"Die Matrizentheorie der Statik," Ingenieur-Archiv **25** (1957), 174-192.

The matrix theory of statics

By **J. H. Argyris**

Translated by D. H. Delphenich

1. Introduction. – The following article ([1]) gives an overview and a considerable extension to the matrix theory of statics of elastic structures that the author had developed in an earlier treatise ([2]).

Several years ago, we already proved that none of the usual methods of statics were actually suitable for determining the stress distribution and compliance matrices of the highly statically-indeterminate systems of modern aircraft constructions. Similar complications also appear in other domains of application of statics. The iteration processes can be useful in certain cases, but in general they are too lengthy and have not proved themselves for the membrane-like and shell-like aircraft structures.

We can overcome those difficulties with the matrix formulation of statics in conjunction with electronic digital computers. Not only does the matrix formulation allows us to give a much clearer form to the calculations, but it is also the ideal notation for digital computers ([3]). In addition, the theoretical derivations of matrix theory are so transparent and elegant that new and practically-worthwhile relations that would be impossible (or at least hard) to understand in the usual notation would now prove to be quite simple.

The theory will be developed for both the force and deformation procedures here. I will show that the two methods are dual: Any relation in the one process has a corresponding relation in the other process that can be obtained by a simple "translation." We also refer to the previous work ([2]), in which the advantages and disadvantages of the two methods are treated thoroughly. For continuous structures like aircraft shells, the force procedures are to be preferred, in general.

In Section **2**, the laws of unit loads and unit displacements are recalled, which enable one to develop an especially elegant derivation of the force and deformation procedures. Here, we shall confine ourselves to small displacements and distortions, for which the usual equilibrium conditions and distortion expressions will be valid. On the other hand, the law of elasticity can be nonlinear. The laws of units take an especially-simple form for structures.

([1]) Extended version of the lecture by the author at the 1956 Whitsuntide at the Society for Applied Mathematics and Mechanics in Stuttgart.

([2]) **J. H. Argyris**, Aircraft Engineering **26** (1954), pps. 347, 383, and **27** (1955), pps. 42, 80, 125, 145. See also **J. H. Argyris** and **S. Kelsey**, Wissenschaftliche Gesellschaft für Luftfahrt, Jahrbuch 1956.

([3]) **P. M. Hunt**, Aircraft Engineering **28** (1956), pps. 70, 111, 155.

Section **3** gives an extended formulation of the laws of unit loads and unit displacements for the special case of linearly-elastic systems. After that, the compliance and stiffness of a structure will be treated thoroughly. The actual static calculation of systems by force and displacement procedures will then be developed in Section **4**. All types of loading, as well as temperature effects, will be considered. It will be shown that we can obtain the compliance (or stiffness) of the structure quite simply as the end result of determining the stresses. In order to do that, we have all of the information that is needed to carry out the dynamical calculations, and in particular, the investigation of natural vibrations. *In all cases, it is unnecessary to determine the compliance or stiffness separately from the static calculations.* With the methods that are given here, it is possible in practice to investigate complicated systems like aircraft wings ([2,3]) and fuselages systematically for all static and dynamical cases. It is also self-evident that the treatment of problems for beams and frameworks is a simple problem, even when one considers shear deformations. It is noteworthy that we will require only three basically-simple matrices and one column matrix of loads in all static calculations.

The general theory of Section **4** can be applied to any structure, so to one with cutout sections, as well. However, the practical calculations with such systems will generally be more complicated and less transparent than they will be for the corresponding structure without cutout sections. That will obviate a special method (Section **5**) that appeared before in the aforementioned work (2), and which makes it possible to derive the stress distribution in the structure with cutout sections from the static calculations for continuous systems. That procedure is amazingly simple and is also suitable for trusses and frameworks whose regular construction is perturbed by only a few missing components.

In practice, it always happens that individual members in a structure will be modified after completing the static calculation. In order to avoid repeating the entire study of elasticity, a method will be derived in Section **6** for determining the stress distribution of modified systems from the one in the original system. That new procedure is a generalization of the method of Section **5** and is much shorter than the direct calculation with the modified structure in most cases.

Works on the matrix theory of statics, and the force method in particular, have been published before ([1]). However, it seems that none of those treatises are as general and as simple as the present theory.

2. General foundations. –

a) *Introduction.* – In this section, we shall treat the basic laws of the matrix theory of statics. As we mentioned, we shall confine ourselves to the case of small displacements and deformations. However, the laws of elasticity can still be nonlinear, as long as they are single-valued (Fig. 1). In order to make the duality between the force and deformation procedures especially clear, we

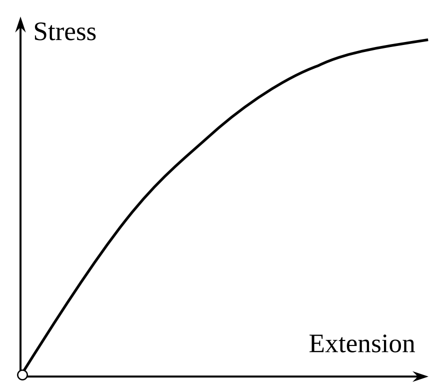


Figure 1. Nonlinear law of elasticity.

([1]) **B. Langfors**, Aeronautical Sciences **19** (1952), pp. 451; **H. Falkenheiner**, La Recherche Aeronautique **17** (1950).

shall exhibit the laws and their derivations in two corresponding columns, and indeed the left column will always describe the force procedures. The law that is dual to each relation can be easily derived with the use of the following "dictionary":

Force procedures	**Deformation procedures**
Force (stress)	Displacement (deformation)
Displacement (deformation)	Force (stress)
Compliance = $\dfrac{\text{displacement}}{\text{force}}$	Stiffness = $\dfrac{\text{force}}{\text{displacement}}$

The duality of compliance and stiffness was already treated thoroughly by **C. B. Beizeno** and **R. Grammel** ([1]).

The concept of force is taken in its general sense here. Therefore, it can also refer to a group of forces or a moment. Similarly, the term displacement can also include a group of displacements or a rotation.

In a continuum, the stresses and deformations of an element *dx dy dz* are written as column matrices:

$$\boldsymbol{\sigma} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}\} \qquad \text{and} \qquad \boldsymbol{\varepsilon} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}\}\ . \tag{1}$$

Observe that {……} will always mean a column matrix. The general notation for matrices will involve square brackets [……].

In the following study, which refers to structures, the concept of stress (deformation, resp.) is understood in its general sense, and also includes internal forces, moments, etc. (changes in in length, rotations, etc.). The notations for stresses and deformations in structures will then be S, v.

b) *The unit laws.*

The law of unit load

We let $\boldsymbol{\varepsilon}$ denote the state of deformation and let r denote an associated displacement of a body that results from given loads, temperature expansion, or any sort of applied deformations, such as ones that are created by, e.g., manufacturing defects or support displacements ([2]) (Fig. 2a).

The relationship between $\boldsymbol{\varepsilon}$ and r is a kinematical one and can be derived from the compatibility conditions. In what follows, we

The law of unit displacement

We let $\boldsymbol{\sigma}$ denote the state of stress and let R denote an associated force in a body that results from given displacements, temperature stresses, or any sort of applied constraint stresses, such as ones that are created by, e.g., manufacturing defects or support displacements ([2]) (Fig. 2b).

The relationship between $\boldsymbol{\sigma}$ and R is a static one and can be derived from the equilibrium conditions. In what follows, we will refer to the

([1]) **C. B. Biezeno** and **R. Grammel**, *Technische Dynamik*, Chap. II, numbers **9** and **10**, 1st ed., Berlin 1939.
([2]) In the following, we shall use the collective notation: given strains for the various types of loads.

will refer to the compatibility conditions as kinematical compatibility conditions.

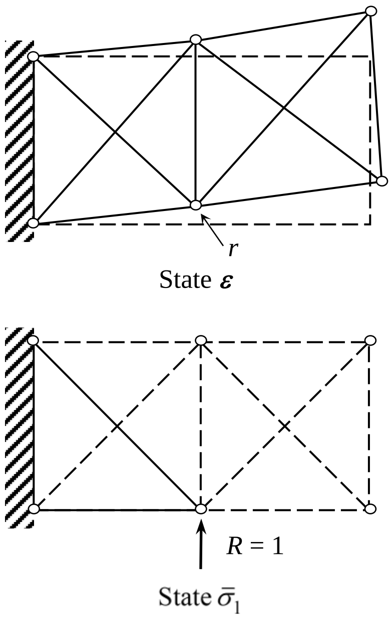


Figure 2a. The law of unit load

We also introduce:

$\bar{\boldsymbol{\sigma}}_1$ is a (virtual) stress state that results from a force $R = 1$ in the direction of r, but which needs to satisfy only the static, but not the kinematical, compatibility conditions. The stress state is chosen such that no displacements of the state $\boldsymbol{\varepsilon}$ in the direction of the forces that are found to be in equilibrium with $R = 1$ are present (Fig. 2a).

We refer to $\bar{\boldsymbol{\sigma}}_1$ as a statically-compatible stress state. In all of the following developments, quantities with overbars will refer to statically-compatible stress distributions.

The law of unit loads now replaces the kinematical relationship between $\boldsymbol{\varepsilon}$ and r with the use of statically-compatible $\bar{\boldsymbol{\sigma}}_1$. It is written ([1]):

$$1 \cdot r = \int \bar{\boldsymbol{\sigma}}_1' \boldsymbol{\varepsilon} \, dV , \qquad (2.a)$$

equilibrium conditions as static compatibility conditions.

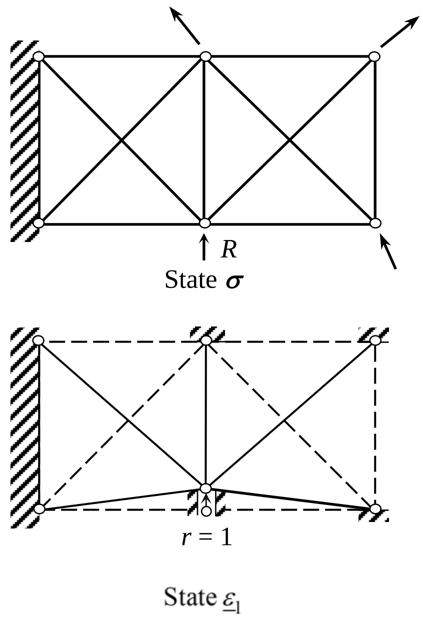


Figure 2b. The law of unit displacement

We also introduce:

$\underline{\boldsymbol{\varepsilon}}_1$ is a (virtual) deformation state that results from a displacement $r = 1$ in the direction of R, but which needs to satisfy only the kinematical, but not the static, compatibility conditions. The displacement state is chosen such that no forces in the state $\boldsymbol{\sigma}$ act in the direction of the displacements that are associated with $r = 1$ (Fig. 2b).

We shall refer to $\underline{\boldsymbol{\varepsilon}}_1$ as a kinematically-compatible deformation state. In all of the following developments, underlined quantities will refer to kinematically-compatible deformation distributions.

