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On surface transformations

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I.

In recent years, I have posed the question of whether there are, amongst the surface transformations of a triply-extended space, ones for which, not the contact of first order, but contact of second order – viz., osculation – plays the role of an invariant. I have treated this question in a paper in volume X of the Jahresschrift der Universität Lund (Sept. 1874), and there I arrived at the result that those transformations for which first-order contact is already an invariant relation – i.e., Lie's contact transformations – are also the only ones that leave contact of higher order invariant. At the same time, a paper of Lie [*]) appeared in volume VIII of the Mathematischen Annalen, in which the actual question of osculation transformations was raised. For that reason, I would like to undertake the aforementioned investigation of what I provisionally posed, as well as examining more closely some of the points were only touched upon there. I thus commence here with the proof of the non-existence of any special osculation transformations of planar curves, and will first carry out this proof in a purely geometric way and then in a purely analytic one (§ 1.2). Thus, I shall first give a more precise overview of the question that is to be treated later on.

§ 1.

Geometric proof of the absence of any special osculation transformations of planar curves.

1. The osculation transformations will convert every curve in the plane into one or more, but not infinitely many, curves in that plane, and furthermore, any two mutually osculating curves into two likewise mutually osculating curves. Thus, if one applies an osculation transformation to a figure that consists of a curve C and two infinitely close curves C', C'' that osculate C in two neighboring points then what must result consists of a curve Γ that is the transform of C and two neighboring curves Γ', Γ'' that are the transforms of C', C'' and osculate Γ in two neighboring points. Since C', C'' osculate one and the same curve in two neighboring points, they must then contact each other, and on the same basis, they must also contact Γ', Γ''. *This means that any osculation*

[*]) Begründung einer Invariantentheorie der Berührungstransformationen, by Sophus Lie, pp. 233, note.

transformation must possess the property that any two infinitely neighboring curves must be converted into two curves of the same type.

However, *as I will likewise show*, this property belongs to Lie's contact transformations alone, and thus we achieve the proof of the absence of any special osculation transformations.

2. If one has a transformation of the stated type in the (x, y) plane, which then takes any curve in that plane to another curve and any two infinitely close mutually contacting curves to two likewise infinitely close, mutually contacting curves then if $\lambda_1, \lambda_2, \lambda_3$ are the parameters of any three-fold system of curves $\psi(x, y, \lambda_1, \lambda_2, \lambda_3) = 0$, and if:

(1) $$\varphi(\lambda_1, \lambda_2, \lambda_3, d\lambda_1, d\lambda_2, d\lambda_3) = 0$$

expresses the condition for two neighboring curves (λ), $(\lambda + d\lambda)$ to contact each other, which are the curves that arise from the curve (λ) by the stated transformation, or, more briefly, which are the curves that correspond to the curve (λ) and that can be represented by an equation:

(2) $$f(x, y, \lambda_1, \lambda_2, \lambda_3) = 0,$$

which is so arranged that when one eliminates x, y, p from this equation and the three following ones:

(3) $$\begin{cases} f'(x) + pf'(y) = 0, \\ \sum \dfrac{df}{d\lambda} d\lambda = 0, \\ \sum \dfrac{df'(x)}{d\lambda} d\lambda + p \sum \dfrac{df'(y)}{d\lambda} d\lambda = 0, \end{cases}$$

then one comes back to equation (1). The latter equation then defines the contact condition for two consecutive curves (λ), as well as for two consecutive curves (2), such that when two infinitely close curves (λ) contact each other, the corresponding curves (2) must also contact each other.

Conversely, if any two three-fold systems of curves give rise to the same differential equation as the condition of contact then this itself will be the basis for a transformation that has precisely the aforementioned character.

Having established that, it would give a three-fold systems of curves – perhaps, the first system (λ) – so one will be led to an essential relationship between this system and any other one (2) in the following way:

If the parameters λ are the point coordinates of a space R_3 with three dimensions, and the variables x, y are regarded as arbitrary constants then equation (2) represents a system of ∞^2 surfaces in R_3. Equation (1) associates any point of R_3 with an elementary complex cone, and since (1) results from (2) by means of equations (3), any two infinitely close surfaces (2) must intersect in a curve whose line elements $(\lambda, d\lambda)$ all satisfy equation (1), so they belong to the elementary complex cone (1); in other words, these intersection curves shall be curves of the complex (1). ∞^1 surfaces (2) go through any point of R_3 . Their tangent planes at each point will be enveloped by a cone that coincides in the vicinity of the point with the cone that is generated by the line elements of the intersection curve of any two neighboring surfaces in this ∞^1 that are determined by this point. This is, however, precisely the elementary complex cone (1) that goes through the point. Thus:

The surfaces (2) *define a solution, with two arbitrary constants x, y, of the partial differential equation of first order whose elementary complex cone is represented by equation* (1).

Now, two neighboring integrals of the partial differential equation of first order – I shall call it $\Phi = 0$, for brevity – shall always intersect on a characteristic of this equation. It must then be the curve that is represented by the equations:

$$f = 0, \qquad f'(x) + p\, f'(y) = 0$$

as a characteristic of $\Phi = 0$, and indeed each system of values x, y, p will correspond to a definite characteristic *); i.e., a simple infinitude of curves (2) that possess a common element x, y, p, corresponding to ∞^1 points (λ) that define a characteristic of $\Phi = 0$. If we represent the *system of curves* (λ) by its equation in x, y point coordinates:

$$\psi(x, y, \lambda_1, \lambda_2, \lambda_3) = 0 \tag{4}$$

then – when $x, y, \lambda_1, \lambda_2, \lambda_3$ is interpreted in the way above – it will be a solution of $\Phi = 0$ with x, y as two arbitrary constants, and any family of curves λ (i.e., curves (4)) that contact it at one and the same point must then correspond to a characteristic of $\Phi = 0$. Now, there are no more than ∞^3 characteristics of $\Phi = 0$, so, in general, a characteristic envelopes no family of ∞^1 characteristics, and thus, conversely, the points (λ) of an arbitrary characteristic of $\Phi = 0$ must correspond to infinitely many curves (λ), namely, curves (4), that contact it at one and the same point. Therefore, a simple infinitude of curves (2) that contact at a point correspond to a simple infinitude of curves (l) that likewise contact at a point, such that the two families of curves correspond to the same characteristic.

