get the equations of motion upon first writing out the four Lagrange equations that relate to the four parameters ξ, η, θ, ψ. Since neither T nor U contain those parameters, the corresponding Lagrange equations will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \xi'}\right) = 0, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \eta'}\right) = 0, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \varphi'}\right) = 0, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \psi'}\right) = 0 .$$

Hence, one concludes four first integrals immediately upon equating $\dfrac{\partial T}{\partial \xi'}, \dfrac{\partial T}{\partial \eta'}, \dfrac{\partial T}{\partial \varphi'}, \dfrac{\partial T}{\partial \psi'}$ to constants. One will then have the integrals:

$$\xi' = \xi'_0 , \qquad \eta' = \eta'_0 , \qquad \varphi' + \psi' \cos\theta = r_0 ,$$

$$A\psi' \sin^2 \theta + C\,(\varphi' + \psi' \cos\theta)\cos\theta = K\,.$$

It then remains for one to write out the last Lagrange equation, which relates to θ; however, one can replace it with the *vis viva* integral $T = U + h$.

One will obtain the equations that were established directly in no. **407** in that fashion.

448. Painlevé integral, which is analogous to that of *vis viva* in certain cases in which the constraints depend upon time. – One can construct an integral that is analogous to the *vis viva* integral in certain cases where the constraints depend upon time. Under that hypothesis, the expressions for x, y, z in terms of $q_1, q_2, \ldots, q_k$ will contain t : The semi-*vis viva* T will no longer be homogeneous in $q'_1, q'_2, \ldots, q'_k$; we can write it as $T = T_2 + T_1 + T_0$, T_2 , where T_2 denotes the set of terms that have degree two in $q'_1, q'_2, \ldots, q'_k$, T_1, the terms that have degree one with respect to those quantities, and T_0 denotes the terms that are independent of those quantities. By a calculation that is analogous to the preceding one (no. **445**), one will again have:

$$Q_1 q'_1 + \cdots + Q_k q'_k = \frac{d}{dt}\left(\sum q'_\alpha \frac{\partial T}{\partial q'_\alpha}\right) - \sum q''_\alpha \frac{\partial T}{\partial q'_\alpha} - \sum q'_\alpha \frac{\partial T}{\partial q_\alpha}\,.$$

However, on the one hand:

$$\sum q'_\alpha \frac{\partial T}{\partial q'_\alpha} = 2T_2 + T_1\,,$$

because T depends upon t *directly* and by the intermediary of the q_α , q'_α .

$$\frac{d}{dt}(T_2 - T_0) = Q_1 q'_1 + \cdots + Q_k q'_k - \frac{\partial T}{\partial t}\,.$$

If the quantity $Q_1 q'_1 + \cdots + Q_k q'_k - \frac{\partial T}{\partial t}$ is equal to $\frac{d}{dt} V(q_1, \ldots, q_k, t)$ then one will finally have:

$$T_2 - T_0 = V + h\,.$$

That is what happens, for example, when $\partial T / \partial t$ depends upon only t, and if $Q_1\,dq_1 + \ldots + Q_k\,dq_k$ is an exact total differential $dU(q_1, \ldots, q_k)$ of a function that does not contain t. In that case, the integral will be written:

$$T_2 - T_0 = U + F(t) + h\,,$$

in which $\partial T / \partial t$ is supposed to be equal to $F'(t)$. (*See* PAINLEVÉ, *Leçons sur l'intégration des équations de la Mécanique*, pp. 89 ; Hermann, 1895.)

III. – SMALL MOTIONS AROUND A POSITION OF STABLE EQUILIBRIUM.

449. Stability of equilibrium. – When the constraints on a holonomic system are independent of time and the given forces are derived from a force function, one knows (no. **173**) that the necessary and sufficient conditions for equilibrium are:

$$\frac{\partial U}{\partial q_1} = 0, \qquad \frac{\partial U}{\partial q_2} = 0, \qquad \ldots, \qquad \frac{\partial U}{\partial q_k} = 0,$$

in which $q_1, q_2, \ldots, q_k$ denote the k independent parameters that define the position of the system. Those equalities are the necessary, but not sufficient conditions for U to present a maximum or a minimum. *If U is a maximum in a position of the system then it will be a stable equilibrium position*. That theorem, which was already stated by Lagrange, was proved by Lejeune-Dirichlet in the following manner (Journal de Liouville):

We can always suppose that the values of the parameters that correspond to the equilibrium position are $q_1 = 0$, $q_2 = 0$, …, $q_k = 0$, and that U is zero for those values, because U is defined only up to a constant. Equilibrium is stable when perturbing the system from the equilibrium position in an arbitrary manner and giving the various points very small initial velocities will produce a motion in which the system deviates very little from that equilibrium position. More precisely, let ε be a positive number that is given in advance that is as small as one desires. One can find a positive number η that is small enough that when the initial values of the parameters $q_1, q_2, \ldots, q_k$, and the velocities of the various points are less than η in absolute value, the values of $q_1, q_2, \ldots, q_k$ will remain less than ε in absolute value during all the entire duration of the motion.

Having recalled that definition, assume that U is zero and a maximum for the values $q_1 = 0$, $q_2 = 0$, …, $q_k = 0$; one can show that the equilibrium is stable. Since U is a maximum, one can find a positive number ε that is small enough that for all systems of values $q_1, q_2, \ldots, q_k$ that are found between $-\varepsilon$ and $+\varepsilon$ or equal to those limits, the function U will be *negative*, except for just the combination $q_1 = q_2 = \ldots = q_k = 0$, which makes it zero. In particular, give the limiting values to one of the variables q_ν and then give the other ones $q_1, \ldots, q_{\nu-1}, q_{\nu+1}, \ldots, q_k$ all possible systems of values that are found between $\pm\varepsilon$ or equal to those limits. Let $-P_\nu$ be the largest value of U for those values of the parameters. P_ν is a non-zero positive number, because if q_ν is equal to $\pm\varepsilon$ then U cannot be zero for any values that might be given to the other parameters at the indicated limits. There will then exist k positive numbers $P_1, P_2, \ldots, P_k$ that are obtained by setting $q_1, q_2, \ldots, q_k$ equal to $\pm\varepsilon$, in succession. Call the smallest of them P; one will necessarily have:

$$U \leq -P, \qquad U + P \leq 0,$$

once one of the parameters becomes equal to $\pm\varepsilon$, while the other ones remain between $\pm\varepsilon$ or equal to those limits.

Having said that, perturb the system from its equilibrium position by giving the parameters values $q_1^0, q_2^0, \ldots, q_k^0$ that are found between $\pm\varepsilon$, and then assign initial velocities $v_1^0, v_2^0, \ldots, v_k^0$ to the various points. Upon applying the *vis viva* theorem to the motion that arises, we will have:

$$\sum \frac{mv^2}{2} = U + \sum \frac{mv_0^2}{2} - U_0 .$$

Since U_0 is negative, the quantity (…) will be positive. Moreover, it can be made as small as one pleases, because it is continuous and will be annulled when all of the initial velocities and all of the initial values of the parameters are zero. More precisely, one can determine a number η that is less than ε and small enough that when the values (…) and (…) are smaller than η in absolute value, one will have:

$$\sum \frac{mv_0^2}{2} - U_0 < P.$$

The *vis viva* equation will then give:

$$\sum \frac{mv^2}{2} < U + P.$$

Since the parameters $q_1, q_2, \ldots, q_k$ start with values that are found between $\pm\,\varepsilon$, none of the parameters can attain those limits during the motion, because as soon as one of them is attained, $U + P$ will become *negative,* and likewise the *vis viva* $\sum mv^2$, which is impossible.

Limiting velocities. – One can also assign the upper limits to the velocities during the entire duration of the motion. Indeed, since U is negative, because the parameters remain between $\pm\,\varepsilon$, one will have:

$$\sum mv^2 < 2P .$$

If one lets v_i denote the velocity of the point of mass m_i then one will have:

$$m_i\, v_i^2 < 2P, \quad v_i < \sqrt{\frac{2P}{m_i}}.$$

That limit is very small along with ε because P will go to zero when ε goes to zero.

One will obtain narrower limits for the velocities when one remarks that $\sum mv^2$ is a positive-definite quadratic form $2T$ in $q_1', q_2', \ldots, q_k'$: That form $2T$ will remain less than $2P$, so it will result that the absolute values of $q_1', q_2', \ldots, q_k'$ will remain less than a certain limit that one can determine in each particular case.

Remark I. – The proof supposes essentially that U depends upon *all* of the parameters $q_1, q_2, \ldots, q_k$. If U depends upon only some of them – q_1, q_2, q_3, for example – and is a maximum and zero for $q_1 = q_2 = q_3 = 0$ then the position that corresponds to $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, $q_4 = a_4, \ldots, q_k = a_k$, where $a_4, a_5, \ldots, a_k$ are arbitrary constants, will be an equilibrium position, but it will not be stable. Upon perturbing the system very slightly from that position and giving very small velocities to the points, one will have a motion in which q_1, q_2, q_3 will remain very close to zero, but the other parameters q_4, q_5, …, q_k will not remain close to a_4, a_5, …, a_k . Furthermore, the velocities will remain very small. For example, imagine a heavy body of revolution that is suspended by a point on its axis and take the notations of no. **395**. Presently, there is a force function that one can write:

$$U = - Mg\, z \cos \theta,$$

which depends upon only θ, whereas the position of the body depends upon the three Euler angles θ, φ, ψ. The function U is a maximum for $\theta = \pi$. The corresponding positions of the body, which are infinite in number, are equilibrium positions, which is obvious *a priori*, moreover, since the axis is vertical, and the center of gravity is above the suspension point. However, those conditions are not stable in the strict sense of the word, because when one imparts an initial rotation around the vertical to the body, no matter how small, one will get a motion in which the points become distant from their equilibrium positions by finite quantities.

Remark II – Converse of the Lejeune-Dirichlet theorem – Consider an equilibrium position of a system in which the derivatives $\frac{\partial U}{\partial q_1}$, $\frac{\partial U}{\partial q_2}$, …, $\frac{\partial U}{\partial q_k}$ are all zero *without U being a maximum*. It is *probable* that the correspond position will be *unstable*.

However, that proposition can be proved rigorously only under certain restrictions. [*See* LIAPOUNOFF, J. de Math. de Jourdan (1906); HADAMARD, paper presented to the Academy in 1896 and published in its Recueil in 1897; PAINLEVÉ, C. R. Acad. Sci., t. CXXV, pp. 1021; HAMEL, Math. Ann., Bd. LVII, pp. 541; L. SILLA, Rend. della R. Accad. Lincei (5) **17** (1908), pp. 347] – Some other authors, such as FEJÉR (Crelle, Bd. 131) and RÉTHY (*ibid.*, Bd. 133), have studied the stability in a resisting medium.

450. Small motions. – Imagine, as above, a system of constraints that are independent of time and whose position depends upon k geometric parameters q_1, q_2, …, q_k . Suppose that the applied forces are derived from a force function $U(q_1, q_2, …, q_k)$ that depends upon all of the variables q_ν and that that function will be a maximum and zero when all of the variables q_ν are annulled ($\nu = 1, 2, …, k$). The corresponding equilibrium position will be stable. We propose to study the small motions of the system around that position. q_1, q_2, …, q_k will remain very small under those motions, while the velocities also remain very small. Therefore, the derivatives q'_1, q'_2, …, q'_k will remain very small, because the *vis viva* is a homogeneous function of degree two in the derivatives q'_ν that is essentially positive. We shall commence with the simplest case in which the system has *complete constraints.*

1. *System with complete constraints.* – The position of the system will then depend upon a parameter q that is supposed to be *zero* in the equilibrium position; the number k =

1. The semi-*vis viva* T will then be a homogeneous function of degree two in q' that has the form:

$$T = q'^2 f(q) = q'^2 \left[f(0) + \frac{q}{1} f'(0) + \cdots \right],$$

in which one supposes that the function $f(q)$ can be developed into a Maclaurin series. Suppose that the first term in the development $f(0)$ is non-zero: That first term $f(0)$ is necessarily positive, because since q is very small, T will have the sign of $f(0)$, and a *vis viva* is essentially positive. Upon setting $f(0) = a$, we can write:

$$T = a\, q'^2 + T_1 ,$$

where T_1 is very small compared to the first term, because T_1 contains $q\, q'^2$ as a factor.

Now take the force function U: By hypothesis, it is a function of q that is zero and maximal for $q = 0$. Therefore, if one sets $U = F(q)$ and develops $F(q)$ by the Maclaurin formula then one will see that $F(0)$ and $F'(0)$ are zero, while $F''(0)$ is *negative*, in general. Upon setting $\frac{1}{2}F''(0) = -\alpha$, $\alpha > 0$, one can write:

$$U = -\alpha\, q^2 + U_1 ,$$

in which U_1 is the sum of the terms that follow in the Maclaurin development. U_1 is thus very small with respect to the term $-\alpha\, q^2$, because it contains q^3 as a factor.

In order to study the small oscillations, one can assume that one can neglect T_1 and U_1 and take:

$$T = \alpha\, q'^2, \qquad U = -\alpha\, q^2.$$

From Lagrange, since $\partial T / \partial q$ is zero, the equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = \frac{\partial U}{\partial q}$$

will become:

(1) $$a\, q'' = -\alpha\, q, \qquad q'' = -r^2 q$$

upon setting $\alpha / a = r^2$. The integral of that equation is:

$$q = \lambda \cos(rt + \rho),$$

in which λ and r denote two arbitrary constants that one can determine when one knows the initial position (i.e., q_0) and the initial velocity, namely, q'_0. The period of an oscillation of the system is $2\pi / r$. The constant r has a physical significance that is obviously independent of the choice of the parameter q.

The initial values of q and q' for $t = 0$ are a_1 and b_1, resp., so one will have:

$$q = a_1 \cos rt + \frac{b_1}{r} \sin rt .$$

If the initial values of q and q' in a second experiment are a_2 and b_2, resp., one will similarly have:

$$q = a_2 \cos rt + \frac{b_2}{r} \sin rt$$

for the motion.

Finally, if the initial values of q and q' in a third experiment are $a_1 + a_2$ and $b_1 + b_2$, resp., then the corresponding expression for q will be:

$$q = (a_1 + a_2) \cos rt + \frac{b_1 + b_2}{r} \sin rt ;$$

i.e., it will be the sum of the *preceding two:* That fact, which amounts to saying that the equation of motion is linear, constitutes what one calls the *superposition of small motions.*

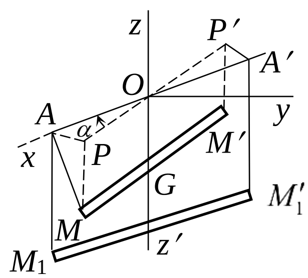

Figure 262.

Example. – Two massless strings AM, $A'M'$ of the same length l are attached to two fixed points A and A' that are located along the horizontal axis Ox at equal distances $OA = OA' = a$ from the origin O, and they support a homogeneous heavy bar MM' of length $2a$ that is equal to AA'. That bar is pierced at its midpoint G by an infinitely-small opening through passes the axis Oz, which is supposed to be vertical and pointing upwards. The system is perturbed very slightly from its equilibrium position M_1M_1' and left to itself with no initial velocity: We shall study the small oscillations.

Let θ denote the angle between the string AM and Oz' at an arbitrary initial instant, and let a denote the angle that the projection PP' of the bar MM' into the plane xOy makes with Ox. The isosceles triangle AOP and the rectangular triangle AMP give the relation:

$$l \sin \theta = 2a \sin \frac{\alpha}{2} .$$

The position of the system depends upon only one parameter θ, which is annulled in the equilibrium position. If the only given force is weight then when one lets z denote the ordinate $\overline{OG} = - l \cos \theta$ of the center of gravity G, one will have:

$$U = Mgl\,(\cos\,\theta - 1),$$

in which the constant is determined in such a fashion that U will be zero for $\theta = 0$. U is obviously maximal for that value. If one develops U with the Maclaurin formula then one will have:

$$U = -\,Mgl\,\frac{\theta^2}{2} + U_1\,,$$

in which U_1 has an order that is higher than that of θ by two. One calculates T using Koenig's theorem:

$$T = \tfrac{1}{2}M(\zeta'^2 + k^2\alpha'^2) = \tfrac{1}{2}M\left(l^2\theta'^2\sin^2\theta + \tfrac{1}{3}a^2\alpha'^2\right),$$

because the moment of inertia $M\,k^2$ of a homogeneous bar of length $2a$ with respect to its center is $\tfrac{1}{3}M\,a^2$. Now, the geometric relation above gives:

$$\alpha = 2\arcsin\left(\frac{l}{2a}\sin\theta\right),$$

so one can infer α' by derivation, and:

$$T = \tfrac{1}{2}M\left(l^2\sin^2\theta + \frac{4}{3}\frac{a^2\,l^2\cos^2\theta}{4a^2 - l^2\sin^2\theta}\right)\theta'^2.$$

The finite equation of motion will then be $T = U + h$, from the *vis viva* theorem. However, for infinitely-small oscillations, we must reduce the coefficient of θ'^2 in T to what it becomes for $\theta = 0$, and take:

$$T = \tfrac{1}{6}M\,l^2\,\theta'^2, \qquad U = -\tfrac{1}{2}Mgl\,\theta^2,$$

approximately.

From the Lagrange equation, the equation of motion is then:

$$\theta'' = -\frac{3g}{l}\theta\,.$$

The period of small oscillations is $2\pi\sqrt{\dfrac{l}{3g}}$.

Remark. – In the preceding theory, we supposed that $f(0)$ is non-zero, since T has the form $q'^2\,f(q)$. If that condition is not realized then one can realize it by a change of variable. Indeed, suppose that one has:

$$f(q) = q^n\,\varphi(q)$$

for small values of q, where $\varphi(0)$ is non-zero.

One then makes the substitution:

$$\frac{n+2}{2} q^{n/2} q' = s', \quad q = s^{\frac{2}{n+2}},$$

in which s denotes a new variable, and one will have:

$$T = q'^2 q^n \varphi(q) = \frac{4}{(n+2)^2} \varphi\left(s^{\frac{2}{n+2}}\right) s'^2,$$

in which the coefficient of s'^2 is no longer zero for $s = 0$.

We have likewise supposed that the since the coefficient of q'^2 in T is non-zero for $q = 0$, the development of $U(q)$ by the Maclaurin formula will begin with a term in q^2. However, it can happen that $U(q)$ is maximal for $q = 0$, with the derivatives of U up to an arbitrary odd order that is greater than 1 being annulled, and the first derivative that is not annulled has even order and is negative. For example, to take the simplest case, one can have:

$$U(q) = -\alpha q^4 + U_1,$$

in which U_1 contains q^5 as a factor, and α is positive. Upon reducing T to the form $a\,q'^2$ and neglecting U_1, one will have:

$$a\,q'' = -2\alpha\,q^2 \tag{2}$$

for the equation of motion.

One is then dealing with a situation that does not depend upon the choice of variable: *The period of small oscillations around the equilibrium position will vary with their amplitude.* Indeed, when one places the system in the position that corresponds to q_0 and releases it with zero velocity, upon integrating (2), one will have:

$$\left(\frac{dq}{dt}\right)^2 = \frac{\alpha}{a}(q_0^4 - q^4).$$

One can then infer t as a function of q by an elliptic quadrature; q oscillates from $-q_0$ to $+q_0$. One-fourth of an oscillation will have the duration:

$$\sqrt{\frac{\alpha}{a}} \int_0^{q_0} \frac{dq}{\sqrt{q_0^4 - q^4}} = \frac{1}{q_0} \sqrt{\frac{\alpha}{a}} \int_0^1 \frac{ds}{\sqrt{1-s^4}},$$

when one sets $q = s\,q_0$. That duration in inversely proportional to q_0 and will become infinitely large when q_0 tends to zero.

2. *Systems with two degrees of freedom*. – Imagine a system with constraints that are independent of time whose position depends upon two parameters q_1 and q_2. One will have:

$$T = A q_1'^2 + 2B q_1' q_2' + C q_2'^2,$$

in which A, B, C are functions of q_1 and q_2.

We suppose that the parameters are chosen in such a fashion that the discriminant $AC - B^2$ are non-zero for $q_1 = q_2 = 0$. Upon developing the coefficients A, B, C with the Maclaurin formula and letting a, b, c denote the values of the coefficients for $q_1 = q_2 = 0$, one will have:

$$T = A q_1'^2 + 2B q_1' q_2' + C q_2'^2 + T_1,$$

in which T_1 have order three with respect to q_1, q_2, (…), (…), and will be annulled when q_1 and q_2 are zero. When q_1 and q_2 are very small, T will then have the sign of the trinomial that is composed of the terms that precede T_1, and since T is essentially positive for any q_1' and q_2', one will have:

$$a > 0, \quad c > 0, \quad b^2 - 4ac < 0.$$

Now take the force function $U(q_1, q_2)$: Since that function is zero and maximal for $q_1 = q_2 = 0$, upon developing it with the Maclaurin formula, one will generally have:

$$U = -(\alpha q_1^2 + 2\beta q_1 q_2 + \gamma q_2^2) + U_1,$$

in which U_1 has order three in q_1 and q_2. Since U must be negative for sufficiently-small arbitrary values of q_1 and q_2, one will have:

$$\alpha < 0, \quad \gamma > 0, \quad \beta^2 - \alpha\gamma < 0,$$

in general.

In order to obtain the small motions around the equilibrium position, we shall neglect T_1 and U_1 and take:

$$T = \quad \alpha q_1'^2 + 2\beta q_1' q_2' + \gamma q_2'^2,$$
$$U = -(\alpha q_1^2 + 2\beta q_1 q_2 + \gamma q_2^2).$$

The two Lagrange equations then become:

$$\text{(3)} \qquad \begin{cases} \alpha q_1'' + b q_1'' = -(\alpha q_1 + \beta q_2), \\ b q_1'' + c q_1'' = -(\beta q_1 + \gamma q_2), \end{cases}$$

which are linear equations with constant coefficients. In order to integrate them, one sets:

$$\text{(4)} \qquad q_1 = \lambda_1 \cos(rt + \rho), \quad q_2 = \lambda_2 \cos(rt + \rho),$$

in which λ_1, λ_2, r, ρ denote constants. Upon substituting and dividing by $\cos(rt + \rho)$, one will have:

(5) $\lambda_1 (a\,r^2 - \alpha) + \lambda_2 (b\,r^2 - \beta) = 0, \qquad \lambda_1 (b\,r^2 - \beta) + \lambda_2 (c\,r^2 - \gamma) = 0,$

so, upon eliminating λ_1 and λ_2 :

(6) $(a\,r^2 - \alpha)(c\,r^2 - \gamma) - (b\,r^2 - \beta)^2 = 0,$

which is a quartic equation that gives two real and positive values to r^2. Indeed, when one substitutes the values 0 and $-\infty$ for r^2 in the left-hand side, one will get positive results, while the values α / a and γ / c will give negative results. One can always suppose that r is positive, because the solution (4) will not change when one changes the signs of r and ρ: One can then take r to be the two positive roots r_1 and r_2 of equation (6).