The law of unit displacements now replaces the static relation between $\boldsymbol{s}$ and R with the use of kinematically-compatible $\underline{\boldsymbol{\varepsilon}}_1$. It is written ([1]):

$$1 \cdot R = \int \underline{\boldsymbol{\varepsilon}}_1' \boldsymbol{\sigma} \, dV , \qquad (2.b)$$

([1]) Cf., foonote ([2]), pp. 1.

in which the integration extends over the entire volume [1]. Equation (2.b) can be derived most simply from the principle of virtual forces.

Since the law (2.a) replaces only kinematical, but not static relations, it is actually unnecessary since $\boldsymbol{\varepsilon}$ is a true deformation state. It is sufficient that one has:

$$\boldsymbol{\varepsilon} = \underline{\boldsymbol{\varepsilon}},$$

in which $\underline{\boldsymbol{\varepsilon}}$ is a kinematically-compatible deformation state that results from the given strain.

in which the integration extends over the entire volume [1]. Equation (2.b) can be derived most easily from the principle of virtual displacements.

Since the law (2.b) replaces only static, but not kinematical relations, it is actually unnecessary, since $\boldsymbol{\sigma}$ is a true stress state. It is sufficient that one has:

$$\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}},$$

in which $\bar{\boldsymbol{\sigma}}$ is a statically-compatible stress state that results from the given strain.

c) *Application to structures.* – Here, we are interested in the application of the matrix theory of statics to structures, in particular. Any framework will be decomposed into a finite number of elements for practical calculations. We assume that the stresses and deformations in each element are determined completely when certain forces or stresses S and displacements or deformations v are known on the boundaries of the element. The simplest example of an element is a rod in an ideal framework. In that case, it is sufficient to prescribe a stress or deformation in each element in order to know the entire stress or deformation field. In a continuous structure, like an aircraft shell, we subdivide the system into a net of lines and again refer to the parts of the structure that lie between two intersecting pairs of neighboring lines of the net as elements. The points of intersection of the lines of the net [2] will be referred to as nodes. It is generally assumed that the external forces act at the nodes [3], and that the longitudinal stresses vary linearly between neighboring points [3]. We cannot go into the details of that procedure here, but simply refer to the previous articles [4].

We now consider a structure that consists of l elements. External forces $R_1, \ldots, R_m$ or displacements $r_1, \ldots, r_m$ can be prescribed at m nodes of the system.

The following notations will be introduced:

$$\left.\begin{array}{ll} \mathbf{R} = \{R_1, \ldots, R_m\} & \text{external forces,} \\ \mathbf{S} = \{S_1, \ldots, S_m\} & \text{stresses or forces in the } l \text{ elements that result from } \mathbf{R}, \\ \mathbf{r} = \{r_1, \ldots, r_m\} & \text{displacements in the direction of } \mathbf{R}, \\ \mathbf{v} = \{v_1, \ldots, v_m\} & \text{deformations of } l \text{ elements that result from } \mathbf{r}. \end{array}\right\} \qquad (3)$$

---

(1) A prime denotes the transpose or reflected matrix.
(2) The lines of the net do not need to be orthogonal.
(3) These assumptions are not necessary, see Section 4.a).
(4) Cf., footnote (2) on pp. 1.

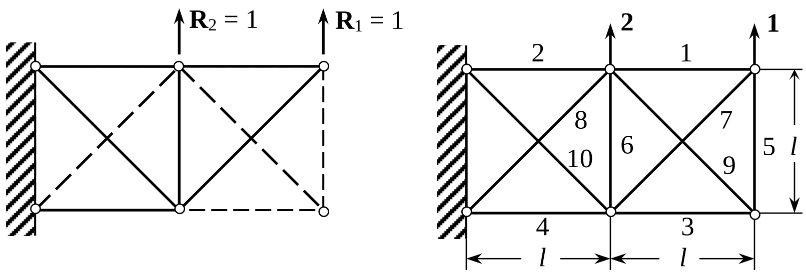


Figure 3a. Example of the calculation of an $\bar{\mathbf{b}}$-matrix.

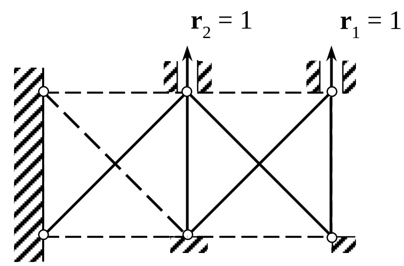


Figure 3b. Example of the calculation of a $\underline{\mathbf{a}}$-matrix.

When more than one stress or deformation of the element is prescribed, S_1, v_1, etc., will denote submatrices. *The displacement state* (**r**, **v**) *and the stress state* (**R**, **S**) *can be independent of each other.* One sees that **R**, **S**, **R**, and **v** are column matrices.

The unit load law

If $\bar{\mathbf{S}}$ is a statically-compatible stress state that corresponds to the forces **R** then we can always set:

$$\bar{\mathbf{S}} = \bar{\mathbf{b}}\,\mathbf{R}\,, \tag{4.a}$$

in which $\bar{\mathbf{b}}$ is a rectangular matrix that is calculated from merely static relationships (see also the definition of $\bar{\boldsymbol{\sigma}}_1$). One sees that (4.a) is also valid for nonlinear systems.

As an example of a $\bar{\mathbf{b}}$ matrix, we consider the statically-compatible stress state in Fig. 3a. Here, we have:

$$\bar{\mathbf{b}} = \begin{array}{c} \begin{array}{cc} R_1 & R_2 \end{array} \\ \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 2 & 1 \\ 0 & 0 \\ 0 & 1 \\ \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ -\sqrt{2} & -\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \tag{5.a}$$

The unit displacement law

If $\underline{\mathbf{v}}$ is a kinematically-compatible deformation state that corresponds to the displacements **r** then we can always set:

$$\underline{\mathbf{v}} = \underline{\mathbf{a}}\,\mathbf{r} \tag{4.b}$$

in which $\underline{\mathbf{a}}$ is a rectangular matrix that is calculated from merely kinematical relationships (see also the definition of $\underline{\boldsymbol{\varepsilon}}_1$). One sees that (4.b) is also true for nonlinear systems.

As an example of an $\underline{\mathbf{a}}$ matrix, we consider the kinematically-compatible deformation state in Fig. 3b. Here, we have:

$$\underline{\mathbf{a}} = \begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \tag{5.b}$$

The associated force is given over each column. The number in the column next to each row refers to the enumeration of the rods in Fig. 3.

We can now easily show that the unit load law (2.a) for structures can be written in the following simpler form:

$$\mathbf{r} = \bar{\mathbf{b}}' \mathbf{v} . \qquad (6.a)$$

We can introduce a kinematically-compatible state $\underline{\mathbf{v}}$ in place of **v**, and with the use of (6.b), we will get:

The associated displacement is given over each column. The number in the column next to each row refers to the enumeration of the rods in Fig. 3.

We can now easily show that the unit displacement law for structures can be written in the following simpler form:

$$\mathbf{R} = \underline{\mathbf{a}}' \mathbf{S} . \qquad (6.b)$$

We can introduce a kinematically-compatible state $\bar{\mathbf{S}}$ in place of **S**, and with the use of (6.b), we will get:

$$\bar{\mathbf{b}}' \underline{\mathbf{a}} = \mathbf{E} = \underline{\mathbf{a}}' \bar{\mathbf{b}} , \qquad (7)$$

in which **E** is a unit matrix. The interesting relation (7) can also be derived directly. It is obvious that (7) assumes that the number and directions of the forces **R** in (6.b) and the displacements **r** in (6.a) coincide. We easily confirm that equations (5) fulfill the relation (7).

Statically-determinate structures

If **S** are the true stresses that result from **R** then one will always have:

$$\mathbf{S} = \mathbf{b}\, \mathbf{R} \qquad (8.a)$$

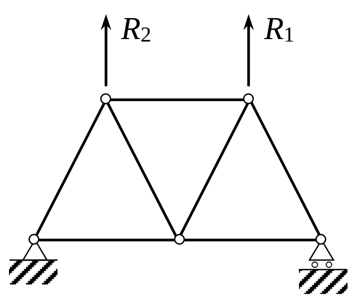


Figure 4.a. Statically-determinate system.

for statically-determinate structures, in which the matrix **b** can be determined from just the equilibrium conditions. Therefore, one has:

$$\bar{\mathbf{b}} \equiv \mathbf{b} .$$

Fig. 4a illustrates an example of a statically-determinate system.

Kinematically-determinate structures

If **v** are the true deformations that result from **R** then one will always have:

$$\mathbf{v} = \mathbf{a}\, \mathbf{r} \qquad (8.b)$$

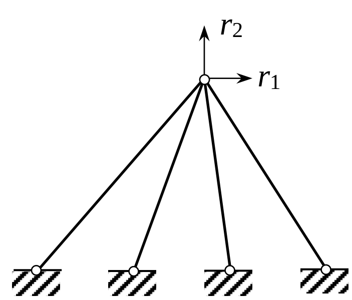


Figure 4.b. Kinematically-determinate system.

for kinematically-determinate structures, in which the matrix **a** can be determined from just the compatibility conditions. Therefore, one has:

$$\underline{\mathbf{a}} \equiv \mathbf{a} .$$

Fig. 4b illustrates an example of a kinematically-determinate system.

3. Application to linearly-elastic systems. – In this and the following sections, we shall confine ourselves to linearly-elastic systems, so ones for which **Hooke**'s law is valid.

a) *The unit laws.*

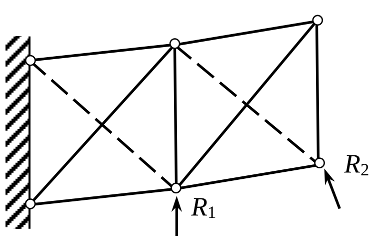
The unit load law

For linearly-elastic systems, the application of the principle of virtual forces leads to a second formulation of the unit load law that is equivalent to the one in (2.a). We find that:

$$1 \cdot r = \int \bar{\boldsymbol{\sigma}}_1' \boldsymbol{\varepsilon} \, dV = \int \boldsymbol{\sigma}_1' \bar{\boldsymbol{\varepsilon}} \, dV \,, \qquad (9.a)$$

in which the first expression was explained for the true displacement r in section 2.a), and $\bar{\boldsymbol{\varepsilon}}$ is a deformation state that corresponds to a statically-compatible stress state that results from the given strain [one can then find $\bar{\boldsymbol{\varepsilon}}$ in every suitable statically-equivalent – i.e., statically-determinate – subsystem (Fig. 5a)], and $\boldsymbol{\sigma}_1$ is the true stress state that results from $R = 1$.

As in the case of the nonlinear system, (9.a) assumes that no displacements that are associated with r are found to point in the direction of the forces that are in equilibrium with $R = 1$.



