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On the theory of surface transformations.

By

A. V. BÄCKLUND in Lund.

Translated by D. H. Delphenich

Any transformation that converts any surface into a certain surface, and conversely, the latter surface, in turn, into the former, is an ordinary (Lie) contact transformation. I proved that in my treatise in volume IX of these Annals. In volumes XI and XIII of these Annals, I discussed a certain class of transformations that lead from a surface to infinitely many families of surfaces, namely, the class that consists of those surface transformations that are based in three equations of the form:

$$\begin{aligned} X &= F(z, x, y, p, q, \dots\ ), \\ Y &= F_1(\qquad\qquad\quad), \\ Z &= F_2(\qquad\qquad\quad). \end{aligned}$$

Any transformation that belongs to this class is characterized by the fact that it converts any surface in the domain (x, y, z) into just one surface in (X, Y, Z), while a surface in the latter domain will be converted into infinitely many surfaces in the former. Later on, in volume XVII of these Annals, I discussed those transformations that convert any surface in the domains (x, y, z), (X, Y, Z) into first-order partial differential equations. They are given by three equations:

$$(\alpha) \qquad \left\{ \begin{aligned} &F_1(z, x, y, p, q, Z, X, Y, P, Q) = 0, \\ &F_2(\qquad\qquad\qquad\qquad\qquad\quad) = 0, \\ &F_3(\qquad\qquad\qquad\qquad\qquad\quad) = 0. \end{aligned} \right.$$

Among them, there are some that include the penultimate transformations, which I treated quite laboriously in volume XI of these Annals (†) (M. A., Bd. XVII, pp. 308). Transformations that are determined by more than three equations in $z, x, y, p, q, Z, X, Y, P, Q$ will not be, in general, surface transformations for the entire domains (x, y, z), (X, Y, Z). If the number of equations that the transformation determines is four then there will exist broadly inclusive families of surfaces for which the transformation becomes a

(†) Translator: In all of what follows, we will use the abbreviation “M. A.” for Mathematische Annalen, instead of “d. A.” (diese Annalen), which the author used.

surface transformation (M. A., Bd. XVII, pp. 313). A special transformation of this type is expressed by the four equations:

$$X = x,\quad Y = y,\quad f(Z, z, x, y, p, q, P, Q) = 0,\qquad \varphi(Z, z, x, y, p, q, P, Q) = 0.$$

The problem of determining surfaces that are transformed into other surfaces by this transformation is equivalent to the problem of determining the solutions:

$$\begin{array}{lll} z = F(x, y), & p = F'(x), & q = F''(y), \\ Z = \Phi(x, y), & P = \Phi'(x), & Q = \Phi''(y) \end{array}$$

of $f = 0$, $\varphi = 0$. The discussions in nos. 5 and 6 of my treatise in volume XVII of these Annals related to the resolution of this problem. Here, I will again go into the characterization of these families of surfaces, and thus also consider some special cases of the equations $f = 0$, $\varphi = 0$.

In the realm in which a transformation that is determined by four equations between z, x, y, p, q, Z, X, Y, P, Q becomes a surface transformation, there is, in general, a single-valued surface transformation. However, there are cases in which it becomes infinitely-valued, either in such a way that it converts any surface in one domain (x, y, z), (X, Y, Z) of the realm in question into a certain surface in the other one, while converting any surface of the latter domain into infinitely many in the former, or in such a way that it transforms any surface of the domain into infinitely many surfaces. Lie commented upon a transformation with the latter character in a treatise on surfaces of constant curvature (in Archiv für Mathematik und Naturwissenschaft, Bd. 5, Christiania, 1880). He was led to it by his study of a method that had recently been given by Bianchi for generating new surfaces of constant curvature from a given one. In connection with my general theorems, I have sought to briefly summarize some of this theory of Bianchi and Lie at the end of the present treatise.

The third paragraph is concerned with some special surface transformations of the category (α).

§ 1.

Some remarks on the figure that is defined analytically by two equations in z, z', x, y, and the first derivatives of z, z' with respect to x, y.

1. Two surface elements that have z, x, y, p, q (z', x, y, p', q', resp.) for their parameters shall, in the event that the parameter values $(z', z, x, y, p, q, p', q')$ satisfy the two equations:

$$(1)\qquad \begin{cases} f(z, z', x, y, p, q, p', q') = 0, \\ \varphi(\qquad\qquad\qquad\qquad) = 0, \end{cases}$$

be called two *corresponding* elements of this system of equations. Infinitely many surface elements can be added to two arbitrarily-chosen corresponding elements of the system that are infinitely close to them (are united with them, resp.), and define mutually corresponding elements of the system (1), moreover. Namely, if (z, x, y, p, q), (z', x, y, p', q') are the parameters of the first two elements, and dz, dz', dp, dq, dp', dq' are set equal to any values that satisfy the equations:

$$(a) \qquad dz = p\,dx + q\,dy, \qquad dz' = p'\,dx + q'\,dy, \qquad df = 0, \qquad d\varphi = 0,$$

then $(z + dz, x + dx, \dots, q + dq)$ become parameters of two elements of the stated kind, precisely. Each such totality of two corresponding pairs of united elements belongs to one – and in general, *only one* – system of values for (r, s, t, r', s', t') that simultaneously associates a solution $z = F(x, y)$, $z' = \Phi(x, y)$ of (1) with a system of values for the second derivatives of F and Φ. This system of values is the one that satisfies the following six equations:

$$(b) \qquad \begin{cases} dp = r\,dx + s\,dy, & dp' = r'\,dx + s'\,dy, & [f, \varphi]_{zxy} = 0, \\ dq = s\,dx + t\,dy, & dq' = s'\,dx + t'\,dy, & [f, \varphi]_{z'xy'} = 0, \end{cases}$$

of which the last two provide the conditions for any values of r, s, t, r', s', t' to produce a solution of (1), at all. However, one must conclude from this that *one can always lay one – and in general, only one – surface pair* $z = F(x, y)$, $z' = \Phi(x, y)$ *that represents a solution of* (1) *through any two strips whose surface elements define mutually corresponding elements of the system* (1),

The agreement between this theorem and the theorem on pp. 291 of my treatise in vol. XVII of these Annals is obvious. There, it said that a simply-infinite family of integrals: $z' = \Phi(x, y)$ goes through any strip of elements (z', x, y, p', q'). Now, as one sees from (1) and equations (*a*), there are simply-infinitely many strips of elements (z, x, y, p, q) that correspond to the elements (z', x, y, p', q') of a given strip. From what we just said, any one of these ∞^1 strips, when combined with the given one, must determine a surface pair $z = F(x, y)$, $z' = \Phi(x, y)$ that defines a solution of (1). Therefore, in total, a simply-infinite family of integral surfaces $z' = \Phi(x, y)$ that go through the given strips of elements (z', x, y, p', q') gets added to the given strips, as was previously remarked by myself in the cited place.

We can also formulate the developments here thus: *A completely-determined pair of surfaces that define a solution of* (1) *will go through any pair of curves that are represented by three equations* $z = f(x)$, $z' = \varphi(x)$, $y = \psi(x)$. Namely, equations (2), together with the first two of equations (*a*), determine the parameters p, q, p', q' of mutually corresponding surface elements that, from the foregoing, define strips that determine the surfaces of the pair in question unambiguously, along with the two curves.

If we regard z, z', x, y as the coordinates of the points in a (four-dimensional) space R_4 (M. A., Bd. XVII, pp. 289) then we can also say: *A completely-determined integral* M_2^0 *of* (1) *goes through any* M_1^0.

2. However, there are mutually-corresponding pairs of united elements of (1) that are associated with infinitely many systems of values of (r, s, t, r', s', t'), in the above sense. The two elements (z, x, y, p, q), (z', x, y, p', q') of the two pairs are, indeed, to be chosen from the elements of (1) completely arbitrarily. Namely, one can determine the ratios *dx*, *dy*, *dp*, *dq* in such a way that the sheaf of (*r*, *s*, *t*) that is expressed by the equations of the first column of (*b*), which are completely associated with the equation $[f, \varphi]_{z'xp'} = 0$, and at the same time, the sheaf of (r', s', t') that satisfy the equations of the next column in equations (*b*) – these equations being applied to the element ($z' + dz'$, …, $q' + dq'$) that corresponds to the element ($z + dz$, …, $q + dq$) – are completely included in the equation $[f, \varphi]_{zxp} = 0$. Ultimately, the following quadratic equation must be true for dy / dx:

$$(2)\qquad \left(\frac{dy}{dx}\right)^2\left(\frac{\partial f}{\partial p'}\frac{\partial \varphi}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial \varphi}{\partial p'}\right) - \frac{dy}{dx}\left(\frac{\partial f}{\partial p'}\frac{\partial \varphi}{\partial p}+\frac{\partial f}{\partial q'}\frac{\partial \varphi}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial \varphi}{\partial p'}-\frac{\partial f}{\partial q}\frac{\partial \varphi}{\partial p'}\right) + \left(\frac{\partial f}{\partial q'}\frac{\partial \varphi}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial \varphi}{\partial q'}\right) = 0,$$

and an equation must exist between dp / dx, dq / dx that comes about from $[f, \varphi]_{z'xp'} = 0$ by eliminating *r*, *s*, *t* by means of (*b*) and considering of the last equation for dy / dx to be written down in (2).

Each of these ∞^1 possible systems of values for dp / dx, dq / dx, together with a value for dy / dx in (2), determines a surface element ($z + dz$, $x + dx$, …, $q + dq$) that defines a pair with the element (*z*, *x*, *y*, *p*, *q*) that determines ∞^1 systems of values for (*r*, *s*, *t*, r', s', t'), in conjunction with its corresponding pairs, in the manner that was set down in the previous number.

We thus see that *for any surface-pair* $z = F(x, y)$, $z' = \Phi(x, y)$ *that defines a solution of* (1), *there are two families of mutually-corresponding strip-pairs, along which, this surface pair contacts infinitely many other surface-pairs that likewise define solutions of* (1). ([1])

One pp. 290, 291 of M. A., Bd. XVII, I showed that the surfaces $z' = \Phi(x, y)$ [or $z = F(x, y)$] that define a part of the solutions of (1) must satisfy two linear, third-order partial differential equations as integrals, and whose first derivatives also reduce to only three. The integral surfaces of such pairs of third-order partial differential equations are comprised of any two families of characteristics (M. A., Bd. XVII, pages 91-94). Those of the aforementioned contact strips that lie on the surface $z' = \Phi(x, y)$ [or $z = F(x, y)$] will define the characteristics of any pair of third-order equations, precisely. However, the characteristics of this pair of equations that belongs to (1) will be strips along which first-order contact of the integral surfaces is already possible.