With this, we have proved:

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*) The result of this argument is simply the following: Every ordinary nonlinear differential equation $\varphi(\lambda, d\lambda) = 0$ can be replaced in an unbounded number of ways by ∞^3 curves $\psi(x, y, p, \lambda_1, \lambda_2, \lambda_3) = 0$, $\chi(x, y, p, \lambda_1, \lambda_2, \lambda_3) = 0$, or else it is, according to Lie, based in a certain curve complex by means of $\varphi(\lambda, d\lambda) = 0$. Now, there is just one system of curves in this complex that is representable by a system of equations of the form:

$$\psi(x, y, \lambda_1, \lambda_2, \lambda_3) = 0, \qquad \psi'(x) + p\ \psi'(y) = 0.$$

These are the characteristics of the partial differential equation of first order that is connected with $\varphi(\lambda, d\lambda) = 0$.

It might also be remarked that it follows from the above that any ordinary nonlinear differential equation $\varphi(\lambda, d\lambda) = 0$ can be interpreted as the condition for the contact of two neighboring curves of a three-fold system of curves.

If a correspondence is established between two three-fold systems of curves, in such a way that any two neighboring, mutually contacting curves of the one system correspond to two such curves in the other system then all curves of the one systems that contact at a point must also correspond to a system of such curves in the other one.

As a consequence, the transformation that leads from one curve system to the other one is a transformation of line elements (x, y, p). It must further be the case that any two united elements must go to two other such elements, if two united line elements always belong to a (real or imaginary) curve of the one system and the corresponding elements are linked to the corresponding curve. – *Every transformation of the initially given type is then a Lie contact transformation.* Q. E. D.

§ 2.

Analytical proof of the same theorem.

3. Since any curve in the plane possesses a system of values $x, y, p, p', \dots$ *), by which it is conversely completely determined, any curve transformation of two spaces (x, y), (X, Y) – these spaces are thought of as extended over each other – must, in the first place, be a transformation of the systems of values $x, y, p, p', \dots$ and $X, Y, P, P', \dots$ In particular, an osculation transformation will take a system of values (x, y, p, p') to a system (X, Y, P, P'), and naturally all of the systems of values that belong to a curve in (x, y) must be converted into a system that belongs to a curve in (X, Y). Every osculation transformation will then be defined by equations of the form:

$$\begin{aligned} x &= F(X, Y, P, P'), \\ y &= F_1(\qquad\qquad), \\ p &= \Phi_1(\qquad\qquad), \\ p' &= \Phi_2(\qquad\qquad), \end{aligned}$$

where the $F, \dots, \Phi_2$ are determined in such a way that the system of equations:

$$\text{(a)} \qquad dy - p\,dx = 0, \qquad dp - p'\,dx = 0,$$

always goes to the similar system:

$$\text{(b)} \qquad dY - P\,dX = 0, \qquad dP - P'\,dX = 0.$$

This is then the analytical condition for two neighboring elements $(x, \dots, p')$, $(x + dx, \dots, p' + dp')$ that belong to a curve to yield corresponding elements $(X, \dots, P')$, $(X + dX, \dots, P' + dP')$ that likewise belong to a curve.

I now consider the following series of ∞^1 consecutive elements (x, y, p, p'):

*) $p = dy / dx$, $p' = dp / dx$, …

$$x_0,\ y_0,\ p_0,\ p',$$
$$x_0,\ y_0,\ p_0,\ p' + dp',$$
$$x_0,\ y_0,\ p_0,\ p' + 2\ dp',$$
$$\dots$$

If any two of these neighboring elements satisfy equations (a) – so one then has $dx = dy = dp = 0$ – then any two neighboring elements of the corresponding ∞^1 elements (X, Y, P, P') must then satisfy equations (b); i.e., these ∞^1 elements will be coupled by a curve. By eliminating P, P' from the equations:

$$\begin{aligned} x_0 &= F(X, Y, P, P'), \\ y_0 &= F_1(\qquad\qquad), \\ p_0 &= \Phi_1(\qquad\qquad), \end{aligned}$$

it is then self-explanatory that one will obtain the equation of any curve *that corresponds to the line element* (x_0, y_0, p_0), and in a similar way, the equation of a curve in (x, y) *that corresponds to the line element* (X, Y, P). Alternatively, when the quantity P' is eliminated from the transformation equation, every osculation transformation must lead to two equations:

(c) $$f(x, y, p, X, Y, P) = 0, \qquad \varphi(x, y, p, X, Y, P) = 0,$$

that possess a common integral in the variables x, y, p, as well as in the X, Y, P. Conversely, two equations (c) that have stated relationship to each other will determine an osculation transformation – *assuming that* the equations:

$$f = 0, \quad \varphi = 0, \ \frac{df}{dx} + p\frac{df}{dy} + p'\frac{df}{dp} = 0, \qquad \frac{df}{dX} + P\frac{df}{dY} + P'\frac{df}{dP} = 0 \quad ^*)$$

for every arbitrary system of values (X, Y, P, P') [(x, y, p, p'), resp.] yield a system of values (x, y, p, p') [(X, Y, P, P'), resp.], or some such system of values.

However, I will show that systems of equations with the property (c) *cannot lead to all systems of values* (x, y, p, p') *in the plane*, since any equations in the three-fold infinitude (X, Y, P) are associated with only a two-fold infinitude of curves, so by the calculation described, only the ∞^3 elements (x, y, p, p') can appear on these curves. With that, it is then proved that no special osculation transformation can exist.

4. If P (p, resp.) were eliminated from equations (c) then this would yield two equations:

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*) Combined, if necessary, with the equations:

$$\frac{d\varphi}{dx} + p\frac{d\varphi}{dy} + p'\frac{d\varphi}{dp} = 0, \qquad \frac{d\varphi}{dX} + P\frac{d\varphi}{dY} + P'\frac{d\varphi}{dP} = 0,$$

which, due to the connection that exists between f and φ, can always be combined with the equations above.