State $\bar{\boldsymbol{\varepsilon}}$

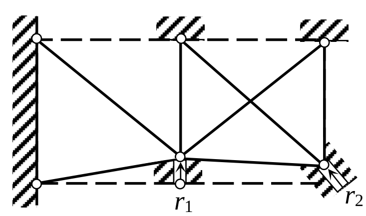
The unit displacement law

For linearly-elastic systems, the application of the principle of virtual displacements leads to a second formulation of the unit displacement law that is equivalent to the one in (2.b). We find that:

$$1 \cdot R = \int \underline{\boldsymbol{\varepsilon}}_1' \boldsymbol{\sigma} \, dV = \int \boldsymbol{\varepsilon}_1' \underline{\boldsymbol{\sigma}} \, dV \,, \qquad (9.b)$$

in which the first expression was explained for the true force R in section 2.a), and $\underline{\boldsymbol{\sigma}}$ is a stress state that corresponds to a kinematically-compatible displacement state $\underline{\boldsymbol{\varepsilon}}$ that results from the given strain [one can then find $\underline{\boldsymbol{\sigma}}$ in every suitable kinematically-equivalent – i.e., kinematically-determinate – subsystem (Fig. 5b)], and $\boldsymbol{\varepsilon}_1$ is the true stress state that results from $r = 1$.

As in the case of the nonlinear system, (9.b) assumes that no forces that are in equilibrium with $R = 1$ are found to point in the direction of the displacements that are associated with $r = 1$.



State $\underline{\boldsymbol{\sigma}}$

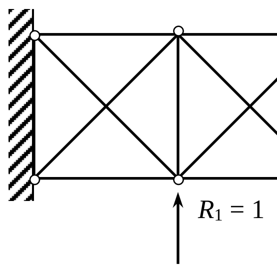


Figure 5a. The unit load law for linear systems.

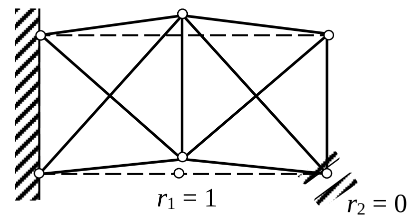


Figure 5b. The unit displacement law for linear systems.

b) *Application to structures.*

The unit load law

If **S** once more means the stress that results from the load **R** then one will have:

$$\mathbf{S} = \mathbf{b}\,\mathbf{R} \qquad (8.a)$$

for all linear structures. The unit load law (2.a) and (9.a) will now assume the equivalent forms:

$$\mathbf{r} = \bar{\mathbf{b}}'\,\mathbf{v} = \mathbf{b}'\,\mathbf{v} = \mathbf{b}'\,\bar{\mathbf{v}}, \qquad (10.a)$$

in which $\bar{\mathbf{v}}$ corresponds to the definition of $\bar{\boldsymbol{\varepsilon}}$. The displacements **r** in the direction of **R** can obviously arise as a result of an arbitrary strain as in (9.a). The restriction that was mentioned in 3.a) is true for the forces that are found in equilibrium with **R**.

The unit displacement law

If **v** once more means the column matrix of deformations that results from the displacements **r** then one will have:

$$\mathbf{v} = \mathbf{a}\,\mathbf{r} \qquad (8.b)$$

for all linear structures. The unit load law (2.b) and (9.b) will now assume the equivalent forms:

$$\mathbf{R} = \underline{\mathbf{a}}'\,\mathbf{S} = \mathbf{a}'\,\mathbf{S} = \mathbf{a}'\,\underline{\mathbf{S}}, \qquad (10.b)$$

in which $\underline{\mathbf{S}}$ corresponds to the definition of $\underline{\boldsymbol{\sigma}}$. The displacements $\bar{\mathbf{R}}$ in the direction of **r** can obviously arise as a result of an arbitrary strain as in (9.b). The restriction that was mentioned in 3.a) is true for the displacements that are associated with **r**.

If the forces **R** in (10.b) and the displacements **r** in (10.a) act on the same nodes and in the same directions then one will have [see (7)]:

$$\bar{\mathbf{b}}'\,\underline{\mathbf{a}} = \mathbf{b}'\,\mathbf{a} = \mathbf{E} = \mathbf{a}'\,\mathbf{b} = \underline{\mathbf{a}}'\,\bar{\mathbf{b}}\,. \qquad (11)$$

The compliance of the structure

The stiffness of the structure

We shall denote the stress and deformation of an element p in a structure by $\mathbf{S}_p$ and $\mathbf{v}_p$, resp. The following relations exist:

$$\mathbf{v}_p = \mathbf{f}_p\,\mathbf{S}_p\,, \qquad (1.2a)$$

in which $\mathbf{f}_p$ denotes the compliance of the element p. If we prescribe more than one stress

$$\mathbf{S}_p = \mathbf{k}_p\,\mathbf{v}_p\,, \qquad (12.b)$$

in which $\mathbf{k}_p$ denotes the stiffness of the element p. If we prescribe more than one deformation and deformation on an element then $\mathbf{S}_p$, $\mathbf{v}_p$, $\mathbf{f}_p$ will be matrices. The matrix $\mathbf{f}_p$ is always symmetric [i.e., the **Maxwell** reciprocity theorem ([1])].

on an element then $\mathbf{v}_p$, $\mathbf{S}_p$, $\mathbf{k}_p$ will be matrices. The matrix $\mathbf{k}_p$ is always symmetric [i.e., the dual **Maxwell** reciprocity theorem ([1])].

The simplest example of $\mathbf{f}_p$ and $\mathbf{k}_p$ is given by a rod in an ideal framework. If S_p is the force in the rod and v_p is its change in length then:

$$f_p = \frac{l}{E\,A}\,, \tag{13.a}$$

$$k_p = \frac{E\,A}{l}\,, \tag{13.b}$$

in which E, A, l are self-evident notations. The compliance and stiffness of complicated elements was examined thoroughly in the previous work ([*]), which also considered shear deformations. In general, the definitions of the stress $\mathbf{S}_p$ and deformation $\mathbf{v}_p$ in an element differ between the force and deformation procedures.

With the use of (3), we can now set:

$$\mathbf{v} = \mathbf{f}\,\mathbf{S}\,, \tag{12.c}$$

in which:

$$\mathbf{f} = \begin{bmatrix} f_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & 0 & f_p & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_l \end{bmatrix} \tag{14.a}$$

is a diagonal matrix. **f** will be referred to as the compliance of the l uncoupled elements of the structure.

With the use of (3), we can now set:

$$\mathbf{S} = \mathbf{k}\,\mathbf{v}\,, \tag{12.d}$$

in which:

$$\mathbf{k} = \begin{bmatrix} k_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & 0 & k_p & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & k_l \end{bmatrix} \tag{14.b}$$

is a diagonal matrix. **k** will be referred to as the stiffness of the l uncoupled elements of the structure.

If the same stresses (forces) and deformations are prescribed in (12.c) and (12.d) then the relation will exist:

$$\mathbf{f}\,\mathbf{k} = \mathbf{E} = \mathbf{k}\,\mathbf{f}\,. \tag{15}$$

An application of equations (10.a) and (12.c) will yield the m displacements **r** that result from **R** in its direction:

$$\mathbf{r} = \bar{\mathbf{b}}'\,\mathbf{f}\,\mathbf{b}\,\mathbf{R} = \mathbf{b}'\,\mathbf{f}\,\mathbf{b}\,\mathbf{R} = \mathbf{b}'\,\mathbf{f}\,\bar{\mathbf{b}} = \mathbf{F}\,\mathbf{R}. \tag{16.a}$$

An application of equations (10.b) and (12.d) will yield the m forces **R** that result from **r** in its direction:

$$\mathbf{R} = \underline{\mathbf{a}}'\,\mathbf{k}\,\mathbf{a}\,\mathbf{r} = \mathbf{a}'\,\mathbf{k}\,\mathbf{a}\,\mathbf{r} = \mathbf{a}'\,\mathbf{k}\,\underline{\mathbf{a}}\,\mathbf{r} = \mathbf{K}\,\mathbf{r}. \tag{16.b}$$

([1]) Cf., **C. B. Biezeno** and **R. Grammel**, *loc. cit.*, Chap. II, no. 9.
([*]) See footnote ([2]) on page 1.

In that:

$$\mathbf{F} = \bar{\mathbf{b}}'\mathbf{f}\,\mathbf{b} = \mathbf{b}'\mathbf{f}\,\mathbf{b} = \mathbf{b}'\mathbf{f}\,\bar{\mathbf{b}} \qquad (17.a)$$

is the compliance of the structure in the prescribed m directions. Equation (17.a) shows that the square matrix is symmetric, as the **Maxwell** reciprocity theorem would dictate ([1]).

In that:

$$\mathbf{K} = \underline{\mathbf{a}}'\mathbf{k}\,\mathbf{a} = \mathbf{a}'\mathbf{k}\,\mathbf{a} = \mathbf{a}'\mathbf{k}\,\underline{\mathbf{a}} \qquad (17.b)$$

is the stiffness of the structure in the prescribed m directions. Equation (17.b) shows that the square matrix is symmetric, as the dual **Maxwell** reciprocity theorem would dictate ([1]).

The relation (15) generalizes to:

$$\mathbf{F}\,\mathbf{K} = \mathbf{E} = \mathbf{K}\,\mathbf{F} \qquad (15.a)$$

for a structure.

The construction of a structure from its various elements results from the matrix **b** in (17.a), which expresses a static relation. In that way, we can consider the construction (17.a) to be a generalized series connection of springs or elements. Fig. 6.a shows a simple example of a series connection.

The construction of a structure from its various elements results from the matrix **a** in (17.b), which expresses a kinematic relation. In that way, we can consider the construction (17.b) to be a generalized parallel connection of springs or elements. Fig. 6.b shows a simple example of a parallel connection.

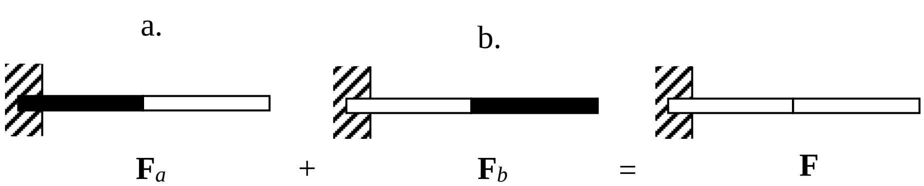


Figure 6a. Example of a series connection of elements. The black pieces of the cantilever are each assumed to be rigid.

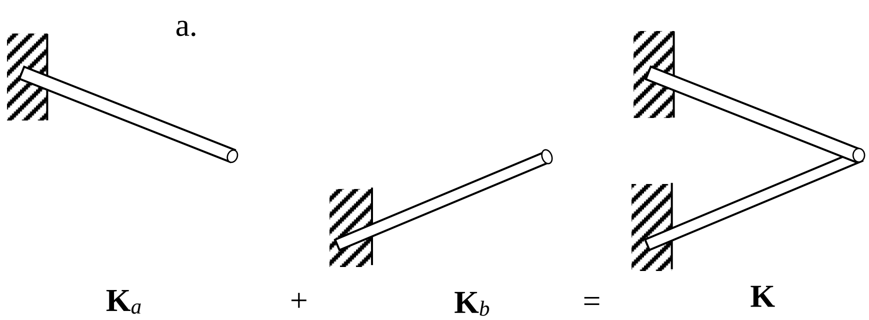


Fig. 6b. Example of a parallel connection of elements.

Example. – As an application of the compliance and stiffness matrices **F** and **K**, resp., we consider the natural vibrations of a structure when m masses M_i act at the m nodes. We let:

([1]) Cf., footnote ([2]) on pp. 1.