If one replaces r with one of those roots in equations (5) then they will reduce to one – the first one, for example. Upon setting $r = r_1$, one will then have:

$$\frac{\lambda_1}{b\,r_1^2 - \beta} = \frac{\lambda_2}{\alpha - a\,r_1^2} = \mu_1,$$

in which μ_1 denotes an arbitrary constant. One will then have the solution:

$$q_1 = \mu_1 (b\,r_1^2 - \beta) \cos(r_1\,t + \rho_1), \quad q_2 = \mu_1 (\alpha - a\,r_1^2) \cos(r_1\,t + \rho_1).$$

The second root r_2 gives an analogous solution, with some other constants μ_2 and ρ_2, and the general integrals of the equations of motion are finally:

(7) $$\begin{cases} q_1 = \mu_1 (b\,r_1^2 - \beta) \cos(r_1\,t + \rho_1) + \mu_2 (b\,r_2^2 - \beta) \cos(r_2\,t + \rho_2), \\ q_2 = \mu_1 (\alpha - a\,r_1^2) \cos(r_1\,t + \rho_1) + \mu_2 (\alpha - a\,r_2^2) \cos(r_2\,t + \rho_2), \end{cases}$$

with four arbitrary constants μ_1, μ_2, ρ_1, and ρ_2, which one will determine when one knows the initial values of q_1, q_2, and their derivatives (…) and (…).

One sees that the motion in the neighborhood of the equilibrium position is the resultant motion of two oscillations whose periods are $2\pi / r_1$ and $2\pi / r_2$, respectively. If those periods are mutually commensurable then there will exist a period for the motion, and otherwise the system would not pass through the same configuration at any point in time. One has already seen an example of that in no. **272**.

The quantities r_1 and r_2, which thus have a physical significance, are obviously independent of the choice of the parameters q_1 and q_2. They are invariants of the problem.

Particular case. – When we showed that the equation (6) in r^2 had two positive roots, we had assumed that if we were to substitute α / a and γ / c in the left-hand side then we would have at least one negative result. That would not happen if one had:

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = k^2,$$

in which k^2 is a positive constant. Equations (6) will then be:

$$(r^2 - k^2)^2 = 0 .$$

It will have equal roots: Nevertheless, the general integrals do not contain time except inside the sines and cosines. Indeed, the equations of motion (3) are then written:

$$a(q_1'' + k^2 q_1) + b(q_2'' + k^2 q_2) = 0,$$
$$b(q_1'' + k^2 q_1) + c(q_2'' + k^2 q_2) = 0,$$

and since $b^2 - ac$ is positive, they will give:

$$q_1'' + k^2 q_1 = 0, \quad q_2'' + k^2 q_2 = 0,$$

whose general integrals are:

$$q_1 = \mu_1 \cos(k\,t + \rho_1), \qquad q_2 = \mu_2 \cos(k\,t + \rho_1) .$$

There is only one period for each oscillation then, namely, $2\pi / k$.

Another method. – Those results can be obtained in a different way when one makes use of the properties of quadratic forms. Consider the two quadratic forms:

$$S = a\,q_1^2 + 2b\,q_1\,q_2 + c\,q_2^2, \quad U = -(\alpha\,q_1^2 + 2\beta\,q_1\,q_2 + \gamma\,q_2^2),$$

with the aid of which one can write the equations of motion in the form:

$$\frac{d^2}{dt^2}\left(\frac{\partial S}{\partial q_1}\right) = \frac{\partial U}{\partial q_1}, \quad \frac{d^2}{dt^2}\left(\frac{\partial S}{\partial q_2}\right) = \frac{\partial U}{\partial q_2}.$$

Make a change of variables that is linear with constant coefficients:

$$q_1 = k_1\,s_1 + h_1\,s_2 , \qquad q_2 = k_2\,s_1 + h_2\,s_2 ,$$

in which s_1 and s_2 are new parameters, and k_1, h_1, k_2, h_2 are constants.

One can determine the coefficients of the substitution in such a fashion that one simultaneously reduces the two forms to the sum of squares. When one regards q_1 and q_2 as Cartesian coordinates, that will amount to taking the axes to be the lines that are conjugate to both the pairs of lines $S = 0$, $U = 0$. One will then have:

$$S = s_1^2 + s_2^2, \quad U = -(r_1^2\,s_1^2 + r_2^2 s_2^2).$$

The semi-*vis viva* becomes $T = s_1'^2 + s_2'^2$, and the equations of motion become:

$$s_1'' = -r_1^2\, s_1, \quad s_2'' = -r_2^2 s_2\,,$$

so upon integrating them, one will have:

$$s_1 = \mu_1 \cos (r_1\, t + \rho_1), \qquad s_2 = \mu_2 \cos (r_2\, t + \rho_2)\,.$$

One can remark that the quartic equation in r^2 is obtained by equating the discriminant of the form $U + r^2\, S$ to zero.

The variables s_1 and s_2 are called the *principal* variables.

Application. – Imagine a heavy, homogeneous bar AB of length $2a$ that is suspended by a string of length l that is attached to a fixed point O. The system is capable of being displaced in a vertical plane xOy. One wishes to study its infinitely-small oscillations around the vertical, which is the equilibrium position.

The position of the system depends upon the two angles θ and φ that the vertical Ox makes with the directions of the string and the bar, resp., which are parameters that are indeed zero in the equilibrium position. There is a force function here:

$$U = Mg\, \xi + C,$$

in which ξ is the abscissa of the center of gravity G. In order to satisfy the conditions of the preceding theory, one must choose C in such a manner that U is annulled in the equilibrium position. The coordinates of the center of gravity have the expressions:

$$\xi = l \cos \theta + a \cos \varphi, \qquad \eta = l \sin \theta + a \sin \varphi,$$

in which the force function will be:

$$U = -\, Mg\, [l\, (\cos \theta - 1) + a\, (\cos \varphi - 1)]\,.$$

Now calculate the semi-*vis viva* T. The *vis viva* of the mass that is concentrated at G is:

$$M\, (\xi'^2 + \eta'^2) = M\, [l^2\, \theta'^2 + a^2\, \varphi'^2 + 2al\, \theta'\, \varphi' \cos (\theta - \varphi)]\,.$$

Since the *vis viva* of the rotation around the center of gravity is (…) $M^2\, \varphi'^2$, from the value of the moment of inertia of a homogeneous bar with respect to its center, the semi-*vis viva* will be:

$$T = \frac{M}{2}\Big[l^2\theta'^2 + \tfrac{4}{3}a^2\varphi'^2 + 2al\,\theta'\,\varphi'\cos(\theta - \varphi)\Big].$$

In order to find infinitely-small oscillations, it will suffice to consider the second-order terms in U and T:

$$U = -\frac{Mg}{2}(l\,\theta^2 + a\,\varphi^2),$$
$$T = \quad \frac{M}{2}(l^2\theta'^2 + \tfrac{4}{3}a^2\,\varphi'^2 + 2al\,\theta'\,\varphi').$$

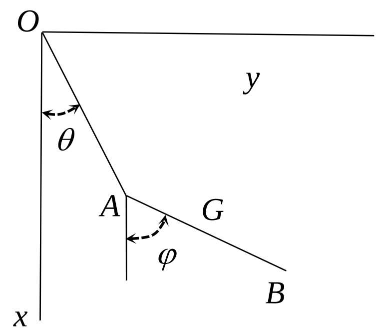

Figure 263.

The Lagrange equations are then:

$$l''^2\ \theta'' + a\ l\ \varphi'' = -\,g\ l\ \theta, \qquad \tfrac{4}{3}a^2\ \varphi + a\ l\ \theta'' = -\,g\ a\ \varphi\ .$$

One integrates those equations by setting:

$$q = \lambda_1 \cos{(r\ t + \rho)}, \qquad \varphi = \lambda_2 \cos{(r\ t + \rho)},$$

in which λ_1 and λ_2 must satisfy the conditions:

$$(l\ r^2 - g)\ \lambda_1 + a\ r^2\ \lambda_2 = 0, \qquad l\ r^2\ \lambda_1 + \ \lambda_2 = 0\ .$$

The equation for r^2 will then be:

$$(l\ r^2 - g)\ \left(\tfrac{4}{3}ar^2 - g\right) - a\ l\ r^4 = 0,$$

which an equation that has degree two in r^2 that gives two positive real roots r_1^2 and r_2^2, so one will have a system of particular solutions for each of them. Upon adding those solutions, one will get the general integrals:

$$\theta = a\,r_1^2\,\mu_1 \cos{(r_1\,t + \rho_1)} + a\,r_2^2\,\mu_2 \cos{(r_2\,t + \rho_2)},$$
$$T = (g - l\,r_1^2)\,\mu_1 \cos{(r_1\,t + \rho_1)} + (g - l\,r_2^2)\,\mu_2 \cos{(r_2\,t + \rho_2)},$$

with the four constants μ_1 , μ_2 , ρ_1 , ρ_2 .

For example, if one supposes that $a = \frac{3}{4} l$ then one will find values r_1^2 and r_2^2 for $\frac{2g}{l}(2+\sqrt{3})$ and $\frac{2g}{l}(2-\sqrt{3})$, resp., which will immediately give the periods $2\pi / r_1$, $2\pi / r_2$, respectively, for the two oscillations that comprise the small motion.

Remark. – In the preceding theory, we supposed that $ac - b^2$ that is non-zero. If that discriminant were zero then one would have to make another choice of parameters in such a manner that the new approximate expression for T would have a non-zero discriminant. For example, if one takes a point that moves in a plane that has the origin as a stable equilibrium position then one will have:

$$T = \frac{m}{2}(r'^2 + r^2\,\theta'^2)$$

in polar coordinates r and θ, which is an expression whose discriminant r^2 is annulled in the equilibrium position. Upon taking Cartesian coordinates, one will have the new expression:

$$T = \frac{m}{2}(x'^2 + y'^2),$$

whose discriminant is non-zero.

We have also supposed that the development of the force function U begins with second-order terms in q_1 and q_2. Since $U = 0$ is a maximum, it can happen that the development begins with terms of an arbitrary even order; for example, fourth order:

$$U = -(\alpha q_1^4 + \beta q_1^2 q_2 + \cdots + \delta q_2^4) + U_1 .$$

In that case, the study of small oscillations will become more complicated: The equations that are obtained by neglecting U_1 are not linear.

The general oscillation is no longer the resultant of two special oscillations that each have a well-defined period.

3. *General case*. – Imagine a system that is subject to constraints that are independent of time, and the forces that act upon that system are derived from a force function U. We suppose that it is in a stable equilibrium position in which the function U is a maximum. Let $q_1, q_2, \ldots, q_k$ be the parameters that define the position of the system. We assume that they are zero, as well as U, in the equilibrium position. Since the equilibrium is stable, when one displaces the system and then leaves it to itself, the parameters q and their derivatives will remain very small during all of the period of the motion. We consider $q_1, q_2, \ldots, q_k$ and their derivatives to be quantities that are small of first order. The total semi-*vis viva* will then be a homogeneous quadratic function of the q':

$$T = \sum A_{ij}\, q_i' q_j' \quad \begin{pmatrix} i = 1, 2, \ldots, k \\ j = 1, 2, \ldots, k \end{pmatrix}, \qquad A_{ij} = A_{ji}.$$

Each of the coefficients A_{ij} is a function of q that takes the value a_{ij} when the parameters q are annulled. One can then write:

$$T = \sum a_{ij}\, q'_i\, q'_j + T_1 \quad \begin{pmatrix} i = 1, 2, \ldots, k \\ j = 1, 2, \ldots, k \end{pmatrix}, \qquad a_{ij} = a_{ji},$$

in which the function T_1 is a sum of quantities that are small of order higher than 2.

On the other hand, we know that the value 0 is a maximum of U. We can then write:

$$U = - \sum b_{ij}\, q_i\, q_j + U_1 ,$$

in which U_1 has order higher than 2. We remark that since the lower-order terms give their signs to the expression for T and U, the sums that are composed of the second-order terms must remain constantly positive for T and U, because we have specified the sign of U.

Our approximation consists of neglecting the terms in U_1 and T_1 . The Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = \frac{\partial U}{\partial q_\nu}$$

are

$$a_{\nu 1}\, q''_1 + a_{\nu 2}\, q''_2 + \cdots + a_{\nu k}\, q''_k = -(b_{\nu 1}\, q_1 + \cdots + b_{\nu k}\, q_k) \qquad (\nu = 1, 2, \ldots, k)$$

here.

The k simultaneous differential equations that we obtain are linear of first order and have constant coefficients. We can then integrate them by setting:

$$q_1 = \lambda_1 \cos (r\, t + \rho), \quad \ldots, \qquad q_k = \lambda_k \cos (r\, t + \rho) . \tag{8}$$

In order for those values to satisfy the Lagrange equations, one must have that:

$$\lambda_1 (b_{11} - r^2\, a_{11}) + \lambda_2 (b_{12} - r^2\, a_{12}) + \ldots + \lambda_k (b_{1k} - r^2\, a_{1k}) = 0 ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_1 (b_{k1} - r^2\, a_{k1}) + \lambda_2 (b_{k2} - r^2\, a_{k2}) + \ldots + \lambda_k (b_{kk} - r^2\, a_{kk}) = 0 .$$

In order for the λ to be non-zero, it is necessary that the determinant of those homogeneous linear equations should be zero:

$$\begin{vmatrix} b_{11} - r^2 a_{11} & b_{12} - r^2 a_{12} & \cdots & b_{1k} - r^2 a_{1k} \\ \vdots & \cdots & \cdots & \vdots \\ b_{k1} - r^2 a_{k1} & b_{k2} - r^2 a_{k2} & \cdots & b_{kk} - r^2 a_{kk} \end{vmatrix} = 0. \tag{9}$$

When that equality is considered to be an equation in r^2, it will generally give k values for that unknown, and $2k$ values for r that are pair-wise equal and opposite in sign. However, one can always suppose that r is positive, because the solution (8) will not change when r and ρ change signs. Upon adopting one of those k values for r (r_ν, for example) one can determine all of the λ as functions of an arbitrary μ_ν, and one will get the system of solutions (1), which contains the arbitrary quantity ρ_ν, in addition to μ_ν. One then has k systems of particular solutions to the differential equations, and their sum will give the general solution, which will contain $2k$ arbitrary constants, as it must.

The general oscillation is then the resultant motion of k partial oscillations that have periods $2\pi / r_1$, $2\pi / r_2$, …, $2\pi / r_k$, respectively. The roots $r_1, r_2, \ldots, r_k$ are invariants: Their values are independent of the choice of parameters.

Since the equations are linear, if one has two systems of particular solutions to them $q_\nu = f_\nu(t)$ and $q_\nu = \varphi_\nu(t)$ then the functions $q_\nu = f_\nu(t) + \varphi_\nu(t)$ will once more be solutions. One then has what one calls the *superposition of small motions.*

Without invoking the theory of quadratic forms, one can prove that the roots of the equations in r are *real*. Indeed, if that equation admits an imaginary root $a + ib$ then it will admit the conjugate root $a - ib$: The corresponding values of the constants λ_ν will also be conjugate imaginaries. One will then find a system of real particular solutions for the q_ν that have the form:

$$q_\nu = (A_\nu + i\,B_\nu)\cos(a + i\,b)\,t + (A_\nu - i\,B_\nu)\cos(a - i\,b)\,t\,,$$

or, in real form:

$$q_\nu = A_\nu\,(e^{bt} + e^{-bt})\cos a\,t + B_\nu\,(e^{bt} - e^{-bt})\sin a\,t\,.$$

One will then have a motion of the system in which the variables q_ν and their derivatives will begin by being as small as one desires and conclude by becoming infinitely large with t, which contradicts the fact that the equilibrium is *stable*.

One sees by an analogous argument that if the equation in r has multiple roots then time will not appear outside of the sine and cosine, because expressions of the form:

$$\mu\,t\cos(r\,t + \rho)$$

will become infinitely large with t.

The theory of quadratic forms will lead to the same results. If one sets:

$$S = -\sum a_{ij}\,q_i\,q_j\,, \qquad U = -\sum b_{ij}\,q_i\,q_j$$

then one of those two quadratic forms S will be essentially positive, while the other one U will be essentially negative for all values of the variables and can become zero only if all of the variables are annulled. The equations of small motions can be written:

$$\frac{d^2}{dt^2}\left(\frac{\partial S}{\partial q_\nu}\right) = \frac{\partial U}{\partial q_\nu}\,,$$

and equation (9) that gives r^2 is obtained by equating the discriminant of $U + r^2 S$ to zero. One can always reduce the two quadratic form S and U into ones that are sums of squares:

$$S = s_1^2 + s_2^2 + \cdots + s_k^2, \qquad U = -(r_1^2 s_1^2 + r_2^2 s_2^2 + \cdots + r_k^2 s_k^2)$$

by a linear change of variables that substitutes new variables $s_1, s_2, \ldots, s_k$ for the old ones $q_1, q_2, \ldots, q_k$.

The semi-*vis viva* will then be:

$$T = s_1'^2 + s_2'^2 + \cdots + s_k'^2 .$$

The equations of small motions will become:

$$\frac{d^2}{dt^2}\left(\frac{\partial S}{\partial s_\nu}\right) = \frac{\partial U}{\partial s_\nu};$$

i.e.:

$$\frac{d^2 s_\nu}{dt^2} = -r_\nu^2 s_\nu, \qquad s_\nu = \mu_\nu \cos(r_\nu t + \rho_\nu).$$

One will then immediately have the equations of small motions in finite form, with $2k$ arbitrary constants μ_ν and ρ_ν . The variables $s_1, s_2, \ldots, s_k$ that one must choose in order to reduce T, S, and U to sums of squares are called the *principal variables.*

In conclusion, we shall cite three notes by BETH that appeared in Comptes rendus de l'Académie royale des Sciences d'Amsterdam (1910) and (1911) and an article by HORN in Crelle's Journal, Bd. 131, pp. 224.

Remark. – We have supposed that the determinant of the a_{ij}, which is the discriminant of the form S, is not zero. If it were zero then one would have to choose another system of parameters. We have also supposed that the development of U in powers of $q_1, q_2, \ldots, q_k$ begins with second-order terms. If that development began with terms of higher order (say, fourth or sixth) then the equations of small motions would no longer be linear.

451. Small motions perturbed by a periodic perturbing force. – Consider a system like the one whose small motions around a stable equilibrium position that corresponded to:

$$q_1 = q_2 = \ldots = q_k = 0$$

we just studied.

Suppose that the constitutive forces of the system are derived from the force function U, which is maximal and zero in equilibrium, and they are combined with very small perturbing forces during the motion that are functions of time and also generally of $q_1, q_2, \ldots, q_k$, and their derivatives.

Let X, Y, Z denote the forces that act upon the point of the system whose coordinates are x, y, z . From the general theory of Lagrange equations, if one sets:

$$R_\nu = \sum \left(X \frac{\partial x}{\partial q_\nu} + Y \frac{\partial y}{\partial q_\nu} + Z \frac{\partial z}{\partial q_\nu} \right)$$

then the equations of motion will become:

(10)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = \frac{\partial U}{\partial q_\nu} + R_\nu \qquad (\nu = 1, 2, \ldots, k) .$$

We suppose that T and U are reduced to the same quadratic forms as above.

Since the perturbing forces are independent of the ones that determine the equilibrium, they will not generally be annulled in the equilibrium position, and consequently, when the term R_ν is developed in powers of $q_1, q_2, \ldots, q_k$, and their derivatives, it will contain a term that is independent of those variables, and the terms that follow it can be considered to be negligibly-small quantities compared to it. The R_ν will then be functions of only time; we suppose that they are periodic.

The equations of motion (10), rather than being linear with vanishing right-hand sides, as before, will now have the periodic functions R_ν for their right-hand sides. Those functions can be developed in a sum of sines and cosines:

$$R_\nu = 2A_\nu \cos (at + \alpha) + 2B_\nu \cos (bt + \beta) + \ldots + 2L_\nu \cos (lt + \lambda),$$

in which A_ν, B_ν, …, a, b, …, α, β, … denote constants. We say that each term in R_ν represents a simple perturbing force, namely, the first one is a perturbing force of period $2\pi / a$, the second one is a force of period $2\pi / b$, etc.

Suppose, to simplify, that one has chosen $q_1, q_2, \ldots, q_k$ to be the principal variables. As we just saw, the approximate values of T and U will then be:

$$T = q_1'^2 + q_2'^2 + \cdots + q_k'^2, \qquad U = -(r_1^2 q_1^2 + r_2^2 q_2^2 + \cdots + r_k^2 q_k^2).$$

The equations of perturbed motion are then:

(11)
$$\begin{cases} q_\nu''^2 + r_\nu^2 q_\nu^2 = A_\nu \cos(at+\alpha) + B_\nu \cos(bt+\beta) + \cdots + L_\nu \cos(lt+\lambda) \\ \qquad (\nu = 1, 2, 3, \ldots, k). \end{cases}$$

The general integrals of those equations will take on a different analytical form according to whether one of the quantities a, b, …, l is equal to r_ν or not.

First suppose that none of the quantities a, b, …, l are not equal to one of the roots r_1, r_2, …, r_k : The general integrals of equations (11) are:

(12)
$$\begin{cases} q_\nu = \mu_\nu \cos(r_\nu t + \rho_\nu) + \dfrac{A_\nu}{r_\nu^2 - a^2}\cos(at+\alpha) + \dfrac{B_\nu}{r_\nu^2 - b^2}\cos(bt+\beta) + \cdots + \dfrac{L_\nu}{r_\nu^2 - b^2}\cos(lt+\lambda) \\ \qquad (\nu = 1, \ldots, k), \end{cases}$$

in which μ_ν and ρ_ν denote arbitrary constants. Therefore, in this case, the simple perturbing force will give rise to terms such as:

$$2\, A_\nu \cos(at + \alpha)$$

in R_ν, which will introduce a simple oscillation into the system:

$$\frac{A_\nu}{r_\nu^2 - a^2}\cos(at + \alpha)$$

whose period is that of the force and whose amplitude is independent of the initial conditions, which influence only μ_ν and ρ_ν. If a is close to r_ν (i.e., if the period $2\pi / a$ of the simple perturbing force is close to the period $2\pi / r_\nu$ of a natural oscillation of the system when left to itself) then the coefficient $\frac{A_\nu}{r_\nu^2 - a^2}$ will become a large number, and the amplitude of the oscillation that is introduced by that perturbing force will become considerable. That remark foreshadows what will happen when one of the quantities a, b, …, l is equal to one of the roots r_ν.

For example, suppose that a is equal to r_1, but different from r_2, r_3, …, r_k, and none of the quantities b, …, l is equal to one of the roots r_1, r_2, …, r_k. The general integrals of equations (11) for $r = 2, 3, \ldots, k$ will then keep the form (12) that was found before. However, the first equation:

$$\frac{d^2 q_1}{dt^2} = A_1 \cos(at + \alpha) + \ldots + L_1 \cos(lt + \lambda),$$

in which $a = r_1$, will have the integral:

$$q_1 = \mu_1 \cos(r_1 t + \rho_1) + \frac{A_1 t}{2r_1}\sin(r_1 t + \alpha) + \frac{B_1}{r_1^2 - b^2}\cos(bt + \beta) + \ldots + \frac{L_1}{r_1^2 - l^2}\cos(lt + \lambda)\,.$$

Time t will then appear as a factor in the terms in the integral that is produced by the perturbing force whose period $2\pi / a$ is equal to the period $2\pi / r_1$ of one of the natural oscillations of the system. Hence, *when the period of one of the perturbing forces tends to that of one of the simple proper oscillations of the system, the amplitude of the perturbation will become gradually larger. In the limit, the perturbation will agree with the corresponding simple oscillation, whose amplitude, which is proportional to t, will increase indefinitely, or at least exceed the limits within which the linear equations are sufficient as an approximation.*

That theorem explains a great number of phenomena, such as the way that a musical string will vibrate when the air vibrates in unison with it, but not otherwise, the selective absorption of light rays and heat by a medium that is capable of generating rays of the same wave length, etc.