If we regard *z*, z', *x*, *y* as point coordinates in R_4 then we must conclude from what we just developed that *two families of* M_1^0 *lie on any integral* M_2^0 *of* (1), *along which, a* M_2^0 *of infinitely many other integral-*M_2^0 *of the same pair of equations* (1) *will contact.* I call

([1]) If $z = F(x, y)$, $z' = \Phi(x, y)$; $z = F^{(1)}(x, y)$, $z' = \Phi^{(1)}(x, y)$ define two such contacting pairs, and the two surfaces $z = F(x, y)$, $z = F^{(1)}(x, y)$ osculate along their contact curve, moreover, then the two surfaces $z' = \Phi(x, y)$, $z' = \Phi^{(1)}(x, y)$ must also osculate along their contact curves. Then, a value of (*r*, *s*, *t*) that satisfies the equation $[f, \varphi]_{z'xp'} = 0$ will correspond to a single value of (r', s', t') by means of three of the equations that are defined by setting the first derivatives of *f* and φ in (1) with respect to *x*, *y* equal to zero.

these M_1^0, along with their sheaf of ∞^1 tangent planes, characteristic strips (*), or more briefly, the characteristics of (1), as well as the M_1^0 themselves. *One can lay, not just one, as through any M_1^0, but ∞^∞ integral- M_2^0 of* (1) *through any characteristic.*

The characteristics are the only M_1^0 with that property.

3. On pp. 296 of M. A., Bd. XVII, I proved that the most general equation $x - \chi(y, p, q, p', q') = 0$ that has ∞^3 integral- M_2^0 in common with (1) is defined by a linear, second-order partial differential equation. It follows from this that, in general, there is one *and only one* equation $x - \chi(y, p, q, p', q') = 0$ that, first of all, will be satisfied by those systems of values of (x, y, p, q, p', q') that will be separate from any two equations (in addition to $f = 0$, $\varphi = 0$) between these quantities, and secondly, will satisfy an arbitrary relation between the differential quotients of x (or χ) for this same system of values. One can arrive at this theorem in another way, which I would like to give here. From it, one can then reason conversely that the equation for c must be a second-order partial differential equation.

If we would like to apply equations (10) in M. A., Bd. XVII, pp. 291, namely:

$$[f, \varphi]_{zxp} = 0, \quad [\varphi, \psi]_{zxp} = 0, \quad [\psi, f]_{zxp} = 0,$$

to the ∞^4 systems of special values of $(z, z', x, y, p, q, p', q')$ that are in question here then we would have to first introduce certain values of ψ for the ratios of the differential quotients that are determined in the following way: y is the function $x - \chi(y, p, q, p', q')$, which is free of z, z', and which we will think of as being eliminated everywhere by means of equations (1). If we write the two arbitrary equations in x, y, p, q, p', q' that are added to (1) in the form:

$$x - F(p, q, p', q') = 0, \qquad y - \Phi(p, q, p', q') = 0$$

then the values in question of:

$$-\frac{\partial\psi}{\partial y} : \frac{\partial\psi}{\partial x}, \ldots, -\frac{\partial\psi}{\partial q} : \frac{\partial\psi}{\partial x},$$

i.e.:

$$\frac{\partial x}{\partial y}, \ldots, \frac{\partial x}{\partial q},$$

will be determined uniquely by the assumed linear relation between these differential quotients and through the following equations (M. A., Bd. XVII, pp. 413):

$$F'(p) - \frac{\partial x}{\partial y}\Phi'(p) - \frac{\partial x}{\partial p} = 0, \ldots, F'(q') - \frac{\partial x}{\partial y}\Phi'(q') - \frac{\partial x}{\partial q'} = 0.$$

(*) Two infinitely-neighboring M_1^0 will define a strip on any M_2^0 that goes through them.

From the equations (10), M. A., Bd. XVII, we then come to a well-defined system of values of (r', s', t') for any system of values of $(z, z', x, y, p, q, p', q')$ that we spoke of, and then, from the first derivatives of $f = 0$, $\varphi = 0$, to a likewise completely determined system of values of (r, s, t).

The values of the differentials of z, z', p, q, p', q' that are represented by:

$$p\,dx + q\,dy = dz, \ldots, r\,dx + s\,dy = dp, \ldots, s'\,dy + t'\,dx = dq'$$

will satisfy the equations $df = 0$, $d\varphi = 0$, $d\psi = 0$ identically on the basis of equations (10) above (M. A., Bd. XVII). Due to the expressions for $\partial x / \partial p$, etc. – i.e., $\partial \psi / \partial x$, etc. – that are given by the equations $F'(p) - \frac{\partial x}{\partial y}\Phi'(p) - \frac{\partial x}{\partial p} = 0$, etc., the two equations:

$$d(x - F) = 0, \quad d(y - \Phi) = 0$$

will give only one new equation. This equation, which we shall retain, gives a well-defined value for dy / dx in terms of $z, z', x, y, p, q, p', q'$. Corresponding to these values of dy / dx and the values of $\frac{dp}{dx}\left(= r + s\frac{dy}{dx}\right), \ldots, \frac{dq'}{dx}\left(= s' + t'\frac{dy}{dx}\right)$ that follow from them, the ∞^4 elements $(z, z', x, y, p, q, p', q')$ that fulfill the equations:

$$f = 0, \quad \varphi = 0, \;\; x = F, \;\; y = \Phi$$

will be associated with ∞^3 completely-determined M_1^0. From no. 1, any of these M_1^0 will determine an integral-M_2^0 that is common to $f = 0$, $\varphi = 0$, and which will also be an integral of an equation $\psi = x - \chi = 0$ with the property stated above. We then have, in total, ∞^3 M_2^0 that satisfy equations (1) and a certain equation $\psi = x - \chi = 0$ with the aforementioned property that is so determined that it also belongs to it as an integral. Thus, as we previously stated briefly, the equation $\psi = 0$ will be determined completely by the stated conditions.

4. Equations $\psi \equiv x - \chi(y, p, q, p', q') = 0$ of this character define precisely the totality of all integrals of a second-order partial differential equation. In M. A., Bd. XVII, I explained how one could present this equation, and in addition, confirm the existence of intermediary integrals, each of which is expressed in terms of two involutory, first-order partial differential equations. In this no., I will treat the case of a first integral of this linear, second-order partial differential equation for ψ that is expressed in terms of a first-order partial differential equation – viz., $\Omega = 0$.

We first consider a simply-infinite family of solutions $\psi(x, y, p, q, p', q', \lambda) = 0$ (*) of the first-order partial differential equation $\Omega = 0$. Its enveloping structure (*Umhüllungsgebilde*) is called $\Psi = 0$. This equation is also a solution of $\Omega = 0$, and

(*) The quantities z, z' shall be thought of as having been eliminated using (1).

therefore has ∞^3 integral-M_2^0 in common with (1), like any solution of this equation. On any M_2^0 that satisfies (1) and the equation $\psi^0 \equiv \psi(x, y, p, q, p', q', \lambda^0) = 0$ as an integral, a strip will be distinguished by the equation $\psi^{(1)} \equiv \psi^0 + d\lambda\ \psi'(\lambda^0) = 0$. It also belongs to the equation $\Psi = 0$, because it envelops the equations $\psi(\lambda) = 0$. The elements (*) of the strip:

$$\frac{\partial \Psi}{\partial x}, \quad \frac{\partial \Psi}{\partial y}, \ldots, \frac{\partial \Psi}{\partial q'}$$

will be proportional to:

$$\frac{\partial \psi^0}{\partial x}, \quad \frac{\partial \psi^0}{\partial y}, \ldots, \frac{\partial \psi^0}{\partial q'},$$

resp. Therefore, systems of values of (r, s, t, r', s', t') that are determined by $\Psi = 0$, in conjunction with (1), will be associated with the elements of the strip, and indeed the same thing will be true for $\psi^0 = 0$, in conjunction with (1). As a result, an integral-M_2^0 that is common to (1) and $\Psi = 0$ (M. A., Bd., XVII, nos. 6, 7) must go through the aforementioned strip. This must be considered to be a different M_2^0 from the previous one, because otherwise $\Psi = 0$ would have the same ∞^3 integral-M_2^0 in common with (1) as $\psi^0 = 0$. If we were to choose another family of ∞^1 solutions $\psi = 0$ that, in fact, included the two equations $\psi^0 = 0$, $\psi^{(1)} = 0$ then we would find another solution $\Psi' = 0$ as an enveloping structure to the latter ones. We would then also find a new, third, integral-M_2^0 of (1) – namely, an integral-M_2^0 that is common to (1) and $\Psi' = 0$ – that would go through the strip in question. In this way, we recognize that ∞^∞ integral-M_2^0 of (1) will go through any strip, *and the strip will therefore (no. 2) be a characteristic of* (1) (**).

(*) I briefly refer to the M_2^0-elements $(z, z', x, y, p, q, p', q')$ that are connected with some M_1^0 as elements of a strip that starts on that M_1^0.

(**) We see more as follows: Because the strip that is now considered is a characteristic of (1), the first derivatives of (1) must be satisfied by ∞^1 of the systems of values of (r, s, t, r', s', t') that are associated with the elements of the strip by its equations: $dp = r\,dx + s\,dy$, $dq = s\,dx + t\,dy$, $dp' = r'\,dx + s'\,dy$, $dq' = s'\,dx + t'\,dy$, namely, by the unique corresponding system of values of (r', s', t') through the sheaf of (r, s, t) that is determined by the first two equations, combined with the individual systems of values (r, s, t) of this sheaf by means of the derivatives of equations (1), which reduce to three mutually-independent equations, on the basis of $[f, \varphi]_{z'xp'} = 0$. The element $(z, z', x, y, p, q, p', q')$ of the strip belongs to the equation $\psi^{(1)} = 0$, and it therefore suffices to apply one of the equations:

$$\frac{d\psi^{(1)}}{dx} = 0, \qquad \frac{d\psi^{(1)}}{dy} = 0,$$

in which r', s', t' are replaced with their values in terms of r, s, t that are provided by the first derivatives of (1), as was just mentioned, in order to obtain a system of values of (r, s, t, r', s', t') that simultaneously satisfies equations (1) and $\psi^{(1)} = 0$. However, from nos. 6, 7, M. A., Bd. XVII, an integral-M_2^0 of (1) and