(d) $$p = f(x, y, X, Y), \qquad P = \varphi(x, y, X, Y)$$

that completely replace them, and will possess a common integral in the space (x, y), as well as in the space (X, Y). This relationship between equations (d) will be expressed algebraically by the relations:

$$\frac{d\varphi}{dx} + f\,\frac{d\varphi}{dy} = 0, \qquad \frac{df}{dX} + \varphi\frac{df}{dY} = 0,$$

from which, by eliminating f, an equation for the determination of φ emerges:

$$\frac{d}{dx}\left(\frac{d\varphi}{dx} : \frac{d\varphi}{dy}\right) + \varphi\frac{d}{dY}\left(\frac{d\varphi}{dx} : \frac{d\varphi}{dy}\right) = 0.$$

This can be brought into the form:

$$\frac{d}{dx}\left(\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY}\right) - \left(\frac{d\varphi}{dx} : \frac{d\varphi}{dy}\right)\cdot\frac{d}{dy}\left(\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY}\right) = 0;$$

i.e., if $\frac{d\varphi}{dx} + p\,\frac{d\varphi}{dy} = 0$ then one must also have:

$$\left(\frac{d}{dx} + p\frac{d}{dy}\right)\left(\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY}\right) = 0.$$

The differentials of φ and $\left(\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY}\right)$, when both regarded as functions of x, y, shall then vanish simultaneously, so:

$$\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY} = \psi(X, Y, \varphi).$$

The integral of this equation is of the form:

$$\text{an arbitrary function of } (\Psi_1(X, Y, \varphi), \Psi_2(X, Y, \varphi), x, y) = 0.$$

Here, if one sets P in place of φ then one has the second of equations (d). It is only doubly infinite relative to X, Y, P, so the ∞^3 (X, Y, P) will thus be associated with only ∞^3 curves in (x, y), and thus, from what we just set down, there is no special osculation transformation of the curves in a plane. Q.E.D.

II.

All of this begs the question: To what extent can any result that was established for the plane be extended to spaces of more dimensions? I will resolve this question by generally treating the problem of exhibiting all of the transformations of space of $n + 1$ dimensions that take the manifolds of n dimensions – i.e., the surfaces in this space – to each other. Regarding such transformations, it is clear, *a priori*, that there must be two essentially different classes of them: The one subsumes all transformations that take any surface in a domain (z, x_1, x_2, …, x_n) of space in general into only one surface (some surfaces, resp.) in the other domain (Z, X_1, X_2, …, X_n), while the second class subsumes the ones that make any surface in the one domain correspond to infinitely many of them in the other one.

I direct my attention, in turn, to spaces of 2 dimensions – viz., plane. Since a curve in the plane is completely determined by a system of values (x, y, p, p', …), and since the condition for this is that two infinitely close systems of values of this type belong to one and the same curve should be expressed by the following equations:

(A) $$dy - p\,dx = 0, \qquad dp - p'\,dx = 0, \ldots,$$

any curve transformation of two domains (x, y), (X, Y) must be characterized by equations between x, y, p, p', …, X, Y, P, P', … that take the systems of equations (A) to the similar system:

(B) $$dY - P\,dX = 0, \qquad dP - P'\,dX = 0, \ldots$$

I. e., in order to exhibit a curve transformation, one must define two arbitrary equations:

(C) $$\begin{cases} X = F(x, y, p, p', \cdots, p^k), \\ Y = F_1(x, y, p, p', \cdots, p^l), \end{cases}$$

and, when one satisfies the aforementioned condition of the simultaneous existence of the systems of equations (A), (B), derives the following equations from them:

(D) $$\begin{cases} P = \Phi(x, y, p, p', \cdots), \\ P' = \Phi_1(\qquad\qquad), \\ \quad \cdots \end{cases}$$

In general, it will then be the case that equations (C), (D) cannot be solved for x, y, p, p', …, in which case, the transformation belongs to the second class above: It will be a *multi-valued* transformation. Indeed, any curve in (x, y) will, in fact, be converted into only one curve in the domain (X, Y), but a curve in the latter domain will correspond to infinitely many curves in the former, namely, all integrals of a certain differential equation *). However, when equations (C) are chosen in such a way that they, along with

*) Or possibly, a system of several differential equations.

perhaps the first k of equations (D), define a system that can be solved for $x, y, p, \ldots, p^{k-1}$, such that these equations are also representable in the form:

$$\begin{aligned} x &= f(X, Y, P, \ldots), \\ y &= f_1(\qquad\qquad), \\ p &= \varphi(\qquad\qquad), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

then the transformation will belong to the first class above: It will be a *single-valued* (i.e., finitely multi-valued) transformation. Then, first and foremost, it will be a transformation of same-named curve segments $(x, y, p, \ldots, p^{k-1})$ and $(X, Y, P, \ldots, P^{k-1})$, and furthermore, those segments $(x, y, p, \ldots, p^{k-1})$ that unite into a curve correspond to same-named segments that likewise unite into a curve.

In regard to the transformation of this class, there exists the theorem that they are exclusively contact transformations, as Lie defined them. All single-valued curve transformations are thus transformations of (x, y, p) into (X, Y, P). It was already proved that no other curve transformation of (x, y, p, p') to (X, Y, P, P') exists. In the following paragraphs, it will also be shown that no special contact transformations of higher order exist.

We go on to spaces with an arbitrary number $n + 1$ of dimensions. In order to define a transformation in the most general way that takes all surfaces (M_n), in turn, to surfaces, one may take $n + 1$ arbitrary equations:

$$(\mathrm{C}') \qquad \begin{cases} Z = F(z, x_1, \cdots, x_n, p_1, \cdots, p_n, p_{11}, p_{12}, \cdots, p_{klm}, \cdots), \\ X_1 = F_1(\qquad\qquad\qquad\qquad\qquad), \\ \quad\cdots \\ X_n = F_n(\qquad\qquad\qquad\qquad\qquad), \end{cases}$$

and derive the following ones from them by differentiation and elimination:

$$(\mathrm{D}') \qquad \begin{cases} P_k = \Phi_k(z, \cdots, x_k, \cdots, p_k, \cdots p_{kl}, \cdots, p_{klm}, \cdots), \\ P_{kl} = \Phi_{kl}(\qquad\qquad\qquad\qquad), \\ \cdots \\ \qquad\qquad (k, l, m, \cdots = 1, 2, \cdots, n), \end{cases}$$

such that the system of equations:

$$(\mathrm{A}') \qquad dz - \sum p_k\, dx_k = 0, \qquad dp_k - \sum p_{kl}\, dx_l = 0, \qquad \ldots, \textit{ad inf.}$$

remains invariant.

In general, a multi-valued transformation will be established by (C′). It is only when (C′) is a Lie contact transformation – and this theorem will be treated in paragraph 4 – that the surface transformation is single-valued (finitely multi-valued).