One encounters another important application in the perturbations of the motion of locomotives. The mass of the machine that is carried by the supports forms a system that is subject to oscillations of a well-defined period τ. The perturbing forces that are produced by the inertia of the moving pieces (such as pistons and crankshafts) will give sums of projections or moments that have the period of a one turn of the wheel for their principal period. The corresponding perturbations must then pass through a maximum amplitude when the velocity of the locomotive is such that it will make one turn of the wheel during the period τ of an oscillation. (VICAIRE, C. R. Acad. Sci. Paris, t. CXII, pp. 82).

IV. – OSCILLATIONS AROUND A STABLE MOTION.

452. General method. – The Lagrange equations permit one to likewise study the small oscillation of a system around a stable motion. Upon following a method that is similar to the one that we have employed for the study of small oscillations around a stable equilibrium position, one will once more be led to integrate linear equations, but those equations will no longer have constant coefficients.

Let a system be given in which the constraints can depend upon time and whose position is defined by k parameters $q_1, q_2, \ldots, q_k$ that are geometrically independent. The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu \qquad (\nu = 1, 2, \ldots, k) .$$

Suppose that one has found a particular solution to those equations:

$$q_1 = f_1(t), \qquad q_2 = f_2(t), \qquad \ldots, \qquad q_k = f_k(t),$$

in which the integration constants have well-defined values. One will then have the particular motion that the system takes on when $q_1, q_2, \ldots, q_k$ take the values $f_1(0), f_2(0), \ldots, f_k(0)$ at the instant $t = 0$, and the derivatives $q'_1, q'_2, .., q'_k$ will take the values $f'_1(0)$, $f'_2(0), \ldots, f'_k(0)$. One says that the motion is stable when upon assigning *arbitrary* initial conditions to the system that are infinitely close to the preceding ones, the system takes on a motion that is infinitely-close to the particular motion that is being considered. One can recognize whether the motion in question is stable, and at the same time, find the infinitely-close motions by the following method: Replace the parameters $q_1, q_2, \ldots, q_k$ with new parameters $s_1, s_2, \ldots, s_k$ that are defined by the relations:

$$q_1 = f_1(t) + s_1, \qquad q_2 = f_2(t) + s_2, \ldots, \qquad q_k = f_k(t) + s_k .$$

According to Lagrange, the equations of motion will become:

$$(1) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial s'_\nu}\right) - \frac{\partial T}{\partial s_\nu} = S_\nu ,$$

in which T and S_ν are functions of $s_1, s_2, \ldots, s_k$ and $s'_1, s'_2, \ldots, s'_k$.

With this new choice of parameters, the particular motion whose stability was just studied will be:

$$s_1 = 0,\ s_2 = 0,\ \ldots,\ s_k = 0\ .$$

One will obtain it by supposing that the parameters s_ν and their derivatives s'_ν have *zero* values at time $t = 0$. The problem comes down to see whether giving initial values to those parameters and their derivatives that are *arbitrary, but infinitely close* will produce an infinitely-close motion; i.e., a motion under which the quantities $s_1, s_2, \ldots, s_k$ and $s'_1, s'_2, \ldots, s'_k$ will remain infinitely small.

Suppose that this is the case and assume that $T, S_1, S_2, \ldots, S_k$ can be developed in increasing positive powers of $s_1, s_2, \ldots, s_k$ and $s'_1, s'_2, \ldots, s'_k$, keep only terms in both sides of the equations that have order one with respect to those quantities and $s''_1, s''_2, \ldots, s''_k$. Since the equations thus-obtained are verified, by hypothesis, for:

$$s_1 = 0,\ s_2 = 0,\ \ldots,\ s_k = 0,$$

they will be homogeneous and linear with respect to the unknowns s_ν and their first and second derivatives.

453. Example. – Consider a point of unit mass that is attracted to a fixed center O in proportion to the n^{th} power of the distance:

$$F = -\mu r^n, \qquad \mu > 0\ .$$

When one calls the polar coordinates r and θ and applies the Lagrange equations, the equations of motion will be:

$$r'' - r\,\theta'^2 = -\mu r^n, \qquad (\ldots)\,(r^2\,\theta') = 0\ . \tag{2}$$

They admit the particular solution:

$$r = r_0, \quad \theta' = \sqrt{\mu r_0^{\prime n-1}}, \quad \theta = \sqrt{\mu r_0^{\prime n-1}}\,t, \tag{3}$$

for which the trajectory will be a circle with its center at O that is traversed with constant velocity. Let us see whether that particular motion is stable. In order to do that, set:

$$r = r_0 + \varepsilon, \quad \theta = \sqrt{\mu r_0^{\prime n-1}}\,t + \eta, \tag{4}$$

and see whether ε and η will remain very small when one has supposed that ε, η, and their derivatives ε', η' were very small to begin with. Under that hypothesis, regard ε, η, and their derivatives as quantities that are small of order one and neglect their squares and products. Upon substituting the values (4) in the equations of motion (2) and letting ω denote the constant quantity $\sqrt{\mu r_0^{n-1}}$, we will have:

(5) $\varepsilon'' - \omega^2\,\varepsilon - 2r_0\,\omega\,\eta' = -\,n\,\omega^2\,\varepsilon\,, \qquad r_0\,\eta'' + 2\omega\,\varepsilon' = 0\,.$

The left-hand side of the first equation is the term in ε in the development of $\mu\,(r_0 + \varepsilon)^n$. The second of those equations can be integrated and will give:

(6) $r_0\,\eta'' + 2\omega\,\varepsilon' = a\,\omega,$

in which a denotes a very small arbitrary constant, since ε and μ' are very small for $t = 0$. If one eliminates η' from (5) and (6) then one will get:

$$\varepsilon'' + (n + 3)\,\omega^2\varepsilon = -\,2\,a\,\omega^2,$$

which is a linear equation with constant coefficients. If $(n + 3)$ is negative or zero then the general integral of that equations will contain exponentials or algebraic terms that increase indefinitely with t, and the circular motion considered will not be stable. Therefore, suppose that $(n + 3)$ is positive. One will then have:

$$\varepsilon = b\cos(\omega t\sqrt{n+3}+\alpha)+\frac{2a}{n+3},$$

in which b and α are arbitrary constants, the first of which is very small. Therefore, ε will remain very small, and as a result $r = r_0 + \varepsilon$ will remain close to r_0. Now take equation (6): Upon replacing ε with the value that was just found for it and integrating, one will have:

(7) $$r_0\,\eta = -\frac{2b}{\sqrt{n+3}}\sin(\omega t\sqrt{n+3}+\alpha)+\frac{n-1}{n+3}\,a\,\omega t + c,$$

in which c is a very small constant. One sees that μ contains a term in t. Therefore, μ will increase indefinitely with t, and as a result, the circular motion will not be stable. There is an exception for $n = 1$, because the term in t will then disappear. If n is not equal to 1 then in order for η to remain very small, it would be necessary to choose the initial conditions in such a fashion that *it is zero*. That condition means that under the perturbed motion, the area constant must be equal to ωr_0^2 *as in the circular motion.* Indeed, if we write out the area integral for the perturbed motion:

$$(r_0 + \varepsilon)^2\,(\omega + \eta') = C$$

then that integral will give:

$$r_0\,\eta' + 2\varepsilon\omega = \frac{C-\omega r_0^2}{r_0}$$

when we neglect ε^2 and $\varepsilon\,\eta'$, which is an equation that is identical to (6). In order for a to be zero, it will then be necessary that $C = \omega r_0^2$. In summary, the circular motion is not stable, except for the case of $n = 1$. If $(n + 3)$ is positive then it will become stable when one modifies the initial conditions very slightly in such a manner that the area constant *remains the same*.

The scope of this book does not permit us to elaborate upon that question of stable motions, moreover. We shall refer to Routh's *Mechanics* (Advanced Part, Chap. III) for a deeper study of that.

V. – APPLICATION OF THE LAGRANGE EQUATIONS TO RELATIVE MOTION.

454. First method, independent of the theory of relative motion. – In order to find the relative motion of a system with respect to axes *Oxyz* that are animated with a known motion, it will suffice to apply the Lagrange equations of absolute motion by choosing the parameters to be the variables $q_1, q_2, \ldots, q_k$, which define the position of the system with respect to the moving axes: Those same parameters obviously define the position of the system with respect the fixed axes $O_0\, x_0\, y_0\, z_0$, because the axes *Oxyz* have a known motion.

The absolute semi-*vis viva* T_a of the system will be a function of $q_1, q_2, \ldots, q_k$, q_1', q_2', …, q_k', and possibly t. On the other hand, if one imparts a virtual displacement on the system that is compatible with the constraints that exist at the instant t, so the displacement is obtained by keeping t constant and giving arbitrary infinitely-small increments $\delta q_1, \delta q_2, \ldots, \delta q_k$ to $q_1, q_2, \ldots, q_k$, resp., then the sum of the works done by the applied forces, besides the constraint forces, will have the expression $Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k$. The equations of motion will then be:

$$\frac{d}{dt}\left(\frac{\partial T_a}{\partial q_\nu'}\right) - \frac{\partial T_a}{\partial q_\nu} = Q_\nu \quad (\nu = 1, 2, \ldots, k) .$$

If the given forces are derived from a force function U then the quantity Q_ν will be equal to $\partial U / \partial q_\nu$.

In order to calculate T_a, it is not necessary to form the expression for the absolute coordinates as functions of the $q_1, q_2, \ldots, q_k$, and t. The absolute velocity $\mathbf{v}_a$ of m is the resultant of its relative velocity $\mathbf{v}_r$ with respect to the axes *Oxyz* and its guiding velocity $\mathbf{v}_c$, which is due to the motion of those axes. The projections of the velocity $\mathbf{v}_r$ onto *Oxyz* are x', y', z', where x, y, z are the coordinates of m, and the primes denote the derivatives with respect to t. As for the guiding velocity $\mathbf{v}_e$, it is the velocity that the point m would possess if it were fixed in the moving axis. It will then be the resultant of a velocity that is due to a translation $\mathbf{V}^0$, which is equal and parallel to the velocity of the point O, and a velocity that is due to a rotation $\boldsymbol{\omega}$ around an axis that passes through O. Upon calling the projections of $\mathbf{V}^0$ onto the moving axes V_x^0, V_y^0, V_z^0, and letting p, q, r denote the projections of $\boldsymbol{\omega}$, one will have that the projections of the guiding velocity $\mathbf{v}_c$ onto the three axes *Oxyz* (no. **51**) are $V_x^0 + q\,x - r\,y, \ldots$ The projections of the absolute velocity $\mathbf{v}_a$ of the point m onto those axes will then be $x' + V_x^0 + q\,x - r\,y, \ldots$, and one will have:

$$T_a = \tfrac{1}{2}\sum m\,[(x' + V_x^0 + q\,x - r\,y)^2 + (y' + V_y^0 + r\,x - p\,z)^2 + (z' + V_z^0 + p\,y - q\,x)^2] .$$

That expression will permit one to calculate T_a as a function of $q_1, q_2, \ldots, q_k$, q_1' , q_2' , …, q_k' , and t, because the coordinates x, y, z of the different points are functions of $q_1, q_2, \ldots, q_k$, q_1' , q_2', …, q_k' , and possibly t, while V_x^0 , V_y^0 , V_z^0 , p, q, r are known functions of time.

455. Example. – Consider a fixed vertical axis Oy and a plane P that passes through that axis and turns around it with a constant angular velocity ω . Find the motion of a heavy, homogeneous bar that moves without friction in that plane.

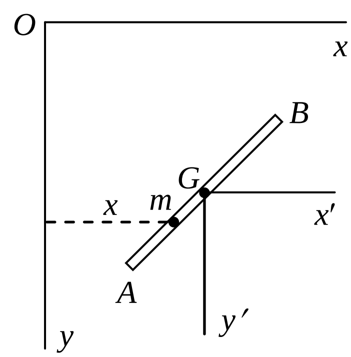

Figure 264.

That comes down to finding the relative motion of the bar with respect to the axes Ox and Oy, which are drawn in the moving plane P. The position of the bar with respect to those axes is defined by independent parameters: viz., the coordinates ξ, η of the center of gravity G and the angle θ that the bar GA makes with the parallel Gx_1 to Ox . The absolute velocity $\mathbf{v}_a$ of a point m on the bar is the resultant of its relative velocity $\mathbf{v}_r$, which is located in the plane xOy, and its guiding velocity $\mathbf{v}_c$. The latter velocity is the one that the mass m would possess if it were fixed in the moving plane: It will then be equal to $\omega\, x$ and perpendicular to the plane xOy, where x is the abscissa of the point m . The relative velocity and the guiding velocity are then rectangular, and one will have:

$$\mathbf{v}_a^2 = \mathbf{v}_r^2 + \mathbf{v}_c^2 ;$$

the absolute semi-*vis viva* of T_a will then be:

$$T_a = \tfrac{1}{2}\sum m\,\mathbf{v}_a^2 = \tfrac{1}{2}\left(\sum m\,\mathbf{v}_r^2 + \sum m\,\mathbf{v}_c^2\right).$$

Let us calculate those two terms separately. The relative motion of the bar is the motion of a bar in the plane xOy : From Koenig's theorem, its *vis viva* under that motion will be:

$$\sum m\,\mathbf{v}_r^2 = M\,(\xi'^2 + \mu'^2 + k^2\,\theta'^2) ,$$

in which $M k^2$ is the moment of inertia of the bar with respect to its center G. On the other hand, $\sum m \mathbf{v}_c^2$ is equal to $\omega^2 \sum m x^2$. The sum $\sum m x^2$ is the moment of inertia with respect to Oy, which is equal to the moment of inertia $\sum m x_1^2$ with respect to the parallel axis Gy_1, plus the product of the total mass with the square of the distance from the axes Oy and Gy_1, namely, $M \xi^2$. If one lets r denote the distance mG then the distance x_1 from a point m to the axis Gy_1 will be $x_1 = \pm r \cos \theta$, and the sum $\sum m x_1^2$ will be $\cos^2\theta \sum m r^2$ or $Mk^2 \cos^2 \theta$; hence:

$$\sum m x_1^2 = M \omega^2 (k^2 \cos^2 \theta + \xi^2) .$$

Finally, the absolute semi-*vis viva* is then:

$$T_a = \tfrac{1}{2} M (\xi'^2 + \eta'^2 + k^2 \theta'^2 + \omega^2 k^2 \cos^2 \theta + \omega^2 \xi^2) .$$

The only given force is the weight Mg that is applied at G, so there will exist a force function $U = Mg \eta$. Upon suppressing the factor M and successively applying the Lagrange equations to the parameters ξ, η, θ, the three equations of motion will then be:

$$\frac{d}{dt}(\xi') - \omega^2 \xi = 0, \qquad \frac{d}{dt}(\eta') = g , \qquad \frac{d}{dt}(k^2\theta') + k^2 \omega^2 \sin \theta \cos \theta = 0,$$

which are equations that give ξ, η, θ as functions of t. One first has:

$$x = A e^{\omega t} + B e^{-\omega t}, \qquad h = \tfrac{1}{2} g t^2 + C t + D,$$

which are equations that give the relative motion of a point G. The third equation will then give θ as a function of t : That is the equation that one encountered in a problem that was treated before in no **366**.

It should be pointed out that T_a is not homogeneous in ξ', η', θ' here: That amounts to the fact that the constraints that were imposed upon the system depend upon time; namely, the bar slides on a plane that is animated with a known motion.

Remark. – In the preceding, we supposed that the bar was free to move in a plane that turned. Suppose that its two extremities A and B are subject to sliding on the axes Ox and Oy as in the problem in no. **420**; ξ, η, θ will no longer be independent then. Upon letting $2l$ denote the length of the bar:

$$\xi = l \cos \theta, \qquad \eta = l \sin \theta, \qquad k^2 = \tfrac{1}{3} l^2.$$

One must express T_a and U as functions of the single independent parameter θ, and upon replacing ξ and η in the values that were found above with their present expressions, that will give:

$$T_a = \tfrac{2}{3} M l^2 (\theta'^2 + \omega^2 \cos^2 \theta), \qquad U = M gl \sin \theta,$$

and the equation of motion will be:

$$\frac{d}{dt}\left(\tfrac{4}{3}l^2\theta'\right)+\tfrac{4}{3}\,\omega^2\,l^2\sin\theta\cos\theta=gl\cos\theta,$$

as we found in a different way (no. **420**).

456. Second method, inferred from the theory of relative motion. – Suppose that one would like to find the relative motion of a system with respect some axes *Oxyz* that are animated with a known motion. The position of the system with respect to those axes depends upon certain geometrically-independent parameters $q_1, q_2, \ldots, q_k$. On the other hand, the system is subjected to given forces, and if one imparts a virtual displacement on the system that is compatible with the constraints by varying the parameters $\delta q_1, \delta q_2, \ldots, \delta q_k$ then the sum of the elementary works done by those forces will have the form:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k\,.$$

One can regard the moving axes as fixed on the condition that one must add the centrifugal force and the composite centrifugal force to the forces that really act upon each force m ; let:

$$R_1\,\delta q_1 + R_2\,\delta q_2 + \ldots + R_k\,\delta q_k$$

be the sum of the virtual works done by those fictitious forces for a displacement $\delta q_1, \delta q_2, \ldots, \delta q_k$.

One then applies the Lagrange equations of motion of the system with respect to the axes *Oxyz*, when they are regarded as fixed. In order to do that, one forms the semi-*vis viva* T_r of the system in its motion with respect to those axes. It will be a function of $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$, and possibly t. The equations of motion will be:

$$\frac{d}{dt}\left(\frac{\partial T_r}{\partial q'_\nu}\right)-\frac{\partial T_r}{\partial q_\nu}=Q_\nu+R_\nu \qquad (\nu=1, 2, \ldots, k)\,.$$

One will effortlessly apply that method to the examples that were treated before in the theory of relative motion.

457. Gilbert's mixed method. – While appealing, in part, to the theory of relative motion, Gilbert employed the following method ["Application de la méthode de Lagrange à divers problèmes de mouvement relatif," Annales de la Société scientifique de Bruxelles (1883)].

As before, one seeks the motion of a system with respect to the axes *Oxyz*, which are animated with a known motion. The position of the system with respect to those axes is supposed to depend upon k parameters $q_1, q_2, \ldots, q_k$, and the sum of the virtual works done by the applied forces for a displacement $\delta q_1, \delta q_2, \ldots, \delta q_k$ is again supposed to equal:

$$Q_1\ \delta q_1 + Q_2\ \delta q_2 + \ldots + Q_k\ \delta q_k .$$

Draw auxiliary axes $Ox_1y_1z_1$ through the moving origin O that are parallel to the fixed axes $O\ x_0\ y_0\ z_0$.

One can consider the axes $Ox_1y_1z_1$ to be fixed on the condition that one must add only the centrifugal forces to the forces that are actually applied, because the axes $Ox_1y_1z_1$ are animated with a translational motion (no. **416**). If we let **J** denote the acceleration of the moving origin O then the centrifugal force that must be applied to each point is $-\ m$ **J**. Let J_x, J_y, J_z denote the projections of **J** onto the axes $Oxyz$: the projections of $-\ m$ **J** onto those axes will be:

$$-\ m\ J_x\ ,\ -\ m\ J_y\ ,\ -\ m\ J_z\ ,$$

and for a virtual displacement that is imparted upon the system, the sum of the works done by the centrifugal forces is:

$$-\sum m\ (J_x\ \delta x + J_y\ \delta y + J_z\ \delta z)\ ,$$

where the sum is extended over all points. The quantities J_x, J_y, J_z are known functions of t. Upon setting:

$$K = -\sum m\ (x\ J_x + y\ J_y + z\ J_z) = -\ M\ (\xi\ J_x + \ \eta\ J_y + \ \zeta\ J_z)\ ,$$

one will see that the sum of the virtual works done by the centrifugal forces is δK . We can write the function K in a different way by introducing the total mass M of the system and the coordinates ξ, η, ζ of the center of gravity G with respect to the axes $Oxyz$. We will then see that:

$$K = -\ M\ \mathbf{J}\ \overrightarrow{OG} = -\ M\ J \cdot OG\ \cos \cdot JOG\ .$$

Thanks to the introduction of those centrifugal forces, one can regard the axes $Ox_1y_1z_1$ as fixed and apply the Lagrange equations to the motion with respect to those axes, which then becomes an absolute motion. Let T denote the semi-*vis viva* of the system under that motion with respect to the axes $O\ x_1\ y_1\ z_1$; the equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu + \frac{\partial K}{\partial q_\nu}\ .$$

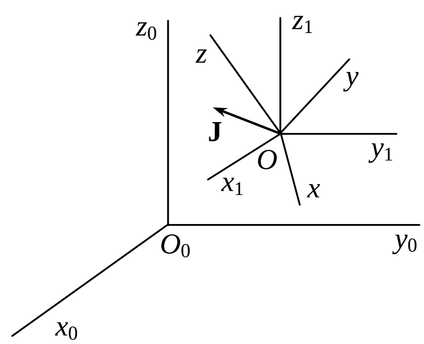

Figure 265.

The term $\partial K / \partial q_\nu$ provides the centrifugal forces. The virtual work done by those forces as a function of the variables $q_1, q_2, \ldots, q_k$, which is equal to δK, will become:

$$\frac{\partial K}{\partial q_1}\delta q_1 + \frac{\partial K}{\partial q_2}\delta q_2 + \cdots + \frac{\partial K}{\partial q_k}\delta q_k .$$

If the given forces are derived from a force function U :

$$Q_\nu = \frac{\partial K}{\partial q_\nu}$$

then the right-hand side will be:

$$\frac{\partial (U+K)}{\partial q_\nu} .$$

Calculating T. – The velocity $\mathbf{v}_1$ of a point m with respect to the axes $O\,x_1\,y_1\,z_1$, which are regarded as fixed, is the resultant of its relative velocity $\mathbf{v}_r$ with respect to the axes $Oxyz$ and its guiding velocity $\mathbf{v}'_c$ due to those axes.

The velocity $\mathbf{v}_r$ will have projections onto $Oxyz$ that are the derivatives x', y', z'. The velocity $\mathbf{v}'_c$ will have projections onto the same axes that are $qz - ry$, $rx - pz$, $py - qx$, because under the motion of the trihedron $Oxyz$ with respect to $O\,x_1\,y_1\,z_1$, since the origin O is fixed, p, q, r will denote the components of the instantaneous rotation $\boldsymbol{\omega}$ of the moving trihedron $Oxyz$, as above.

One will then have:

$$T = \tfrac{1}{2}\sum m\,[(x' + qz - ry)^2 + (y' + rx - pz)^2 + (z' + py - qx)^2],$$

which one can write:

$$T = T_r + \mathcal{G} + \mathcal{V},$$

when one sets:

$$T_r = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2),$$
$$\mathcal{G} = \tfrac{1}{2}\sum m\,[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2],$$
$$\mathcal{V} = \tfrac{1}{2}\sum m\,[x'(qz - ry) + y'(rx - pz) + z'(py - qx)] .$$

The quantity T_r is the semi-*vis viva* of the system under its relative motion with respect to the axes $Oxyz$. It is expressed directly by means of the variables q_ν and their derivatives q'_ν .

The quantity $\mathcal{G}$ represents the semi-*vis viva* of the system that is due to its guiding rotation around the instantaneous axis $\overrightarrow{O\omega}$ of the trihedron $Oxyz$; it will then have the expression:

$$\tfrac{1}{2} H\,\omega^2,$$

in which H is the moment of inertia at the instant t of the material system with respect to the axis $\overrightarrow{O\omega}$, and ω is the magnitude of the guiding rotation.