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & 0 & M_i & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & M_m \end{bmatrix}$$

the diagonal mass matrix. The m natural frequencies ω and the forms $\mathbf{r} = \mathbf{r}_0\, e^{i\omega t}$ of the natural vibrations of the structure are calculated from:

either [1]:

$$\left. \begin{aligned} &\left| \mathbf{FM} - \frac{1}{\omega^2}\mathbf{E} \right| = 0, \\ \text{and} \quad &\left[\mathbf{FM} - \frac{1}{\omega^2}\mathbf{E} \right] \mathbf{r}_0 = 0. \end{aligned} \right\} \qquad (18.a)$$

or [1]:

$$\left. \begin{aligned} &\left| \mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{E} \right| = 0, \\ \text{and} \quad &\left[\mathbf{M}^{-1}\mathbf{K} - \omega^2\mathbf{E} \right] \mathbf{r}_0 = 0. \end{aligned} \right\} \qquad (18.b)$$

More general mass distributions can be considered with no further analysis. Since the compliance (stiffness) refers to not only forces (displacements), but also to moments (rotations), it is easy to substitute the rotational inertia in the calculations.

The investigations of vibrations in beams and frames, whose **F** or **K** (when one includes shear deformations) are simple to ascertain, give interesting practice problems.

4. The calculation of structures. –

In this section, we shall investigate a series of problems in structures whose stresses cannot be determined by static considerations. We shall refer to such structures as *statically-indeterminate*. The undetermined quantities are called statically-indeterminate forces or stresses. Fig. 7a shows a two-fold statically-indeterminate framework.

In this section, we shall investigate a series of problems in structures whose deformations cannot be determined by kinematical considerations. We shall refer to such structures as *kinematically-indeterminate*. The undetermined quantities are called kinematically-indeterminate displacements or deformations. Fig. 7b shows a six-fold kinematically-indeterminate framework. (r_1 and r_2 are prescribed, U_1 to U_6 are unknown.)

---

(1) Obviously, the determinant equation in (18.a) [(18.b), resp.] is included in the following matrix equation.

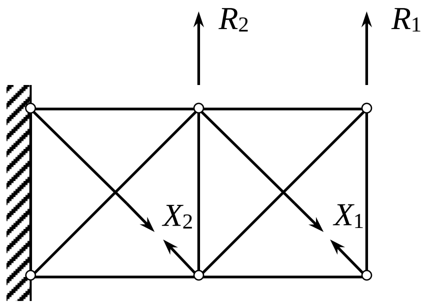


Figure 7a. Two-fold statically-indeterminate system

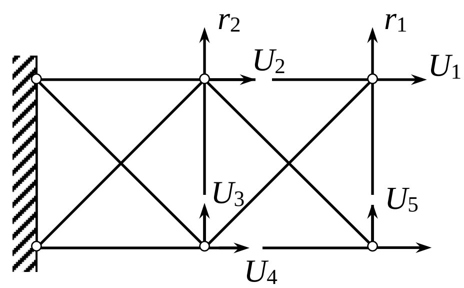


Figure 7b. Six-fold kinematically-indeterminate system

a) **Problem I.** –

Let m forces R be given in an n-fold statically-indeterminate ([1]) structure (Fig. 7a). Determine the statically-indeterminate force or stress matrix:

$$\mathbf{X} = \{X_1, X_2, \ldots, X_n\} \qquad (19.a)$$

and the stresses **S**.

The complete force matrix can be written as:

$$\{\mathbf{R}\ \mathbf{X}\}. \qquad (20.a)$$

We can now put the stresses **S** into the form:

$$\mathbf{S} = \mathbf{b}_0\,\mathbf{R} + \mathbf{b}_1\,\mathbf{X} = \mathbf{b}\,\mathbf{R}\,, \qquad (21.a)$$

in which the matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ are calculated from static considerations, and **b** is still unknown. We shall call the subsystem in which one ascertains $\mathbf{b}_0$ the *fundamental static system*.

In order to determine **X**, we apply the method of unit loads (10.a) to the compatibility condition in the direction of **X**. We find that:

$$\mathbf{b}_1'\,\mathbf{v} = \mathbf{b}_1'\,\mathbf{f}\,\mathbf{S} = \mathbf{b}_1'\,\mathbf{f}\,\mathbf{b}_0\,\mathbf{R} + \mathbf{b}_1'\,\mathbf{f}\,\mathbf{b}_1\,\mathbf{X} = 0$$

or

$$\mathbf{X} = -\,\mathbf{D}^{-1}\,\mathbf{D}_0\,\mathbf{R}\,, \qquad (22.a)$$

Let m displacements r be given in an n-fold kinematically-indeterminate ([1]) structure (Fig. 7b). Determine the kinematically-indeterminate displacement matrix:

$$\mathbf{U} = \{U_1, U_2, \ldots, U_n\} \qquad (19.b)$$

and the deformations **v**.

The complete displacement matrix can be written:

$$\{\mathbf{r}\ \mathbf{U}\}\,. \qquad (20.b)$$

We can now put the deformations **U** into the form:

$$\mathbf{v} = \mathbf{a}_0\,\mathbf{r} + \mathbf{a}_1\,\mathbf{U} = \mathbf{a}\,\mathbf{r}\,, \qquad (21.b)$$

in which the matrices $\mathbf{a}_0$ and $\mathbf{a}_1$ are calculated from kinematical considerations, and **a** is still unknown. We shall call the subsystem in which one ascertains $\mathbf{a}_0$ the *fundamental kinematical system*.

In order to determine **U**, we apply the method of unit displacements (10.b) to the equilibrium condition in the direction of **U**. We find that:

$$\mathbf{a}_1'\,\mathbf{S} = \mathbf{a}_1'\,\mathbf{k}\,\mathbf{v} = \mathbf{a}_1'\,\mathbf{k}\,\mathbf{a}_0\,\mathbf{r} + \mathbf{a}_1'\,\mathbf{k}\,\mathbf{a}_1\,\mathbf{U} = 0$$

or

$$\mathbf{U} = -\,\mathbf{C}^{-1}\,\mathbf{C}_0\,\mathbf{r}\,, \qquad (22.b)$$

([1]) The number of statically and kinematically-indeterminate quantities in a structure is obviously different in general; cf., also sec. 7.a and 7.b.

in which:

$$\mathbf{D} = \mathbf{b}_1' \, \mathbf{f} \, \mathbf{b}_1 \quad \text{and} \quad \mathbf{D}_0 = \mathbf{b}_1' \, \mathbf{f} \, \mathbf{b}_0 \,. \tag{23.a}$$

It will then follow from (21.a) and (22.a) that:

$$\mathbf{b} = \mathbf{b}_0 - \mathbf{b}_1 \, \mathbf{D}^{-1} \, \mathbf{D}_0 \,, \tag{24.a}$$

in which the stresses and deformations in the structure are known.

In order to determine the displacements **r** in the directions of the forces **R**, we again employ (10.a) and obtain:

$$\mathbf{r} = \bar{\mathbf{b}}' \, \mathbf{v} = \bar{\mathbf{b}}' \mathbf{f} \, \mathbf{b} \, \mathbf{R} = \mathbf{F} \, \mathbf{R} \,. \tag{25.a}$$

We can replace $\bar{\mathbf{b}}$ (the statically-compatible system) with $\mathbf{b}_0$. Therefore, the compliance of the system is:

$$\mathbf{F} = \mathbf{F}_0 - \mathbf{D}_0' \, \mathbf{D}^{-1} \, \mathbf{D}_0 \,. \tag{26.a}$$

In that, $\mathbf{F}_0 = \mathbf{b}_0' \, \mathbf{f} \, \mathbf{b}_0$ is the compliance of the basic system.

It should be observed, in particular, that the calculation of the system using the force procedure requires only the matrices $\mathbf{b}_0$, $\mathbf{b}_1$, **f**, and the column **R**.

Example. – For the doubly statically-indeterminate framework in Fig. 7a, determine the matrix **b** and the compliance **F** under the condition that all rods possess the same compliance $l \,/\, E\,A$.

The chosen statically-indeterminate quantities are given in Fig. 7a. The corresponding matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ follow from elementary static considerations. One has:

$$\mathbf{X} = \{X_1, X_2\},$$

in which:

$$\mathbf{C} = \mathbf{a}_1' \, \mathbf{k} \, \mathbf{a}_1 \quad \text{and} \quad \mathbf{C}_0 = \mathbf{a}_1' \, \mathbf{k} \, \mathbf{a}_0 \,. \tag{23.b}$$

It will then follow from (21.b) and (22.b) that:

$$\mathbf{a} = \mathbf{a}_0 - \mathbf{a}_1 \, \mathbf{C}^{-1} \, \mathbf{C}_0 \,, \tag{24.b}$$

in which the stresses and deformations in the structure are known.

In order to determine the forces **R** in the directions of the displacements **r**, we again employ (10.b) and obtain:

$$\mathbf{R} = \underline{\mathbf{a}}' \, \mathbf{S} = \underline{\mathbf{a}}' \mathbf{k} \, \mathbf{a} \, \mathbf{r} = \mathbf{K} \, \mathbf{r} \,. \tag{25.b}$$

We can replace $\underline{\mathbf{a}}$ (the kinematically-compatible system) $\mathbf{a}_0$. Therefore, the stiffness of the system is:

$$\mathbf{K} = \mathbf{K}_0 - \mathbf{C}_0' \, \mathbf{C}^{-1} \, \mathbf{C}_0 \,. \tag{26.b}$$

In that, $\mathbf{K}_0 = \mathbf{a}_0' \, \mathbf{k} \, \mathbf{a}_0$ is the stiffness of the basic system.

It should be observed, in particular, that the calculation of the system using the displacement procedure requires only the matrices $\mathbf{a}_0$, $\mathbf{a}_1$, **k**, and the column **r**.

Example. - For the six-fold kinematically-indeterminate framework in Fig. 7b, determine the matrix **a** and the stiffness **K** under the condition that all rods possess the same stiffness $E\,A \,/\, l$.

The chosen kinematically-indeterminate quantities are given in Fig. 7b. The corresponding matrices $\mathbf{a}_0$ and $\mathbf{a}_1$ follow from elementary kinematic considerations. One has:

$$\mathbf{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\} \,,$$

$\mathbf{a}_0 = \underline{\mathbf{a}}$ from (5.b), since the basic system is an identity,

$\mathbf{a}_1 =$

$$\mathbf{b}_0 = \begin{array}{c} \begin{matrix} R_1 & R_2 \end{matrix} \\ \begin{bmatrix} -1 & 0 \\ -2 & -1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \mathbf{b}_1 = \begin{array}{cc} \begin{matrix} R_1 & \quad R_2 \end{matrix} & \\ \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} \end{array} \qquad (27.a)$$

$$\begin{array}{cc} \begin{matrix} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \end{matrix} & \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{matrix} \end{array} \qquad (27.b)$$

The number in the column next to each matrix refers to the numbering of the rods in Fig. 3 and Fig. 8. The associated force or displacement is given over each column.