A complete solution of $\Omega = 0$ has five arbitrary constants. It is therefore of the form:

$$\psi(x, y, p, q, p', q', \lambda_1, \lambda_2, \ldots, \lambda_5) = 0,$$

and can be, in particular, linear in the λ if $\Omega = 0$ is a linear, first-order partial differential equation. I will now consider an equation $\psi(\lambda_1^0, \lambda_2^0, \ldots, \lambda_5^0) = 0$ that is included in the complete solution that was written down, along with all ∞^3 of its infinitely-close equations in the same family that contain one and the same element (x, y, p, q, p', q'):

$$(c) \qquad \psi(\lambda_1^0, \lambda_2^0, \ldots, \lambda_5^0) + d\lambda_1\psi'(\lambda_1^0) + d\lambda_2\psi'(\lambda_2^0) + \cdots + d\lambda_5\psi'(\lambda_5^0) = 0,$$

and along with it, an integral-M_2^0 of (1) and $\psi(\lambda_1^0, \lambda_2^0, \ldots, \lambda_5^0) = 0$, that likewise possesses the same element (x, y, p, q, p', q'). From the previous discussion, any intersection between this integral-M_2^0 and any equation (c) must define a characteristic. Only two strips start from the element $(z, z', x, y, p, q, p', q')$ that run through the integral-M_2^0 and are associated with equations (1) as characteristics. (No. 2) All of equations (c) must yield the same characteristics. Thus, if two of these equations – say:

$$\begin{aligned}
&\psi(\lambda_1^0, \lambda_2^0, \ldots, \lambda_5^0) + d\lambda_1^0\,\psi'(\lambda_1^0) + \cdots + d\lambda_5^0\,\psi'(\lambda_5^0) = 0, \\
&\psi(\qquad\qquad\quad) + d\lambda_1^0\,\psi'(\quad) + \cdots + d\lambda_5^0\,\psi'(\quad) = 0,
\end{aligned}$$

produce different characteristics then the equation:

$$\psi(\lambda_1^0, \lambda_2^0, \ldots, \lambda_5^0) + \frac{d\lambda_1^0 + \mu\, d\lambda_1'}{1+\mu}\psi'(\lambda_1^0) + \cdots + \frac{d\lambda_5^0 + \mu\, d\lambda_5'}{1+\mu}\psi'(\lambda_5^0) = 0$$

will yield a third strip of that type, and therefore ∞^1 strips – corresponding to the ∞^1 values of μ – will start from $(z, z', x, y, p, q, p', q')$, which all lie on the same integral-M_2^0, and would be characteristics of (1), which is impossible for an arbitrary element $(z, z', x, y, p, q, p', q')$ of (1) (*). Therefore, one ultimately has: One and the same characteristic of (1) will be expressed by the equations:

$\psi^{(1)} = 0$ must then be able to go through any strip. If we proceed in the same way with the other equations of the first (arbitrarily selected) simply-infinite family of solutions:

$$\psi(x, y, p, q, p', q', \lambda) = 0$$

then we will conclude that the aforementioned integral-M_2^0 of $\Psi = 0$ must define an osculating enveloping structure of ∞^1 integral-M_2^0 of the ∞^1 enveloping solutions $\psi(\lambda) = 0$ of $\Psi = 0$.

(*) In fact, I omit the case in which one has $f = F(z', z, x, y, \varphi)$ in equations (1), or the case in which the two equations $f = 0$, $\varphi = 0$ simultaneously lack p, q or p', q'.

$$\psi(\lambda_1^0, \lambda_2^0, \dots, \lambda_5^0) = 0, \qquad \psi'(\lambda_1^0)\, d\lambda_1 + \psi'(\lambda_2^0)\, d\lambda_2 + \dots + \psi'(\lambda_5^0)\, d\lambda_5 = 0,$$

where three of the ratios $\frac{d\lambda_1}{d\lambda_5}, \frac{d\lambda_2}{d\lambda_5}, \frac{d\lambda_3}{d\lambda_5}, \frac{d\lambda_4}{d\lambda_5}$ are arbitrary. Since the stated equations define a characteristic of $\Omega = 0$, one can also say: *The characteristics of* $\Omega = 0$ *will now be characteristics of* (1).

It follows from this, moreover, that of the elements $(z, z', x, y, p, q, p', q')$ of (1), *only one* characteristic series of elements $(z, z', x, y, p, q, p', q')$ emerges for $\Omega = 0$. We then consider any integral $\psi^0 = 0$ of $\Omega = 0$, an integral-M_2^0 of (1) and $\psi^0 = 0$, and any strip of this integral-M_2^0 that does not define a characteristic of (1). We can lay ∞^∞ other solutions $\psi' = 0$ of $\Omega = 0$ through these strips. From the foregoing, the intersection of any M_2^0 and any solution $\psi' = 0$ that is infinitely close to $\psi^0 = 0$ must consist completely of characteristics of (1) and $\Omega = 0$. The stated strip is also contained in the same intersection. However, it is not a characteristic strip for the systems of equations (1). Therefore, all of the characteristics of $\Omega = 0$ that go through the element $(z, z', x, y, p, q, p', q')$ of the strip and lie on the M_2^0 must, at the same time, belong to the latter equation $\psi' = 0$, and this to all equations $\psi' = 0$. We further consider any element $(z, z', x, y, p, q, p', q')$ of the strip and a solution $\psi'' = 0$ that includes this element, but not other elements of the strip, and which is infinitely close to $\psi^0 = 0$. $\psi'' = 0$ must also include the characteristics of $\Omega = 0$ that lie on M_2^0 and start from $(z, z', x, y, p, q, p', q')$. An integral-$M_2^0$ of (1) and $\psi'' = 0$ goes through any characteristic of ($\Omega = 0$ and) (1) (*). When we reason in the same way as before with this M_2^0, we will see that all of the solutions $\psi = 0$ of $\Omega = 0$ that possess the specified element $(z, z', x, y, p, q, p', q')$ will contain one and the same characteristic of (1) and $\Omega = 0$ in common. *We thus find that for every element* $(z, z', x, y, p, q, p', q')$ *there is only one sequence of elements* $(z, z', x, y, p, q, p', q')$ *that is a characteristic sequence for* [(1) *and*] $\Omega = 0$.

I have been able to omit the case in which infinitely many integral-M_2^0 of (1) and $\psi^0 = 0$ go through the element $(z, z', x, y, p, q, p', q')$. This can therefore not enter into consideration for an arbitrary element of (1) and an arbitrary integral $\psi^0 = 0$ of $\Omega = 0$. In fact, those equations $\psi = 0$ that have such a special relationship with a particular system (1) that ∞^1 values of r, s, t would be associated with any element $(z, z', x, y, p, q, p', q')$ of (1), and $\psi = 0$ will satisfy *at least two* first-order partial differential equations.

However, if the characteristic strips of $\Omega = 0$ that start from any element $(z, z', x, y, p, q, p', q')$ of (1) are linked to one and the same sequence of elements $(z, z', x, y, p, q, p', q')$ that is therefore determined by five equations in:

$$z, z', x, y, p, q, p', q': \qquad \psi = C, \quad \psi' = C', \dots, \psi^{\mathrm{IV}} = C^{\mathrm{IV}}$$

(*) See the beginning of the third note of this number.

then any equation $\Omega = 0$ *will be a linear, first-order partial differential equation.* One can then lay a completely-determined integral: Funct($\psi, \psi', \ldots, \psi^{\mathrm{IV}}$) = 0 of $\Omega = 0$ through an arbitrary quadruply-infinite manifold that is represented by (1) and:

$$F(z, z', x, y, p, q, p', q') = 0, \quad \Phi(z, z', x, y, p, q, p', q') = 0.$$

Now, the characteristics that start from the element $(z, z', x, y, p, q, p', q')$ and are common to (1) and $\Omega = 0$ obey the equations:

$$r + ms = \mu, \qquad s + mt = \nu, \qquad r' = as + b, \qquad s' = a's + b', \quad t' = a''s + b'',$$

so we will have completely-determined values of $r, s, \ldots, t', dy : dx$ when we substitute these values for r, s, t, r', s', t' and substitute the values $dz = p\,dx + q\,dy$, $dz' = p'\,dx + q'\,dy$, $dp = r\,dx + s\,dy$, …, $dq' = s'\,dx + t'\,dy$ in the equations $dF = 0$, $d\Phi = 0$. Given the values of $r, s, \ldots, t'$ above, the differentials of (1) will be fulfilled identically. By applying the values of $r, s, \ldots, t'$ that we have now obtained, we determine a certain integral strip of (1) at every element $(z, z', x, y, p, q, p', q')$ of our four-fold manifold: (1) and $F = 0$, $\Phi = 0$, all of whose elements belong to the aforementioned four-fold manifold. Thus, this four-fold manifold is decomposed into a completely-determined family of ∞^3 integral strips of (1). The ∞^3 integral-M_2^0 of (1) that, from no. 1, go through these strips also satisfy the equation Funct($\psi, \psi', \ldots, \psi^{\mathrm{IV}}$) = 0, of which it was assumed that it was an integral of $\Omega = 0$ and that it generated all characteristics of $\Omega = 0$.

We can recognize the possibility of *expressing such a thing by a linear, first-order partial differential equation* that is a first integral of the second-order partial differential equation for ψ. A second-order partial differential equation in R_3 of the form:

$$F(x, y, p, q, r, s, t) = 0$$

is a special case of a system of equations (1) (M. A., Bd. XVII, no. 32). If it is related to a third-order equation in such a way that for any element (x, y, p, q, r, s, t) of the second-order equation there exists a characteristic that is common to that equation and the third-order equation then these characteristics will define a system with precisely the same behavior as system of characteristics of $\Omega = 0$ above. Those second and third-order equations that are derived from a pair of equations $f_1(x, y, p, q, r, s, t) = 0$, $f_2(x, y, p, q, r, s, t) = 0$ with common characteristics and ∞^∞ common integral surfaces by means of a surface transformation:

$$X = F_1(x, y, p, q, r, s, t), \qquad Y = F_2(x, y, p, q, r, s, t),$$
$$P = F_3(x, y, p, q, r, s, t), \qquad Q = F_4(x, y, p, q, r, s, t)$$

(see M. A., Bd. XIII, pp. 76) will define a special system of that type.

5. What is the image of the equation $\Omega = 0$ in the space of (x, y, z)? Any element $(z, z', x, y, p, q, p', q')$ of (1) shall be associated with a characteristic by way of $\Omega = 0$, so, as we remarked previously, the same element will also be associated with a certain simply-infinite family (i.e., sheaf) of values of (r, s, t, r', s', t'). p', q' will be determined as functions of z, z', x, y, p, q by equations (1). Therefore, any surface element (z, x, y, p, q) will be associated with a simply-infinite family of simple sheaves of (r, s, t) that correspond to the values of z' by means of $\Omega = 0$. The totality of all of these families of values for (r, s, t) that are found on the ∞^5 surface elements of the space (x, y, z) will be represented by an equation $F(z, x, y, p, q, r, s, t) = 0$. When r, s, t are interpreted as point coordinates in a space R', it will represent a line surface in this space. It is now the image of $\Omega = 0$.

Moreover, this second-order partial differential equation $F = 0$ must have ∞^∞ integral surfaces in common with the two linear, third-order partial differential equations for which one part $z = F(x, y)$ of the solutions of (1) will be common integrals, in such a way that ∞^1 characteristics that are common to all three equations will start from any surface element. If two elements (z, x, y, p, q, r, s, t) are united into two infinitely-close characteristics then the latter will themselves be united in their entire extent.