Finally, the value of $\mathcal{V}$ can be written:

$$\mathcal{V} = p \sum m\,(y\,z' - z\,y') + q \sum m\,(z\,x' - x\,z') + r \sum m\,(x\,y' - y\,x')\;.$$

The vector $\overrightarrow{O\sigma}$, whose projections onto the moving axes are:

$$\sum m\,(y\,z' - z\,y')\,, \qquad \sum m\,(z\,x' - x\,z')\,, \qquad \sum m\,(x\,y' - y\,x')\,,$$

is the resultant kinetic moment relative to the various points with respect to the point O; one will then have:

$$\mathcal{V} = \overrightarrow{O\sigma} \cdot \overrightarrow{O\omega} = \omega\sigma \cos\,\omega\sigma$$

immediately.

The advantage of those geometric forms that are given to the quantities K, T_r, $\mathcal{G}$, $\mathcal{V}$ consists of the fact that in each particular problem, they provide expressions for those quantities as functions of the q_ν and the q'_ν directly, without one having to pass to them by way of coordinate transformations.

458. Application to the relative motion of a heavy system with respect to the Earth by taking into account the motion of the Earth. – Imagine a heavy system S that is subject to given constraints at a point O of the terrestrial surface. We propose to study its relative motion with respect to the axes $Oxyz$, which are fixed in the Earth and carried by it in its rotational motion around the polar line PP'. If, following Gilbert's method, we draw the axes $O\,x_1\,y_1\,z_1$ through O, which are directions that are fixed in space, then the motion of the trihedron $Oxyz$ with respect to those axes will be a rotation ϖ that is equal to that of the Earth, which takes place around an axis $O\omega$ that is parallel to the South-North direction $P'P$.

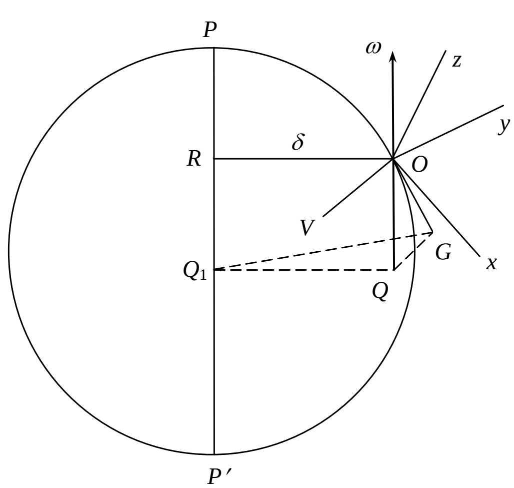

Figure 266.

The quantities T_r, $\mathcal{G}$, $\mathcal{V}$ are calculated as we explained above; in particular, $\mathcal{G}$ is equal to $\frac{1}{2} H\,\omega^2$, where H is the moment of inertia at the instant t of the material system S with respect to the axis $O\omega$. On the other hand, we calculate U, which is a force function for the forces that are actually applied (viz., attraction to the Earth), and K. We know that the weight mg of an arbitrary point of the system S is the resultant of the attraction and the centrifugal force $\mathbf{\Phi}$ of magnitude $m\omega^2\rho$ (no. **424**). Upon adopting Gilbert's viewpoint, *we regard the acceleration* **g** *as constant in magnitude and direction with respect to the Earth over the entire extent of the system S,* whose dimensions are supposed to be very small. The constant direction of **g** is the descending vertical – or nadir – OV at the point O. The forces that are actually applied are the attractions **A** of the Earth at the various points m of the system S. Now, since m **g** is the geometric sum of **A** and $\mathbf{\Phi}$, **A** will be the geometric difference of m **g** and $\mathbf{\Phi}$. For an arbitrary displacement that is imparted to the point m, the work done by **A** will be the difference between the work done by m **g** and the work done by $\mathbf{\Phi}$, so finally, the force function U from which the forces **A** that are actually applied is derived will be the difference between the force function from which the weight is derived and the one from which the force $\mathbf{\Phi}$ is derived. The height of the center of gravity G above the horizontal plane of the point O is $\overline{OG}\cos GOV$, so the weight will be derived from the force function $Mg\,\overline{OG}\cos GOV$, where M is the total mass of the system.

The force $\mathbf{\Phi}$ is normal to the terrestrial axis PP', while ρ shall denote the distance from the point m to that axis. The elementary work done by the force $\mathbf{\Phi}$ is:

$$m\,\omega^2\,\rho\,d\rho = d\,\frac{m\,\omega^2\rho^2}{2}.$$

The sum of the forces is then derived from the force function:

$$\tfrac{1}{2}\sum m\,\omega^2\rho^2 = \tfrac{1}{2}H_1\,\omega^2,$$

in which H_1 denotes the moment of inertia of the system (viz., $\sum m\,\rho^2$) with respect to the axis PP' of the Earth. Hence, the function U, which is the difference between the preceding two, will be:

$$U = Mg\,\overline{OG}\cos GOV - \tfrac{1}{2}H_1\,\omega^2.$$

However, we can calculate H_1, which is the moment of inertia with respect to PP', as a function of H, which is the moment of inertia with respect to the parallel $O\omega$ to PP'. Indeed, from a known theorem, if one lets d_1 and d denote the distances GQ_1 and GQ, resp., from the center of gravity to the parallel axes PP' and $O\omega$, resp., then one will have (no. **317**):

$$H_1 - H = M(d_1^2 - d^2).$$

On the other hand, in the triangle GQQ_1, when one lets δ denote the distance QQ_1, which is obviously equal to the distance OR from the point O to the Earth's axis, one will have:

$$d_1^2 - d^2 = \delta^2 - 2d\delta\cos GQQ_1 .$$

The quantity $d\cos GQQ_1$ is the projection of QG onto QQ_1. It will also be the projection of OG onto QQ_1 then, as well as onto its parallel OR ; i.e., $OG\cos GQQ_1$. Hence:

$$H_1 = H + M(\delta^2 - 2d\delta\cos GQQ_1) .$$

From that:

$$U = Mg\overline{OG}\cos GOV - \tfrac{1}{2}H_1\omega^2 + M\omega^2\delta\overline{OG}\,GOR - \tfrac{1}{2}\omega^2 M\delta^2 .$$

In order to evaluate K, observe that the origin O of the comparison system $Oxyz$ describes a circle of radius δ around PP' with an angular velocity ω due to the rotation of the planet. The acceleration **J** then has a magnitude of $\omega^2\,\delta$, and it points from O to R. Hence, from the general value of K, namely, $-M\omega^2\delta\overline{OG}\,GOR$, one will have:

$$K = -M\omega^2\delta\overline{OG}\,GOR .$$

Finally, one will have:

$$U + K = Mg\overline{OG}\cos GOV - \tfrac{1}{2}H_1\omega^2 - \tfrac{1}{2}\omega^2 M\delta^2 ,$$

in which the last term is a *constant* that will disappear under differentiation. Furthermore, one will have:

$$T = T_r + \mathcal{G} + \mathcal{V} = T_r + \mathcal{V} + \tfrac{1}{2}H_1\omega^2 ,$$

from the value of $\mathcal{G}$. If one lets $q_1, q_2, \ldots, q_k$ denote the parameters that define the position of the heavy system with respect to the axes $Oxyz$ the equations of relative motion will be:

$$(a) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = \frac{\partial (U+K)}{\partial q_\nu} \quad (\nu = 1, 2, \ldots, k) .$$

If one replaces T and $U + K$ with their values then that will produce some important reductions once more. First, $\mathcal{G} = \tfrac{1}{2}H\omega^2$ depends upon only the current positions of the points of the system, and not upon their velocities. That quantity will not contain $q'_1, q'_2, \ldots, q'_k$ then, and $\partial\mathcal{G}/\partial q'_\nu$ will be zero. As a result, the term $-\partial\mathcal{G}/\partial q_\nu$ in the left-hand side of (a) will be equal to the term $-\dfrac{1}{2}\dfrac{\partial H\omega^2}{\partial q_\nu}$ in the right-hand side. What will then remain is the equation:

$$(b) \qquad \frac{d}{dt}\frac{\partial(T_r+\mathcal{V})}{\partial q'_\nu}-\frac{\partial(T_r+\mathcal{V})}{\partial q_\nu}=Mg\,\frac{\partial(\overline{OG}\cos GOV)}{\partial q_\nu} \qquad (\nu = 1, 2, \ldots, h)\,.$$

These are the definitive equations of relative motion of a heavy system on the surface of the Earth. In order to write them out, one sees that it will suffice to calculate T_r, $\mathcal{V}$, and $\overline{OG}\cos GOV$.

459. Example. – *A heavy body of revolution is suspended by a point on its axis OZ. Moreover, that axis is subject to remain in a plane that is fixed with respect to the Earth. Describe the motion of the solid body with respect to terrestrial objects, while taking into account the rotational motion of the Earth.*

Let *OXYZ* be the principal axes of inertia of a solid body, which are carried with it, and let *Oxyz* be axes that are fixed in the Earth, and with respect to which one seeks the motion. We choose the *xy*-plane to be the plane in which *OZ* moves and the *Ox* axis to be the projection onto that plane of a parallel $O\omega$ to the representative vector of the terrestrial rotation. $O\omega$ is parallel to the South-North direction of the axis of the world. We make *Oz* point to the same side as $O\omega$ with respect to the plane *xOy* . The center of gravity *G* is supposed to be on the positive part *OZ* of the axis of revolution at a distance of $OG = l$ from the fixed point.

The position of the solid body with respect to the axes *Oxyz* depends upon two parameters: for example, the Euler angles φ and ψ that the axes *XYZ* make with *xyz*. The angle θ is equal to $\pi / 2$ here, because $zOZ = \pi / 2$.

Let us calculate T_r and $\mathcal{V}$. The quantity T_r is the semi-*vis viva* of the solid body with respect to the axes *Oxyz* . The motion of the solid body with respect to those axes is the motion of a solid body around a fixed point. If we then let *P*, *Q*, *R* denote the components along the principal axes *OXYZ* of the instantaneous rotation $\mathbf{\Omega}$ of the body with respect to the axes *Oxyz* then, from the general formulas (no. **382**), we will have:

$$\sin\theta = 1, \qquad P = \psi'\sin\varphi, \qquad Q = \psi'\cos\varphi, \qquad R = \varphi',$$

$$2T_r = A\,(P^2 + Q^2) + CR^2 = A\,\psi'^2 + C\,\varphi'^2.$$

We likewise calculate $\mathcal{V}$. Upon letting σ denote the magnitude of the resultant moment $\boldsymbol{\sigma}$ with respect to *O* of the relative quantities of motion with respect to the axes *Oxyz*, we will have:

$$\mathcal{V} = \omega\,\sigma\cos\,\omega,\boldsymbol{\sigma}\,.$$

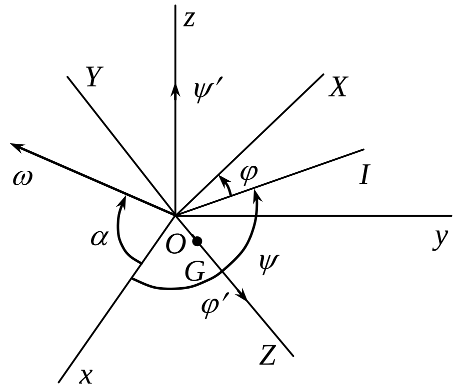

Figure 267.

Let the *OI* be the intersection of the plane *XOY* and the plane *xOy* ; one will have:

$$xOI = \psi, \qquad IOX = \varphi .$$

The vector $\boldsymbol{\sigma}$ will have components along the axes *OXYZ* that equal *AP*, *AQ*, *CR*, resp. Its projection onto *OI* will be $A\,(P\cos\varphi - Q\sin\varphi)$ – i.e., *zero*, from the values of *P*, *Q*, *R* – and its projection onto *OZ* will be $A\,(P\cos\varphi + Q\sin\varphi)$ – i.e., $A\,\psi'$.

The vector $\boldsymbol{\sigma}$ is the sum of a vector whose measure is $A\psi'$ that is carried by *Oz* and a vector whose measure is *CR* or $C\varphi'$ that is carried by *OZ*. Upon letting α denote the constant angle ωOx, its projection onto $O\omega$ will then be:

$$\sigma\cos\,\omega,\sigma = A\,\psi'\sin\alpha + C\,\varphi'\cos\alpha\sin\psi ,$$

and $\mathcal{V}$ is equal to the product of that quantity with ω.

The quantity $T_r + \mathcal{V}$, which we shall call Θ to abbreviate, will then have the expression:

$$\Theta = T_r + \mathcal{V} = \tfrac{1}{2}(A\,\psi'^2 + C\,\varphi'^2) + \omega\,(A\,\psi'\sin\alpha + C\,\varphi'\cos\alpha\sin\psi) .$$

On the other hand, let *a*, *b*, *c* denote the cosines of the constant angles that the nadir *OV* makes with the axes *Oxyz*, resp., and note that the coordinate *z* of the center of gravity *G* is zero, because that point is on the axis of revolution *OZ*, so it will be in the plane *xOy*. We will then have that the projection of *OG* onto the nadir is:

$$\overline{OG}\cos GOV = a\,\xi + b\,\eta = l\,(a\sin\psi - b\cos\psi),$$

because in the plane *xOy*, the axis *OGZ* makes an angle of $\psi - \pi/2$ with *Ox*, and $\overline{OG}$ is called *l*. The coordinates ξ and η are then $l\sin\psi$ and $-\,l\cos\psi$. From (*b*), if one remarks that neither Θ nor $\overline{OG}\cos GOV$ contains φ then the two equations of motion will be:

(1) $$\frac{d}{dt}\left(\frac{\partial\Theta}{\partial\varphi'}\right) = 0, \qquad \frac{d}{dt}\left(\frac{\partial\Theta}{\partial\psi'}\right) - \frac{\partial\Theta}{\partial\psi} = M\,gl\,(a\cos\psi + b\sin\psi)\,.$$

One can obtain two first integrals of those equations. One will first have immediately that $\partial\Theta / \partial\varphi' =$ const.; i.e.:

(2) $$\varphi' + \omega\cos\alpha\sin\psi = k\,.$$

One can then form the combination of equations (1) that gives Painlevé's generalized *vis viva* integral (no. **448**) upon multiplying the first one by φ' and the second one by ψ' and adding them. One will obtain a relation that one can write:

(3) $$\left\{\begin{aligned} &\frac{d}{dt}\left(\varphi'\frac{\partial\Theta}{\partial\varphi'} + \psi'\frac{\partial\Theta}{\partial\psi'}\right) - \left(\varphi''\frac{\partial\Theta}{\partial\varphi'} + \psi''\frac{\partial\Theta}{\partial\psi'} + \varphi\frac{\partial\Theta}{\partial\varphi} + \psi\frac{\partial\Theta}{\partial\psi}\right) \\ &\quad = Mgl\,(a\cos\psi + b\sin\psi)\,\psi'. \end{aligned}\right.$$

Since Θ does not contain t, the last of the terms on the left-hand side will be $d\Theta / dt$. On the other hand, upon separating the terms Θ_2 of degree two in φ' and ψ' and the terms of degree one Θ_1 in Θ, one will have:

$$\Theta = \Theta_2 + \Theta_1\,, \qquad \varphi'\frac{\partial\Theta}{\partial\varphi'} + \psi'\frac{\partial\Theta}{\partial\psi'} = 2\Theta_2 + \Theta_1\,,$$

and equation (3) will be written:

$$\frac{d}{dt}(2\Theta_2 + \Theta_1) - \frac{d}{dt}(\Theta_2 + \Theta_1) = M\,gl\,(a\cos\psi + b\sin\psi)\,\psi',$$

so upon integrating:

$$\Theta_2 = M\,gl\,(a\cos\psi - b\sin\psi) + \text{const.}\,;$$

i.e.:

(4) $$A\,\psi'^2 + C\,\varphi'^2 = 2\,M\,gl\,(a\cos\psi - b\sin\psi) + h\,.$$

Obviously, that first integral can be obtained independently of Gilbert's method. It is the *vis viva* integral, when it is applied to the relative motion with respect to the axes *Oxyz*.

We shall apply these formulas to two particularly simple cases.

460. Foucault's gyroscopic compass. – Suppose that the body is suspended *by its center of gravity*. That will make $\overline{OG} = l = 0$, and the two first integrals will become:

$$\varphi' + \omega\cos\alpha\sin\psi = k, \qquad A\,\psi'^2 + C\,\varphi'^2 = h\,.$$

Those equations are integrated by elliptic quadratures. However, since the angular velocity ω of the Earth is very small, one can neglect ω^2. Upon inferring φ' from the first one in order to substitute it in the second and neglecting ω^2, one will then have:

$$A\,\psi'^2 - 2C\,k\omega\cos\alpha\sin\psi = f, \tag{5}$$

in which f denotes a new constant. We can always suppose that one has chosen the positive sense OZ of the body axis to be the one such that the initial value R_0 or φ'_0 of the rotation of the body around OZ is positive. Suppose, moreover, that this value is sufficiently large: k will then be positive. Equation (5) will then immediately reduce to the equation of motion for a simple pendulum. Indeed, if the angle xOZ is called u then one will have $\psi = u + \pi/2$, and equation (5) will become:

$$u'^2 = \frac{2Ck\omega}{A}\cos\alpha\cos u + \frac{f}{A},$$

which is identical to the equation of motion of a simple pendulum for which u is the angle of deflection from the vertical.

The axis of the gyroscope OZ is then animated with a pendulum motion around Ox. In reality, as a result of the air resistance and friction, the axis OZ will stop after a certain amount of time at Ox – i.e., along the projection of the axis of the world O_w onto the fixed plane xOy.

The name *gyroscopic compass* was given to that apparatus. If the plane xOy in which the axis of the gyroscope is constrained to move is the horizontal plane at the place of observation then the relative equilibrium position of the axis OZ will be the direction of the meridian: The apparatus can serve as a declination compass. If the plane xOy coincides with the meridian plane then the axis will be placed along $O\omega$, which will then coincide with Ox ; the apparatus will serve as an inclination compass.

One can summarize the discussion by saying that the axis of the gyroscopic compass will tend to make the smallest possible angle with the Earth' axis.

Figure 268. Gilbert's baro-gyroscope (see original manuscript).

461. Gilbert's baro-gyroscope. – In Foucault's gyroscopic compass that we just studied, the center of gravity of the body of revolution is supposed to be placed at the point of suspension O and the axis OZ of the body will be constrained to describe a plane that is fixed on the Earth and passes through O. The condition that the center of gravity must be found at O is very difficult to realize experimentally, so Gilbert looked for the influence of the rotation of the Earth on the motion of a heavy body of revolution that is suspended from a fixed point O on its axis OZ when that axis OZ is constrained to move in a vertical plane that is fixed in the Earth, and the center of gravity G is no longer at O. Gilbert experimentally realized the conditions that we just indicated in the following apparatus, which he called the *baro-gyroscope.*

Imagine a bronze torus D whose steel axis a pivots freely in the hollow conical journals in steel screws v and v' that traverse a steel screed (*chape*) CC that is supported by the knives A and A' on tempered steel surfaces of cylindrical form whose knives occupy the base. That system will present an exact symmetry with respect to the plane that passes through the axis of the torus and the edges of the knives, and its mobility around them will be such that a slight puff of wind will suffice to provoke oscillations.

Once one has ensured the horizontality of the suspension axis AA' by leveling screws V, V', V'', the torus will consist of a solid body of revolution that moves around a fixed point O that is placed at the intersection of its axis of revolution vv' and the suspension axis AA'. Furthermore, the axis of revolution vv' of the torus can move only in a vertical plane that is fixed with respect to the Earth.

Upon acting on the screws v and v', the other screws u and u', and on a cursor p that slides with hard friction along a needle that defines the lower prolongation of the axis of the torus vv', one will succeed in placing the center of gravity G of the moving system along the axis of the torus vv' slightly below the point O. *Since the torus is at rest*, one will then have a composite pendulum that is suspended by the axis AA', which will be in stable equilibrium when the needle $v'p$ (i.e., the axis of the torus) is vertical. One then moves the screed by using a geared motor and imparts a very rapid rotation upon the torus around its axis, after which, one replaces it on its support while guiding it by the forks F in order for the edges of the knives A, A' to occupy exactly the horizontal position that was assigned to them. At that instant, delicate, but very well-defined, phenomena will develop that are due to the rotation of the Earth. The system will take up a new apparent position of stable equilibrium in which the axis of the torus is not vertical, but makes a small angle E with the vertical that will become larger, and with equal velocity, as the vertical plane in which the axis of the torus can move gets closer to the meridian plane. If one enjoys the most favorable conditions by putting the plane in which the axis of the torus moves in the meridian plane then the angle of deviation E between the axis of the torus and the vertical will be observed sharply. It will become larger as the proper rotation of the torus becomes larger and the distance OG from the center of gravity of the axis AA' become smaller. Furthermore, the deviation will happen towards the North or the South according to the sense of rotation of the torus. One can explain that easily by applying the general formulas that we applied above to the present case.

The plane in which the axis of the body OZ moves (Fig. 269) is the plane of the meridian at the point O here, namely, POP'. In order to apply the general formulas, we must take that plane to be the xy-plane by taking the x-axis to be the projection of the rotation $O\omega$, which is equal and parallel to the rotation of the Earth in the plane xOy. Presently, $O\omega$ coincides with Ox.

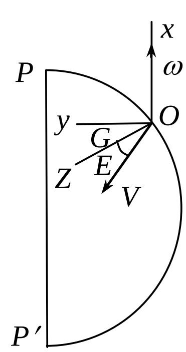

Figure 269.

Draw the descending vertical OV. It is in the plane yOx and makes an angle of:

$$xOV = \frac{\pi}{2} + \lambda$$

with Ox, where λ is the latitude of the point O. The cosines a, b, c of the angles that the vertical OV makes with the axes $Oxyz$ will then be:

$$a = -\sin\lambda\,, \quad b = \cos\lambda\,, \quad c = 0,$$

resp., and the term $a \sin\psi - b\cos\psi$, which enters into the integral (4) will have the value:

$$-\cos(\lambda - \psi)\,.$$

Since α is zero, the two first integrals that were obtained above (2) and (4) will now be:

$$\text{(6)} \qquad \begin{cases} \varphi' + \omega \sin\psi = k = n + \omega \sin\psi_0, \\ A\psi'^2 + C\varphi'^2 = -2M\,gl\cos(\lambda - \psi) + h, \end{cases}$$

in which n denotes the initial value of φ' – i.e., the rotation of the torus.

If we eliminate φ', while neglecting ω^2, then we will have the equation:

$$\text{(7)} \qquad A\,\psi'^2 - 2n\,C\,\omega\sin\psi = -2M\,gl\cos(\lambda - \psi) + f$$

for determining ψ, in which f denotes a new constant.