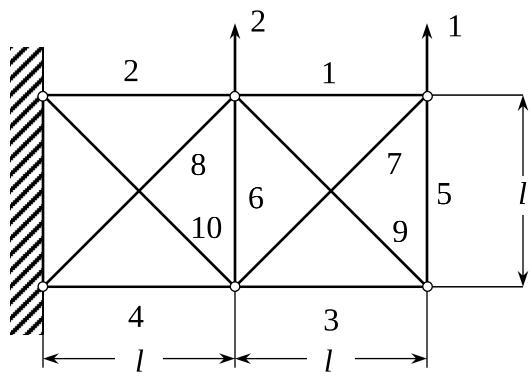


Figure 8. Numbering of the rods.

The compliance of the unconstrained element is:

$$\mathbf{F} = \frac{l}{E\,A}\mathbf{E}_{10}, \qquad (28.a)$$

The stiffness of the unconstrained element is:

$$\mathbf{k} = \frac{E\,A}{l}\mathbf{E}_{10}, \qquad (28.b)$$

in which the index 10 gives the number of the diagonal element in the unit matrix.

An application of equation (24.a) gives ([1]):

An application of equation (24.b) gives ([1]):

([1]) For the sake of economy of space, the matrices **C**, $\mathbf{C}^{-1}$, and **D**, $\mathbf{D}^{-1}$ are not reproduced here.

$$
\mathbf{b} = \frac{1}{55}\begin{array}{c} \begin{array}{cc} R_1 & R_2 \end{array} \\ \begin{bmatrix} -31 & -3 \\ -82 & -31 \\ 24 & -3 \\ 83 & 24 \\ 24 & -3 \\ -3 & 21 \\ 31/\sqrt{2} & 3/\sqrt{2} \\ 27/\sqrt{2} & 31/\sqrt{2} \\ -24/\sqrt{2} & 3/\sqrt{2} \\ -28/\sqrt{2} & -24/\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \qquad (29.a)
$$

$$
\mathbf{a} = \frac{1}{335}\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \begin{bmatrix} -45 & 55 \\ -62 & -21 \\ 48 & -81 \\ 81 & -32 \\ 48 & -81 \\ -57 & 159 \\ 45/\sqrt{2} & -55/\sqrt{2} \\ -31/\sqrt{2} & 157/\sqrt{2} \\ -48/\sqrt{2} & 81/\sqrt{2} \\ 12/\sqrt{2} & -104/\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \qquad (29.b)
$$

For the compliance of the structure, we find from (26.a) that:

$$\mathbf{F} = \frac{l}{55\,E\,A}\begin{bmatrix} 397 & 136 \\ 136 & 93 \end{bmatrix}. \qquad (30.a)$$

For the stiffness of the structure, we find from (26.b) that:

$$\mathbf{K} = \frac{E\,A}{355\,l}\begin{bmatrix} 93 & -136 \\ -136 & 397 \end{bmatrix}. \qquad (30.b)$$

We easily confirm that the relations $\mathbf{b}'\,\mathbf{a} = \mathbf{E}_2$ and $\mathbf{F}\,\mathbf{K} = \mathbf{E}_2$ are fulfilled.

b) **Problem** II.

Let the displacements **r** be given in an *n*-fold indeterminate structure. Calculate **X** and **S**. The forces **K** are unknown here, but they can be calculated from (25.a). One has:

$$\mathbf{R} = \mathbf{F}^{-1}\,\mathbf{r}\,. \qquad (31.a)$$

In that way, the problem is reduced to Problem I.

Let the forces **R** be given in an *n*-fold indeterminate structure. Calculate **U** and **v**. The displacements **r** are unknown here, but they can be calculated from (25.b). One has:

$$\mathbf{r} = \mathbf{K}^{-1}\,\mathbf{R}. \qquad (31.b)$$

In that way, the problem is reduced to Problem I.

c) **Problem III.**

We refer to the deformations that are given to the *unconstrained* elements as a result of temperature changes, manufacturing defects, support displacements, and loads *on* the elements, etc., as the *initial deformations H*. A simple example of an initial deformation occurs in a framework whose rod *p* of length l_p

We refer to the stresses that are given to the *fixed* nodes as a result of temperature changes, manufacturing defects, support displacements, and loads *on* the elements, etc., as the *constraint stresses J*. A simple example of constraint stress occurs in a framework whose rod *p* of cross-section A_p is exposed to a is exposed to a temperature change Θ_p. The change in length of the rod will then be:

$$H_p = l_p\, \alpha_p\, \Theta_p ,$$

in which α_p is the thermal expansion coefficient. When more than one deformation is prescribed in the element, H_p will become a corresponding column matrix ([1]) $\mathbf{H}_p$.

The construction of the structure from the deformed unconstrained elements will generally require an additional stress state when the system is statically-indeterminate.

We shall now examine the following problem: Suppose that we are given:

1) The matrix of initial deformations of the unconstrained elements:

$$\mathbf{H} = \{\mathbf{H}_1, \ldots, \mathbf{H}_p, \ldots, \mathbf{H}_l\} \qquad (32.a)$$

and

2) The m forces $\mathbf{R} = 0$ at the nodes.

Calculate $\mathbf{X}$ and $\mathbf{S}$ once more.

The total deformation of the element is the sum of the elastic and applied initial deformations. Therefore:

$$\mathbf{v} = \mathbf{f}\, \mathbf{b}_1\, \mathbf{X} + \mathbf{H} . \qquad (33.a)$$

Applying (10.a) and (33.a) will yield:

$$\mathbf{b}_1' \mathbf{v} = \mathbf{D}\, \mathbf{X} + \mathbf{b}_1' \mathbf{H} = 0$$

and

$$\mathbf{X} = -\, \mathbf{D}^{-1} \mathbf{b}_1' \mathbf{H} . \qquad (34.a)$$

With that, one has:

$$\left.\begin{aligned} \mathbf{S} &= \mathbf{b}_1 \mathbf{X} = -\mathbf{b}_1\, \mathbf{D}^{-1} \mathbf{b}_1' \mathbf{H} , \\ \text{and} \quad \mathbf{v} &= -\mathbf{f}\, \mathbf{b}_1\, \mathbf{D}^{-1} \mathbf{b}_1' \mathbf{H} + \mathbf{H} . \end{aligned}\right\} \qquad (35.a)$$

temperature change Θ_p. The constraint force in the rod when the nodal endpoints are fixed will then be:

$$J_p = -\, E\, A_p\, \alpha_p\, \Theta_p ,$$

in which α_p is the thermal expansion coefficient. When more than one constraint stress is prescribed in the element, J_p will become a corresponding column matrix ([1]) $\mathbf{J}_p$.

The freeing of the fixed nodes of the structure under constraint stresses will generally produce an additional deformation state when the system is kinematically-indeterminate.

We shall now examine the following problem: Suppose that we are given:

1) The matrix of the constraint stresses in the elements with fixed nodes:

$$\mathbf{J} = \{\mathbf{J}_1, \ldots, \mathbf{J}_p, \ldots, \mathbf{J}_l\} \qquad (32.b)$$

and

2) The m displacements $\mathbf{r} = 0$ of the nodes.

Calculate $\mathbf{U}$ and $\mathbf{v}$ once more.

The total stress in the elements is the sum of the elastic and applied constraint stresses. Therefore:

$$\mathbf{S} = \mathbf{k}\, \mathbf{a}_1\, \mathbf{U} + \mathbf{J} . \qquad (33.a)$$

Applying (10.b) and (33.b) will yield:

$$\mathbf{a}_1' \mathbf{S} = \mathbf{C}\, \mathbf{U} + \mathbf{a}_1' \mathbf{J} = 0$$

and

$$\mathbf{U} = -\, \mathbf{C}^{-1} \mathbf{a}_1' \mathbf{J} . \qquad (34.b)$$

With that, one has:

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{a}_1 \mathbf{U} = -\mathbf{a}_1\, \mathbf{C}^{-1} \mathbf{a}_1' \mathbf{J} , \\ \text{and} \quad \mathbf{S} &= -\mathbf{k}\, \mathbf{a}_1\, \mathbf{C}^{-1} \mathbf{a}_1' \mathbf{J} + \mathbf{J} . \end{aligned}\right\} \qquad (35.b)$$

([1]) Cf., footnote ([2]) on pp. 174.

The displacements [1] $\mathbf{r}$ of the *m* nodes that result from $\mathbf{H}$ are once more calculated from (10.a). We find that:

$$\mathbf{r} = \mathbf{b}'_0\,\mathbf{v} = \mathbf{b}'\,\mathbf{H}\,, \tag{36.a}$$

in which $\mathbf{b}_0$ and $\mathbf{b}$ were defined in Problem I. The second formula (36.a) shows that the displacements $\mathbf{r}$ that result from the initial deformations $\mathbf{H}$ can be determined without the corresponding stress calculation. It is sufficient that the true stresses (so the matrix $\mathbf{b}$) that result from the loads $\mathbf{R}$ are known in the direction of $\mathbf{r}$.

The forces [2] $\mathbf{R}$ of the *m* nodes that result from $\mathbf{J}$ are once more calculated from (10.b). We find that:

$$\mathbf{R} = \mathbf{a}'_0\,\mathbf{S} = \mathbf{a}'\,\mathbf{J}\,, \tag{36.b}$$

in which $\mathbf{a}_0$ and $\mathbf{a}$ were defined in Problem I. The second formula (36.b) shows that the forces $\mathbf{R}$ that result from the constraint stresses $\mathbf{J}$ can be determined without the corresponding deformation calculation. It is sufficient that the true deformations (so the matrix $\mathbf{b}$) that result from the displacement $\mathbf{r}$ are known in the direction of $\mathbf{R}$.

d) **Problem IV.**

Suppose that one is given:

1) The initial deformations:

$$\mathbf{H} = \{\mathbf{H}_p\}$$

as in Problem III, and:

2) The displacements:

$$\mathbf{r} = 0\,.$$

With the use of Problems I and III, the compatibility condition $\mathbf{r} = 0$ will become:

$$\mathbf{r} = \mathbf{F}\,\mathbf{R} + \mathbf{b}'\,\mathbf{H} = 0\,,$$

so

$$\mathbf{R} = -\,\mathbf{F}^{-1}\,\mathbf{b}'\,\mathbf{H}\,. \tag{37.a}$$

The stresses $\mathbf{S}$ and deformations $\mathbf{v}$ are now given by (21.a) and (34.a) as:

$$\left.\begin{aligned} \mathbf{S} &= -[\mathbf{b}\,\mathbf{F}^{-1}\,\mathbf{b}' + \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_1]\,\mathbf{H}\,, \\ \mathbf{r} &= \mathbf{f}\,\mathbf{S} + \mathbf{H}\,. \end{aligned}\right\} \tag{38.a}$$

Suppose that one is given:

1) The constraint stresses:

$$\mathbf{J} = \{\mathbf{J}_p\}$$

as in Problem III, and:

2) The forces:

$$\mathbf{R} = 0\,.$$

With the use of Problems I and III, the equilibrium condition $\mathbf{R} = 0$ will become:

$$\mathbf{R} = \mathbf{K}\,\mathbf{r} + \mathbf{a}'\,\mathbf{J} = 0\,,$$

so

$$\mathbf{r} = -\,\mathbf{K}^{-1}\,\mathbf{a}'\,\mathbf{J}\,. \tag{37.b}$$

The deformations $\mathbf{v}$ and stresses $\mathbf{S}$ are now given by (21.b) and (34.b) as:

$$\left.\begin{aligned} \mathbf{v} &= -[\mathbf{a}\,\mathbf{K}^{-1}\,\mathbf{a}' + \mathbf{a}_1\,\mathbf{D}^{-1}\,\mathbf{a}'_1]\,\mathbf{J}\,, \\ \mathbf{S} &= \mathbf{k}\,\mathbf{v} + \mathbf{J}\,. \end{aligned}\right\} \tag{38.b}$$

(1) Here, it is assumed that the displacements *r* take place in the directions of the forces *R* of Problem I.
(2) Here, it is assumed that the forces *R* take place in the directions of the displacements *r* of Problem I.