As was remarked, I have previously discussed the question of whether there are no other single-valued surface transformations than just the Lie contact transformations, and the proof of this character of the latter transformations was presented in a paper in the Jahresschrift der Universität Lund for two and three dimensions. At the same time, Lie addressed the same question in a treatise in the Mathematischen Annalen, and then added another one: the question of whether partial differential equations of higher order admit transformations that are not contact transformations. The proof of the non-existence of contact transformations of higher order that was carried our in my previous paper, which referred to the totality of all surfaces, likewise showed, as Lie communicated to me in a letter, that no transformations of the sort could exist for the totality of integral surfaces of partial differential equations of a partial differential equation of higher order; in the present paper, I have presented this as a corollary to my earlier theorem.

In paragraph 5, I will mention a map of a partial differential equation of first order on a space of $n + 1$ dimensions to a space of n dimensions that flows out of the aggregate of the foregoing paragraphs in order to deduce a conclusion that relates to the transformation of equations of first order.

Such a map was already based on a contact transformation, and it was also deduced by Lie, as I must infer from a remark in his treatise "Allgemeine Theorie partieller Differentialgleichungen 1. Ordnung," Abh. der Gesellschaft der Wissenschaften zu Christiania für 1874, pp. 218, that was used as an aid to his synthetic investigations.

In conclusion, one finds brief remarks on a class of remarkable transformations of space of three dimensions.

In addition, I must remark here, as I also in did in my earlier paper, that in the summer of the previous year I spoke with Felix Klein in Munich on the subject of osculation transformations of the plane, in particular, and that when the problem was not resolved by these conversations, the solution of it was essentially facilitated by assuming a new viewpoint that he suggested for regarding the question.

§ 3.

On the single-valued transformations of plane curves.

5. I shall first take up the considerations of the second number in a somewhat extended form. Instead of a three-fold system of curves (4), I will treat a system with $k + 1$ arbitrary parameters λ:

$$f(x, y, \lambda_1, \lambda_2, \ldots, \lambda_{k+1}) = 0, \tag{5}$$

and apply to it the previous process that was used in the second number for system (4). $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ will be regarded as point coordinates for a space R_{k+1} with $k + 1$ dimensions and x, y as arbitrary constants. By eliminating x, y, p from the equations:

$$(6) \qquad \begin{cases} f = 0, \quad f'(x) + pf'(y) = 0, \\ \sum \frac{df}{d\lambda} d\lambda = 0, \quad \sum \frac{df'(x)}{d\lambda} d\lambda + p \sum \frac{df'(y)}{d\lambda} d\lambda = 0, \end{cases}$$

the following equation results:

$$(7) \qquad \psi(\lambda \, d\lambda) = 0,$$

which is now the condition for the contact of two neighboring curves λ, viz., curves (5).

Above all, we might make the following remarks about this equation: When one sets x, y, p, λ equal to constants – the value of λ is then chosen such that the first two of equations (6) are satisfied – one recognizes from equations (6) that every such system of values corresponds to ∞^{k-2} values of $d\lambda_i \, / \, d\lambda_{k+1}$, and indeed these values are of the form:

$$d\lambda_i = \alpha_1 d\lambda_i^{(1)} + \alpha_2 d\lambda_i^{(2)} + \cdots + \alpha_{k-1} d\lambda_i^{(k-1)}, \qquad (i = 1, 2, \ldots, k + 1),$$

where the α are taken arbitrarily. The rays of the cone $\psi = 0$ that belongs to any point arrange themselves into a singly-infinite family of planar pencils of k- 2 dimensions, and the cone itself shall thus be represented in plane coordinates by $k – 1$ equations; let:

$$(8) \qquad \begin{cases} \psi_1(\lambda_1, \cdots, \lambda_{k+1}, \pi_1, \cdots, \pi_{k+1}) = 0, \\ \psi_2(\qquad\qquad\qquad\qquad) = 0, \\ \qquad\qquad\qquad\qquad\qquad \cdots \\ \psi_{k-1}(\qquad\qquad\qquad\qquad) = 0, \end{cases}$$

be these $k – 1$ equations, which are homogeneous in π. Next, the contact condition (7) will be replaced by this system of partial differential equations of first order in R_{k+1} . However, any equation (7) is still not characterized completely. Namely, one further has that the surface elements (λ, π) of the manifolds (5) – which are M_k in the space R_{k+1} – satisfy equations (8) for the manifold system (5) as a common solution that possesses two arbitrary constants x, y.

Conversely, any system of $k – 1$ *partial differential equation of first order in* R_{k+1} *that admits a common solution with two arbitrary constants will lead to an equation* $\psi(\lambda \, d\lambda) = 0$ *that can be interpreted as the contact condition for two neighboring curves of a* k+1-*fold system:*

$$f(x, y, \lambda_1, \ldots, \lambda_{k+1}) = 0,$$

and when x, y, *are considered to merely be arbitrary constants, the equation of any such system of curves will always represent a common complete solution of the system of partial differential equations.*

By means of the map (8) of the system of equations onto the plane that comes out of this, every line element (x, y, p) in the plane will correspond to a characteristic M_{k-1} that is the intersection manifold of $k - 1$ dimensions of two infinitely close integrals M_k , and any element (x, y, p, p') will correspond to a characteristic M_{k-2} that is the intersection of three consecutive integrals M_k , etc. The points of a characteristic M_{k-1} thus correspond to those curves (5) that contact a point, and the points of a characteristic M_{k-2} thus correspond to those curves (5) that osculate at a point, etc.

Furthermore, as might emerge here especially, when such a correspondence is established between two $k + 1$-fold systems of curves:

$$f(x, y, \lambda_1, \ldots, \lambda_{k+1}) = 0,$$
$$\varphi(\qquad\qquad\qquad) = 0,$$

two neighboring mutually contacting curves of the one system correspond to two neighboring, likewise mutually contacting, curves of the other system, so all curves $f = 0$ must then contact at a point and curves $\varphi = 0$ correspond to ones with the same property. Then, both systems of curves give rise to the same system of partial differential equations (8), and the both families of $f = 0$ ($\varphi = 0$, resp.) that contact at a point correspond to one and the same characteristic M_{k-1} . What comes out of this is the fact that for all transformations of the plane for which any two neighboring, mutually contacting curves are converted into similar curves, contact of first order must be an invariant relation; thus, all such transformations are Lie contact transformations, which was already proved in the second number.