6. The part $z = F(x, y)$ of the solutions of (1) does not always have to define a system of two third-order partial differential equations. For example, if the quantity z' is absent from the two equations (1) then the integrals of the second-order partial differential equation $[f, \varphi]_{z'xp'} = 0$, from which one imagines that p', q' have been eliminated by using (1), will represent that part of the solutions of (1), precisely. One now has ∞^1 functions z': $\int (p'\,dx + q'\,dy) = z'$ that correspond to any integral $z = F(x, y)$.

If z', as well as z, is missing from both equations (1) then the part $z = F(x, y)$, as well as the part $z' = \Phi(x, y)$, of the solutions of (1) will be integrals of second-order partial differential equations – namely, the equations $[f, \varphi]_{z'xp'} = 0$, $[f, \varphi]_{zxp} = 0$ – which have p', q' eliminated by using (1) in the in the first case, and p, q, in the second. Every solution of (1) thus has the form: $z = F(x, y)$ + an arb. const., $z' = \Phi(x, y)$ + an arb. const.

§ 2.

On the transformation of certain families of surfaces that is based upon the two equations (1).

7. Each of the two equations $z = F(x, y)$, $z' = \Phi(x, y)$ that defines a solution of (1) will be, in general, represented by two linear, third-order partial differential equations. The two equations of a solution represent two surface in the spaces (x, y, z), (x, y, z'), resp. A one-to-one correspondence exists between them (M. A., Bd. XVII, no. 22), and furthermore, a one-to-one correspondence exists between the elements (z, x, y, p, q, r, s, t), $(z', x, y, p', q', r', s', t')$ of the two mutually-corresponding surfaces, namely, one that makes any system of values of (z, x, y, p, q, r, s, t) correspond to the system of values of $(z', x, y, p', q', r', s', t')$ that results by elimination from (1), $[f, \varphi]_{z'xp'} = 0$, and any three of the equations $\frac{df}{dx} = 0$, $\frac{df}{dy} = 0$, $\frac{d\varphi}{dx} = 0$, $\frac{d\varphi}{dy} = 0$. As a result, there now exists a transformation of the space (x, y, z) to the space (x, y, z') that is a surface transformation for certain pairs of linear, third-order partial differential equations, and under which, second-order contact remains preserved, in addition. In M. A., Bd. IX (*), I proved that there is no special second-order contact transformation that is a surface transformation of all of the space of (x, y, z). In the same article, I also remarked that there is no transformation that takes all integral surfaces of a higher-order partial differential equation (**), in turn, into surfaces, and verified the basis for that in the case of a second-order partial differential equation (*loc. cit.*, pp. 312, no. 10). However, we have now seen that such transformations can come about for systems of several differential equations. I would now like to seek to explain this situation more thoroughly.

We first consider *one* partial differential equation of order m – viz., $F = 0$ –, and we assume that there is a transformation that takes the integral surfaces of it, in turn, to surfaces – in particular, it takes integral surfaces that have contact of order m at a point to surfaces with contact of order m – so we see the following: If C means an arbitrary integral surface, and p, any point on it, and one lets p_1, p_2 denote the values of first differential quotients of z, and $p_{k_1k_2}$, those of the second, …, while $p_{k_1k_2\cdots k_n}$ ($k_1, k_2, \ldots, k_n = 1$ or 2) denote those of order n that belong to C at the point p then one has determined a system of values of $p_{k_1k_2\cdots k_{m+1}}$ by means of the $m + 3$ equations:

$$\delta p_{k_1k_2\cdots k_m} = p_{k_1k_2\cdots k_m1}\,dx + p_{k_1k_2\cdots k_m2}\,dy, \qquad \frac{dF}{dx} = 0, \qquad \frac{dF}{dy} = 0,$$

if one denotes the values of the m^{th} differential quotients of z by $p_{k_1k_2\cdots k_m} + \delta p_{k_1k_2\cdots k_m}$ that belong to the point ($x + dx$, $y + dy$) of an integral surface C' that is infinitely close to C, and has a contact of order $m - 1$ with it at the stated point ($x + dx$, $y + dy$). This system of values of $(m + 1)^{\text{th}}$ differential quotients of z belongs to an integral surface C'' of $F = 0$ that possesses a contact of order m with C at the point p and with C' at the point ($x + dx$, $y + dy$). Now, the assumed transformation takes our three surfaces C, C', C'' to three surfaces Γ, Γ', Γ'', of which the latter has a contact of order m with the first two infinitely-

(*) See also M. A., Bd. XI, pp. 213.

(**) M. A., Bd. IX, pp. 306. (In regard to the transformations of first-order partial differential equations, see § 5 of the cited article, especially.) – For two-dimensional spaces, things take on a different form. For example, we consider the system of equations (M. A., Bd. XVII, pp. 297):

$$f_0\left(z, z', x, \frac{dz}{dx}, \frac{dz'}{dx}\right) = 0, \qquad \varphi_0\left(z, z', x, \frac{dz}{dx}, \frac{dz'}{dx}\right) = 0.$$

When we eliminate the quantities z, dz / dx from these equations and the equation $[f_0, \varphi_0]_{zx\,dz/dx} = 0$, we will get a second-order equation for z'. We will get a second second-order equation that is an equation for z from $[f_0, \varphi_0]_{z'x\,dz'/dx} = 0$, by eliminating z', dz' / dx by means of $f_0 = 0$, $\varphi_0 = 0$. A one-to-one correspondence exists between any two integral curves of the same equations $z' = \Phi(x)$, $z = F(x)$, which collectively define a solution of $f_0 = 0$, $\varphi_0 = 0$, and this is the case especially for the elements (z', x, dz' / dx), (z, x, dz / dx) of the two second-order equations. Thus, here we have a transformation that is not a transformation of arbitrary curves in the plane (z, x) into curves, but which takes all integral curves of the one of the aforementioned second-order equations to integral curves of the other second-order equation in a single-valued way. (Cf., M. A., Bd. IX, pp. 300.) However, this does not actually characterize all contact transformations, since we are only concerned with the ∞^2 integral curves of second-order equations, no two of which will contact each other, in general.

close surfaces Γ, Γ′ at two infinitely-close points. However, any two infinitely-close surfaces Γ, Γ′ must then have a contact of order $m - 1$ with each other, such that the assumed transformation converts any two infinitely-close integral surfaces C, C' of $F = 0$ that have a contact of order $m - 1$ into two similar surfaces Γ, Γ′.

One can now construct two integral surfaces of $F = 0$ that are infinitely close to each other of second order and have contact of order $m - 2$ with each other, and then a third integral surface that is infinitely close to the first two of second order and has contact of order $m - 1$ with them at two infinitely-close points. This is due to the fact that the equations $\delta p_{k_1k_2\cdots k_{m-1}} = p_{k_1k_2\cdots k_{m-1}1}\,dx + p_{k_1k_2\cdots k_{m-1}2}\,dy$, $F = 0$ can be satisfied for all infinitely small values of $\delta p_{k_1k_2\cdots k_{m-1}}$ by values of $p_{k_1k_2\cdots k_m}$. From the remarks that we just made, these three surfaces will be converted into three new surfaces of that type. Therefore, our transformation must convert any two integral surfaces that are infinitely close to each other of second order and possess contact of order $m - 2$ at some point into two similar surfaces. When we pursue the same line of reasoning further, we will ultimately come to the theorem that any two integral surfaces of $F = 0$ that possess first-order contact and are infinitely close of order $m - 1$ will be converted into just such surfaces by means of the aforementioned transformation.

However, in M. A., Bd. IX, I proved that for the surfaces of an infinite system that fills all of space at least four times, if any two of them that are infinitely close and have first-order contact are converted into similar surfaces by a transformation then that transformation will be an ordinary (i.e., Lie) contact transformation that is a surface transformation for all of space (*). Indeed, I have not especially emphasized the idea that when the contacting surfaces are infinitely close of order r, one will arrive first-hand at pairs of surfaces by my proof that are infinitely close of order $r - 1$ and contact each other, and that one will then go from the latter pairs to pairs of surfaces that are infinitely close of order $r - 2$ and contact each other, etc. However, this is self-explanatory when

(*) I shall take this opportunity to fill a hole in the argument that was carried out on pp. 311 of M. A., Bd. IX. It was proved there that two families of surfaces $f(z, x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n+2}) = 0$, $\varphi(z', x'_1, \ldots, x'_n, \lambda_1, \ldots, \lambda_{n+2}) = 0$ that are associated with each other in such a way that any two surfaces $f(\lambda) = 0$, $f(\lambda + d\lambda) = 0$ that contact each other will correspond to two likewise contacting surfaces $\varphi(\lambda) = 0$, $\varphi(\lambda + d\lambda) = 0$ that are complete solutions (where z, x, z, x are now arbitrary constants) of one and the same first-order partial differential equation $\Phi(\lambda_1, \lambda_2, \ldots, \lambda_{n+2}, \pi_1, \ldots, \pi_{m+2}) = 0$. However, in order to conclude from this that the surfaces $f = 0$, $\varphi = 0$ must be related to each other in the same way that they are for an ordinary contact transformation, it is perhaps easiest to employ the following argument: An *arbitrary* equation $U(z, x_1, \ldots, x_n) = 0$ corresponds to a certain integral-M_{n+1} of $\Phi = 0$ that is generated by ∞^n characteristics of this equation. The same M_{n+1} is an enveloping structure of ∞^n integrals $\varphi = 0$. The values of the constants (z', x') that are valid for them are determined by an equation $V(z', x'_1, \ldots, x'_n) = 0$. The surface elements (z, x, p), (z', x', p') of $U = 0$, $V = 0$ correspond to each other in a one-to-one way, after what we just proved, and for that reason, *any two* of the united surface elements (z, x, p) must correspond to two likewise united surface elements (z', x', p'). Therefore, etc.

[Previously, I applied another argument instead of this one that is completely similar to the one on pp. 300 of the cited article. Namely, if one considers an (arbitrary) surface in (z, x) to be the enveloping structure of all the families of surfaces $f = 0$ that have stationary contact with it then it is quite clear that the corresponding surfaces $\varphi = 0$ will envelope a surface in (z', x') whose surface elements will correspond to the surface elements of the surface in (z, x).]

one (M. A., Bd. IX, pp. 310) considers systems of surfaces that are infinitely close to each other, instead of systems that fill up all space.

For that reason, the assumed transformation of the partial differential equation of order m, $F = 0$, can be nothing but an ordinary contact transformation that encompasses all surfaces in space and under which first-order contact will remain invariant.