In general, the forces are prescribed in statics problem, but not the displacements. Therefore, Problems I and III (II and IV, resp.) are especially interesting in the application of the force (deformation, resp.) method.

e) **Problem V.**

In a highly statically-indeterminate structure, it is often advantageous to choose a basic system that is itself statically-indeterminate. We let **Z** denote the column matrix of the statically-indeterminate quantities in the basic systems. The complete force matrix will then be:

$$\{\mathbf{R}\ \mathbf{Z}\ \mathbf{X}\}\ .$$

We shall now examine the following generalization of Problem I:

Suppose that we are given:

1) the forces **R** and
2) the undetermined quantities **Z**

for every $X = 1$ and $R = 1$. Determine **X** and **S**.

We once more assume that the true stresses **S** take the form:

$$\mathbf{S} = \mathbf{b}_0\ \mathbf{R} + \mathbf{b}_1\ \mathbf{X} = \mathbf{b}\ \mathbf{R}\ , \qquad (21.\text{a})$$

in which the matrices $\mathbf{b}_0$ and $\mathbf{b}_1$ can be calculated from static considerations alone, since **Z** is known for every R and X. We also introduce the stress matrix $\bar{\mathbf{S}}$, which only needs to be statically compatible with **R** and **X**. We have:

$$\bar{\mathbf{S}} = \bar{\mathbf{b}}_0\,\mathbf{R} + \bar{\mathbf{b}}_1\,\mathbf{X}\ , \qquad (39.\text{a})$$

in which $\bar{\mathbf{b}}_0$ ($\bar{\mathbf{b}}_1$, resp.) is a stress matrix whose columns are each statically-compatible with the corresponding $R = 1$ ($X = 1$, resp.) and $\mathbf{X} = 0$ ($\mathbf{R} = 0$, resp.). For example, it is possible to find $\bar{\mathbf{b}}_0$ ($\bar{\mathbf{b}}_1$, resp.) in a statically-determinate

In a highly kinematically-indeterminate structure, it is often advantageous to choose a basic system that is itself kinematically-indeterminate. We let **W** denote the column matrix of the kinematically-indeterminate quantities in the basic systems. The complete displacement matrix will then be:

$$\{\mathbf{r}\ \mathbf{W}\ \mathbf{U}\}\ .$$

We shall now examine the following generalization of Problem I:

Suppose that we are given:

1) the displacements **r** and
2) the undetermined quantities **W**

for every $U = 1$ and $r = 1$. Determine **U** and **v**.

We once more assume that the true deformations **v** take the form:

$$\mathbf{v} = \mathbf{a}_0\ \mathbf{r} + \mathbf{a}_1\ \mathbf{U} = \mathbf{a}\ \mathbf{r}\ , \qquad (21.\text{b})$$

in which the matrices $\mathbf{a}_0$ and $\mathbf{a}_1$ can be calculated from kinematic considerations alone, since **W** is known for every r and U. We also introduce the deformation matrix $\underline{\mathbf{v}}$, which only needs to be kinematically compatible with **r** and **U**. We have:

$$\underline{\mathbf{v}} = \underline{\mathbf{a}}_0\,\mathbf{r} + \underline{\mathbf{a}}_1\,\mathbf{U}\ , \qquad (39.\text{b})$$

in which $\underline{\mathbf{a}}_0$ ($\underline{\mathbf{a}}_1$, resp.) is a deformation matrix whose columns are each kinematically-compatible with the corresponding $r = 1$ ($U = 1$, resp.) and $\mathbf{U} = 0$ ($\mathbf{r} = 0$, resp.). For example, it is possible to find $\underline{\mathbf{a}}_0$ ($\underline{\mathbf{a}}_1$, resp.) in a subsystem, in which case the definition of $\bar{\mathbf{b}}_0$ will coincide with that of $\mathbf{b}_0$ in Problem I.

With the use of the two alternative forms of (10.a), the compatibility condition in the direction of **X** gives:

$$\bar{\mathbf{b}}_1' \, \mathbf{v} = \bar{\mathbf{b}}_1' \, \mathbf{f} \, \mathbf{S} = \bar{\mathbf{b}}_1' \, \mathbf{f} \, \mathbf{b}_0 \, \mathbf{R} + \bar{\mathbf{b}}_1' \, \mathbf{f} \, \mathbf{b}_1 \, \mathbf{X} = 0 \,,$$

or

$$\mathbf{b}_1' \, \bar{\mathbf{v}} = \mathbf{b}_1' \, \mathbf{f} \, \bar{\mathbf{S}} = \mathbf{b}_1' \, \mathbf{f} \, \bar{\mathbf{b}}_0 \, \mathbf{R} + \mathbf{b}_1' \, \mathbf{f} \, \bar{\mathbf{b}}_1 \, \mathbf{X} = 0 \,.$$

Therefore, **X** can again be expressed using (22.a):

$$\mathbf{X} = - \mathbf{D}^{-1} \, \mathbf{R}_0 \, \mathbf{D} \,, \qquad (22.a)$$

in which one now has:

$$\text{and} \quad \left. \begin{aligned} \mathbf{D} &= \bar{\mathbf{b}}_1' \, \mathbf{f} \, \mathbf{b}_1 = \mathbf{b}_1' \, \mathbf{f} \, \mathbf{b}_1 = \mathbf{b}_1' \, \mathbf{f} \, \bar{\mathbf{b}}_1 \,, \\ \mathbf{D}_0 &= \bar{\mathbf{b}}_1' \, \mathbf{f} \, \mathbf{b}_0 = \mathbf{b}_1' \, \mathbf{f} \, \mathbf{b}_0 = \mathbf{b}_1' \, \mathbf{f} \, \bar{\mathbf{b}}_0 \,. \end{aligned} \right\} \quad (40.a)$$

The stress matrix **S** and the compliance **F** are now ascertained from (21.a) and (26.a). The introduction of the statically-compatible matrices $\bar{\mathbf{b}}_0$, $\bar{\mathbf{b}}_1$ can lead to an appreciable simplification in the calculations.

statically-determinate subsystem, in which case the definition of $\underline{\mathbf{a}}_0$ will coincide with that of $\mathbf{a}_0$ in Problem I.

With the use of the two alternative forms of (10.b), the equilibrium condition in the direction of *U* gives:

$$\underline{\mathbf{a}}_1' \, \mathbf{S} = \underline{\mathbf{a}}_1' \, \mathbf{k} \, \mathbf{v} = \underline{\mathbf{a}}_1' \, \mathbf{k} \, \mathbf{a}_0 \, \mathbf{r} + \underline{\mathbf{a}}_1' \, \mathbf{k} \, \mathbf{a}_1 \, \mathbf{U} = 0 \,,$$

or

$$\mathbf{a}_1' \, \underline{\mathbf{S}} = \mathbf{a}_1' \, \mathbf{k} \, \underline{\mathbf{v}} = \mathbf{a}_1' \, \mathbf{k} \, \underline{\mathbf{a}}_0 \, \mathbf{r} + \mathbf{a}_1' \, \mathbf{k} \, \underline{\mathbf{a}}_1 \, \mathbf{U} = 0 \,.$$

Therefore, **U** can again be expressed using (22.b):

$$\mathbf{U} = - \mathbf{C}^{-1} \, \mathbf{C}_0 \, \mathbf{r} \,, \qquad (22.b)$$

in which one now has:

$$\text{and} \quad \left. \begin{aligned} \mathbf{C} &= \underline{\mathbf{a}}_1' \, \mathbf{k} \, \mathbf{a}_1 = \mathbf{a}_1' \, \mathbf{k} \, \mathbf{a}_1 = \mathbf{a}_1' \, \mathbf{k} \, \underline{\mathbf{a}}_1 \,, \\ \mathbf{C}_0 &= \underline{\mathbf{a}}_1' \, \mathbf{k} \, \mathbf{a}_0 = \mathbf{a}_1' \, \mathbf{k} \, \mathbf{a}_0 = \mathbf{a}_1' \, \mathbf{k} \, \underline{\mathbf{a}}_0 \,. \end{aligned} \right\} \quad (40.b)$$

The deformation matrix **v** and the stiffness **K** are now ascertained from (21.b) and (26.b). The introduction of the kinematically-compatible matrices $\underline{\mathbf{a}}_0$, $\underline{\mathbf{a}}_1$ can lead to an appreciable simplification in the calculations.

In conclusion, we remark that Problems II to IV can be similarly generalized.

5. Application of the force method to systems with cutouts. – For the practical application of matrix statics, in conjunction with electronic computers, one of the most important problems is that of the careful and repeated verification of the elements (viz., the coefficient matrix) of the basic matrices $\mathbf{b}_0$, $\mathbf{b}_1$, and **f**. In order to simplify the checking and obtain matrices that are easy to understand, it is important to avoid all special cases that require special consideration. One such case appears, e.g., in a system that takes the form of a membrane, such as a wing when individual elements are missing between the mesh lines (viz., structures with cutouts). One will then find that, in general, one must choose a basic system that is more complicated (for the calculation of the $\mathbf{b}_0$-matrix) than it is for the corresponding structure without the cutouts. It is also necessary to introduce special statically-indeterminate stress systems **X** in the neighborhood of the cutout that will perturb the otherwise-regular structure of the $\mathbf{b}_1$-matrix and make it harder to check. Finally, it can happen that some of the diagonal coefficients in the equations for the unknowns **X** have the same order of magnitude as the remaining coefficients in the corresponding equations, and that the precise solution of the equations by the digital computer will be made more difficult by that.

In order to avoid that difficulty in structures with cutouts, it is advantageous to apply a gimmick that the author developed in the aforementioned work ([1]). That method is also the ideal procedure for finding the new distribution of stresses that results from a subsequent introduction of cutouts without needing to repeat the entire static calculation.

The principle of the procedure us simple. In order to obtain a regular (continuous) construction and corresponding regular schema of elements for the basic matrices, we introduce as many additional elements as are necessary in the system. Although the dimensions (cross-sections or thicknesses) of those new elements are arbitrary, it is recommended that they should be consistent with those of the surrounding parts of the structure. Of course, the original system and the new (continuous) system will exhibit different stresses for the same strain. However, it is possible to achieve identical stresses in the two structures when we impose initial deformations on the new (i.e., fictitious) elements of the continuous system such that the total stresses (that result from the given strain and initial deformation) will reduce to zero in those elements. In that way, the new elements will, in fact, be eliminated, although the regular schema of the matrices and equations will remain preserved. That also shows that the matrix formulation allows us to derive the necessary initial deformations, and what is even more important, the associated stresses, very simply from the stress calculation of continuous systems under the exclusive action of the given strains. The true stresses in the original system can now be determined very simply from the stress analysis of the new system by superimposing the stresses that result from the given strain and the initial deformations. One observes that only a statically-indeterminate calculation is necessary, namely, that of the continuous system under the given strain. Fig. 9 explains the principle that is applied.