Introduce the angle E in place of ψ, which is the angle that the axis OZ of the gyroscope makes with the vertical. That angle is counted as positive from Ox to Oy, so one will have:

$$xOZ = \psi - \frac{\pi}{2}, \qquad xOV = \frac{\pi}{2} + \lambda,$$

$$E = xOZ - xOV = \psi - \lambda - \pi\,.$$

The equation will become:

$$A\left(\frac{dE}{dt}\right)^2 = -2n\,C\,\omega\sin(E + \lambda) + 2\,Mgl\cos E + f\,.$$

We can easily reduce this to something that is identical to the equation of motion of a simple pendulum by a new change of the origin of the angles. We confine ourselves to seeking the equilibrium position of the axis. We will obtain it by looking for the values of E that annul d^2E / dt^2 ; i.e., the derivative of the right-hand side. We will then have:

$$n\,C\,\omega\cos(E + \lambda) + M\,gl\sin E = 0,$$

(8)
$$\tan E = -\frac{n\,\omega C \cos\lambda}{M\,gl - n\omega C \sin\lambda}.$$

One will then have the angle E that it makes with the vertical axis of the torus after several oscillations on one side and the other of the direction that is defined by that axis.

Since ω is very small, the sign of the denominator will be equal to $M\,gl$ – i.e., positive. $\tan E$ will then have the same sign as $-n$, if n is positive; i.e., if the torus turns in the positive sense around the axis OG then the deviation will occur towards the North, because E is negative, while if n is negative then the deviation will occur towards the South. One sees that the deviation will be larger when n is positive and have a velocity of rotation that is equal in absolute value.

VI. – NON-HOLONOMIC SYSTEMS.

462. Form of the constraint equations in non-holonomic systems. – We have already said that a system is called *non-holonomic* when some of the constraints that are imposed cannot be expressed in *finite terms*, but translate analytically into differential relations, such as for the hoop and the bicycle.

That situation presents itself whenever a solid body is constrained to roll and pivot on a fixed surface. Indeed, the position of an entirely-free solid body depends upon six coordinates, which are, for example, the three coordinates of the center of gravity and the three Euler angles. In order to express the idea that the body rolls and pivots on a fixed surface, one must write down that the velocity of the molecule at the point of contact is zero. Now, upon calling the six coordinates $q_1, q_2, q_3, q_4, q_5, q_6$, that condition will be expressed by relations of the form:

$$A_1\,dq_1 + A_2\,dq_2 + \ldots + A_6\,dq_6 = 0,$$

whose coefficients are functions of $q_1, q_2, q_3, \ldots, q_6$, but *whose left-hand side is not, in general, an exact differential and does not admit an integrating factor.*

The constraint that is imposed upon the body cannot be expressed by relations *in finite terms* between the coordinates then. Some special difficulties will result from that for the application of the theorems of analytical mechanics, the most salient of which is that the Lagrange equation *cannot be applied* when one takes those exceptional constraints into account in order to modify the expression for the *vis viva* $2T$.

From this standpoint, the difficulties that result from this type of constraint have been pointed out by C. Neumann ["Grundzüge de Analytischen Mechanik," Berichte der königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig (1888), pp. 32], by Vierkandt ["Ueber gleitende und rollende Bewegung," Monastheft für Mathematik und Physik **3** (1892)], by Hadamard ["Sur les mouvements de roulement," Société des Sciences de Bordeaux (1895)], by Carvallo in a paper that was inserted into the Journal de l'École Polytechnique in 1900, and by Korteweg [Nieuw Archief (1899)].

First example. – Take, for example, a homogeneous sphere of radius a that is constrained to roll on a fixed plane. Take the fixed axes to be two axes $O\xi$, $O\eta$ in the plane and a perpendicular axis $O\zeta$ on the side where the sphere is found. Let ξ, η, ζ be the coordinates of the center G of the sphere with respect to those axes ($\zeta = a$). Draw three axes $G\,x_1\,y_1\,z_1$ through G that are parallel to the axes $O\xi\eta\zeta$, and let p_1, q_1, r_1 denote the components of the instantaneous rotation of the sphere along those axes. Upon writing out that the point of sphere that is in contact has zero velocity, one will have:

$$(1) \qquad \frac{d\xi}{dt} - a\,q_1 = 0, \qquad \frac{d\eta}{dt} + a\,p_1 = 0, \qquad \frac{d\zeta}{dt} = 0\,.$$

Furthermore, if θ, φ, ψ are the Euler angles of a system of axes $Gxyz$ that is fixed in the sphere with respect to the axes $G\,x_1\,y_1\,z_1$ then one will have:

$$(2) \qquad \begin{cases} p_1 = \theta' \cos\psi + \varphi' \sin\theta \sin\psi, \\ q_1 = \theta' \sin\psi - \varphi' \sin\theta \cos\psi, \\ r_1 = \psi' + \varphi' \cos\theta, \end{cases}$$

in which θ', φ', ψ' are the derivatives $d\theta / dt$, $d\varphi / dt$, $d\psi / dt$, resp. The relations (1), which express the idea that the real displacement is a rolling motion, will then be written:

$$(3) \qquad \begin{cases} d\xi - a \sin\psi \, d\theta + a \sin\theta \cos\psi \, d\varphi = 0, \\ d\eta + a \cos\psi \, d\theta + a \sin\theta \sin\psi \, d\varphi = 0. \end{cases}$$

Similarly, the virtual displacements that are compatible with the constraints are characterized by:

$$(4) \qquad \begin{cases} \delta\xi - a \sin\psi \, \delta\theta + a \sin\theta \cos\psi \, \delta\varphi = 0, \\ \delta\eta + a \cos\psi \, \delta\theta + a \sin\theta \sin\psi \, \delta\varphi = 0. \end{cases}$$

Since the coordinate ζ is constant, the position of the system will depend upon five parameters ξ, η, θ, φ, ψ, which are coupled by the relations (4), *whose left-hand sides are not exact total differentials and cannot be integrated.* The system has three degrees of freedom, because $\delta\theta$, $\delta\varphi$, $\delta\psi$ remain arbitrary, while $\delta\xi$, $\delta\eta$ are determined from the relations (4).

Second example. Hoop. – Consider a hoop of radius a that is constrained to roll and pivot on a fixed horizontal plane Π as in no. **411**.

Take two fixed axes $O\xi$, $O\eta$ in that plane and a fixed axis $O\zeta$ that points vertically upwards. Let ξ, η, ζ denote the coordinates of the center of gravity G of the hoop with respect to those axes, while θ, φ, ψ are the Euler angles that were defined in no. **411** and determine the position of the hoop with respect to the axes $G\,x_1\,y_1\,z_1$, which are parallel to the fixed axes $O\xi\eta\zeta$. The velocity **V** of the center of gravity G has projections onto the fixed axes $O\xi\eta\zeta$, and as a result, onto the parallel axes $G\,x_1\,y_1\,z_1$ that are equal to:

$$\frac{d\xi}{dt}, \quad \frac{d\eta}{dt}, \quad \frac{d\zeta}{dt}.$$

On the other hand, the contact point H of the hoop with the plane Π (Fig. 244) will have coordinates with respect to $G\,x_1\,y_1\,z_1$ of:

(5) $$x_1 = a \cos\theta \sin\psi, \qquad y_1 = -a\cos\theta\cos\psi, \qquad z_1 = -a\sin\theta.$$

In order to express the idea that the hoop rolls and pivots on the plane Π, one must write down that the velocity of the material point that is placed at H is *zero*. Let p_1, q_1, r_1 denote the components of the instantaneous rotation $\boldsymbol{\omega}$ of the hoop along the axes $G\,x_1\,y_1\,z_1$, and note that the velocity of the material point that is placed at H is the resultant of the velocity that is due to the motion of the axes $G\,x_1\,y_1\,z_1$ and the velocity that is due to the rotation ω around G: Upon writing out that the three projections of the velocity of the material point are zero, one will then have:

(6) $$\begin{cases} \dfrac{d\xi}{dt} + q_1\,z_1 - r_1\,y_1 = 0, \\ \dfrac{d\eta}{dt} + r_1\,x_1 - p_1\,z_1 = 0, \\ \dfrac{d\zeta}{dt} + p_1\,y_1 - q_1\,x_1 = 0. \end{cases}$$

From the expressions for p_1, q_1, r_1 above (2) and the values of x_1, y_1, z_1 in (5), one will see that the preceding conditions (6) will give:

(7) $$\begin{cases} d\xi - a\sin\psi\sin\theta\; d\theta + a\cos\psi\cos\theta\, d\psi + a\cos\psi\, d\varphi = 0, \\ d\eta + a\cos\psi\sin\theta\, d\theta + a\sin\psi\cos\theta\, d\psi + a\sin\psi\, d\varphi = 0. \\ d\zeta - a\cos\theta\, d\theta = 0 \end{cases}$$

as a result of the obvious reductions.

Those relations express the idea that the real displacement is one of rolling.

Similarly, when one expresses the idea that the virtual displacements that are compatible with the constraints are the rolling of the hoop on the plane, one will have the following condition equations for the differentials $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\varphi$, $\delta\psi$:

(8) $$\begin{cases} \delta\xi - a\sin\psi\sin\theta\,\delta\theta + a\cos\psi\cos\theta\,\delta\psi + a\cos\psi\,\delta\varphi = 0, \\ \delta\eta + a\cos\psi\sin\theta\,\delta\theta + a\sin\psi\cos\theta\,\delta\psi + a\sin\psi\,\delta\varphi = 0, \\ \delta\zeta - a\cos\theta\,\delta\theta = 0 \end{cases}$$

The last of the relations (7) or (8) is equivalent to the relation in finite terms:

$$\zeta = a\sin\theta,$$

which is geometrically obvious when one evaluates the distance ζ from the point G to the plane Π. However, the first two relations (8) cannot be integrated and written in finite form. One will then see that the system considered is not holonomic. It is a system with *three degrees of freedom*, because the most general virtual displacement that is compatible with the constraints is obtained by giving arbitrary values to $\delta\theta$, $\delta\varphi$, $\delta\psi$; $\delta\xi$, $\delta\eta$, $\delta\zeta$ will then be determined from the relations (8).

463. Use of Lagrange equations, combined with the method of multipliers. – Imagine, in general, a system that is initially subject to constraints *that can be expressed by relations in finite terms between the coordinates of the various points.* Let k be the number of independent parameters $q_1, q_2, \ldots, q_k$ that fix the position of the system when one takes those constraints into account. If one supposes that the constraints are independent of time then one will have the following coordinates for an arbitrary point of the system:

$$(9) \qquad \begin{cases} x = f(q_1, q_2, \ldots, q_k), \\ y = \varphi(q_1, q_2, \ldots, q_k), \\ z = \psi(q_1, q_2, \ldots, q_k). \end{cases}$$

One will obtain a virtual displacement that is compatible with those constraints upon varying $q_1, q_2, \ldots, q_k$ by $\delta q_1, \delta q_2, \ldots, \delta q_k$, resp. The general equation of dynamics will then take the form (no. **411**):

$$(10) \qquad (P_1 - Q_1)\,\delta q_1 + (P_2 - Q_2)\,\delta q_2 + \ldots + (P_k - Q_k)\,\delta q_k = 0,$$

where:

$$P_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}.$$

If there are no other constraints then the $\delta q_1, \delta q_2, \ldots, \delta q_k$ will be arbitrary, and equation (10) will provide k equations, which are the Lagrange equations.

However, now suppose that one combines the preceding constraints with some new constraints that are independent of time and expressible in terms of *non-integrable differential relations between the parameters* $q_1, q_2, \ldots, q_k$. For a virtual displacement that is compatible with those constraints, one will have:

$$(11) \qquad \begin{cases} A_1\,\delta q_1 + A_2\,\delta q_2 + \ldots + A_k\,\delta q_k = 0, \\ B_1\,\delta q_1 + B_2\,\delta q_2 + \ldots + B_k\,\delta q_k = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_1\,\delta q_1 + L_2\,\delta q_2 + \ldots + L_k\,\delta q_k = 0, \end{cases}$$

in which the left-hand sides *are not exact differentials and do not admit integrable combinations.*

Under those conditions, equations (10) must be true for all displacements δq_1, δq_2, …, δq_k that verify the conditions (11). With the method of Lagrange multipliers, the equations of motion will then be:

$$(12)\qquad \left\{\begin{array}{l} P_1 = Q_1 + \lambda_1 A_1 + \lambda_2 B_1 + \ldots + \lambda_p L_1, \\ P_2 = Q_2 + \lambda_1 A_2 + \lambda_2 B_2 + \ldots + \lambda_p L_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_k = Q_k + \lambda_1 A_k + \lambda_2 B_k + \ldots + \lambda_p L_k, \end{array}\right.$$

in which P_α has the expression above. When those equations are combined with the p equations:

$$(13)\qquad \left\{\begin{array}{l} A_1\, dq_1 + A_2\, dq_1 + \ldots + A_p dq_k = 0, \\ B_1\, dq_1 + B_2\, dq_1 + \ldots + B_p dq_k = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_1\, dq_1 + L_2\, dq_1 + \ldots + L_p dq_k = 0, \end{array}\right.$$

which express the idea that the real displacement is compatible with the constraints, that will determine the q_1, q_2, …, q_k and λ_1, λ_2, …, λ_p .

That method was employed by Routh (*Advanced Rigid Dynamics*, pp. 132) and Vierkandt (*loc. cit.*, pp. 47-50).

464. Impossibility of applying the Lagrange equations directly to the minimum number of equations ([1]). – We just saw how one can utilize the Lagrange equations while taking the relations (11) into account by the method of multipliers.

However, one can try to reduce the parameters to the lowest possible number by appealing to the relations (11) in order to keep the minimum number of parameters in the expression for the virtual displacement and equations (13) in order to keep the minimum number of parameters in the expression for the semi-*vis viva* :

$$T = \tfrac{1}{2} \sum m\, (x'^2 + y'^2 + z'^2) \,.$$

After those modifications, the *Lagrange equations will no longer be applicable.* That is what we shall rapidly prove.

A virtual displacement that is compatible with all of the constraints imposed upon the system is defined in terms of x, y, z by:

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x}{\partial q_k}\delta q_k \,,$$

([1]) P. APPELL, "Les mouvements de roulement en Dynamique," Collection Scientia, Gauthier-Villars.

$$\delta y = \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2 + \cdots + \frac{\partial y}{\partial q_k}\delta q_k ,$$

$$\delta z = \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_2}\delta q_2 + \cdots + \frac{\partial z}{\partial q_k}\delta q_k ,$$

in which $\delta q_1, \delta q_2, \ldots, \delta q_k$ are coupled by the p relations (11). One can infer p of the variations $\delta q_k , \delta q_{k-1}, \ldots, \delta q_{k-p+1}$ from those relations as linear, homogeneous functions of the others. Upon substituting $\delta x, \delta y, \delta z$ and setting $n = k - p$ in them:

$$(14) \qquad \begin{cases} \delta x = a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_n\,\delta q_n, \\ \delta y = b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_n\,\delta q_n, \\ \delta z = c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_n\,\delta q_n, \end{cases}$$

in which $\delta q_1, \delta q_2, \ldots, \delta q_k$ are arbitrary now. When one substitutes those values for $\delta x, \delta y, \delta z$ in the general equation of dynamics, one will get a relation in which the coefficients of $\delta q_1, \delta q_2, \ldots, \delta q_k$ must be zero, and one will then have the equations of motion (no. **433**):

$$(15) \qquad \sum m\left(a_\alpha\frac{d^2x}{dt^2} + b_\alpha\frac{d^2y}{dt^2} + c_\alpha\frac{d^2z}{dt^2}\right) = \sum (a_\alpha X + b_\alpha + c_\alpha Z) = Q_\alpha \quad (\alpha = 1, 2, \ldots, n),$$

whose right-hand sides will be denoted by Q_α .

Furthermore, since the real displacement is presently compatible with the constraints, from (14), one will have:

$$dx = a_1\,dq_1 + a_2\,dq_2 + \ldots + a_n\,dq_n ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

Or, upon adopting Lagrange's notation for the derivatives:

$$x' = a_1\,q_1' + a_2\,q_2' + \cdots + a_n\,q_n' ,$$
$$y' = b_1\,q_1' + b_2\,q_2' + \cdots + b_n\,q_n' ,$$
$$z' = c_1\,q_1' + c_2\,q_2' + \cdots + c_n\,q_n' .$$

Let us try to pursue the method that led to the Lagrange equations with the first of equations (15). We suppose, to simplify, that the coefficients of $a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots, a_n, b_n, c_n$ *depend upon only* $q_1, q_2, \ldots, q_k$. One can write the first equation in (15) (α = 1) as:

$$(16) \qquad \frac{d}{dt}\sum m\,(a_1\,x' + b_1\,y' + c_1\,z') - R_1 = Q_1 ,$$

in which R_1 denotes the quantity:

$$R_1 = \sum m\left(x'\frac{da_1}{dt} + y'\frac{db_1}{dt} + z'\frac{dc_1}{dt}\right).$$

Now, a_1, b_1, c_1 are obviously equal to $\frac{\partial x'}{\partial q_1'}$, $\frac{\partial y'}{\partial q_1'}$, $\frac{\partial z'}{\partial q_1'}$, resp., so the first term in equation (16) will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right),$$

as in the Lagrange equations; however, the second one R_1 is not, in general, equal to $\frac{\partial T}{\partial q_1}$. Indeed, one will have:

$$\frac{\partial T}{\partial q_2} = \sum m\left(x'\frac{\partial x'}{\partial q_1} + y'\frac{\partial y'}{\partial q_1} + z'\frac{\partial z'}{\partial q_1}\right).$$

Therefore:

$$R_1 - \frac{\partial T}{\partial q_1} = \sum m\left[x'\left(\frac{da_1}{dt} - \frac{\partial x'}{\partial q_1}\right) + y'\left(\frac{db_1}{dt} - \frac{\partial y'}{\partial q_1}\right) + z'\left(\frac{dc_1}{dt} - \frac{\partial z'}{\partial q_1}\right)\right]. \tag{17}$$

Now, the coefficients a_1, b_1, … are supposed to be functions of q_1, q_2, …, q_n, so one will have:

$$\frac{da_1}{dt} = \frac{\partial a_1}{\partial q_1}q_1' + \frac{\partial a_1}{\partial q_2}q_2' + \cdots + \frac{\partial a_1}{\partial q_n}q_n',$$

and upon differentiating the expression above for x' with respect to q_1 :

$$\frac{\partial x'}{\partial q_1} = \frac{\partial a_1}{\partial q_1}q_1' + \frac{\partial a_1}{\partial q_2}q_2' + \cdots + \frac{\partial a_1}{\partial q_n}q_n',$$

.

The coefficient of x' in the difference $R_1 - \frac{\partial T}{\partial q_1}$ will then be:

$$\left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1}\right)q_2' + \left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right)q_3' + \cdots + \left(\frac{\partial a_1}{\partial q_n} - \frac{\partial a_n}{\partial q_1}\right)q_n' \;; \tag{18}$$

it is not zero, in general. The coefficients of y' and z' have analogous forms. From the values of x', y', z' as functions of q_1', q_2', …, the difference $R_1 - \partial T / \partial q_1$ will then be a quadratic form in the q_1', q_2', …, in general. In order for R_1 to be equal to $\partial T / \partial q_1$ – i.e., in order for the Lagrange equation to apply to the parameter q_1 – it is necessary and sufficient that this quadratic form should be identically zero for any q and q'.

Special cases:

1. If the expressions (14) for δx, δy, δz are exact total differentials then all quantities such as:

$$\frac{\partial a_i}{\partial q_\nu}-\frac{\partial a_\nu}{\partial q_i}, \quad \frac{\partial b_i}{\partial q_\nu}-\frac{\partial b_\nu}{\partial q_i}, \quad \frac{\partial c_i}{\partial q_\nu}-\frac{\partial c_\nu}{\partial q_i}$$

will be zero. Expressions such as (17) will be zero, and the Lagrange equations will apply to all of the parameters. In this case, one can integrate the expressions (14) and express x, y, z in finite form as functions of $q_1, q_2, \ldots q_n$. The system is holonomic.

2. Here is a case in which the Lagrange equation applies to the parameter q_1. Suppose that one has:

$$(19)\qquad \left\{\begin{array}{llll} \dfrac{\partial a_1}{\partial q_2}=\dfrac{\partial a_2}{\partial q_1}, & \dfrac{\partial a_1}{\partial q_3}=\dfrac{\partial a_3}{\partial q_1}, & \cdots & \dfrac{\partial a_1}{\partial q_n}=\dfrac{\partial a_n}{\partial q_1}, \\ \dfrac{\partial b_1}{\partial q_2}=\dfrac{\partial b_2}{\partial q_1}, & \dfrac{\partial b_1}{\partial q_3}=\dfrac{\partial b_3}{\partial q_1}, & \cdots & \dfrac{\partial b_1}{\partial q_n}=\dfrac{\partial b_n}{\partial q_1}, \\ \dfrac{\partial c_1}{\partial q_2}=\dfrac{\partial c_2}{\partial q_1} & \dfrac{\partial c_1}{\partial q_3}=\dfrac{\partial c_3}{\partial q_1} & \cdots & \dfrac{\partial c_1}{\partial q_n}=\dfrac{\partial c_n}{\partial q_1} \end{array}\right.$$

for all points.

Quantities such as (18) that define the coefficients of x', y', z' in $R_1 - \partial T / \partial q_1$ *are zero,* and R_1 will be equal to $\partial T / \partial q_1$. The Lagrange equation will then apply to the parameter q_1. One can characterize this case in a different way. If the conditions (19) are supposed to be fulfilled then one can determine the functions $q_1, q_2, \ldots q_n$ from the conditions:

$$U_1 = \int_{q_1^0}^{q_1} a_1\, dq_1, \qquad V_1 = \int_{q_1^0}^{q_1} b_1\, dq_1, \qquad W_1 = \int_{q_1^0}^{q_1} c_1\, dq_1,$$

in which q_1^0 is an arbitrary constant, and the integration is performed over q_1. From the conditions (19), one immediately finds that:

$$\frac{\partial U_1}{\partial q_2} = \int_{q_1^0}^{q_1} \frac{\partial a_1}{\partial q_2}\, dq_1 = \int_{q_1^0}^{q_1} \frac{\partial a_2}{\partial q_1}\, dq_1 = a_2 - a_2^0,$$

in which a_2^0 is what a_2 will become when one replaces q_1 with the constant q_1^0 in it. Similarly:

$$\frac{\partial U_1}{\partial q_3} = a_3 - a_2^0, \qquad \ldots, \qquad \frac{\partial U_1}{\partial q_n} = a_n - a_n^0.$$

One will have analogous relations for V_1 and W_1. One can then write:

(20)
$$\begin{cases} \delta x = \delta U_1 + a_2^0\,\delta q_2 + a_3^0\,\delta q_3 + \cdots + a_n^0\,\delta q_n, \\ \delta y = \delta V_1 + b_2^0\,\delta q_2 + \cdots + b_n^0\,\delta q_n, \\ \delta z = \delta W_1 + c_2^0\,\delta q_2 + \cdots + c_n^0\,\delta q_n. \end{cases}$$

Hence, the Lagrange equation will apply to q_1 when δx, δy, δz can be put into the form of *an exact total differential, followed by a differential expression that does not contain* q_1 for an arbitrary point of the system.