8. However, if we consider a system of *two* partial differential equations of order m, $F = 0$, $\Phi = 0$, whose first derivatives reduce to only three mutually-independent equations, and we assume that a transformation exists that takes all ∞^∞ common integral surfaces of the two equations to surfaces that have contact of order m – in particular, integral surfaces – to surfaces with contact of order m then we will first find that when C means an integral surface, (x, y), a point on it, and C' is an integral surface that is infinitely close to C and has contact of order $m – 1$ with it at the point $(x + dx, y + dy)$, one can always determine values of $p_{k_1 k_2 \cdots k_{m+1}}$ that fulfill the following equations:

$$\delta p_{k_1 k_2 \cdots k_m} = p_{k_1 k_2 \cdots k_m 1}\, dx + p_{k_1 k_2 \cdots k_m 2}\, dy ,$$

$$\frac{dF}{dx} = 0, \qquad \frac{dF}{dy} = 0, \qquad \frac{d\Phi}{dx} = 0, \qquad \frac{d\Phi}{dy} = 0,$$

so often that $p_{k_1 k_2 \cdots k_m} + \delta p_{k_1 k_2 \cdots k_m}$ – which means the value of the m^{th} differential quotient of z that is associated with the point $(x + dx, y + dy)$ of the integral surface C' – then consolidates the last four of the equations that we just wrote down into a single new equation. Consequently, in this case, our transformation must also convert any two infinitely-close integral surfaces C, C' that have contact of order $m – 1$ into new surfaces of that sort.

If C, C' now refer to infinitely-close integral surfaces that possess contact of order $m – 2$ at the point $(x + dx, y + dy)$ then if a third integral surface should exist that contacts C at the point (x, y) and C' at the point $(x + dx, y + dy)$ to order $m – 1$ then one must be able to determine the values of $m + 1$ quantities $p_{k_1 k_2 \cdots k_m}$ that satisfy the $m + 2$ equations $\delta p_{k_1 k_2 \cdots k_{m-1}} = p_{k_1 k_2 \cdots k_{m-1} 1}\, dx + p_{k_1 k_2 \cdots k_{m-1} 2}\, dy$, $F = 0$, $\Phi = 0$. However, that is impossible for general values of δp, dx, dy. Now, it can happen, as in the case of a common family of *first* integrals (with an arbitrary constant) of $F = 0$, $\Phi = 0$, that a complete family of integral surfaces can be split into ∞^1 groups of surfaces such that the δp that belong to an integral surface C' that is included in the same group as C will satisfy the relation that results from the equations that were written down above by the elimination of $p_{k_1 k_2 \cdots k_m}$. Our transformation then takes any two integral surfaces of one and the same group that are infinitely close of second-order and have contact of order $m – 2$ with each other to two surfaces that likewise have contact of order $m – 2$. However, such a distribution of the surfaces of a complete family of integral surfaces does not necessarily exist for all systems as it does for the one that is defined by $F = 0$, $\Phi = 0$. Therefore, our transformation does not necessarily need to take the two united elements $(z, x, y, p_1, \ldots, p_{\lambda_1 \lambda_2 \cdots \lambda_{m-1}})$, $(z + dz_1, \ldots, p_{k_1 k_2 \cdots k_{m-1}} + \delta p_{k_1 k_2 \cdots k_{m-1}})$ of C at (x, y) (C' at $(x + dx, y + dx)$, resp.)

(*) to likewise united elements of that kind (**). Our transformation then needs even less to be an ordinary (i.e., Lie) contact transformation.

The number of arbitrary constants of a complete solution of the system $F = 0$, $\Phi = 0$ must be reducible to a number that is less than the number of first, second, …, up to $(m - 1)^{\text{th}}$ differential quotients of z, increased by 2. One then has the following theorem:

If, of the surfaces of two k-fold infinite systems, any two surfaces of one system that are infinitely-close to each other and have contact of order r with each other (k = 2 + *the number of first, second, ..., up to* r^{th} *differential quotients of* z) *are converted into two*

(*) These elements are united because they can be constructed on one and the same surface, if not also an integral surface.

(**) We can explain the fact that, in general, there also exist no integral surfaces that have contact of order $m - 1$ with C' at the point $(x + dx, y + dy)$ and with C at any point $(x + d'x, y + d'y)$ by an example. Let two linear, third-order partial differential equations be given whose first derivatives consolidate to three mutually-independent equations, in which case, the given equations cannot necessarily be brought into the form:

$$u + Bv + Cw + E = 0,$$
$$v + Bw + C\varpi + E' = 0,$$
(where B, C, E, E' are functions of z, x, y, p, q, r, s, t)

The condition for the following equations:

$$\delta r = u\,dx + v\,dy,$$
$$\delta s = v\,dx + w\,dy,$$
$$\delta t = w\,dx + \varpi\,dy,$$
$$u + Bv + Cw + E = 0$$
$$v + Bw + C\varpi + E' = 0$$

to exist together is then expressed by the equation:

$$\begin{vmatrix} \delta r - dr & dx & dy & 0 & 0 \\ \delta s - ds & 0 & dx & dy & 0 \\ \delta t - dt & 0 & 0 & dx & dy \\ 0 & 1 & B & C & 0 \\ 0 & 0 & 1 & B & C \end{vmatrix} = 0,$$

if dr, ds, dt mean any values of δr, δs, δt that are possible for the existence of those equations. However, after dropping the common factor $C\,dx^2 - B\,dx\,dy + dy^2$, the condition equation will assume the simple form:

$$\delta r - dr + B(\delta s - ds) + C(\delta t - dt) = 0,$$

which is an equation that is independent of dx, dy.

This proves that when C, C' are two infinitely-close integral surfaces that contact at the point $(x + dx, y + dy)$, and $(r + dr, s + ds, t + dt)$ means the second differential quotient of z at any point $(r + \delta r, s + \delta s, t + \delta t)$ of the surface C' that is associated with the same point, the relation in question can be satisfied only for or special surface C'. It follows further from this that there is, in general, no integral surface that has second-order contact with C' at the point $(x + dx, y + dy)$ and with C at any infinitely-close point. Any two infinitely-close integral surfaces C, C' that contact each other will thus not be converted into contacting surfaces by a second-order contact transformation that concerns the integral surfaces of the third-order partial differential equation.

surfaces with just the same properties in the other system by some transformation then that transformation will certainly be an ordinary (i.e., Lie) contact transformation.

The proof of this theorem is completely analogous to the proof of the theorem that was cited above in M. A., Bd. IX, pp. 311.

We especially direct our attention to the case of two third-order equations:

$$[f, [f, \varphi]]_{zxp} = 0, \qquad [\varphi, [f, \varphi]]_{zxp} = 0$$

(M. A., Bd. XVII, pp. 290). As we remarked above, we have a transformation for which second-order contact is preserved, and which takes the integral surfaces of the equations to other surfaces. Now, this transformation is determined completely as a solution of (1) by the equation $x' = x$, $y' = y$, together with the equations $z = F(x, y, \lambda, \mu, \nu, \rho)$, $z' = \Phi(x, y, \lambda, \mu, \nu, \rho)$. However, any two fourfold-infinite families of surfaces are of a general sort, and two contacting surfaces of the family $z = F(x, y, \lambda, \mu, \nu, \rho)$ thus do not correspond to two contacting surfaces of the other family, as it would have to be if the transformation were a contact transformation.

9. Moreover, this easily resolves the question of whether it is possible that the two third-order partial differential equations:

$$[f, [f, \varphi]]_{zxp} = 0, \qquad [\varphi, [f, \varphi]]_{zxp} = 0$$

will admit a common first integral with any arbitrary constants. Should that be the case, then, from the previous discussion, the transformation that we are dealing with between these equations and the following two:

$$[f, [f, \varphi]]_{z'xp'} = 0, \qquad [\varphi, [f, \varphi]]_{z'xp'} = 0$$

would have to be such that any two infinitely-close, contacting, common integral surfaces of one of the first integrals of the first two equations would have to be converted into two contacting, common, integral surfaces of the last two. Otherwise, the latter equations must also possess a common first integral with arbitrary constants, corresponding to the integrals of that kind that were previously assumed for the first two equations, so the transformation in question must be an ordinary contact transformation. Now, two of the equations that are valid for this transformation read thus: $x' = x$, $y' = y$, and therefore there must be a third equation for the transformation of the form: $z' = F(z, x, y)$. This shows that the two equations (1) must now be able to be brought into the form:

$$f(z', z, x, y) = 0, \qquad \varphi(z', z, x, y, p, q, p', q') = 0.$$

However, one will then have $[f, \varphi]_{z'xp'} = \frac{df}{dx}\varphi'(p') + \frac{df}{dy}\varphi'(q')$, and the quantities r, s, t will thus be missing from the equation $[f, \varphi]_{z'xp'} = 0$. The equations $[f,[f,\varphi]]_{z'xp'} = 0$, $[\varphi,[f,\varphi]]_{z'xp'} = 0$ cannot give rise to any third-order equations then. *Therefore, the two third-order partial differential equations in question cannot have a common first integral with any arbitrary constants.*

10. A single-valued transformation between the two spaces (x, y, z), (x, y, z') will then be determined by equations (1) only when they lead to two pairs of third-order partial differential equations in (x, y, z) [(x, y, z'), resp.]. If one of the systems (1) that were considered in no. 6 is present then the transformation will take on a different form. It cannot be a single-valued transformation, if it either takes any integral surface of a certain second-order partial differential equation (viz., $[f, \varphi]_{z'xp'} = 0$) to a simply-infinite family of integral surfaces of a system of two third-order partial differential equations (viz., $[f,[f,\varphi]]_{z'xp'} = 0$, $[\varphi,[f,\varphi]]_{z'xp'} = 0$) that takes any of the latter surfaces to a certain surface of the form, or it takes any integral surface of a certain second-order partial differential equation (viz., $[f, \varphi]_{z'xp'} = 0$) to ∞^1 corresponding integral surfaces of another second-order partial differential equation (viz., $[f, \varphi]_{zxp} = 0$), and vice versa. Now, in the event that the variable z' is missing from equations (1), one will also have that for any element (z, x, y, p, q, r, s, t) that satisfies the equation $[f, \varphi]_{z'xp'} = 0$ there are ∞^1 corresponding elements (z', x, y, p', q'), each of which has a certain system of values for (r', s', t'); by contrast, each element $(z', x, y, p', q', r', s', t')$ corresponds to a single system of values for (z, x, y, p, q, r, s, t) (or some system of values of them). In the case that z, as well as z', is missing from (1), every element (z, x, y, p, q, r, s, t) of a second-order partial differential equation $[f, \varphi]_{z'xp'} = 0$ will correspond to ∞^1 elements $(z', x, y, p', q', r', s', t')$ of another second-order partial differential equation $[f, \varphi]_{zxp} = 0$, and conversely, every element $(z', x, y, p', q', r', s', t')$ of the latter equation will correspond to ∞^1 elements (z, x, y, p, q, r, s, t) of the former.

The transformation that is now based in equations (1) is, in the vicinity in which it is a surface transformation, a multi-valued (viz., infinitely-valued) surface transformation. A generalization of this shall be treated in no. 15.

§ 3.

Derivation of some special systems of second-order partial differential equations.