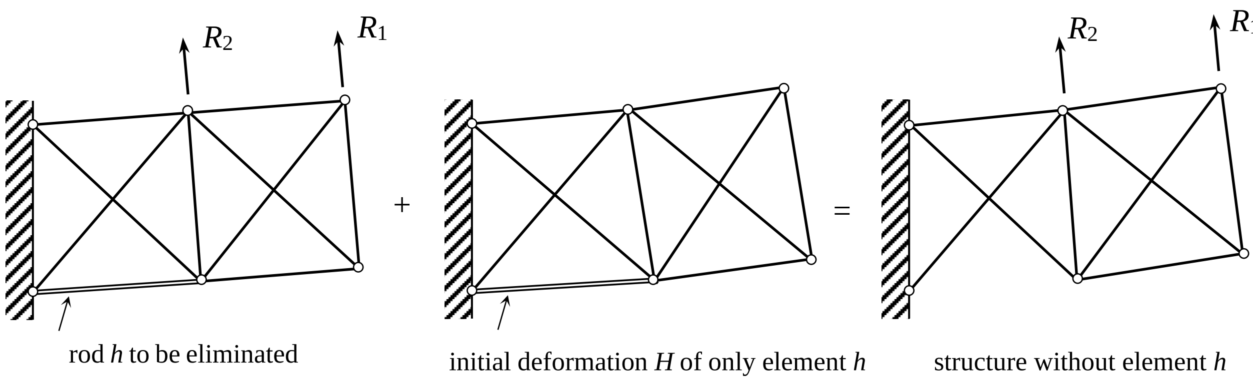


Figure 9. Calculating a structure with cutouts.

A simple argument will show that the dual method to this force procedures, which is the method of the deformation procedure, will yield the calculation for a structure with some elements that are (infinitely) rigid. In this section, we shall develop only the force procedure, which is more important in practice, but refer to the next section for the results of the dual procedure.

The original structure might have h cutouts that are filled in by the introduction of corresponding elements. Now, let **S** be the stress matrix in the continuous system that results from

([1]) See footnote ([2]) on pp. 1.

the given strain. The total stresses $\mathbf{S}_a$, including the stresses that result from the initial deformations $\mathbf{H}$ of the h elements, are determined from (21.a) and (39.a) to be:

$$\mathbf{S}_a = \mathbf{S} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{H}\ , \tag{41}$$

in which $\mathbf{b}_{1h}$ is the submatrix of $\mathbf{b}_1$ that corresponds to the h elements. We now get the condition for the total stress in the h elements to vanish as:

$$\mathbf{S}_{ah} = \mathbf{S}_h - \mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{H} = 0$$

or

$$\mathbf{H} = [\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}]^{-1}\,\mathbf{S}_h\ . \tag{42}$$

Therefore, the stresses in the original system with the cutouts will be:

$$\mathbf{S}_a = \mathbf{S} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,[\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}]^{-1}\,\mathbf{S}_h\ . \tag{43}$$

It should be noted that we need to invert the matrix:

$$[\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}]$$

with this method. The number r of rows or columns in that matrix is equal to the number of stresses ([1]) S (and therefore the initial deformations H, as well) in the elements h. That is why in a highly statically-indeterminate system like an airplane wing, one should expect that r will generally be much smaller than the number of unknowns in the original system. That also shows that in the case of a subsequent introduction of cutouts into a construction that was already calculated, the new procedure can be much faster than the direct investigation of the system with cutouts.

When the structure is affected by only forces $\mathbf{R}$, we can put (43) into the form:

$$\mathbf{S}_a = \mathbf{b}_a\,\mathbf{R}\ , \tag{44}$$

in which we have:

$$\mathbf{b}_a = \mathbf{b} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,[\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}]^{-1}\,\mathbf{b}_h\ . \tag{44.a}$$

The procedure also admits a simple derivation of the compliance $\mathbf{F}_a$ of the original system when the compliance of the continuous system is known already. In fact, an application of (25.a) and (36.a) to both structures ($\mathbf{r}_a$ are the displacements of the former system) will give:

$$\mathbf{r}_a = \mathbf{F}_a\,\mathbf{R} = \mathbf{r} + \mathbf{b}'_h\,\mathbf{H} = \mathbf{F}\,\mathbf{R} + \mathbf{b}'_h\,[\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}]^{-1}\,\mathbf{b}_h$$

([1]) Strictly speaking, we must only set those stresses S in the elements h equal to zero that are linearly independent. Otherwise, the matrix above would be singular. The number r of rows would then be equal to the number of linearly-independent S is the elements h.

or

$$\mathbf{F}_a = \mathbf{F} + \Delta\mathbf{F} \,. \tag{45}$$

In that:

$$\Delta\mathbf{F} = \mathbf{b}'_h \, [\mathbf{b}_{1h} \, \mathbf{D}^{-1} \, \mathbf{b}'_{1h}]^{-1} \, \mathbf{b}_h \, \mathbf{R}^{-1} \tag{45.a}$$

is the increment in the compliance that results from the cutout.

Of course, the introduction of elements into the cutouts means that one is raising the degree of static indeterminacy of the system. However, that is irrelevant when one uses an electronic digital computer. The method has proved to be very simple and expeditious in practice.

Example. – For the framework that is illustrated in Fig. 3a, determine the stress matrix **b** and the compliance **F** for the loads R_1 and R_2 when rod 4 is missing (see also Fig. 8). All of the remaining rods shall again possess the same compliance $l / E A$.

The continuous framework with the element h (4) included was calculated in Section 4.a. We infer equations (27.a) and (27.b):

$$\mathbf{b}_{1h} = [0, -1/\sqrt{2}] \,, \qquad \mathbf{b}_h = \frac{1}{55}[83, 24] \,. \tag{46}$$

With the use of the matrix $\mathbf{D}^{-1}$ that was determined in 4 a), we will find that:

$$[\mathbf{b}_{1h} \, \mathbf{D}^{-1} \, \mathbf{b}'_{1h}]^{-1} = \frac{55}{8} \frac{l}{E A} \,. \tag{47}$$

In conclusion, we determine $\mathbf{b}_a$ and $\Delta\mathbf{F}$ from (44.a) and (46.a):

$$\mathbf{b}_a = \frac{1}{55} \begin{array}{c} \begin{array}{cc} R_1 & R_2 \end{array} \\ \left[\begin{array}{cc} -20.625 & 0 \\ -165.0 & -55 \\ 34.375 & 0 \\ 0 & 0 \\ 34.375 & 0 \\ -75.675 & 0 \\ 20.675 \cdot \sqrt{2} & 0 \\ 110 \cdot \sqrt{2} & 55 \cdot \sqrt{2} \\ -34.375 \cdot \sqrt{2} & 0 \\ 55 \cdot \sqrt{2} & 0 \end{array} \right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} , \qquad \Delta\mathbf{F} = \frac{l}{55 E A} \begin{bmatrix} 861 & 249 \\ 249 & 72 \end{bmatrix} , \tag{48}$$

The compliance $\mathbf{F}_a$ is obtained from (45) with the use of (30.a) for **F**. Of course, the direct investigation of the single-indeterminate framework in Fig. 8 is trivially simple. The calculation above is also understood to mean only an explanation of the procedure.

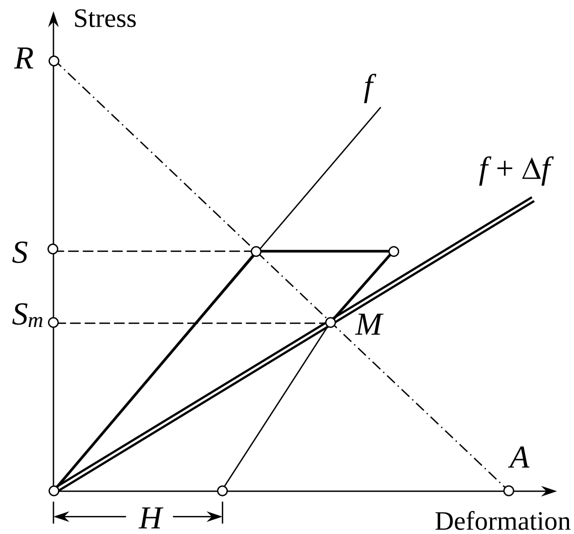


Figure 10a. Calculating a modified system by the force procedure.

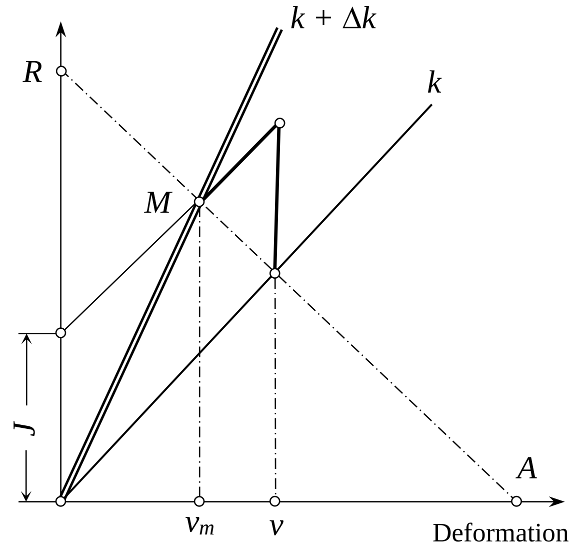


Figure 10b. Calculating a modified system by the deformation procedure.

6. Calculating modified systems. – We shall now generalize the method of Section **5** to the calculation of modified structures. We understand a modified structure to mean a system that is obtained from an original system by changing the compliance or stiffness of individual elements. We shall then treat the following problem here: Under the assumption that the stress distribution in a structure results from a given strain that was determined already, determine the stresses when individual elements are subsequently modified. The solution to that problem by the force procedure results in the same way that it did in Section **5**. Therefore, the elements to be modified will be given initial deformations **H** in such a way that the total deformation will result from the given strain and **H** in the same way as it would in the modified system under the given strain alone. In what follows, the deformation procedure will also be applied to the same problem, and Secs. 10.a and 10.b explain the physical principle of the solution by the two dual procedures. The mathematical development will confirm that the method of Section **5** is only a special case of the new method.

We assume that the original system has $g + h$ elements, h of which are subsequently modified. The index m will refer to quantities in the modified system.

The force procedure

The compliance of the unconstrained element is:

The deformation procedure

The stiffness of the unconstrained element is:

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_g & 0 \\ 0 & \mathbf{f}_h \end{bmatrix} \tag{49.a}$$

in the original system and:

$$\mathbf{f}_m = \begin{bmatrix} \mathbf{f}_g & 0 \\ 0 & \mathbf{f}_h + \Delta\mathbf{f}_h \end{bmatrix} \tag{50.a}$$

in the modified structure.

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_g & 0 \\ 0 & \mathbf{k}_h \end{bmatrix} \tag{49.b}$$

in the original system and:

$$\mathbf{k}_m = \begin{bmatrix} \mathbf{k}_g & 0 \\ 0 & \mathbf{k}_h + \Delta\mathbf{k}_h \end{bmatrix} \tag{50.b}$$

in the modified structure.