6. A curve transformation that leaves second-order contact invariant is, as we already showed, an ordinary contact transformation. A transformation that leaves third-order contact invariant will convert any two neighboring curves with second-order contact into two similar curves, or, I assert, it should also be a transformation of the class discussed in the foregoing number that converts two neighboring curves with first-order contact into other similar ones. Namely, when C', C'' refer to any two infinitely close curves with first-order contact, one can always draw a C that is infinitely close to C', C'' and which osculates these curves at two points that are close to the contact points of these curves. Such a transformation of the stated type will convert C', C'', C into Γ', Γ'', Γ, and of these curves, the latter shall osculate the former two at two neighboring points. However, since Γ', Γ'' osculates one and the same curve Γ at two neighboring points, they have first-order contact with each other. Thus, any two neighboring, contacting curves C', C'' go to two similar ones Γ', Γ'', which was what we asserted.

However, it was already shown that a transformation of the latter type is a Lie contact transformation. Thus, there is no other transformation for which contact of third-order is an invariant relation.

In the same way, it follows that no special contact transformations of order 4, 5, …, exist. However, as was proved above, any single-valued curve transformation must be a transformation of same-named curve segments $(x, y, p, \ldots, p^k)$, $(X, Y, P, \ldots, P^k)$, and therefore, a contact transformation of order 1, 2, 3, 4, … Thus, we finally have: *Any single-valued transformation of curves in a plane must be a Lie contact transformation.*

§ 4.

On transformations of manifolds M_n of n dimensions in a space of $n + 1$ dimensions.

7. Firstly, I remark that if a surface is to have r^{th} order contact with an M_n with two infinitely close surface at two infinitely close points p, p' then the latter two surfaces must have $(r - 1)^{\text{th}}$-order contact at the point p'. Conversely, when two infinitely close surfaces have $(r - 1)^{\text{th}}$-order contact, it is possible in an unbounded number of ways to construct surfaces that have r^{th}-order contact in the vicinity of the contact point. Every surface transformation that takes surfaces that have second-order contact to other such surfaces will then convert any two infinitely close surfaces with first-order contact into two other such surfaces, and on this basis a transformation for which contact of third order remains invariant must convert any two infinitely neighboring surfaces that have second-order contact into two infinitely neighboring surfaces that likewise have second-order contact. By repeating the reasoning that was carried out in the foregoing number – by constructing a surface C that has a second-order contact with two arbitrary neighboring surfaces C', C'' that have first-order contact at a point in neighborhood of the contact point, and which is itself infinitely close to this pair of surfaces C', C'' – one sees that this transformation also must convert infinitely close, contacting surfaces into two surfaces with exactly the same property, etc., in such a way that ultimately any two infinitely close surfaces with first-order contact are converted into similar ones. Now, every single-valued surface transformation must be a transformation of same-named curve segments $(z, x_k, p_k, p_{kl}, \ldots)$, $(Z, X_k, P_k, P_{kl}, \ldots)$, for which contact of some order is an invariant relation. Thus, *any single-valued surface transformation must be a transformation that leaves the first-order contact of two infinitely close surfaces invariant.*

8. We consider an $n + 2$-fold system of surfaces, say:

$$f(z, x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n+2}) = 0. \tag{9}$$

The condition equation for two surfaces that correspond to the parameters λ, $\lambda + d\lambda$ to contact each other will be obtained by eliminating x, x, p from the following $2n + 2$ equations:

$$\left\{\begin{array}{c} f = 0, \quad f'(x_k) + p_k f'(z) = 0, \\ \sum \frac{df}{d\lambda} d\lambda = 0, \quad \sum \frac{df'(x_k)}{d\lambda} d\lambda + p_k \sum \frac{df'(z)}{d\lambda} d\lambda = 0, \\ (k = 1, 2, \cdots, n), \end{array}\right. \tag{10}$$

which gives an ordinary differential equation:

$$\varphi(\lambda, d\lambda) = 0 \tag{11}$$

as the desired condition equation.

If one regards z, x as arbitrary constants and $\lambda_1, \lambda_2, \ldots, \lambda_{n+2}$ as point coordinates in a space R_{n+2} then equation (11) represents a system of elementary cones in this space, and equation (9) represents an $n + 1$-fold system of M_{n+1} in the same space, each of which (from equations (10)) will be cut at each of its points by ∞^{n-1} neighboring M_{n+1} along a manifold of dimension one whose line elements define rays of elementary cones (11). As a result of this, the ∞^n manifolds that go through one and the same point (λ) will generate a cone (11) by means of their surface elements. Thus, if we let $\Phi = 0$ be the partial differential equation of first order whose characteristic cone (or elementary complex cone) is exhibited by equation (11) then we recognize that *a system of surfaces* (9) *for which* $\varphi = 0$ *is a contact condition defines a complete solution with* $n + 1$ *arbitrary constants* $z, x_1, \ldots, x_n$ *of the partial differential equation* $\Phi = 0$.

Therefore, if the surfaces are associated with two $(n + 2)$-fold systems of surfaces:

(12) $$f(z, x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n+2}) = 0, \qquad \varphi(z, x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n+2}) = 0,$$

in such a way that two surfaces $f(\lambda^{(1)}) = 0$, $f(\lambda^{(1)} + d\lambda) = 0$ that contact each other correspond to two likewise contacting surfaces $\psi(\lambda^{(1)}) = 0$, $\psi(\lambda^{(1)} + d\lambda) = 0$ then each of the two equations must be complete solution of one and the same partial differential equation $\Phi = 0$ when z, x are interpreted as constants and the λ as variables. Thus, the parameter λ of those ∞^1 surfaces will be any one of those solutions that contact them at a point, so they possess a common system of values (z, x, p) in the space R_{n+2} of coordinates for the points of a characteristic of $\Phi = 0$, and conversely, such that the system of surfaces (12) that we just wrote down, on the basis of the aforementioned reciprocal relationship itself, must be coupled to each other in such a way that if ∞^1 surfaces of the one system contact each other at a point then the corresponding surfaces of the other system likewise contact at a point. *For that reason, the one system of surfaces must be derivable from the other one by a Lie contact transformation.*

Thus, from what we established in the previous number, *any single-valued surface transformation must be a Lie contact transformation.*