For example, in the motion of the hoop, the Lagrange equation can be applied to the parameter θ, as Ferrers pointed out already [Quarterly Journal of Mathematics (1871-73)]. Indeed, the position of the hoop around its center of gravity G is defined by the values of the angles θ, φ, ψ, so the coordinates x_1, y_1, z_1 of a point on the hoop with respect to the axes $G\,x_1\,y_1\,z_1$ will be functions of θ, φ, ψ :

$$x_1 = f(\theta, \varphi, \psi), \qquad y_1 = f_1(\theta, \varphi, \psi), \qquad z_1 = f_2(\theta, \varphi, \psi) .$$

The absolute coordinates x, y, z of the same point with respect to the fixed axes $O\xi\eta\zeta$ have the form:

$$x = \xi + f\;(\theta, \varphi, \psi),$$
$$y = \eta + f_1(\theta, \varphi, \psi),$$
$$z = \zeta + f_2(\theta, \varphi, \psi) .$$

Impart a virtual displacement that is compatible with the constraints to the system; we will have:

$$\delta x = \delta\xi + \delta f, \qquad \delta y = \delta\eta + \delta f_1, \qquad \delta z = \delta\zeta + \delta f_2 ,$$

in which one must replace $\delta\xi$, $\delta\eta$, $\delta\zeta$ with their values in (8). However, one will see immediately that those values are written:

$$\delta\xi = \delta(-a \sin\psi \cos\theta) - a \cos\psi\, \delta\varphi ,$$
$$\delta\eta = \delta(a \cos\psi \cos\theta) - a \sin\psi\, \delta\varphi ,$$
$$\delta\zeta = \delta(a \sin\theta) .$$

One will finally have the following expressions for δx, δy, δz then:

$$\delta x = \delta(-a \sin\psi \cos\theta + f) - a \cos\psi\, \delta\varphi ,$$
$$\delta y = \delta(a \cos\psi \cos\theta + f_1) - a \sin\psi\, \delta\varphi ,$$
$$\delta z = \delta(a \sin\theta + f_2) ,$$

which indeed have the form (20). In fact, we presently have three arbitrary variations $\delta\theta$, $\delta\varphi$, $\delta\psi$, and we see that δx, δy, δz can each be put into the form of *a total differential*, followed by *a differential expression that does not contain* θ. We can then write the Lagrange equation that relates to the parameter θ.

465. General form of the equations of motion that are convenient for all holonomic and non-holonomic systems ([1]). – Imagine a system that is subject to constraints such that in order to obtain the most general virtual displacement that is compatible with the constraints at the instant t, it will suffice to subject the k parameters $q_1, q_2, \ldots, q_k$ to arbitrary variations $\delta q_1, \delta q_2, \ldots, \delta q_k$. If we then let x, y, z denote the coordinates of any point of the system with respect to some fixed axes then the virtual displacement of that point will have projections onto the axes that are:

$$(1) \qquad \left\{ \begin{aligned} \delta x &= a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_n\,\delta q_n, \\ \delta y &= b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_n\,\delta q_n, \\ \delta z &= c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_n\,\delta q_n, \end{aligned} \right.$$

in which $\delta q_1, \delta q_2, \ldots, \delta q_k$ are arbitrary. In those formulas, the coefficients $a_1, a_2, \ldots, c_k$ can depend upon time t, the parameters $q_1, q_2, \ldots, q_k$, and *some other parameters* $q_{k+1}, q_{k+2}, \ldots, q_{k+p}$, whose variations are coupled with those of $q_1, q_2, \ldots, q_k$ by relations of the form:

$$(2) \qquad \left\{ \begin{aligned} \delta q_{k+1} &= \alpha_1\,\delta q_1 + \alpha_2\,\delta q_2 + \cdots + \alpha_n\,\delta q_n, \\ \delta q_{k+2} &= \beta_1\,\delta q_1 + \beta_2\,\delta q_2 + \cdots + \beta_n\,\delta q_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_{k+p} &= \lambda_1\,\delta q_1 + \lambda_2\,\delta q_2 + \cdots + \lambda_n\,\delta q_n, \end{aligned} \right.$$

in which the coefficients $\alpha_1, \alpha_2, \ldots, \lambda_k$ likewise depend upon t and the set of parameters $q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}$. Under those conditions, the real displacement of the system during the time interval dt will be defined by relations of the form:

$$(3) \qquad \left\{ \begin{aligned} dx &= a_1\,dq_1 + a_2\,dq_2 + \cdots + a_n\,dq_n + a\,dt, \\ dy &= b_1\,dq_1 + b_2\,dq_2 + \cdots + b_n\,dq_n + b\,dt, \\ dz &= c_1\,dq_1 + c_2\,dq_2 + \cdots + c_n\,dq_n + c\,dt, \end{aligned} \right.$$

with

$$(4) \qquad \left\{ \begin{aligned} dq_{k+1} &= \alpha_1\,dq_1 + \alpha_2\,dq_2 + \cdots + \alpha_n\,dq_n + \alpha\,dt, \\ dq_{k+2} &= \beta_1\,dq_1 + \beta_2\,dq_2 + \cdots + \beta_n\,dq_n + \beta\,dt, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dq_{k+p} &= \lambda_1\,dq_1 + \lambda_2\,dq_2 + \cdots + \lambda_n\,dq_n + \lambda\,dt, \end{aligned} \right.$$

in which the coefficients $a_i, b_i, c_i, \alpha_i, \beta_i, \ldots, \lambda_i$ are the same as in equations (1) and (2). When the coefficients $a, b, c, \ldots, \alpha, \beta, \ldots, \lambda$ are multiplied by dt, they will be zero when the constraints do not depend upon time.

One can then obtain the equations of motion as follows:

The general equation of dynamics, which is deduced from *d'Alembert's principle* and the principle of virtual work, is:

([1]) Appell, Comptes rendus, 7 August 1899; Crelle's Journal, v. 121; Journal de Jordan, t. VI, 1900.

(5) $$\sum m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z),$$

in which x'', y'', z'' are the second derivatives of the coordinates with respect to time, and X, Y, Z are the projections of any one of the given forces (no. **431**).

That equation must be true for all displacements (1) that are compatible with the constraints: They will then decompose into the following k equations:

(6) $$\left\{\begin{array}{l} \sum m\,(x''\,a_1 + y''\,b_1 + z''\,c_1) = \sum (X\,a_1 + Y\,b_1 + Z\,c_1), \\ \sum m\,(x''\,a_2 + y''\,b_2 + z''\,c_2) = \sum (X\,a_2 + Y\,b_2 + Z\,c_2), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sum m\,(x''\,a_k + y''\,b_k + z''\,c_k) = \sum (X\,a_k + Y\,b_k + Z\,c_k). \end{array}\right.$$

The left-hand sides of those equations are calculated as they are in the *Lagrange* equations. Upon replacing δx, δy, δz with their values in (1), one will have:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \dots + Q_k\,\delta q_k$$

for the sum of the virtual works done by the applied forces.

The quantities Q_1, Q_2, …, Q_k, are the left-hand sides of equations (6):

$$Q_1 = \sum (X\,a_1 + Y\,b_1 + Z\,c_1)\,,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

In order to calculate the left-hand sides, divide the relations (3), which define the real displacement, by dt, and let x', y', z', q'_1, q'_2, …, q'_k denote the total derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, $\frac{dq_1}{dt}, \frac{dq_2}{dt}, \dots, \frac{dq_k}{dt}$. We will then have:

(7) $$\left\{\begin{array}{l} x' = a_1\,q'_1 + a_2\,q'_2 + \cdots + a_n\,q'_n + a, \\ y' = b_1\,q'_1 + b_2\,q'_2 + \cdots + b_n\,q'_n + b, \\ z' = c_1\,q'_1 + c_2\,q'_2 + \cdots + c_n\,q'_n + c. \end{array}\right.$$

If one takes the total derivatives of the two sides with respect to t one more time then one will have:

(8) $$\left\{\begin{array}{l} x'' = a_1\,q''_1 + a_2\,q''_2 + \cdots + a_k\,q''_k + \cdots, \\ y'' = b_1\,q''_1 + b_2\,q''_2 + \cdots + b_k\,q''_k + \cdots, \\ z'' = c_1\,q''_1 + c_2\,q''_2 + \cdots + c_k\,q''_k + \cdots, \end{array}\right.$$

in which the unwritten terms do not contain q''_1, q''_2, …, q''_k. However, one will obviously have:

$$a_1 = \frac{\partial x''}{\partial q_1''}, \qquad b_1 = \frac{\partial y''}{\partial q_1''}, \qquad c_1 = \frac{\partial z''}{\partial q_1''},$$
$$a_2 = \frac{\partial x''}{\partial q_2''}, \qquad b_2 = \frac{\partial y''}{\partial q_2''}, \qquad c_2 = \frac{\partial z''}{\partial q_k''},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

The equations of motion are then written:

$$(9) \qquad \begin{cases} \sum m \left(x'' \dfrac{\partial x''}{\partial q_1''} + y'' \dfrac{\partial x''}{\partial q_1''} + z'' \dfrac{\partial x''}{\partial q_1''} \right) = Q_1, \\ \sum m \left(x'' \dfrac{\partial x''}{\partial q_2''} + y'' \dfrac{\partial x''}{\partial q_2''} + z'' \dfrac{\partial x''}{\partial q_2''} \right) = Q_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Now consider the function:

$$S = \tfrac{1}{2} \sum m \,(x''^2 + y''^2 + z''^2) = \tfrac{1}{2} \sum m\, J^{\,2},$$

in which J is magnitude of the acceleration of the point m : The equations of motion (9) take the form:

$$(10) \qquad \frac{\partial S}{\partial q_1''} = Q_1 , \qquad \frac{\partial S}{\partial q_2''} = Q_2 , \quad \dots, \qquad \frac{\partial S}{\partial q_k''} = Q_k .$$

One sees that in order to write them, it will suffice to calculate just the function S and to express it in such a fashion that it no longer contains other second derivatives besides those of the parameters $q_1, q_2, \dots, q_k$, whose variations are regarded as arbitrary. It can then happen that when the function S is calculated as a function of the $q_1, q_2, \dots, q_{k+p}$, it will contain their first derivatives q_1', q_2', …, q_{k+p}' and their second derivatives q_1'', q_2'', …, q_{k+p}''. When the relations (4) are divided by dt, they will give q_{k+1}', q_{k+2}', …, q_{k+p}' as linear functions of q_1', q_2', …, q_k', and upon differentiating them with respect to time, one will likewise obtain q_{k+1}'', q_{k+2}'', …, q_{k+p}'' as linear functions of q_1'', q_2'', …, q_k''. Therefore, one can always arrange in some way that the function S no longer contains second derivatives other than q_1'', q_2'', …, q_k''. It will then contain those quantities to degree two, moreover. Once the function S has been thus prepared, one can write out equations (10). Those equations, when combined with the conditions (4), comprise a system of $k + p$ equations that define $q_1, q_2, \dots, q_{k+p}$ as functions of time.

The motion is then characterized when one knows the function S, which one calls ([1]) the *energy of acceleration of the system*, and the quantities $Q_1, Q_2, \ldots, Q_k$, which are calculated as in the Lagrange equations.

The function S has degree two in $q_1'', q_2'', \ldots, q_k''$. It will obviously suffice to calculate the terms in S that contain the second derivatives of the parameters, because the other ones will contribute nothing when one takes their partial derivatives with respect to $q_1'', q_2'', \ldots, q_k''$.

From formulas (7) and (8), one can remark that if one defines the semi-*vis viva* by:

$$T = \frac{1}{2} \sum m\,(x'^2 + y'^2 + z'^2)$$

then the coefficients of the terms in T that have degree two in $q_1', q_2', \ldots, q_k'$ will be identical to the coefficients of the terms in S that have degree two in $q_1'', q_2'', \ldots, q_k''$. The coefficients of the terms in that function S that have degree two in $q_1'', q_2'', \ldots, q_k''$ will depend upon the parameters $q_1, q_2, \ldots, q_{k+p}$, and time; the terms that have first degree in $q_1'', q_2'', \ldots, q_k''$ will contain the first derivatives $q_1', q_2', \ldots, q_{k+p}'$ to degree two, in addition.

466. Examples:

First application. *Planar motion of a material point in polar coordinates*. – Let r and θ be the polar coordinates of a point (x, y) of mass m. One will have:

$$x = r \cos \theta, \qquad y = r \sin \theta,$$

$$S = \frac{m}{2}(x''^2 + y''^2) = \frac{m}{2}[(r'' - r\,\theta'^2)^2 + (r\,\theta'' + 2r'\,\theta')^2].$$

Upon letting P denote the component of the applied force X, Y along the perpendicular to the radius vector, and letting R denote its component along the radius vector, one will immediately see that the virtual work done by the force:

$$X\,\delta x + Y\,\delta y$$

is:

$$P\,r\,\delta\theta + R\,\delta r.$$

The equations of motion are then:

$$\frac{\partial S}{\partial \theta''} = P\,r, \qquad \frac{\partial S}{\partial r''} = R,$$

or

$$mr\,(r\,\theta'' + 2r'\,\theta') = P\,r, \qquad m\,(r'' - r\,\theta'^2) = R.$$

([1]) That terminology was proposed by A. de Saint-Germain (Comptes rendus, t. CXXX).

Remark. – The quantity:

$$r\,\theta'' + 2r'\,\theta'$$

in S is the derivative of $r^2\theta'$, up to a factor of r. Therefore, introduce a parameter λ whose real variation is defined by:

$$d\lambda = r^2\,d\theta,$$

in place of θ and whose virtual variation is:

$$\delta\lambda = r^2\,\delta\theta\,.$$

We have:

$$\lambda' = r^2\,\theta', \qquad \lambda'' = r\,(r\,\theta'' + 2r'\,\theta'),$$

so

$$S = \frac{m}{2}\left[(r'' - r\,\theta'^2)^2 + \frac{1}{r^2}\lambda''^2\right],$$

$$X\,\delta x + Y\,\delta y = \frac{P}{r}\,\delta l + R\,\delta r,$$

and the equations of motion are written:

$$\frac{\partial S}{\partial r''} = R, \qquad \frac{\partial S}{\partial \lambda''} = \frac{P}{r}\,;$$

the second one is:

$$m\,\lambda'' = P\,r\,.$$

If P is zero then λ'' will be constant, which will give the areal theorem.

Second application: *Solid body moving around a fixed point*. – Take a solid body that moves around a fixed point O and calculate the energy of acceleration S while referring the motion to a system of axes $Oxyz$ that move in both the body and in space. Let $\mathbf{\Omega}$ denote the instantaneous rotation of the trihedron $Oxyz$, and let P, Q, R be its components along the axes, let $\boldsymbol{\omega}$ denote the rotation of the body, and let p, q, r be its components. A molecule m of the body whose coordinates are x, y, z will possess an absolute velocity $\mathbf{v}$ whose projections are:

$$v_x = q\,z - r\,y, \quad \ldots$$

That molecule possesses an absolute acceleration $\mathbf{J}$ that has the projections:

$$J_x = \frac{d}{dt}v_x + Q\,v_z - R\,v_y\,, \qquad \ldots, \tag{11}$$

since it will result from this that $\mathbf{J}$ is the absolute velocity of the point whose coordinates are v_x, v_y, v_z. Upon letting p', q', r' denote the derivatives of p, q, r, resp., with respect to time, one will have:

$$\frac{dv_x}{dt} = q\frac{dz}{dt} - r\frac{dy}{dt} + z\,q' - y\,r', \qquad \ldots$$

Now, dx / dt, dy / dt, dz / dt, which are the projections of the relative velocity of the molecule with respect to the axes *Oxyz*, are:

$$\frac{dx}{dt} = q\,z - r\,y - (Q\,z - Ry), \quad \ldots$$

because the relative velocity is the geometric difference between the absolute velocity and the guiding velocity. From that, one will have the following expression for J_x , which we arrange in order of x, y, z :

$$J_x = -x\,(q^2 + r^2) + y\,[q\,(p - P) + p\,Q - r'] + z\,[z\,(p - P) + p\,R + q'] \,. \tag{12}$$

One will get J_y and J_z similarly, and finally:

$$2\,S = \sum m(J_x^2 + J_y^2 + J_z^2) \,.$$

That sum is easily calculated then. One sees that the quantities $\sum m\,x^2$, $\sum m\,y^2$, $\sum m\,z^2$, $\sum m\,y\,z$, $\sum m\,z\,x$, $\sum m\,x\,y$ will appear in the result, which are easy to express with the help of the coefficients A, B, C, D, E, F of the ellipsoid of inertia relative to the point O when they are referred to the axes *Oxyz* .

In order to simplify, we shall write that sum here while supposing that the axes *Oxyz* are the principal axes of inertia at the point O_1 and upon letting A, B, C denote the moments of inertia with respect to those axes; upon confining ourselves to the terms in p', q', r', we will then have:

$$\begin{aligned} 2\,S = A\,p'^2 + B\,q'^2 + C\,r'^2 &+ 2\,[(C - B)\,q\,r + A\,(r\,Q - q\,R)]\,p' \\ &+ 2\,[(A - C)\,r\,p + B\,(p\,Q - r\,P)]\,q' \\ &+ 2\,[(B - A)\,p\,q + C\,(q\,P - p\,Q)]\,r' + \ldots \end{aligned} \tag{13}$$

Euler equations. – Take the moving axes to be three axes that are *invariably coupled* with the body and coincide with the three principal axes of inertia. We will then have:

$$P = p, \qquad Q = q, \qquad R = r,$$

$$2\,S = A\,p'^2 + B\,q'^2 + C\,r'^2 + 2\,(C - B)\,qrp' + 2\,(A - C)\,rpq' + 2\,(B - A)\,pqr' + \ldots$$

Let L, M, N denote the sums of the moments of the applied force with respect to the axes, and let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

be the elementary angles that turn the body about the axes in order to go from one position to an infinitely-close one. We shall make λ, μ, ν play the role of the parameters $q_1, q_2, \ldots, q_k$. On the one hand, we have:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta x) = L\,\delta\lambda + M\,\delta\mu + N\,\delta\nu\,,$$

and on the other hand, the components p, q, r of the instantaneous rotation of the body are:

$$p = \frac{d\lambda}{dt} = \lambda', \qquad q = \frac{d\mu}{dt} = \mu', \qquad r = \frac{d\nu}{dt} = \nu'.$$

The function S will then be:

$$S = \frac{1}{2}(A\,\lambda''^2 + B\,\mu''^2 + C\,\nu''^2) + (C - B)\,\mu'\nu'\lambda'' + (A - C)\,\nu'\lambda'\mu'' + (B - A)\,\lambda'\mu'\nu'' + \ldots,$$

in which the unwritten terms do not contain λ'', μ'', ν''. The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N.$$

For example, the first is written:

$$A\,\lambda'' + (C - B)\,\mu'\,\nu' = L\;;$$

from the values of p, q, r, that is precisely one of the *Euler* equations.

Body of revolution suspended by a point O on its axis. – Draw a fixed axis Oz_1 through O and, as in no. **400** (Fig. 234), take the axis Oz to be the axis of revolution, the axis Ox to be the perpendicular to the plane zOz_1, and the axis Oy to be perpendicular to the plane xOz. When the position of the trihedron $Oxyz$ is known, in order to get that of the body, it will suffice to know the angle φ that Ox makes with a radius that issues from O and is invariably coupled with the body in the xy-plane. The derivative φ' of that angle with respect to time represents the proper rotation of the body around Oz. The rotation $\boldsymbol{\omega}$ of the body is then the resultant of the rotation $\boldsymbol{\Omega}$ of the trihedron and the rotation φ'. One will then have:

$$p = P, \qquad q = Q, \qquad r = R + \varphi'.$$

Since $A = B$, the function S that was defined by the expression (13) will then become:

$$2S = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(pq - qp) + \ldots \tag{14}$$

Once more, let $\delta\lambda$, $\delta\mu$, $\delta\nu$ be the elementary angles through which the body must turn around the axes Ox, Oy, Oz in order to go from one position to a neighboring one, and let L, M, N be the moments of the forces with respect to the axes Ox, y, z. As above, one will have:

$$p' = \lambda'', \qquad q' = \mu'', \qquad r' = \nu'',$$

and the equations of motion will be:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N\,;$$

i.e., since the component R of the rotation $\mathbf{\Omega}$ does not depend upon λ'', μ'', ν'':

$$\begin{aligned} A\,p' - (AR - Cr)\,q &= L\,, \\ A\,q' + (AR - Cr)\,p &= M\,, \\ C\,r' &= N\,. \end{aligned}$$

One will then recover equations (61) of no. **400**.

467. Theorem analogous to Koenig's theorem. Application to the hoop. – Let x, y, z be the absolute coordinates of a point of mass m in some system, let ξ, η, ζ be the coordinates of the center of gravity G, and let x_1, y_1, z_1 be the relative coordinates of the same point with respect to the axes $Gx_1y_1z_1$, which are parallel to the fixed axes and drawn through G. Let $\mathbf{J}_0$ denote the absolute acceleration of the point G, so:

$$\mathbf{J}_0^2 = \xi''^2 + \eta''^2 + \zeta''^2,$$

while $\mathbf{J}_1$ denotes the relative acceleration of the point m with respect to the axes Gx_1, y_1, z_1:

$$J_1^2 = x_1''^2 + y_1''^2 + z_1''^2\,.$$

Finally, let M denote the total mass of the system. One will have:

$$\begin{aligned} x &= \xi + x_1\,, & y &= \eta + y_1\,, & z &= \zeta + z\,, \\ x'' &= \xi'' + x_1''\,, & y'' &= \eta'' + y_1''\,, & z'' &= \zeta'' + z_1''\,, \end{aligned}$$

Let us calculate the *energy of acceleration* then:

$$S = \tfrac{1}{2}\sum m\,\mathbf{J}^2 = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2),$$

while noting that:

$$\sum m\,x_1, \qquad \sum m\,x_2, \qquad \sum m\,x_3$$

are zero, so one also has:

$$\sum m\,x_1'' = \sum m\,y_1'' = \sum m\,z_1'' = 0\,.$$

We then find that:

$$S = \tfrac{1}{2}M\,\mathbf{J}_0^2 + \sum m\,\mathbf{J}_1^2\,,$$

which can be written:

$$S = \tfrac{1}{2} M\,\mathbf{J}_0^2 + S_1,$$

if we let S_1 denote the energy of acceleration that is calculated for the relative motion around the center of gravity.

One will then have a theorem that is analogous to Koenig's theorem for the *vis viva*.

Now, let us take up the problem of the hoop that was treated in no. **411** (Fig. 244) again while using this new method, but keeping the same notations.