11. In no. 23 of my treatise in M. A., Bd. XVII, the surface transformation that was defined by the three general equations:

$$(3) \qquad \begin{cases} F_1(z, x, y, p, q, z', x', y', p', q') = 0, \\ F_2(\qquad\qquad\qquad\qquad) = 0, \\ F_3(\qquad\qquad\qquad\qquad) = 0, \end{cases}$$

and in no. 24 the special case was treated in which the transformation converted any strip of a given fourfold-infinite family into a simply-infinite family of surfaces. However, the transformation can also be so arranged that it converts any integral strip of the pair of equations:

$$(4) \qquad \begin{cases} f(z, x, y, p, q) = C, \\ \varphi(\qquad\qquad) = C', \end{cases}$$

where C, C' denote arbitrary constants, into a family of surfaces. The integral strips of (4) are represented by the equations:

$$(5) \qquad dz - p\,dx - q\,dy = 0, \quad \frac{df}{dx}dx + \frac{df}{dy}dy = 0, \quad \frac{d\varphi}{dx}dx + \frac{d\varphi}{dy}dy = 0.$$

An arbitrary surface element (z, x, y, p, q) determines, first of all, certain values of C, C' in (4), and then ∞^1 directions ($dy : dx$), each of which provides a sheaf of (r, s, t), in conjunction with the stated element, and on the basis of (5). Their equations have the form $r\,dx + s\,dy = \mu\,dx$, $s\,dx + t\,dy = \nu\,dx$, where $\mu\,dx$, $\nu\,dx$ itself depends upon dx, dy. This sheaf gives rise to a surface element ($z + p\,dz + q\,dy$, $x + dx$, …, $p + \mu\,dx$, $q + \nu\,dx$) that belongs to an integral strip of (4) that starts from (z, x y, p, q). If one now introduces the values μ, ν for dp / dx, dq / dx, which take the form $\alpha + \beta\,dy / dx$ relative to $dy : dx$ into the condition equation for the involution of the two first-order partial differential equations that correspond to the strip [M. A., Bd. XVII, pp. 307, eq. (19)]:

$$\frac{dF_1}{dx}[F_2, F_3]_{z'x'p'} + \frac{dF_2}{dx}[F_3, F_1]_{z'x'p'} + \frac{dF_3}{dx}[F_1, F_2]_{z'x'p'} = 0,$$

and demands that they must be independent of the special values of $dy : dx$ then F_1, F_2, F_3 will fulfill two equations. F_1 can be chosen arbitrarily, and F_2, F_3 will then be determined by any two equations, so the transformation (3) will convert all integral strips of the system (4), (5) into families of surfaces.

If we ask how the figure in r' that consists of any family of surfaces is constituted then we will need to visualize only the following from no. 24, M. A., Bd. XVII: Any element (z, x, y, p, q) corresponds to a family of ∞^1 strips in r' that might be briefly denoted by S'. Just as ∞^∞ integral-strips of (4) start from any surface element in r, ∞^∞ surfaces of our family of surfaces in r' will go through any strip S'. ∞^1 sheaves (5) of (r, s, t) belong to any surface element in r that lead to just as many surface elements of integral strips that are united with the element. Correspondingly, any strip S' is united with ∞^1 other such strips. If we further remark that every surface element in r' corresponds to ∞^1 surface elements in r, among which, only one of them will satisfy the equations $f = C_0$, $\varphi = C_0'$ – if C_0, C_0' denote (any) well-defined values of the arbitrary parameters *C*, *C′* in (4) – then we will see that ∞^2 strips *S′*, corresponding to the different values of *C*, *C′*, will go through an arbitrary element (*z′*, *x′*, *y′*, *p′*, *q′*). The figure that is composed of those sheaves of (*r′*, *s′*, *t′*) that belong to *S′* will thus be expressed by two equations:

$$(6) \qquad \begin{cases} F(z', x', y', p', q', r', s', t') = C, \\ \Phi(\qquad\qquad\qquad\qquad) = C'. \end{cases}$$

Here, *C*, *C′* are the same as they were in (4), because, as we already remarked, ∞^∞ surfaces that correspond to the integral of the equations $f = C_0$, $\varphi = C_0'$, and which are therefore integral surfaces of our figure (6), will go through any strip *S′* that corresponds to an element of $f = C_0$, $\varphi = C_0'$, so any strip *S′* will become a common characteristic of the two second-order equations in (6).

We arrive at equations (6) simply, as follows: By differentiating (3), while regarding *z*, *x*, *y*, *p*, *q* as constants, and then eliminating *dx′*, *dy′*, one will arrive at two equations in *z*, *x*, *y*, *p*, *q*, *z′*, *x′*, *y′*, *p′*, *q′*, *r′*, *s′*, *t′*. When we eliminate *z*, *x*, *y*, *p*, *q* from these two equations, equations (3) and (4), we will obtain the equations (6) in question. *From what we just discussed, these two second-order partial differential equations will be coupled to each other in such a way that a strip that starts at every element* (*z′*, *x′*, *y′*, *p′*, *q′*, *r′*, *s′*, *t′*) *will, at the same time, define a characteristic of an equation* $F = C_0$ *and an equation* $\Phi = C_0'$. *These strips will be associated with* ∞^∞ *common integral surfaces of* $F = C_0$, $\Phi = C_0'$. *Of the first derivatives of F and Φ with respect to x′, y′, one of them will therefore be an algebraic consequence of the other one. Furthermore, any characteristics that are precisely the strips S′ above, will be first-contact contact strips, such that there will be* ∞^∞ *common integral surfaces of* $F = C$, $\Phi = C'$ *– in which C, C′ refer to completely arbitrary constants – that possess first-order contact along a strip S′*. The common integral surfaces of $F = C_0$, $\Phi = C_0'$ will be associated with families of ∞^1 surfaces such that each family will correspond to an integral strip of $f(z, x, y, p, q) = C_0$, $\varphi(z, x, y, p, q) = C_0'$.

12. The three equations:

$$(7) \qquad \begin{cases} F_1(z, x_1, x_2, x_3, p_1, p_2, p_3, z', x_1', x_2', x_3', p_1', p_2', p_3') = 0, \\ F_2(\qquad\qquad\qquad\qquad\qquad\qquad) = 0, \\ F_3(\qquad\qquad\qquad\qquad\qquad\qquad) = 0 \end{cases}$$

of a manifold transformation of a four-dimensional space can be related to each other in such a way that they define a transformation that converts any integral strip (viz., integral-M_1) of the system:

$$(8)\qquad \left\{\begin{aligned} &f(z, x_1, x_2, x_3, p_1, p_2, p_3) = C, \\ &\varphi(\qquad\qquad\qquad\qquad) = C', \\ &\psi(\qquad\qquad\qquad\qquad) = C'', \\ &\chi(\qquad\qquad\qquad\qquad) = C''' \end{aligned}\right.$$

into a doubly-infinite integral family that is common to two involutory, first-order partial differential equations. One then has to satisfy only two partial differential equations for F_1, F_2, F_3 . One arrives at these equations as follows: The condition for the two first-order partial differential equations that define a strip to be involutory has a similar form to equation (19) in M. A., Bd. XVII, pp. 307. One now merely replaces dx_3, dp_1, dp_2, dp_5 with their values that one derives from (1) in terms of dx_1, dx_2 [setting $dz = p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3$], and then sets the individual coefficients of dx_1, dx_2 equal to zero. One thus has the two desired equations for F_1, F_2, F_3 . The figure in r' – i.e., in the space of (x_1', x_2', x_3', z') – that consists of those families of surfaces that thus correspond to the system (8) can be characterized as follows: Any surface element (z, x, p) corresponds to a family of ∞^2 M_2 (*) whose envelopes generate all of these M_2 from certain (characteristic) M_1 . (See M. A., Bd. XI, pp. 430.) Since an arbitrary surface element (z', x', p') corresponds to a certain surface element (z, x, p), and one finds ∞^1 directions (dx_1, dx_2, dx_3) for integral strips of the same equations $f = C_0$, etc., in that element, any M_2 that corresponds to an element (z, x, p) and contains the element (z', x', p') must lie on some M_3, along with ∞^1 infinitely-close M_2 . Now, any element (z', x', p') that goes through those M_2 that correspond to the element (z, x, p) will be associated with a family of ∞^2 systems of values of p'_{ik} , and since, as we just remarked, any M_2 will lie on an M_3, along with ∞^1 infinitely-close M_2 , all of these ∞^2 values of p'_{ik} , but only these, must be associated with the figure in r' that corresponds to the system $f = C_0$, $\varphi = C_0'$, $\psi = C_0''$, $\chi = C_0'''$. For that reason, this figure must be defined algebraically by four second-order partial differential equations. We are given those values of p'_{ik} that are associated with one of the M_2 that corresponds to the element (z, x, p) as one of its surface elements by the following equations:

$$\begin{aligned} \frac{\partial F_i}{\partial x_2'} + \left(\frac{\partial F_i}{\partial x_1'} + p_1' \frac{\partial F_i}{\partial z'}\right)\frac{\partial x_1'}{\partial x_2'} + p_2' \frac{\partial F_i}{\partial z'} + \frac{\partial F_i}{\partial p_1'}\left(p_{12}' + p_{11}' \frac{\partial x_1'}{\partial x_2'}\right) \\ + \frac{\partial F_i}{\partial p_2'}\left(p_{22}' + p_{21}' \frac{\partial x_1'}{\partial x_2'}\right) + \frac{\partial F_i}{\partial p_3'}\left(p_{32}' + p_{31}' \frac{\partial x_1'}{\partial x_2'}\right) = 0, \end{aligned}$$

$$\frac{\partial F_i}{\partial x_3'} + \left(\frac{\partial F_i}{\partial x_1'} + p_1' \frac{\partial F_i}{\partial z'}\right)\frac{\partial x_1'}{\partial x_3'} + p_3' \frac{\partial F_i}{\partial z'} + \frac{\partial F_i}{\partial p_1'}\left(p_{13}' + p_{11}' \frac{\partial x_1'}{\partial x_3'}\right)$$

(*) These M_2 manifolds are defined by any ∞^2 united surface elements (z', x', p').

$$+\frac{\partial F_i}{\partial p'_2}\left(p'_{23}+p'_{21}\frac{\partial x'_1}{\partial x'_3}\right)+\frac{\partial F_i}{\partial p'_3}\left(p'_{33}+p'_{31}\frac{\partial x'_1}{\partial x'_3}\right) = 0.$$

When we eliminate $\frac{\partial x'_1}{\partial x'_2}$, $\frac{\partial x'_1}{\partial x'_3}$, we will obtain four equations that are valid for those values of p'_{ik} that correspond to the element (z, x, p) by way of M_2 and include one and the same element (z', x', p') that will be associated with the latter element. As a result, by eliminating z, x, p from these four equations in $z, x_1, x_2, x_3, p_1, p_2, p_3, z', x'_1, \ldots, p'_{11}, p'_{12}, \ldots, p'_{33}$ one simply determines those equations in (7) and (8) that define the figure in r' that corresponds to the system (8). These equations become four second-order partial differential equations for z':

$$\begin{aligned} &F(z', x'_1, \ldots, p'_i, \ldots, p'_{ik}, \ldots) = C, \\ &\Phi(\qquad\qquad\qquad\qquad) = C', \\ &\Psi(\qquad\qquad\qquad\qquad) = C'', \\ &\Xi(\qquad\qquad\qquad\qquad) = C''', \end{aligned}$$

in which one understands C, C', C'', C''' to means the arbitrary constants that were previously introduced into (8).