9. Amongst an $(n + k)$-fold infinitude of surfaces:

(13) $$f(z, x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n+k}) = 0,$$

there are ∞^{k-1} of them that include a given element (z, x, p). If one regards $\lambda_1, \ldots, \lambda_{n+k}$ as point coordinates in a space R_{n+k} then the condition equation for the contact of two neighboring surfaces (13) will be represented by a differential equation:

$$\psi(\lambda, d\lambda) = 0$$

that is based in a system of $k - 1$ equations of first order in R_{n+k} that is homogeneous relative to π:

(14) $$\begin{cases} \Psi_i(\lambda_1, \cdots, \lambda_{n+k}, \pi_1, \cdots, \pi_{n+k}) = 0, \\ \qquad (i = 1, 2, \cdots, k-1), \end{cases}$$

and which admits a common solution with $n + 1$ arbitrary constants. *The equation* $f = 0$ *– in which* z, x, *figure as arbitrary constants – defines such a common solution, along with* *a system of* ∞^{n+k} *manifolds* M_n *in the space* R_{n+1}, *for which* $\psi = 0$ *defines the contact conditions.*

There will thus be a relationship between the space R_{n+1} and the elements (λ, π) in the space that obey the system of equations (14). I will come back to this later.

10. *Corollary to the theorem of number 8.* – For two partial differential equations of second order in the space R_{n+k}, about which it is known, firstly, that each of them allows a k-fold system of integral M_n ($k > n + 2$) (k is large enough that the elements (z, x_k, p_k, p_{kl}) of this system will all be elements of the differential equations), and secondly, that they cannot be derived from each other by an ordinary contact transformation, one knows that no transformation exists that associates all integral M_n of the two equations in such a way that contact of second order remains preserved. Such a transformation would, in fact, take any two infinitely close contacting integrals of one of the k-fold systems to two similar integrals of another one, and, from number 8, it would thus be an ordinary contact transformation.

11. I will make the following remarks in passing: The fact that no special transformation exists in the space of three dimensions – ordinary point-space – such that contact of second order is an invariant relation can be expressed analytically as follows: There is no pair of equations:

$$\begin{aligned} &F(z, x, y, p, q, Z, X, Y, P, Q) = 0, \\ &\Phi(\qquad\qquad\qquad\qquad\qquad\quad) = 0, \end{aligned}$$

that is *five-fold* infinite relative to the $z, \ldots, q$, as well as the $Z, \ldots, Q$, and whose equations possess a single infinitude of common integrals relative to the $z, \ldots, q$ as variables, as well as the $Z, \ldots, Q$.

Indeed, there are unboundedly many pairs of equations with the *latter* property; e.g.:

$$\begin{aligned} &F(\varphi(z, x, y, p, q), \psi(Z, X, Y, P, Q)) = 0, \\ &\Phi(\varphi, \varphi_1, \varphi_2, \varphi_3, \psi, \psi_1, \psi_2, \psi_3) \qquad = 0, \end{aligned}$$

where φ_1, φ_2, φ_3 (ψ_1, ψ_2, ψ_3, resp.) are integrals of the equation $(\varphi, \chi) = 0$ $((\psi, \Theta) = 0)$, but on the same grounds, they are not transformations of all surfaces in space. Thus, e.g., under the stated equations, just the integrals of the equations $\varphi = C$, $\psi = C$ are preserved.

§ 5.

Some transformations of partial differential equations of first order.

12. From number 8, it follows that any partial differential equation $\Phi = 0$ of first order in the space R_{n+k} whose characteristic cone is represented by an equation $\varphi(\lambda_1, \ldots, \lambda_{n+1}, d\lambda_1, \ldots, d\lambda_{n+1}) = 0$ is mapped to the space $R_n(z, x_1, \ldots, x_{n-1})$ by means of any complete solution $f(z, x_1, \ldots, x_{n-1}, \lambda_1, \ldots, \lambda_{n+1}) = 0$. Every surface element of this space corresponds to a characteristic of $\Phi = 0$, every surface M_{n-1} in R_n *) corresponds to an integral M_n of $\Phi = 0$, and in particular, the ∞^{n+1} surfaces $f = 0$ correspond to the conoid of $\Phi = 0$; i.e., the integrals that are generated by characteristics that go through one and the same point.

As a consequence of this map of the partial differential equation $\Phi = 0$ on R_n the most general transformation of this equation into itself that is so arranged that it takes integrals to integrals must be developed from the most general surface transformation of R_n . Now, the latter transformation, when it always associates a surface with a surface (not ∞ surfaces), is necessarily a Lie contact transformation. *Corresponding to this, we obtain a transformation that permutes the characteristics of the partial differential equation* $\Phi = 0$ *with each other as the most general transformation that takes an integral of the equation, in turn, to an integral (not* ∞ *integrals).*

We refer two partial differential equations of first order $\Phi = 0$, $\Psi = 0$ whose characteristic cones are each represented by an equation to one and the same space R_n , and therefore to each other. *The most general transformation that takes an integral of the one equation to an integral of the other one is one that exchanges the characteristics of* $\Phi = 0$ *with those of* $\Psi = 0$. It is the image of the contact transformation of R_n .

In my earlier presentation, I referred to this transformation as Lie's contact transformation of the space (λ), due to the fact that such a transformation can be expressed by an equation:

$$F(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}, \Lambda_1, \Lambda_2, \ldots, \Lambda_{n+1}) = 0,$$

that determines the association of conoids **) of the one equation and integrals of another sort of the other equation. Such a transformation ***) subsumes only the surface elements (λ, π) of the equations $\Phi = 0$, $\Psi = 0$, *but by no means all surface elements of* R_{n+1} *of the space* (λ). Transformations that involve all elements of the R_{n+1} and transform the integral M_n of $\Phi = 0$ into those of $\Psi = 0$, are formulated analytically in the following way:

Instead of writing $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}, \Lambda_1, \Lambda_2, \ldots, \Lambda_{n+1}$ for the coordinates of the points of two regions in R_{n+1}, I will write ; $z x_1, \ldots, x_n$, $z' x'_1, \ldots, x'_n$; furthermore, I will assume that Φ_i, Ψ_i are determined in such a way that each of the two systems of equations:

$$\begin{array}{llll}
X_1 = \Phi & (z x_1, \ldots, x_n, p_1, \ldots, p_n), & X_1 = \Psi & (z' x'_1, \ldots, x'_n, p'_1, \ldots, p'_n), \\
X_2 = \Phi_1 & (\qquad), & X_2 = \Psi_1 & (\qquad), \\
\ldots & \ldots & \ldots & \ldots \\
X_n = \Phi_{n-1} & (\qquad), & X_n = \Psi_{n-1} & (\qquad), \\
Z = \Phi_n & (\qquad), & Z = \Psi_n & (\qquad), \\
P_1 = \Phi_{n+1} & (\qquad), & P_1 = \Psi_{n+1} & (\qquad),
\end{array}$$

*) As well as any manifold of lower dimensions, when considered to be the totality of ∞^{n-1} surface elements (z, x, p).