Take the mass of the hoop to be unity. Let $\mathbf{J}_0$ denote the acceleration of the point G, and let $\mathbf{J}_1$ be the relative acceleration of a point m of the hoop with respect to some axes with fixed directions $G\,x_1\,y_1\,z_1$ that pass through G. Upon applying the preceding theorem, which is the analogue to Koenig's theorem, one will have:

$$S = \tfrac{1}{2} \mathbf{J}_0^2 + S_1 .$$

The relative motion of the hoop around the point G is the motion of a body of revolution that is suspended by a point on its axis. Upon applying the notations of the preceding section to that motion, one will have, from (14):

$$2S_1 = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(pq' - qp') + \ldots$$

It then remains for us to calculate $\mathbf{J}_0^2$. In order for that to be true, let u, v, w denote the projections of the absolute velocity of the point G onto the axes Gx, Gy, Gz : In order to express the idea that the hoop rolls, one must write out that the material point on the hoop that is found to be in contact with the floor at the point H has zero velocity. One will then have:

$$u + a\,r = 0, \qquad v = 0, \quad w - a\,p = 0 . \tag{15}$$

Since the instantaneous rotation of the trihedron $Gxyz$ is $\mathbf{\Omega}$, the absolute acceleration of the point G will have the following projections onto the axes Gy, Gy, Gz :

$$\frac{du}{dt} + Q\,w - R\,v,$$

$$\frac{dv}{dt} + R\,u - P\,w,$$

$$\frac{dw}{dt} + P\,v - Q\,u\,;$$

i.e., from (15):

$$-\,a\,(r' - Q\,p), \qquad -\,a\,(P\,p + R\,r), \qquad a\,(p' + Q\,r),$$

and upon taking the sum of the squares and noting that $P = p$, $Q = q$, one will have:

$$\mathbf{J}_0^2 = a^2 (p'^2 + r'^2) - 2a^2 q (p r' - r p') + \ldots,$$

in which we do not write out the terms that do not contain p', q', r'. Finally, we will have:

$$2S = (A + a^2) p^2 + A q'^2 + (C + a^2) r'^2 + 2 (AR - Cr) (pq' - qp') - 2a^2 q (pr' - rp') + \ldots$$

Once more, let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

denote the infinitely-small angles through which the hoop must turn around the axes Gx, Gy, Gz in order to go from one position to an infinitely-close position. Those quantities are arbitrary and determine the displacement of the hoop completely. We take λ, μ, ν to be the parameters $q_1, q_2, \ldots, q_k$ ($k = 3$), and we will again have:

$$p' = \lambda'', \qquad q' = \mu'', \qquad r' = \nu''.$$

We can then write the left-hand sides of equations of motion such as (10). It remains for us to calculate the right-hand sides. In order to do that, we must calculate the sum of the works done by the applied forces:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

and put it into the form:

$$L'\,\delta\lambda + M'\,\delta\mu + N'\,\delta\nu\,;$$

L', M', N' will be the right-hand sides of the equations. Those quantities have a simple significance: Draw three axes H_x, H_y, H_z that are parallel to the axes G_x, G_y, G_z through the point of contact H with the floor. L', M', N' will then be the sums of the moments of the applied forces when they are taken with respect to the new axes. Indeed, since the velocity of the molecule that is placed at H is zero under a displacement that is compatible with the constraint, the infinitely-small displacement of the hoop will be the resultant displacement of the three elementary rotations $\delta\lambda$, $\delta\mu$, $\delta\nu$ around the axes Hx', Hy', Hz' without displacing H: That proves the proposition.

If the only applied force is the weight g, which is applied to G, then one will obviously have:

$$L' = -ga\cos\theta, \qquad M' = 0, \qquad N' = 0\,.$$

The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = -ga\cos\theta, \qquad \frac{\partial S}{\partial \mu''} = 0, \qquad \frac{\partial S}{\partial \nu''} = 0\,;$$

i.e.: from the value of S :

$$(A + a^2)\, p' - (AR - Cr)\, q + a^2\, q\, r = -g\, a\cos\theta,$$

$$A\,q' + (AR - Cr)\,p = 0,$$

$$(C + a^2)\,r' - a^2\,pq = 0,$$

where the last two are identical to equations (9) and (10) of no. **411**, and the first one is identical to the Lagrange equation that relates to θ (no. **464**).

One can treat the general problem of the rolling of any heavy body of revolution on a plane. [See "Développements sur une forme nouvelle des équations de la Dynamique" by P. Appell, Journal de Mathématiques de M. Jordan **6** (1900), pp. 33.]

The motion of a sphere that is constrained to roll on a surface of revolution was studied by Fritz Noether at Erlangen in a thesis that was presented to the University of Munich in 1909 (edited by Teubner).

468. The equations of motion obtained by looking for the minimum of a function of degree two. – If one forms the function:

$$R = S - (Q_1\,q_1'' + Q_2\,q_2'' + \cdots + Q_k\,q_k''),$$

which contains the quantities q'' to degree two, then one will see that the equations of motion (9) can be written as:

$$\frac{\partial R}{\partial q_1''} = 0, \qquad \frac{\partial R}{\partial q_2''} = 0, \qquad \ldots, \qquad \frac{\partial R}{\partial q_k''} = 0. \tag{16}$$

Those are the equations that one must write in order to find the values of the q_1'', q_2'', …, q_k'' that will make R a minimum. Conversely, the values of q'' that one infers from those equations will make R a minimum, because the homogeneous terms of degree two in R are provided by S and constitute a positive-definite quadratic form. Since the values of q determine the accelerations, one can interpret that result by saying that *the values of the accelerations at each instant make R a minimum.*

In that statement, one can replace the function R with any other function that differs from it only by terms that are independent of the accelerations – for example, with the following two functions:

$$\frac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2) - \sum (X\,x'' + Y\,y'' + Z\,z''),$$

$$\frac{1}{2}\sum \frac{1}{m}\,[(m\,x'' - X)^2 + (m\,y'' - Y)^2 + (m\,z'' - Z)^2]\,.$$

The fact that the accelerations will make the latter function a minimum is a consequence of *Gauss's* principle of least constraint, to which we shall return at the end of the following chapter.

469. On the impossibility of characterizing a non-holonomic system by just the function *T*. – The Lagrange equations will be applicable when the constraints on a system without friction can be expressed in finite terms and one employs parameters that are true coordinates. Suppose, to simplify, that there exists a force function U. One can then write the equations of motion as soon as one knows the expressions for the semi-*vis viva* T and U as functions of the independent parameters.

If, on the contrary, the constraints cannot all be expressed in finite terms then the one can no longer apply the Lagrange equations. In order to write down the equations of motion, it will suffice to know U and the energy of acceleration $S = \frac{1}{2} \sum m\, \mathbf{J}^2$, which is formed from the accelerations in the same way that T is formed from the velocities. Then again, is that necessary?

Might there not exist equations of motion that are more general than Lagrange equations that are applicable to all cases and demand only the knowledge of the two functions T and U in order for one to write them down? We shall show that such equations do not exist. In order to do that, we shall point out two different systems in which the functions T and U are identically the same, but without the equations of motion being the same.

First system. – Imagine a heavy solid body that fulfills the following conditions:

1. The solid body is bounded by a moving edge that has the form of a circle K of radius a.

2. The center of gravity G of the body is situated at the center of the circle K.

3. The ellipsoid of inertia relative to the center of gravity G is one of revolution around the perpendicular Gz to the plane of the circle.

Now suppose that the solid body thus-constituted is constrained to roll without slipping on a fixed horizontal plane that it touches at the circular edge.

As in no. **411**, let Gz_1 be the ascending vertical that is drawn through G. Take the axis Gx to be the perpendicular to the plane zGz_1 and the Gy axis to be the perpendicular to the plane xGz. Gx is then a horizontal in the plane of the circle K, and Gy is a line of greatest slope in that plane that stops at the point where the circle touches the fixed plane. Let θ denote the angle between Gz and the ascending vertical Gz_1, and let ψ be the angle between Gx and a fixed horizontal. Those two angles determine the orientation of the trihedron $Gxyz$. In order to fix the position of the solid body with respect to the trihedron $Gxyz$, it will suffice to know the angle φ that a radius of the circle K, which is invariably coupled with to the body, make with the Gx axis. The instantaneous rotation $\boldsymbol{\omega}$ of the body will then be the resultant of the rotation of the trihedron and a rotation $d\varphi / dt = \varphi'$ around Gz. The components p, q, r are then:

$$p = \theta', \qquad q = \psi' \sin \theta, \qquad r = \psi' \cos \theta + \varphi'.$$

On the other hand, the condition that the circle K rolls shows that the square of the velocity of the center of gravity G is $a^2 (p^2 + r^2)$. By definition, upon taking the mass of the body to be unity and letting A and C denote the moments of inertia with respect to Gx and Gz, one will have:

$$2T = a^2 (p^2 + r^2) + A (p^2 + q^2) + C r^2,$$

so one will have:

$$(1) \qquad \begin{cases} 2T = A\psi'^2 \sin^2\theta + (A + a^2)\,\theta'^2 + (C + a^2)(\psi' \cos\theta + \varphi')^2, \\ U = -g\,a \sin\theta \end{cases}$$

for the defining expressions for the functions T and U.

Second system. – Let a second heavy solid body have the same form and radius a and the same mass as before. Imagine that the distribution of the mass is different, in such a fashion that if one lets A_1 and C_1 denote the moments of inertia that are analogous to A and C then one will have:

$$A_1 = A, \qquad C_1 = C + a^2.$$

We subject that body to the following two constraints: The body touches a fixed horizontal plane P_1 on which it can slide without friction along the circular edge K. The center of gravity G of the body slides without friction on a fixed vertical circumference whose radius is a and whose center O is in the fixed plane P_1 .

In order to express those constraints, take the same moving axes $Gxyz$ and the same notations as above. Let ξ, η, ζ denote the absolute coordinates of the point G with respect to the two axes $O\xi$ and $O\eta$ in the plane P_1 and an ascending vertical $O\zeta$. One can suppose that the fixed vertical circumference that is described by G is in the plane $\xi O \zeta$; one will then have:

First constraint: $\qquad \zeta = a \sin\theta,$

Second constraint: $\qquad \eta = 0, \quad \xi^2 + \zeta^2 = a^2 .$

Hence, one obviously has:

$$\xi = a \cos\theta .$$

Under those conditions, one has:

$$2\,T_1 = \xi'^2 + \eta'^2 + \zeta'^2 + A_1 (p^2 + q^2) + C_1 r^2,$$

or, from the values of ξ, η, ζ, A_1, and C_1 :

$$(2) \qquad \begin{cases} 2T_1 = A\psi'^2 \sin^2\theta + (A + a^2)\,\theta'^2 + (C + a^2)(\psi' \cos\theta + \varphi')^2, \\ U_1 = -g\,a \sin\theta. \end{cases}$$

One sees that the functions T and T_1, U and U_1 are identical. Meanwhile, the equations of motion are different, because the Lagrange equations apply to the second system, but not to the first one. That is what we would like to show.

One can remark that of the three equations of motion, two of them can be put into the same form in the two systems. Indeed, the *vis viva* integral is obviously the same for both of them. Furthermore, one has the right to write down the Lagrange equation that relates to θ (no. **464**) for the first system, which one can obviously do for the second one. However, the third equations are different for the two motions: One has the integral $r = r_0$ for the second system, which does not exist for the first one.

It is obvious that the difference between the two motions will appear immediately when one forms the two functions S and S_1 .

Remark concerning constraints expressed by relations that are nonlinear in the velocity components. – The non-holonomic constraints, such as rolling, that were considered up to now are expressed by relations that are *linear* in the differentials of the coordinates that determine the configuration of the system. However, one can consider more general constraints that are expressed by relations that are *nonlinear* in those differentials. The principle of no. **468** will again permit one to treat those questions (Appell, Comptes rendus, 8 May 1911, and Rendiconti di Palermo, 1911).

VII. – CONSTRAINTS INVOLVING SERVOS.

470. Servos (†). – In a remarkable treatise that was submitted to the Paris Science Faculty in November 1922 and which was entitled *Étude théorique des compas gyrostatiques* ANSCHÜTZ *et* SPERRY, Henri BEGHIN introduced the new notion of a "servo."

There exists an important category of mechanisms that realize their constraints by a method that is entirely different from the one that was just examined. For those mechanisms, *one cannot abstract from the way that the constraints are realized.*

The constraints that are realized by these mechanisms can be arbitrary; most often, they are holonomic. However, instead of those realizations being – so to speak – passive, such as ones that are obtained by simple contact, they use arbitrary forces (e.g., electromagnetic forces, compressed air pressure, etc.) – in a word, *auxiliary energy sources that come into play automatically and are automatically measured out in such a way as to realize this or that constraint at each instant.* One can even imagine an animate being that acts by contact and regulates its action in such a manner as to realize this or that constraint.

Let Σ be a solid body (a disc, for example) that moves around a diameter Δ under the influence of certain given forces. A solid body Σ_1 (a concentric ring, for example) of diameter Δ_1 moves around Δ without having any contact with Σ. The ring Σ_1 carries a toothed wheel a whose axis is Δ that meshes with a pinion b that is attached to the shaft of a motor M. It is easy to image an arrangement (1) that would make the motor turn in one sense or the other without acting directly on either Σ or Σ_1, while Σ and Σ_1 are never in the same plane. If α and α_1 are the azimuths of Σ and Σ_1, respectively, then the constraint:

(†) The French "liaisons par asservissement" literally means "constraints by servitude (or slavery)." However, since the standard modern term is "servo constraints," I will consistently translate "asservissement" as "servo."

(1) *See* the description of the Sperry compass (*The Sperry Gyrocompass*, 7).

$$\alpha = \alpha_1$$

will then be found to be realized in such a way that *the ring* Σ_1 *follows the disc* Σ *in all of its motions around* Δ *without being driven by it.* It is obvious that the manner in which this system behaves has nothing in common with the manner in which would behave if Σ were driven by Σ_1 by direct contact: For example, if a small spring that is fixed to Σ_1 pushes on Σ then the system will take on a uniformly accelerated motion in the case of a servo, while it will obviously remain immobile under the second hypothesis.

What are the forces of constraint in the system in the previous example? If I consider the system $\Sigma\Sigma_1$ then those forces will be, on the one hand, the reactions along the axis Δ, which are ordinary forces of constraint, and the reactions of the pinion *b* on the gear *a*. Those reactions, which play a major role in the problem, have an entirely special character, because the pinion *b* (viz., a foreign obstacle) that exerts them is not fixed, nor is it in a state of motion that is known in advance as a function of *t* : *It is an obstacle whose position is known in advance as a function of the parameters (*α, α_1*, here) upon which the system considered* $\Sigma\Sigma_1$ *depends.*

If I include the rotor *R* of the motor *M* in the system considered then the constraint forces will be the electromagnetic actions to which the rotor is subject on the part of the stator, in addition to the actions of contact between the fixed obstacles and the actions of contact $R\,\Sigma_1$, which are ordinary constraint forces. Indeed, those forces have the character of constraint forces: *They are unknown, but one knows that they have the value that is necessary in order to insure the constraint considered.*

For any elementary displacement that is compatible with the constraint $\alpha = \alpha_1$, the ordinary constraint forces will do zero work. On the contrary, the other constraint forces (whether one means the reactions of the foreign obstacles whose position depends upon parameters α, α_1 or those electromagnetic actions that are exerted at a distance on the rotor) will do non-zero work. That is how the mechanisms that include a servo are distinguished from the other ones.

General study of the mechanisms that include a servo. D'Alembert's principle. – Let Σ be a material system that presents no source of energy dissipation. In addition, suppose that no part of that system can contract or dilate, with the exception that will be assumed below.

Upon taking into account the contacts that are imposed upon it, that system will be supposed to depend upon a limited number *h* of parameters $q_1, q_2, \ldots, q_h$ in such a manner that the coordinates *x*, *y*, *z* of each element of Σ are functions of those parameters that are known in advance, and might also be:

$$(1) \qquad x = f(q_1, q_2, \ldots, q_h, t)\,, \qquad y = \ldots, \qquad z = \ldots$$

at time *t*.

Some of the foreign obstacles that Σ is in contact with are fixed or depend upon *t*. Others, as a result of the contacts imposed, are supposed to depend upon a certain number *k* of the preceding parameters – namely, $q_1, q_2, \ldots, q_k$, and also possibly *t*.

Those contact conditions are *holonomic contact constraints.*

Suppose, in addition that the system is subject to certain non-holonomic constraints; i.e., that the parameters $q_1, q_2, \ldots, q_h$ are coupled by a certain number p of differential relations that express the conditions of rolling without slipping or pivoting at certain contacts. Those relations will permit one to express the p elementary variations:

$$dq_{n+1}\,,\ dq_{n+2}\,,\ \ldots,\ dq_{n+p} \qquad (n + p = h)$$

as functions of $q_1, q_2, \ldots, q_n$, and dt ; they have the form:

$$(2) \qquad (p \text{ relations}) \qquad \begin{cases} A_1\,dq_1 + \cdots + A_h dq_h + A\,dt = 0, \\ B_1\,dq_1 + \cdots + B_h dq_h + B\,dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Those conditions are *non-holonomic contact constraints.* Those are the only two types of constraints that one encounters in modern problems.

For any elementary displacement that is compatible with the constraints that might exist at the instant t (i.e., one for which δt is zero, and $\delta q_1, \ldots, \delta q_n$ are arbitrary), the mutual reactions between the bodies of the system do zero work, as well as the reactions of the fixed obstacles or the ones that depend upon t. I will say that these reactions are *constraint forces of the first kind.*

In addition, the system Σ is supposed to be subject to other constraints that I will call *servo constraints*, which are also expressed by finite equations or linear differential equations, but are realized by means of forces that are completely different: Those forces, which will call *generalized constraint forces*, or *ones of the second kind*, are applied to the bodies in the system: They can be *external* or *internal*.

In the first case (viz., the external forces), they are either actions at a distance, such as electromagnetic ones or other kinds, which are regulated *automatically* in such a manner as to insure the finite or differential constraint that they are supposed to realize, or the contact actions with the foreign obstacles whose position is supposed to depend upon $q_1, q_2, \ldots, q_k, t$ whose motion must regulated *automatically* in such a manner that certain finite or differential equations must be verified at each instant by the parameters q.

In the second case (i.e., if those constraint forces of the second kind are internal), they will be either actions at a distance, such as electromagnetic ones, or internal stresses in the bodies that can contract or dilate (e.g., compressed air, muscles in a living being), which are stresses that are regulated *automatically* – for example, the will of the living being – in such a manner as to realize this or that constraint. Except for that exception, the system will not be supposed to be compressible.

The system Σ can be composed of an electric motor whose velocity ω is independent of the load, which might be, for example a derivative motor (*moteur-dérivation*), within certain limits. The servo constraint will then be realized in the form:

$$d\theta = \omega\,dt\,.$$

The system can be composed of a cyclist and his machine. The cyclist can contract his muscles, not by a given quantity but by a quantity that is measured out in such a way that certain constraints are found to be realized: He will regulate the action of his legs in such a manner as to realize a constant angular velocity, or perhaps he will contract the muscles of his body in such a way to realize an inclination of the frame as a function of t, etc. The methods that will be described below will permit one to study the variation of the unknown parameters.

As an application, one can also imagine a ship Σ, with one part σ of the cargo that is put into motion automatically by a motor in such a manner as to realize certain constraints: For example, as a servo constraint, one might have that the ship must remain constantly vertical, which is realized by a roll stabilizer. A small gyrostatic mechanism that is based upon the principle of the Schlick stabilizer will indicate the true vertical onboard the ship. The servomotor will come into action when that vertical is not in the plane of symmetry of the ship. One can also regulate the motion of σ in such a manner as to realize the motion of σ in such a manner as to realize some relation between its position and the inclinations of the ship. One can then change the period of oscillation of the ship at will and avoid the synchronism of the hull, when appropriate. One can regulate the motion of σ in such a manner as to realize some condition between its position and the angular velocity of the ship that permits one to damp out the oscillations, etc. The forces of constraint of the second kind here will be the mutual actions between Σ and σ.

A material system that presents constraint forces of the second kind will be said to *include a servo.* It is obvious that *the virtual work that is done by constraint forces of the second kind is generally non-zero.*

Having posed those definitions, imagine that there are r servo relations, one of which is finite, while the others are differential relations, and have the form:

$$\text{(3)} \qquad (r \text{ relations}) \quad \begin{cases} g(q_1, \ldots, q_h, t) = 0, \quad \ldots, \\ \varepsilon_1\, dq_1 + \varepsilon_2\, dq_2 + \cdots + \varepsilon_h\, dq_h + \varepsilon\, dt = 0, \quad \ldots \end{cases}$$

The virtual displacements of the system that are compatible with the *contact* constraints that might exist at the instant t ($\delta t = 0$) are obtained by taking $h - p$ of the elementary variations $\delta q_1, \ldots, \delta q_h$ arbitrarily; the other p are defined by the relations (1), which will reduce to:

$$\text{(2}'\text{)} \qquad (r \text{ relations}) \quad \begin{cases} A_1\, \delta q_1 + \cdots + A_h\, \delta q_h = 0, \\ B_1\, \delta q_1 + \cdots + B_h\, \delta q_h = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Among those displacements, there exist ones for which one can confirm *a priori* that the work done by the constraint forces of the second kind is zero, without knowing anything but the way that they act. We shall suppose that they are the ones that simultaneously verify the j relations:

$$\text{(4)} \qquad (j \text{ relations}) \quad \begin{cases} a_1\, \delta q_1 + \cdots + a_h\, \delta q_h = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1\, \delta q_1 + \cdots + l_h\, \delta q_h = 0. \end{cases}$$

D'Alembert's principle, when it is applied to any of those displacements, is expressed by the equation:

$$(5) \qquad \sum m(x''\,\delta x + y''\delta z + z''\,\delta z) = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

here, in which the Σ sign on the left-hand side extends over all elements of the system, while m denotes the mass of one of its elements, and x'', y'', z'' denote the projections of its acceleration, while the Σ on the right-hand side extends over all given forces X, Y, Z. Indeed, it is obvious that for those displacements, the constraint forces, which are either of the first or second kind, will do zero work.

That condition decomposes into $h - p - j$, since the h elementary variations $\delta q_1, \ldots, \delta q_h$ are subject to the p relations (2′) and the j relations (4), so only $h - p - j$ of those variations will be arbitrary.

In order to write those equations effectively, we employ the method of Lagrange multipliers: If x, y, z are expressed as functions of $q_1, \ldots, q_h$, t by equations (1) then the left-hand side of equation (5) will be the sum of h terms of the form:

$$(6) \qquad \delta q \sum m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q} \right) = P\ \delta q\ ,$$

in which q denotes any of the h parameters. The right-hand side is the sum of h terms of the form:

$$(7) \qquad \delta q \sum \left(X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q} \right) = Q\ \delta q\ .$$

D'Alembert's equation is written:

$$(8) \qquad (P_1 - Q_1)\ \delta q_1 + (P_2 - Q_2)\ \delta q_2 + \ldots + (P_h - Q_h)\ \delta q_h = 0\ .$$

That equation must be combined with the p relations (2′), when multiplied by the coefficients A, M, …, respectively, and the j relations (4), when multiplied by λ, μ, …, respectively, where those coefficients A, M, …, λ, μ, … constitute $p + j$ auxiliary unknowns. We will get the equation:

$$(9) \qquad \sum\ (P_i - Q_i + AA_i + MB_i + \ldots + \lambda\, a_i + \mu\, b_i + \ldots)\ \delta q_i = 0,$$

in which i represents the indices 1, 2, …, h. The multipliers A, M, …, λ, μ, … can be chosen in such a manner that the coefficients of $p + j$ of the variations δq_i will be zero, because the relations (2′) and (4) are meant to be independent in the preceding. Equation (9) must be verified for any of the other $h - q - j$ variations δq_i in such a way that the coefficients of those $h - p - j$ variations in equation (9) must also be themselves zero.

In summary, the problem comes down to solving the h equations:

(10)
$$\begin{cases} P_1 - Q_1 + AA_1 + MB_1 + \cdots + \lambda\, a_1 + \mu\, b_1 + \cdots = 0, \\ P_2 - Q_2 + AA_2 + MB_2 + \cdots + \lambda\, a_2 + \mu\, b_2 + \cdots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}$$

to which one must append the p equations (2) that express the non-holonomic contact constraints and the r servo constraints (3), namely, $h + p + r$ equations in $h + p + j$ unknowns (viz., $q_1, \ldots, q_h$, A, M, …, λ, μ, …).

If it happens that r is greater than j then the problem will generally be impossible to solve; i.e., it will not be possible to realize a number of servo constraints that is greater than the number of restrictive conditions that one must impose upon the parameters q in order to annul the virtual work done by forces of the same kind.

If r is equal to j then the problem will be solved by equations (2), (3), and (10).

If r is less than j then the motion will be indeterminate: One can imagine, moreover, that if the function that must replace the forces of the second kind is not sufficiently well-defined then their elimination will become impossible, and that the motion cannot be studied unless one is given some of them.