Any four equations $F = C_0$, $\Phi = C'_0$, $\Psi = C''_0$, $\Xi = C'''_0$ possess ∞^∞ common characteristics M_2 and unboundedly many common intermediate integrals, each of which is expressed by two involutory, first-order partial differential equations. These intermediate integrals correspond to the integral strips of any four equations (8): $f = C_0$, $\varphi = C'_0$, $\psi = C''_0$, $\chi = C'''_0$.

13. Finally, we consider a transformation that is determined by four equations:

$$(9)\qquad \begin{cases} F_1(z, x_1, x_2, x_3, p_1, p_2, p_3, z', x'_1, x'_2, x'_3, p'_1, p'_2, p'_3) = 0, \\ F_2(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad) = 0, \\ F_3(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad) = 0, \\ F_4(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad) = 0 \end{cases}$$

and takes any integral-M_2 of the system of equations:

$$(10)\qquad \begin{cases} f(z, x_1, x_2, x_3, p_1, p_2, p_3) = C, \\ \varphi(\qquad\qquad\qquad\qquad) = C', \\ \psi(\qquad\qquad\qquad\qquad) = C'' \end{cases}$$

to an involutory pair of first-order partial differential equations in r'. The condition for this is expressed by three equations in F_1, F_2, F_3, F_4. One arrives at these equations as follows: One poses the condition for the two first-order partial differential equations that correspond to an M_2 to be involutory. This condition has a similar form to equation (26) in M. A., Bd. XVII, pp. 312. One replaces the $\frac{\partial p_1}{\partial x_2}$, etc., that enter into the symbols (12) with their values in terms of $\frac{\partial x_1}{\partial x_2}$, $\frac{\partial x_1}{\partial x_3}$ from (10), and then expresses the idea that the equation in question should be independent of $\frac{\partial x_1}{\partial x_2}$, $\frac{\partial x_1}{\partial x_3}$. The figure in r' that corresponds to the system (10) will be comprised of those M_2 that correspond to the integral-M_2 of (10) by way of (9). An arbitrary surface element (z', x', p') will be associated with $\infty^3 \cdot \infty^2$ systems of values of p'_{ik} by way of this M_3. Our figure will then encompass only ∞^5 of the ∞^6 systems of values of (z', x', p', p'_{ik}) of the space R' that are associated with an arbitrary (z', x', p'). *One finds them to be determined more precisely by four second-order partial differential equations* $F = 0$, $\Phi = C$, $\Psi = C'$, $\Xi = C''$ *that possess any* ∞^∞ *involutory pairs of equations that correspond to the integral-*M_2 *of* (10) *as common intermediate integrals.*

§ 4.

On the transformation that is based upon four equations in $z, x, y, p, q, z', p', q'$.

14. A transformation that is defined by four equations:

$$(11) \qquad \begin{cases} F_1(z, x, y, p, q, z', x', y', p', q') = 0, \\ F_2(\qquad\qquad\qquad\qquad\qquad) = 0, \\ F_3(\qquad\qquad\qquad\qquad\qquad) = 0, \\ F_4(\qquad\qquad\qquad\qquad\qquad) = 0 \end{cases}$$

is, in general, a surface transformation only for a certain domain that is determined by two third-order partial differential equation. I proved this on pp. 313 of M. A., Bd. XVII. In some situations, this domain can be more restricted. However, before I go into that, I would like to reproduce briefly my earlier proof of the theorem that I mentioned in a somewhat modified form.

We first think of the transformation (11) as having been brought into the form:

$$(12) \qquad \begin{cases} x' = f_1(z', z, x, y, p, q), \\ y' = f_2(\qquad\qquad\qquad), \\ p' = \varphi_1(\qquad\qquad\qquad), \\ q' = \varphi_2(\qquad\qquad\qquad) \end{cases}$$

by solving for x', y', p', q', and then representing any pair of mutually-corresponding surfaces $z = F(x, y)$, $z' = \Phi(x', y')$ in terms of z, z', x, y – perhaps as $z = F(x, y)$, $z' = \bar{\Phi}(x, y)$. I will denote the differential quotients $\bar{\Phi}'(x)$, $\bar{\Phi}'(y)$ by π, κ. One then has:

$$(13) \qquad \pi = p'\frac{dx'}{dx} + q'\frac{dy'}{dx}, \qquad \kappa = p'\frac{dx'}{dy} + q'\frac{dy'}{dy},$$

where

$$\begin{aligned}
\frac{dx'}{dx} &= \frac{df_1}{dx} = \frac{\partial f_1}{\partial x} + p\frac{\partial f_1}{\partial z} + r\frac{\partial f_1}{\partial p} + s\frac{\partial f_1}{\partial q} + \pi\frac{\partial f_1}{\partial z'} = \frac{d'f_1}{dx} + \pi\frac{\partial f_1}{\partial z'},\\
\frac{dy'}{dx} &= \frac{df_2}{dx} = \frac{\partial f_2}{\partial x} + p\frac{\partial f_2}{\partial z} + r\frac{\partial f_2}{\partial p} + s\frac{\partial f_2}{\partial q} + \pi\frac{\partial f_2}{\partial z'} = \frac{d'f_2}{dx} + \pi\frac{\partial f_2}{\partial z'},\\
\frac{dx'}{dy} &= \frac{df_1}{dy} = \frac{\partial f_1}{\partial y} + q\frac{\partial f_1}{\partial z} + s\frac{\partial f_1}{\partial p} + t\frac{\partial f_1}{\partial q} + \kappa\frac{\partial f_1}{\partial z'} = \frac{d'f_1}{dy} + \kappa\frac{\partial f_1}{\partial z'},\\
\frac{dy'}{dy} &= \frac{df_2}{dy} = \frac{\partial f_2}{\partial y} + q\frac{\partial f_2}{\partial z} + s\frac{\partial f_2}{\partial p} + t\frac{\partial f_2}{\partial q} + \kappa\frac{\partial f_2}{\partial z'} = \frac{d'f_2}{dy} + \kappa\frac{\partial f_2}{\partial z'}.
\end{aligned}$$

Equations (13) will then go to the following ones:

$$(14) \qquad \left\{\begin{aligned}
f &\equiv \pi\left(1 - \varphi_1\frac{\partial f_1}{\partial z'} - \varphi_2\frac{\partial f_2}{\partial z'}\right) - \varphi_1\frac{d'f_1}{dx} - \varphi_2\frac{d'f_2}{dx} = 0,\\
\varphi &\equiv \kappa\left(1 - \varphi_1\frac{\partial f_1}{\partial z'} - \varphi_2\frac{\partial f_2}{\partial z'}\right) - \varphi_1\frac{d'f_1}{dy} - \varphi_2\frac{d'f_2}{dy} = 0,
\end{aligned}\right.$$

and the determination of the surface-pair: $z = F(x, y)$, $z' = \Phi(x', y')$ will be equivalent to the determination of the solutions: $z = F(x, y)$, $z' = \bar{\Phi}(x, y)$ of the system of equations $f = 0$, $\varphi = 0$.

For the presentation of these solutions, we proceed exactly as we did in no. 5 of the treatise in M. A., Bd. XVII, pp. 285, in regard to the presentation of solutions to a system of two equations in z, x, y, p, q, z', p', q'. We must satisfy the equation $[f, \varphi]_{z'xp'} = 0$ with the solutions in question. This equation will be free of the third differential quotients u, v, w, v of z, due to the special form of the present equations (14) in relation to r, s, t. For that reason, the two equations:

$$[f, [f, \varphi]]_{z'\chi\pi} = 0, \qquad [\varphi, [f, \varphi]]_{z'\chi\pi} = 0$$

will only be third-order partial differential equations relative to z, and an elimination of z', π, χ from equations (14), along with the equations $[f(\varphi)]_{z'\chi\pi} = 0$, and the equations that we just wrote down, will then lead to two third-order partial differential equations for the determination of the equations $z = F(x, y)$. These third-order equations behave in such a way that their first derivatives with respect to x, y will reduce to only three mutually-independent equations. [This comes about in the same way as it did for the similar theorem that relates to equations (7) on pp. 290 of M. A., Bd. XVII.] However, any third-order equation will admit ∞^∞ common integral surfaces for that very reason. All of the integrals belong to our system of equations (14) as one part $z = F(x, y)$ of the solutions $z = F(x, y)$, $z' = \Phi(x, y)$. By eliminating z', π, χ from the three equations $f = 0$, $\varphi = 0$, $[f(\varphi)]_{z'\chi\pi} = 0$, any of the integrals $z = F(x, y)$ will yield the equation $z' = \bar{\Phi}(x, y)$, which, together with $z = F(x, y)$, will define a solution of (14). All pairs of equations such as $z = F(x, y)$, $z' = \Phi(x', y')$ will represent two surfaces that will be converted into each other by the transformation (12). Any surfaces will correspond to each other in a one-to-one way. There will then exist a one-to-one relationship between the elements (z, x, y, p, q, r, s, t), $(z', x', y', p', q', r', s', t')$ themselves. For that reason, not only must the surfaces $z = F(x, y)$ satisfy two third-order partial differential equations, as we showed, but also the corresponding surfaces $z' = \Phi(x', y')$ must belong to two partial differential equations that are likewise of third order. The aforementioned relation between the elements (z, x, y, p, q, r, s, t), $(z', x', y', p', q', r', s', t')$ is to be thought of as being given by a second-order contact transformation that takes any two pairs of third-order equations to each other. However, it is not a first-order contact transformation, so it is not an ordinary (i.e., Lie) contact transformation (from no. 8).

However, it would be a first-order contact transformation in the event that one of the pairs of third-order equations admitted a first integral with arbitrary constants (cf., no. 8), but in such a case the transformation (12) – which would now be a contact transformation, precisely – would take any surface to any surface, and nothing more could be said of the system of partial differential equations that defined the surface, which could again be transformed into surfaces (cf., no. 9).