The static analysis of the original system

Let the known stresses in the original system result from the given strain be denoted by **S**. In addition, initial deformations **H** will act on the unconstrained elements *h*. The total stress is calculated as in Problems I and III [see (41), as well]:

$$\mathbf{S}_m = \mathbf{S} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{H}\,. \tag{51.a}$$

The total deformation in the elements *h* are then:

$$\mathbf{v}_h = \mathbf{f}_h\,\mathbf{S}_{mh} + \mathbf{H} = \mathbf{f}_h\,[\mathbf{S}_h - \mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{H}] + \mathbf{H} \tag{52.a}$$

Let the known deformations in the original system result from the given strain be denoted by **v**. In addition, constraint stresses **J** will act on the elements *h* with fixed nodes. The total deformations is calculated as in Problems I and III:

$$\mathbf{v}_m = \mathbf{v} - \mathbf{a}_1\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,\mathbf{J}\,. \tag{51.b}$$

The total stresses in the elements *h* are then:

$$\mathbf{S}_h = \mathbf{k}_h\,\mathbf{v}_{mh} + \mathbf{J} = \mathbf{k}_h\,[\mathbf{v}_h - \mathbf{a}_{1h}\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,\mathbf{J}] + \mathbf{J} \tag{52.b}$$

The static analysis of the modified system

Since we prescribe that the stresses and deformations in the modified elements *h* under the given strain alone should be identical to (51.a) [(52.a), resp.], that will imply that:

$$\mathbf{v}_{hm} = [\mathbf{f}_h + \Delta\mathbf{f}_h][\mathbf{S}_h - \mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{H}] = \mathbf{v}_h \tag{53.a}$$

With the use of (52.a), it will then follow that:

$$\mathbf{H} = \mathbf{P}^{-1}\,\mathbf{S}_h\,, \tag{54.a}$$

where:

$$\mathbf{P} = [\mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h} + \Delta\mathbf{f}_h^{-1}]\,. \tag{55.a}$$

One observes that the number of rows in the matrix **P** is equal to the number of stresses *S* in

Since we prescribe that the deformations and stresses in the modified elements *h* under the given strain alone should be identical to (51.b) [(52.b), resp.], that will imply that:

$$\mathbf{S}_{hm} = [\mathbf{k}_h + \Delta\mathbf{k}_h][\mathbf{v}_h - \mathbf{a}_{1h}\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,\mathbf{J}] = \mathbf{S}_h \tag{53.b}$$

With the use of (52.b), it will then follow that:

$$\mathbf{J} = \mathbf{Q}^{-1}\,\mathbf{v}_h\,, \tag{54.b}$$

where:

$$\mathbf{Q} = [\mathbf{a}_{1h}\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h} + \Delta\mathbf{k}_h^{-1}]\,. \tag{55.b}$$

One observes that the number of rows in the matrix **Q** is equal to the number of the h elements (cf., pp. 22). Upon substituting (54.a) in (51.a), we will get:

$$\left.\begin{aligned}\mathbf{S}_m &= \mathbf{S} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{P}^{-1}\,\mathbf{S}_h\,,\\ \mathbf{v}_m &= \mathbf{f}_m\,\mathbf{S}_m\,.\end{aligned}\right\}\tag{56.a}$$

In that way, the stresses and deformations of the modified systems will be expressed in terms of merely the stress distribution in the original system.

If the stresses **S** are due to only the loads **R** (**S** = **b** **R**) then the modified matrix $\mathbf{b}_m$ ($\mathbf{S}_m = \mathbf{b}_m$ **R**) will become:

$$\mathbf{b}_m = \mathbf{b} - \mathbf{b}_1\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,\mathbf{P}^{-1}\,\mathbf{b}_h\,.\tag{57.a}$$

deformations v in the h elements. Upon substituting (54.b) in (51.b), we will get:

$$\left.\begin{aligned}\mathbf{v}_m &= \mathbf{v} - \mathbf{a}_1\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,\mathbf{Q}^{-1}\,\mathbf{v}_h\,,\\ \mathbf{S}_m &= \mathbf{k}_m\,\mathbf{v}_m\,.\end{aligned}\right\}\tag{56.b}$$

In that way, the deformations and stresses of the modified systems will be expressed in terms of merely the stress distribution in the original system.

If the deformations **v** are due to only the displacements **r** (**v** = **a** **r**) then the modified matrix $\mathbf{a}_m$ ($\mathbf{v}_m = \mathbf{a}_m$ **r**) will become:

$$\mathbf{a}_m = \mathbf{a} - \mathbf{a}_1\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,\mathbf{Q}^{-1}\,\mathbf{a}_h\,.\tag{57.b}$$

Compliance of the modified system

The application of (25.a) and (36.a) to the original and modified system under the load **R** will yield:

$$\mathbf{F}_m = \mathbf{F} + \Delta\mathbf{F},\qquad (58.a)$$
$$\Delta\mathbf{F} = \mathbf{b}'_{1h}\,\mathbf{P}^{-1}\,\mathbf{b}_h\,.\qquad (58.a)$$

Δ**F** is the increment (positive or negative) of the compliance that results from the modifications.

The application of (25.b) and (36.b) to the original and modified system under the displacement **r** will yield:

$$\mathbf{K}_m = \mathbf{K} + \Delta\mathbf{K},\qquad (58.b)$$
$$\Delta\mathbf{K} = \mathbf{a}'_{1h}\,\mathbf{Q}^{-1}\,\mathbf{a}_h\,.\qquad (58.b)$$

Δ**K** is the increment (positive or negative) of the stiffness that results from the modifications.

Special cases

1) Eliminating the h elements (cutouts): One has:

$$\Delta\mathbf{f}_h \to \infty \quad\text{and}\quad \mathbf{P} = \mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h}\,.\tag{60.a}$$

That case was treated in Section **5** and correspond to point A in Fig. 9a.

2) Rigidifying the h elements: One has:

$$\Delta\mathbf{f}_h = -\,\mathbf{f}_h \quad\text{and}\quad \mathbf{P} = \mathbf{b}_{1h}\,\mathbf{D}^{-1}\,\mathbf{b}'_{1h} - \mathbf{f}_h^{-1}\,.\tag{61.a}$$

That case corresponds to point R in Fig. 9a.

1) Rigidifying the h elements: One has:

$$\Delta\mathbf{k}_h \to \infty \quad\text{and}\quad \mathbf{Q} = \mathbf{a}_{1h}\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h}\,.\tag{60.b}$$

That case is dual to the method of Section **5** and corresponds to point R in Fig. 9b.

2) Eliminating the h elements: One has:

$$\Delta\mathbf{k}_h = -\,\mathbf{k}_h \quad\text{and}\quad \mathbf{Q} = \mathbf{a}_{1h}\,\mathbf{C}^{-1}\,\mathbf{a}'_{1h} - \mathbf{k}_h^{-1}\,.\tag{61.b}$$

That case corresponds to point A in Fig. 9b.

Example. – For the framework that is illustrated in Fig. 3a, determine the stress matrix $\mathbf{b}_m$ and the compliance $\mathbf{F}_m$ for the loads R_1 and R_2 . The compliance of rod 10 is $l / 6EA$, but for all of the remaining rods, it is l / EA. We consider the framework in 4 a) with the same compliances for the rods as in the original system, and subsequently modify rod 10. We then get from (27.a) and (29.a):

$$\left.\begin{aligned} \mathbf{b}_{1h} &= [0, 1], \\ \mathbf{b}_h &= \tfrac{1}{55}[-28\sqrt{2}, -24\sqrt{2}], \\ \Delta\mathbf{f}_h &= \frac{l}{6EA} - \frac{l}{EA} = -\frac{5}{6}\frac{l}{EA}. \end{aligned}\right\} \tag{62.a}$$

With the use of the matrix $\mathbf{D}^{-1}$ that was determined in 4 a), we will find that:

$$\mathbf{P}^{-1} = -\frac{55}{50}\frac{l}{EA}, \tag{63.a}$$

and therefore:

$$\mathbf{b}_m = \frac{1}{55}\begin{array}{c} \begin{array}{cc} R_1 & R_2 \end{array} \\ \left[\begin{array}{cc} -32.12 & -3.96 \\ -73.04 & -23.92 \\ 22.88 & -3.96 \\ 91.96 & 31.68 \\ 22.88 & -3.96 \\ 4.84 & 27.72 \\ 32.12\sqrt{2} & 3.96\sqrt{2} \\ 18.04\sqrt{2} & 23.32\sqrt{2} \\ -22.88\sqrt{2} & 3.96\sqrt{2} \\ -36.96\sqrt{2} & -31.68\sqrt{2} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array},$$

$$\Delta\mathbf{F} = -\frac{l}{55EA}\begin{bmatrix} 31.4 & 26.9 \\ 26.9 & 23.1 \end{bmatrix}. \tag{64.a}$$

We ascertain $\mathbf{F}_m$ from (58.a).

Example. – For the framework that is illustrated in Fig. 3b, determine the deformation matrix $\mathbf{a}_m$ and the stiffness $\mathbf{K}_m$ for the loads r_1 and r_2 . We consider the framework in 4 a) with the same stiffnesses for the rods as in the original system, and subsequently modify rod 10. We then get from (27.b) and (29.b):

$$\left.\begin{aligned} \mathbf{a}_{1h} &= [0, 0, 0, 0, 1\sqrt{2}, -1\sqrt{2}], \\ \mathbf{a}_h &= \tfrac{1}{335}[12\sqrt{2}, -104\sqrt{2}], \\ \Delta\mathbf{k}_h &= \frac{6EA}{l} - \frac{EA}{l} = 5\frac{EA}{l}. \end{aligned}\right\} \tag{62.b}$$

With the use of the matrix $\mathbf{C}^{-1}$ that was determined in 4 b), we will find that:

$$\mathbf{Q}^{-1} = \frac{355}{226}\frac{EA}{l}, \tag{63.b}$$

and therefore:

$$\mathbf{a}_m = \frac{1}{355}\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \left[\begin{array}{cc} -44.47 & 50.40 \\ -62.85 & -13.64 \\ 49.81 & -96.65 \\ 72.93 & 37.95 \\ 49.81 & -96.65 \\ -65.81 & 235.39 \\ 44.47\sqrt{2} & -50.40\sqrt{2} \\ -31.42\sqrt{2} & 160.68\sqrt{2} \\ -49.81\sqrt{2} & 96.65\sqrt{2} \\ 3.56\sqrt{2} & -30.83\sqrt{2} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array},$$

$$\Delta\mathbf{K} = \frac{EA}{335}\begin{bmatrix} 1.3 & -11.1 \\ -11.1 & 95.8 \end{bmatrix}. \tag{64.b}$$

We ascertain $\mathbf{K}_m$ from (58.b).

We again confirm that $\mathbf{b}'_m \mathbf{a}_m = \mathbf{E}_2$ is fulfilled.

The influence of the increased stiffness in rod 10 on the stress distribution is relatively minor, which was to be expected.

(Received on 4 August 1956)

Author's address: Professor J. H. Argyris D. E., London S. W. 7, Prince Consort Road, Imperial College of Science and Technology (University of London).