**) Regarded as the point λ.

***) The following developments of this number are based in some remarks that Lie communicated to me on the basis of one of my earlier papers.

$$P_2 = \Phi_{n+2}(\qquad\qquad), \qquad P_2 = \Psi_{n+2}(\qquad\qquad),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_n = \Phi_{2n}\ (\qquad\qquad), \qquad P_n = \Psi_{2n}\ (\qquad\qquad),$$

define a contact transformation of the space R_{n+1}. The equation $\Phi = 0$ ($\Psi = 0$, resp.) will now be mapped to the space $R_n : X_1 = 0$ in the aforementioned way by the following equations:

$$X_1 = \Phi_1, \ldots, \quad X_n = \Phi_{n-1}, \quad Z = \Phi_n, \quad P_2 = \Phi_{n+2}, \ldots, P_n = \Phi_{2n},$$

or:

$$X_1 = \Psi_1, \ldots, \quad X_n = \Psi_{n-1}, \quad Z = \Psi_n, \quad P_2 = \Psi_{n+2}, \ldots, P_n = \Psi_{2n},$$

resp. The transformation in question of $\Phi = 0$ to $\Psi = 0$ is then of the form:

$$\text{(a)} \qquad \begin{cases} \Phi_1 = \Psi_1, \cdots, \quad \Phi_n = \Psi_n, \\ \Phi_{n+2} = \Psi_{n+2}, \cdots, \Phi_{2n} = \Psi_{2n}, \end{cases}$$

(b) an arbitrary function of $(zx_1, \ldots, x_n, p_1, \ldots, p_n;\ z'x'_1, \ldots, x'_n, p'_1, \ldots, p'_n) = 0$,

(c) and an equation between Φ and Ψ that makes Ψ vanish when Φ is equal to zero.

The number of equations is $2n + 1$, and $zx_1, \ldots, x_n, p_1, \ldots, p_n$ will then be expressed in terms of $z'x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ by them.

This is, in general, not a surface transformation or a Lie contact transformation of the space R_{n+1} *). We see this most simply when we write for equation (b):

$$\Phi_{n+1} = W(z'x'_1, \ldots, x'_n, p'_1, \ldots, p'_n),$$

where (Ψ_1, W) is non-zero, but (Φ_1, Φ_{n+1}) equals zero. A contact transformation would convert two functions Φ_1, Φ_{n+1} that are in involution into two other such functions, and thus, by no means, into Ψ_1, W.

For the equation (c), we write:

$$\Phi = \Psi,$$

and see from this that *a family of partial differential equations* $\Phi = C$ *can be converted into a family of equations* $\Psi = C$ *by transformations that are not contact transformations of* R_{n+1}*, but still convert any integral of the equation* $\Phi = C$ *into an integral of the corresponding* $\Psi = 0$.

13. The multi-valued surface transformations of R_n are the images of all the other transformations that, in that way, take a partial differential equation of first order on R_{n+1} to itself (to another equation of the same order on the same space, resp.) such that integral

*) Cf., on this, a statement of Lie in his paper "Begründung einer Invariantentheorie der Berührungstransformationen," Math. Annalen, Bd. VIII, pp. 223.

M_n will always, in turn, go to integral M_n . Each such transformation that takes two equations of first order $\Phi = 0$, $\Psi = 0$ to each other, makes any integral of the one equation – e.g., $\Phi = 0$ – correspond to an integral of the other equation $\Psi = 0$, while an integral of the latter one will correspond to infinitely many integrals of the former.

The foregoing can be carried over to partial differential equations of first order of any sort on R_{n+1} in an easily understood way, since any two equations of first order can always be taken to each other by a contact transformation.

14. A system of k partial differential equations of first order with $n + 1$ variables that admit a common solution with $n - k + 1$ *) arbitrary constants will, from number 9, be related to any common complete solution:

$$f(z, x_1, \ldots, x_{n-k}, \lambda_1, \ldots, \lambda_{n+1}) = 0$$

on the space $(z, x_1, \ldots, x_{n-k})$ of $n - k + 1$ dimensions in such a way that any surface element of this R_{n-k+1} corresponds to a characteristic M_k of the system of equations, and any surface of the R_{n-k+1} (i.e., an M_{n-k} in this space) corresponds to an integral M_n of the system. Two systems of equations, each of which consists of k equations, and each of which possesses ∞^{n-k+1} integral M_n, can be mapped to the space R_{n-k+1}, and thus can be related to each other. The most general transformation of one system of equations into the other one that takes any integral M_n of the one system to an integral M_n of the other one will, as a result, be the image of the most general single-valued surface transformation of the space R_{n-k+1} . Under any such transformation of the system of equations to another one, every characteristic M_k of the one system then goes to a characteristic M_k of the other one.

15. In this number, we shall consider, in particular, a system of four partial differential equations of first order with seven variables λ that possess ∞^3 integrals M_6 – say:

$$f(z, x, y, \lambda_1, \ldots, \lambda_7) = 0.$$

The system of equations will be mapped to the space R_3 . Thus, every surface element (z, x, y, p, q) will correspond to a characteristic M_4 and every element (z, x, y, p, q, r, s, t) will correspond to a characteristic M_1 that is the intersection of a single infinitude of neighboring characteristics M_4 that go through one and the same point λ.

Every surface of R_3 corresponds to an integral M_6, and this should include an M_3 (perhaps, as a cuspidal manifold) that consists of ∞^2 characteristic M_1 **).

If, from the ∞^8 elements (z, x, y, p, q, r, s, t) of R_3, ∞^7 of them are distinguished by an equation $F(z, x, y, p, q, r, s, t) = 0$ then this is identical to distinguishing ∞^7 characteristics M_1 . The search for integral surfaces of the partial differential equation of second order $F = 0$ and the search for integral M_3 of the system of equations in R_7 that are each generated

*) $n > k$.