Special cases:

1. Suppose that equations (2′), which express the idea that the virtual displacements are compatible with the non-holonomic contact constraints, and equations (4), which one is led to introduce in order to annul the virtual work done by constraint forces of the second kind, are solved for the $p + j = m$ variations $\delta q_1, \ldots, \delta q_m$:

(11)
$$\begin{cases} \delta q_1 = \mathcal{A}_{m+1}\, \delta q_{m+1} + \cdots + \mathcal{A}_h\, \delta q_h, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_m = \mathcal{L}_{m+1}\, \delta q_{m+1} + \cdots + \mathcal{L}_h\, \delta q_h; \end{cases}$$

the Lagrange multipliers become superfluous. If one replaces the $\delta q_1, \ldots, \delta q_m$ with these expressions in (8) then one will get an equation that is linear in $\delta q_{m+1}, \ldots, \delta q_h$, which must be verified for any variations, and therefore $h - m$ equations of the form:

(12)
$$P_{m+i} - Q_{m+i} + \mathcal{A}_{m+i}\,(P_1 - Q_1) + \ldots + \mathcal{L}_{m+i}\,(P_m - Q_m) = 0,$$

in which i denotes one of the numbers 1, 2, …, $h - m$.

One must combine these equations with the p equations (2) and the r servo equations (3).

2. If the equations (11) reduce to:

(13)
$$\delta q_1 = 0, \qquad \ldots, \qquad \delta q_m = 0$$

then the equations of motion will reduce to the simple form:

(14) $$P_{m+1} = Q_{m+1}\,, \quad \ldots, \qquad P_h = Q_h\,.$$

3. Suppose that the constraint forces of the second kind are uniquely the contact actions of an auxiliary system Σ_1 of moving obstacles whose positions depend upon a certain subset $q_1, \ldots, q_k$ of the parameters $q_1, \ldots, q_h$. In that case, the relations (4) will be:

(15) $$\delta q_1 = 0, \qquad \ldots, \qquad \delta q_k = 0,$$

because it is by leaving those obstacles fixed that one will annul the work done by their actions on the given system Σ. The multipliers $\lambda, \mu, \ldots$ will become superfluous, because equation (8) will no longer contain $\delta q_{k+1}, \ldots, \delta q_h$. Equations (10) will reduce to the following $h - k$ equations:

(16) $$\left\{ \begin{array}{l} P_{k+1} - Q_{k+1} + AA_{k+1} + MB_{k+1} + \cdots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ P_h - Q_h + AA_h + MB_h + \cdots = 0, \end{array} \right.$$

and as in the general case, one must combine these with the p equations (2) and the r relations (3), so one will have $h - k + p + r$ relations in $h + p$ unknowns. The problem will be determinate when the number of servo-equations is equal to the number k of parameters that the auxiliary system Σ_1 depends upon.

4. With the same hypotheses as in the preceding paragraph (3.), we suppose, in addition, that the contact constraints on the system are all holonomic ($p = 0$). The multipliers $A, M, \ldots$ will also become superfluous, and equations (10) will reduce to the following $h - k$ equations:

(17) $$P_{k+1} = Q_{k+1}\,, \quad \ldots, \qquad P_h = Q_h\,,$$

and one must append the r equations (3), which express the servo. The unknowns are uniquely $q_1, \ldots, q_h$.

Remarks:

1. In systems without servos, the virtual displacements to which one applies d'Alembert's equation will be the ones that are compatible with all of the constraints. In the systems that include servos, things will be different for some displacements: There will then exist analytical reasons for the difference that exists between those two categories of systems, and one can understand all of the interest that is attached to the mechanisms that include servos from the industrial standpoint.

2. In the case where the constraint forces of the second kind are uniquely the reactions of the moving obstacles whose positions are functions of some of the parameters q (cases 3. and 4.), the solution of the problem will be independent of the inertia of those bodies and the given forces that are applied to them.

Thus, if one can define two parts Σ, Σ_1 of a system that is subject to r servo constraints such that the partial system Σ is not subject to any constraint force of the second kind, outside of the reactions of the system Σ_1 , and if, on the other hand, the number of parameters that the system Σ_1 depends upon is equal to the number of servo conditions then the inertial forces and the given forces that are applied to Σ_1 will not influence the motion of Σ . The method that was indicated in the special cases 3. and 4. will permit one to put the problem into the form of equations without introducing either inertial forces or given forces. The partial system will then play an auxiliary role. That special case frequently presents itself in the applications.

Equilibrium in systems that include a servo. – D'Alembert's principle will give the equilibrium conditions when one suppresses the P, which are the terms that are provided by the inertial forces in the system considered. Equations (10), which relate to the general case, and equations (12), (14), (16) or (17), which relate to the special cases that were studied, will then give the equilibrium equations if one replaces the P with zero. One must combine those equations with the servo equations, which are finite. The differential equations that express non-holonomic constraints, which are either contact constraints or servo constraints, must obviously not be appended; they are verified identically.

Extending the Lagrange equations. – With the same general conditions that were defined at the outset of this discussion, the coordinates x, y, z of the various elements of the system considered Σ can be expressed by finite expressions [eq. (1)] as functions of time t and the parameters $q_1, \ldots, q_h$ that depend upon the system when one takes into account only the holonomic contact constraints; now, the expression:

$$P = \sum m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right)$$

will now have the value:

$$P = \frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q}.$$

One will then extend the Lagrange equations to the systems that include a servo by replacing $P_1, \ldots, P_h$ with their expressions in equations (10).

It is essential to remark that the *vis viva* must be calculated as a function of the $q_1, \ldots, q_h, q_1', \ldots, q_h', t$ *without taking into account the servo constraints.* The same thing will be true for the elementary work done by the given forces:

$$Q_1\,\delta q_1 + \ldots + Q_h\,\delta q_h .$$

If those forces admit a force function – i.e., if $Q_1, \ldots, Q_h$ are the derivatives $\frac{\partial U}{\partial q_1}, \ldots, \frac{\partial U}{\partial q_h}$ of a function U of $q_1, \ldots, q_h, t$ – then that function U will be calculated without including the servo. It is only in the equations themselves – i.e., in the expressions Q, $\frac{\partial T}{\partial q}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right)$ – that one can take them into account. Meanwhile, when the derivative of $\partial T / \partial q'$ with respect to t is taken for the real motion, which is compatible with the servo constraints, one can carry out all of the simplifications on $\partial T / \partial q'$ that result from those constraints before differentiating with respect to t. In summary: *One can take the servo into account after concluding the calculation of the three categories of expressions* Q, $\frac{\partial T}{\partial q}$, $\frac{\partial T}{\partial q'}$.

Vis viva equation. – Since the contact constraints are not supposed to depend upon t, in particular, equations (2), which represent the non-holonomic constraints, will have no terms in dt ($A = B = \ldots = 0$), because the given forces are supposed to admit the force function $U(q_1, \ldots, q_h)$, we multiply equations (10), which give the motion in the general case, by $dq_1, \ldots, dq_h$, resp., which are elementary variations of the parameters under the real displacement, and the expression:

$$P_1\, dq_1 + \ldots + P_h\, dq_h$$

will give the work done by the inertial forces, with the sign changed:

$$\sum m\,(x''\,dx + y''\,dy + z''\,dz)\;;$$

i.e., the differential dT of one-half the *vis viva*.

The expression:

$$Q_1\, dq_1 + \ldots + Q_h\, dq_h$$

is equal to dU. The multiplier A has the coefficient:

$$A_1\, dq_1 + \ldots + A_h\, dq_h\,,$$

which is zero, since the displacement verifies equations (2). The same thing will be true for the analogous coefficients M, …

One will then have the equation:

$$d(T - U) + \lambda\,(a_1\, dq_1 + \ldots + a_h\, dq_h) + \mu\,(b_1\, dq_1 + \ldots + b_h\, dq_h) + \ldots = 0\,.$$

One sees that $T - U$ is not constant. Since the terms in λ, μ, … represent the elementary work done by the constraint forces of the second kind, which is not zero, in general, the conditions (4) will not be imposed upon the real displacement. According to its sign, that work will correspond to a *gain* or a *loss of mechanical energy* for the system Σ considered.

The same thing will be true in each of the special cases that were defined before: The combination of the *vis vivas* will not be given by the expression $d(T - U)$, because only some of the expressions $P_1, \ldots, P_h$, $Q_1, \ldots, Q_h$ will enter into the equations of motion.

It is interesting to conclude that *the servo can permit one to increase or decrease the desired mechanical energy of a system, and in particular, it can damp out the oscillations of a system that presents no source of dissipation of energy*.

Application. – Let a plate Σ in a fixed plane articulate with a circular base plate Σ_1 that moves around its center O at a point C. A force that is parallel to a fixed line Ox and has a constant magnitude F is exerted on the plate Σ at a point A that is located along the line that joins C to the center of gravity G. A servomotor M acts on the base plate Σ_1 by way of gears, in such a manner as to constantly realize the constraint:

$$\alpha - \beta = \frac{\pi}{2} \tag{1}$$

$$[\alpha = (Ox, OC),\ \beta = (Ox, CA),\ OC = R,\ CA = a,\ CG = b].$$

Since there is just one servo constraint, and on the other hand, the base plate Σ_1 depends upon just one parameter α, the system Σ, taken by itself, will belong to the special case 4 (pp. 7, from the beginning of section VII). One can then apply the Lagrange equations to just the plate Σ. One sees that the mass of the base plate Σ_1 will have no influence on the motion. The *vis viva* of Σ is:

$$2T = M\,(R^2\,\alpha'^2 + b^2\,\beta'^2 + 2R\,b\,\alpha'\,\beta' \cos(\alpha - \beta) + k\,\beta'^2),$$

where Mk^2 denotes the moment of inertia of Σ about G.

The virtual work done by the force F is:

$$d\mathcal{T} = F\,\delta(R\cos\alpha + a\cos\beta),$$

except that the equation that relates to β is written:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \beta'}\right) - \frac{\partial T}{\partial \beta} = -F\,a\sin\beta. \tag{4}$$

Now:

$$\frac{\partial T}{\partial \beta'} = M\,[b^2\,\beta' + 2R\,b\,\alpha' \cos(\alpha - \beta) + k\,\beta'] = M\,(b^2 + k^2)\,\beta',$$

if one takes the servo constraint into account. On the other hand:

$$\frac{\partial T}{\partial \beta} = MR\,b\,\alpha'\,\beta \sin(\alpha - \beta) = MR\,b\,\beta'^2.$$

The equation of motion is then:

$$M\,(b^2 + k^2)\,\beta'' - MR\,b\,\beta'^2 + F\,a\sin\beta = 0. \tag{5}$$

If the constraint $\alpha - \beta = \pi / 2$ is realized by direct contact between Σ and Σ_1 then the motion will be completely different: It will be regulated by the equation:

$$[M (b^2 + k^2 + k^2) + I_1] \beta'' + F (a \sin \beta + R \cos \beta) = 0, \tag{6}$$

in which I_1 denotes the moment of inertia of the base plate about O. Equation (5) will easily give the motion: β'^2 is obtained by adding a term that is sinusoidal in β to a term that is exponential. β varies between two limits, one of which can be pushed out to infinity. On the contrary, equation (6) will give a pendulum motion.

The equilibrium positions are obtained by annulling the right-hand side of equation (4). One will then find the two positions for which CA is parallel to the force. On the contrary, equation (6) will give the positions for which OA is parallel to the force.

Extending the equations in no. **465**. – The equations in no. **465** present the following advantages:

1. They can be applied to systems that are subject to non-holonomic constraints without one having to introduce a system of multipliers as auxiliary unknowns.

2. They permit one use auxiliary parameters that are coupled with the true coordinates $q_1, \ldots, q_h$ by some differential relations.

Therefore, let Σ be a system that fulfills the conditions that were indicated at the beginning of this article (pp. 2). Upon taking into account the *holonomic contact constraints* that are imposed upon its position, which will depend upon h parameters $q_1, \ldots, q_h$, and possibly t, in such a way that the coordinates of each element of matter will be finite functions of the form:

$$x = f(q_1, \ldots, q_h, t), \qquad y = \ldots, \qquad z = \ldots \tag{1}$$

Suppose that these parameters are combined with s auxiliary parameters $q_{h+1}, \ldots, q_{h+s}$ that are coupled with the preceding ones by some differential relations that serve as their definitions, *which are relations that do not, in turn, depend upon any constraint force.* One counts them with the relations that express the non-holonomic contact constraints, because they enter into the formulation of equations in the same way.

We then have p differential relations ($p \geq s$) of the form:

$$(p \text{ relations}) \quad \begin{cases} A_1\, dq_1 + \cdots + A_{h+s}\, dq_{h+s} + A dt = 0, \\ A_1\, dq_1 + \cdots + A_{h+s}\, dq_{h+s} + A dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{2}$$

Suppose that the servo constraints are represented by r finite or differential relations:

$$(3) \qquad (r \text{ relations}) \qquad \begin{cases} g(q_1, \cdots, q_{h+s}, t) = 0, \\ \dots\dots\dots\dots, \\ \varepsilon_1\, dq_1 + \cdots + \varepsilon_{h+s}\, dq_{h+s} + \varepsilon\, dt = 0, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

Finally, the virtual displacements that annul the work done by constraint forces of the second kind are the ones that verify the j relations:

$$(4) \qquad (j \text{ relations}) \qquad \begin{cases} a_1\, \delta q_1 + \cdots + a_{h+s}\, \delta q_{h+s} = 0, \\ b_1\, dq_1 + \cdots + b_{h+s}\, \delta q_{h+s} = 0, \\ \dots\dots\dots\dots\dots \end{cases}$$

Having said that, form the expression:

$$S = \tfrac{1}{2} \sum m (x''^2 + y''^2 + z''^2) \,,$$

which is called the *energy of acceleration*. If we express x'', y'', z'' by means of the parameters $q_1, \ldots, q_h$, which are functions of t, and the first and second derivatives of the parameters q with respect to t then we have seen that the terms P in the d'Alembert equation will have the expressions:

$$P_1 = \frac{\partial S}{\partial q_1''}, \qquad \ldots, \qquad P_h = \frac{\partial S}{\partial q_h''};$$

hence, one establishes the equations of motion.

Case where the differential equations of contact constraint and the definitions (2) *are solved for the p variations dq.* – In order for the equations of motion to appear with their full simplicity, it is useful to solve those p equations (2) for p of the $h + s = n + p$ variations dq. On the one hand, one expresses the p derivatives $q'_{n+1}, \ldots, q'_{n+p}$ as functions of the $q'_1, \ldots, q'_n$ by means of relations of the form:

$$(5) \qquad \begin{cases} q'_{n+1} = \alpha_1\, q'_1 + \cdots + \alpha_n\, q'_n + \alpha, \\ \dots\dots\dots\dots\dots \\ q'_{n+p} = \gamma_1\, q'_1 + \cdots + \gamma_n\, q'_n + \alpha, \end{cases}$$

and on the other hand, one expresses the p virtual displacements $\delta q_{n+p}, \ldots, \delta q_{n+1}$ as functions of the $\delta q_1, \ldots, \delta q_n$:

(6)
$$\begin{cases} \delta q_{n+1} = \alpha_1\, \delta q_1 + \cdots + \alpha_n\, \delta q_n, \\ \dots\dots\dots\dots\dots\dots\dots \\ \delta q_{n+p} = \gamma_1\, \delta q_1 + \cdots + \gamma_n\, \delta q_n; \end{cases}$$

the coefficients α_i , …, α_i are functions of q_1, …, q_{n+p} , t. Of course, those parameters q_1, …, q_n can be chosen from among the true coordinates, as well as from among the auxiliary parameters q_{h+1} , …, q_{n+s} .

Having said that, instead of expressing S as a function of the parameters q_1, …, q_n and their first and second derivatives, as we supposed in the preceding paragraph, it can be interesting to use equations (5), which replace equations (2). Upon differentiating them with respect to t, we will express the second derivatives q''_{n+1}, …, q''_{n+p} as functions of the q''_1, …, q''_n, and the first and derivatives of the parameters q. We can then make the p second derivatives q''_{n+1}, …, q''_{n+p} , disappear from S. S will become a function of the q_1, …, q_{h+s} , t, q'_1, …, q'_{h+s}, and the n second derivatives q''_1, …, q''_n. We know that under those conditions the virtual work that is done by the inertial forces (with the sign changed) will be:

(7)
$$\left(\frac{\partial S}{\partial q''_1}\right)\delta q_1 + \cdots + \left(\frac{\partial S}{\partial q''_n}\right)\delta q_n .$$

On the other hand, if one expresses the virtual work done by the given forces in terms of the δq_1, …, δq_n by using only the relations (6) then one will an expression of the form:

(8)
$$Q_1\ \delta q_1 + \ldots + Q_n\ \delta q_n$$

for that work.

Those two expressions must be equal for any displacement that annuls the work done by constraint forces of the second kind; i.e., one that verifies the j relations (4). Here again, it is interesting to take the relations (6) into account, which will permit one to make the δq_{n+1} , …, δq_{n+p} disappear from equations (4); when those equations are solved for j of the remaining variations δq_1 , …, δq_n , they will be written:

(9)
$$\begin{cases} \delta q_1 = \mathcal{A}_{j+1}\, \delta q_{j+1} + \cdots + \mathcal{A}_n\, \delta q_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \delta q_j = \mathcal{L}_{j+1}\, \delta q_{j+1} + \cdots + \mathcal{L}_n\, \delta q_n. \end{cases}$$

If one replaces δq_1 , …, δq_j with these values in the expressions (7) and (8) and expresses their equality for any of the arbitrary remaining δq_{j+1} , …, δq_n then one will get the equations of motion in the form:

$$(10)\qquad \begin{cases} \left(\dfrac{\partial S}{\partial q''_{j+1}} - Q_{j+1}\right) + \mathcal{A}_{j+1}\left(\dfrac{\partial S}{\partial q''_1} - Q_1\right) + \cdots + \mathcal{L}_{j+1}\left(\dfrac{\partial S}{\partial q''_j} - Q_j\right) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left(\dfrac{\partial S}{\partial q''_n} - Q_n\right) + \mathcal{A}_n\left(\dfrac{\partial S}{\partial q''_1} - Q_1\right) + \cdots + \mathcal{L}_n\left(\dfrac{\partial S}{\partial q''_j} - Q_j\right) = 0. \end{cases}$$

Those equations are simpler than the Lagrange equations that one can write for the same problem [*see* eq. (12), pp. 6], because the number of terms in each of the preceding equations is $j + 1$, instead of $m + 1 = p + j + 1$ in the case of the Lagrange equations. The complication that is thus introduced by the presence of the coefficients $\mathcal{A}$ and $\mathcal{L}$ is provided solely by the relations that express the idea that the work done by constraint forces of the second type is zero, and is not at all provided by the non-holonomic constraints.

One can append the p equations (5) to equations (10), along with the r equations (3) that express the servo constraint.

Case where the displacements that annul the virtual work done by the constraint forces of the second kind are defined by j relations of the form:

$$(11)\qquad \delta q_1 = 0, \qquad \ldots, \qquad \delta q_j = 0 .$$

With the same hypotheses as before, suppose that the conditions that a displacement must fulfill in order to annul the virtual work done by constraint forces of the second kind have the simple form (11). One can, moreover, always place oneself in that case by introducing some conveniently-chosen auxiliary parameters, if necessary.

In that case – which is, in summary, the general case – if one performs the calculations as was just said then equations (10) will be simplified and will take the same form as in the case of a system without servos:

$$(12)\qquad \frac{\partial S}{\partial q''_{j+1}} = Q_{j+1}\,, \quad \ldots, \qquad \frac{\partial S}{\partial q''_n} = Q_n\,.$$

One sees that the equations of no. **465** will give a general solution of the question in a form that is simpler than the Lagrange equations. One must combine those $n - j$ equations with the p equations (5) and the r servo equations (3). If $r = j$ then the number of equations will be equal to the number of unknowns.

Application. – A material plane P can slide by translation on a fixed horizontal plane xOy. A sphere Σ of radius R can roll without slipping on that plane. The motion of the plane P is regulated automatically in such a manner that the center of the sphere turns uniformly around Oz with the velocity ω with respect to the fixed axes Ox, Oy, Oz. Let us study the motion by means of the equations of no. **465**.

Let u, v be the coordinates of a distinguished point A on the plane P with respect to the axes Ox, Oy, Oz. The position of that plane is defined by only those two parameters. The position of the sphere is defined by the coordinates ξ, η of its center, and for example, the Euler angles φ, θ, ψ, which define its orientation.

If p, q, r are the projections onto the axes of the instantaneous rotation of the sphere then the conditions that express the rolling without slipping will be obtained by writing that the material element of the sphere and the material element of the plane, which coincide at the instant t, have the same velocity:

(1) $$\xi' - q\,R = u', \quad \eta' + p\,R = v' .$$

There are two servo constraints:

(2) $$d\xi + \omega\eta\,dt = 0, \qquad d\eta - \omega\xi\,dt = 0 .$$

Since the number of these relations is equal to the number of parameters that the position in the plane P depend upon, one can answer the question by applying the equations of no. **465** to just the sphere Σ .

Upon taking just the holonomic contact constraints into account, the sphere will be considered to depend upon the seven parameters u, v, ξ, η, φ, θ, ψ ($h = 7$). It is interesting to combine them with three auxiliary parameters ($s = 3$) that are coupled with the preceding one by the relations:

(3) $$d\lambda = p\,dt, \qquad d\mu = q\,dt, \qquad d\nu = r\,dt .$$

These $h + s = 10$ parameters are coupled by those three relations and by the two relations (1) that express the non-holonomic contact constraints. Those relations (1) can be written:

(1′) $$d\xi - R\,d\mu = du, \qquad d\eta + R\,d\lambda = d\nu .$$

The relations (3) and (1′) are the p differential relations [eq. (2), pp. 11] of the general theory ($p = 5$).

We keep $h + s - p = n = 5$ parameters from the $h + s = 10$ parameters; we choose u, v, ξ, η, ν . We express the energy of acceleration S of the sphere as a function of the second derivatives of those n parameters by using the $p = 5$ relations (3) and (1′). Now, the value of S is defined by:

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5} M\,R^2\,(p'^2 + q'^2 + r'^2),$$

or, from (3) and (1′):

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5} M\,R^2\,[(v'' - \eta'')^2 + (\xi'' - u'')^2 + R^2\nu''^2] .$$

The virtual displacements that annul the work done by constraint forces of the second kind are defined by the $j = 2$ conditions:

(5) $$\delta u = 0, \qquad \delta v = 0,$$

since those forces are the reactions of the plane on the sphere. Those conditions have the form indicated in the preceding paragraph [eq. (11)], in such a way that the equations of motion have the form [eq. (12)]:

$$(6) \qquad \frac{\partial S}{\partial \xi''} = \Xi , \qquad \frac{\partial S}{\partial \eta''} = \mathrm{H} , \qquad \frac{\partial S}{\partial \nu''} = \mathrm{N} .$$

The right-hand sides are zero, since the given forces (viz., weight of the sphere) do zero work, and we get the equations:

$$(7) \qquad 7\xi'' = 2u'', \qquad 7\eta'' = 2v'', \qquad \nu'' = 0,$$

which will answer the question when they are combined with the servo equations (2). Those five equations can be integrated immediately and will show that the point A describes a cycloid. The formulas (1′) show that the instantaneous rotation vector will remain parallel to the generators of an oblique cone whose base is a horizontal circle that describes the angular velocity ω .

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