**) An integral M_6 can be laid through each M_2 , and just the one characteristic M_4 can be laid through an M_3 when it is generated by ∞^2 strips.

by ∞^2 of the distinguished characteristic strips (M_1) are, as a consequence, equivalent problems.

§ 6.

Some examples of a class of multi-valued surface transformations of the space of three dimensions.

16. As was shown in the introduction, a surface transformation is completely determined by any three equations:

$$(15) \qquad \begin{cases} X = F(z, x, y, p, q), \\ Y = F_1(\qquad\qquad), \\ Z = F_2(\qquad\qquad). \end{cases}$$

It becomes a single-valued transformation in the event that it satisfies the condition that the system of equations:

$$dz = p\,dx + q\,dy, \qquad dp = r\,dx + s\,dy, \qquad dq = s\,dx + t\,dy, \ldots, \text{ad inf.}$$

shall be transformed into the similar one:

$$dZ = P\,dX + Q\,dY, \qquad dP = R\,dX + S\,dY, \qquad dQ = S\,dX + T\,dY, \ldots, \text{ad inf.}$$

in which the quantities P, Q likewise include only z, x, y, p, q, but not higher differential quotients; in the other case, when one obtains from the calculations described:

$$P = \Phi_1(z, x, y, p, q, r, s, t),$$
$$Q = \Phi_2(\qquad\qquad\qquad),$$

the transformation (15) is a multi-valued surface transformation.

This transformation associates any point (X, Y, Z) with a family of ∞^2 elements (z, x, y, p, q) and every surface element (Z, X, Y, P, Q) on an element (z, x, y, p, q) that belongs to each point (X, Y, Z) with a family of ∞^1 systems of values (r, s, t). Any surface in the region (x, y, z) will go to a surface in the region (X, Y, Z) and any surface of the latter region will go to all integrals of a partial differential equation of first order $f(F, F_1, F_2) = 0$.

A partial differential equation of first order $\varphi(Z, X, Y, P, Q) = 0$ corresponds to a partial differential equation of second order that possesses a first integral with two arbitrary constants λ, μ:

$$f(F, F_1, F_2, \lambda, \mu) = 0.\ ^*).$$

*) The transformation (15) was also mentioned by P. du Bois-Reymond in his paper: "Beiträge zur Interpretation der partiellen Differentialgleichungen," Leipzig, 1864, pp. 173.

The linear partial differential equations of first order in the region (X, Y, Z) correspond to partial differential equations of second order in the region (x, y, z) that are linear in r, s, t, $rt - s^2$, and possess a first integral of the form:

$$f(F, F_1, F_2) = \text{an arbitrary function of } \varphi(F, F_1, F_2).$$

17. In particular, due to their application to a certain class of partial differential equations of second order, I would like to draw attention to the following transformations:

$$\begin{cases} X = x, \\ Y = y, \\ Z = q. \end{cases} \tag{16}$$

The remaining equations of this transformation, which are derived in the manner that was set down, become:

$$\begin{cases} P = s, \\ Q = t, \\ R = v, \\ S = w, \\ T = \varpi, \\ \quad \text{etc.} \end{cases} \qquad \left(v = \frac{d^3 z}{dx^2 dy},\ w = \frac{d^3 z}{dx\,dy^2},\ \varpi = \frac{d^3 z}{dy^3} \right). \tag{16′}$$

I consider an equation of second order in the region (x, y, z) that is free of z, p, so it has the form:

$$F(x, y, q, r, s, t) = 0,$$

or, when solved for r:

$$r = f(x, y, q, s, t) \;{}^{*}), \tag{17}$$

and define the corresponding figure of the region (X, Y, Z).

By differentiating equation (17) with respect to y, one obtains:

$$v = \frac{df}{dy} + t\frac{df}{dq} + w\frac{df}{ds} + \varpi\frac{df}{dt},$$

an equation that has the following linear equation as its image in (X, Y, Z):

$$R - S\frac{df}{dP} - T\frac{df}{dQ} = \frac{df}{dY} + Q\frac{df}{dZ} \tag{18}$$

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*) The equation:

$$r = f(x, y, q, s, t) + z\,\varphi(x) + p\,\psi(x)$$

can be treated in the same way.

on the basis of equations (16), (16′). It will then be the image of all equations:

$$r = f(x, y, q, s, t) + \text{an arbitrary } F(x).$$

Every surface in the space (X, Y, Z) corresponds to the integral of an equation:

$$q = F(x, y),$$

whose solution has the form:

(19) $$z = \varphi(x, y) + \Psi(x),$$

where Ψ means an arbitrary function.

Since every integral of equation (18), *inter alia*, must correspond to integrals of equation (17), one might, when $Z = F(X, Y)$ means an integral surface of (18), determine the arbitrary function Ψ in (19) in such a way that the latter equation represents an integral of (17), and the equation that served to determine Ψ gives *) Ψ equal to a well-defined function of x, $F(x)$, increased by $cx + c'$, where c, c' are completely arbitrary.

The problem of integrating the second-order equation (17) *is therefore reduced to the problem of the integration of the linear equation of second order* (18).

By the applied transformation, any two integrals of equation (17) that have n^{th} order contact at a point correspond to two integrals of equation (18) that have $(n-1)^{\text{th}}$ order contact at a point, and accordingly, characteristics of equation (17) will correspond to characteristics of equation (18).

The theory that was established here defines an extension of the well-known theory of Legendre **) of the equations:

$$F(r, s, t) = 0 \quad {}^{***})$$

that do not contain x, y, z, p, q. In order to obtain the Legendre form of the corresponding linear equation (18), one would have to appeal, in place of the transformation (16), to an equation that is derived from it by a reciprocal transformation:

$$X' = P, \qquad Y' = Q, \qquad Z' = PX + QY - Z,$$

namely:

$$X' = s, \qquad Y' = t, \qquad Z' = sx + ty - q.$$

The foregoing theory naturally remains essentially unchanged when an arbitrary contact transformation is applied to the fundamental equations (16), (17).

Helsingborg, 18 July 1875.

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*) By means of a double quadrature.

**) Cf., Boole: Differential Equations, Cambridge, 1859, pp. 369.

***) I was recently made aware of Legendre's theory by Lie.