15. If z' were missing from equations (12) then z' would also be missing from equations (14) and the equation $[f, \varphi]_{z'\chi\pi} = 0$. The determination of the equations $z = F(x, y)$ would then be accomplished by means of a second-order partial differential equation that one would obtain by eliminating π, χ from the aforementioned equations (14) and the equation $[f, \varphi]_{z'\chi\pi} = 0$. Any of the integral surfaces of this second-order partial differential equation would correspond to an involutory pair of first-order partial differential equations in r' by way of (12), or to their integral surfaces $z' = \Phi(x', y', C)$. Moreover, any element (z, x, y, p, q, r, s, t) of any second-order partial differential equation would correspond to a family of ∞^1 elements $(z', x', y', p', q', r', s', t')$. By contrast, one of the latter elements would not need to correspond to ∞^1 of the former. In general, a surface $z' = \Phi(x', y')$ will correspond to only one surface $z = F(x, y)$. The surfaces $z' = \Phi(x', y')$ would generally satisfy a system of two third-order partial differential equations. However, if not only z' were missing from equations (12), but also z, then one would not only have a second-order partial differential equation for the surfaces $z = F(x, y)$, but also a second-order partial differential equation for the surfaces $z' = \Phi(x', y')$. Any element (z, x, y, p, q, r, s, t) [$(z', x', y', p', q', r', s', t')$, resp.] of one of these second-order equations would correspond to an entire family of ∞^1 elements of that kind for the other equation. Any integral surface $z = F(x, y)$ would correspond to ∞^1 integral surfaces, and each of the latter surfaces, to ∞^1 of the former.

16. If z' is missing from our transformation equation then one will obtain the surfaces in r' that correspond to an integral surface $z = F(x, z)$ of the relevant second-order equation by mere quadratures. Namely, one introduces the values $F(x, y)$, $F'(x)$, $F''(y)$ for z, p, q, resp., and then obtains, after eliminating x, y: $p' = \psi_1(x', y')$, $q' = \psi_2(x', y')$, and from these equations, one obtains the equation of the corresponding family of surfaces by performing the quadrature:

$$z' = \int (\psi_1(x', y')\, dx' + \psi_2(x', y')\, dy') .$$

17. However, in general, if the transformation (11) converts any surface $z = f(x, y)$ of the domain r into ∞^1 surfaces in the domain r' then the latter surfaces will *not* be obtained by mere quadratures. Namely, on the one hand, if one substitutes $z = f(x, y)$, $p = f'(x)$, $q = f'(y)$ in (11) and eliminates x, y then one will have two first-order partial differential equations:

$$A(z', x', y', p', q') = 0, \qquad B(z', x', y', p', q') = 0 \tag{15}$$

that correspond to the surface $z = f(x, y)$ and will be involutory from the assumptions that were made. Their common integral surfaces can be denoted by C, C', C'', …, for the moment. Every element (z, x, y, p, q) of the surface $z = f(x, y)$ will correspond to ∞^1 elements (z', x', y', p', q'), one of which lies on C, another, on C', etc. The latter elements can now be regarded as mutually-corresponding elements of the surfaces C, C', C'', etc. Two infinitely-close elements of the surface $z = f(x, y)$ will correspond to ∞^1 pairs of united elements (z', x', y', p', q'), one of which lies on C, another, on C', etc. For that reason, any two united elements of (15) must correspond to ∞^1 pairs of united elements of the same equations (15). An infinitesimal contact transformation of the system of equations (15) into itself, which can be regarded as a contact transformation, precisely, must then come about as follows: When one differentiates equations (11) and then considers the quantities z, x, y, p, q to be constants, one will obtain $\delta x'$, $\delta y'$, $\delta z'$, $\delta p'$, $\delta q'$ as proportional to certain well-defined functions of z', x', y', p', q', z, x, y, p, q. z, x, y, p, q, p', q' will be eliminated by means of the equations $z = f(x, y)$, $p = f'(x)$, $q = f'(y)$. One thus gets the transformation in question as being expressed by equations of the form:

$$\delta x' = \varepsilon\psi A', \quad \delta y' = \varepsilon\psi B', \quad \delta z' = \varepsilon\psi C', \quad \delta p' = \varepsilon\psi D, \quad \delta q' = \varepsilon\psi E',$$

where A', B', …, E' mean completely-determined functions of x', y', z' that are derived from (11) and $z = f(x, y)$, ψ is a still-unknown function of the same quantities, and ε means an infinitely small constant. However, one has to determine ψ in the following way: The transformation in question shall be determined (M. A., Bd. XV, pp. 51) by means of a function Φ:

$$\Phi = \psi(A'\, p' + B'\, q' - C'), \tag{16}$$

and one must have:

$$\delta p' = -\,\varepsilon\left(\frac{\partial\Phi}{\partial x'} + p'\frac{\partial\Phi}{\partial z'}\right), \qquad \delta q' = -\,\varepsilon\left(\frac{\partial\Phi}{\partial y'} + q'\frac{\partial\Phi}{\partial z'}\right).$$

The two equations:

$$\frac{\partial \Phi}{\partial x'} + p'\frac{\partial \Phi}{\partial z'} = -\psi D', \qquad \frac{\partial \Phi}{\partial y'} + q'\frac{\partial \Phi}{\partial z'} = -\psi E',$$

in which one thinks of p', q' as being replaced with their values from (15) or (11), will be two first-order partial differential equations for ψ. If ϖ were used instead of log ψ then these equations would assume the form:

$$(17) \qquad \begin{cases} u\dfrac{\partial \varpi}{\partial x'} + v\dfrac{\partial \varpi}{\partial y'} + w\dfrac{\partial \varpi}{\partial z'} = 1, \\ u'\dfrac{\partial \varpi}{\partial x'} + v'\dfrac{\partial \varpi}{\partial y'} + w'\dfrac{\partial \varpi}{\partial z'} = 1, \end{cases}$$

where u, v, w, u', v', w' would be completely-determined functions of x', y', z'. These two equations must have a solution ϖ in common, so, from the next-to-last discussion, an infinitesimal point transformation of the equations (15) that is characterized by (16) must exist, and such a transformation will always give v in the form $\varpi = \varpi(x', y', z')$ + an arb. funct. of φ, if $\varphi = C$ means the equation of the family of integral surfaces of (15).

For that reason, if the first of the equations (17) is briefly denoted by $A(\varpi) = 1$ and the second one by $B(\varpi) = 1$, then of the three equations:

$$A(\varpi) = 1, \qquad B(\varpi) = 1, \qquad A(B(\varpi)) - B(A(\varpi)) = 0,$$

the last one will always be fulfilled along with the first two.

One then obtains a value of ϖ, and then a value for ψ, by means of the formula $\varpi = \log \psi$, first, by integrating a differential equation (with one arbitrary parameter):

$$\alpha(x, y)\,dx + \beta(x, y)\,dy = 0.$$

The determination of an infinitesimal contact transformation that takes the integral family of our involutory, first-order partial differential equations to itself is therefore, in general, fraught with the same difficulties as the determination of the integral family itself. Our question is also in no way essentially different from that of the determination of the solutions of any two involutory, first-order partial differential equation, since one can always write down two equations:

$$F_1(z, x, y, z', x', y', p', q') = 0, \qquad F_2(z, x, y, z', x', y', p', q') = 0$$

arbitrarily that will make up a system (11), precisely, together with the given partial differential equations.

18. From M. A., Bd. XVII, pp. 312, equation (26), the involutivity condition for the two first-order partial differential equations that correspond to a surface $z = f(x, y)$ by means of (11), reads as follows:

$$\begin{aligned} &(3, 4)\,[F_1, F_2]_{z'x'p'} + (4, 2)\,[F_1, F_3]_{z'x'p'} + (2, 3)\,[F_1, F_4]_{z'x'p'} \\ +\;&(1, 2)\,[F_3, F_4]_{z'x'p'} + (1, 3)\,[F_4, F_2]_{z'x'p'} + (1, 4)\,[F_2, F_3]_{z'x'p'} = 0, \end{aligned}$$

where $[F_m, F_n]_{z'x'p'}$ is the ordinary involution sign, and (m, n) is written, more briefly, instead of:

$$\frac{dF_m}{dx}\frac{dF_n}{dy} - \frac{dF_m}{dy}\frac{dF_n}{dx}$$

$(m, n = 1, 2, 3, 4)$. If the transformation (11) is of the form:

$$(18)\qquad \begin{cases} x' = f_1(z, x, y, p, q), \\ y' = f_2(\qquad\qquad), \\ p' = f_3(\qquad\qquad), \\ q' = f_4(\qquad\qquad) \end{cases}$$

then any condition equation will assume the simpler form:

$$(1, 3) + (2, 4) = 0.$$

It becomes a second-order partial differential equation:

$$(19)\qquad A\,r + B\,s + C\,t + D(rt - s^2) + E = 0,$$

where $A, B, \ldots, E$ are functions of x, y, z, p, q. *This is the most general second-order equation of that form*, so when $A, B, \ldots, E$ are given, and might also be functions of x, y, z, p, q, one will have only four equations for the determination of four functions f_1, f_2, f_3, f_4 whose known terms are A / E, B / E, C / E, D / E. The surface system in r' that corresponds to the integral surfaces of that second-order partial differential equation will be given by two third-order partial differential equations whose first derivatives with respect to x', y' are algebraic consequences of each other. *As a result, from number* 16, *the integral system of this pair of equations will be obtained by solving a second-order partial differential equation* (19) *and carrying out the quadrature.*

19. *If z is missing from the functions f in equations* (18) *then z will also be missing from the functions $A, B, \ldots, E$ in equation* (19), and instead of the aforementioned pair of third-order equations, one will have, as was remarked in no. 15, a partial differential equation of the same form as (19). In particular, it will be identical to equation (19) when the system of equations (18) remains unchanged under a permutation of the primed and unprimed symbols. Any equation (19) will then be distinguished by the fact that any integral surface of it will be transformed into ∞^1 other integral surfaces of it. Consequently, the integral surfaces of the equation will arrange themselves into pairs of ∞^1 surfaces, where the two families of one pair are related to each other in such a way that they go to each other under the given transformation. Insofar as it defines a surface transformation, the transformation will be a single-valued transformation between systems of values (x, y, p, q), (x', y', p', q'), just as it is between (x, y, p, q, r, s, t), ($x', y', p', q', r', s', t'$), etc.

20. I must also mention another case in which the surfaces fulfill a second-order partial differential equation, and for which a transformation (11) becomes a surface transformation. It relates to the theory of surfaces of constant curvature, as was recently developed by Bianchi and Lie (*). Bianchi's and Lie's theories owe their origins to the theorem that surface of curvature centers (viz., the central surface) of any surface whose principal radii of curvature at a point possess a constant difference that is independent of the position of the point will be a surface of constant curvature, or, more precisely, will consist of two surfaces of the same constant curvature. Now, the connection between a surface $z = f(x, y)$, $p = f'(x)$, $q = f'(y)$ and the most general surface $z' = \varphi(x', y')$, $p' = \varphi'(x)$, $q' = \varphi'(y)$ is expressed by the equations:

$$(20) \qquad \begin{cases} (x'-x)p + (y'-y)q - (z'-z) = 0, \\ (x'-x)p' + (y'-y)q' - (z'-z) = 0, \\ 1 + pp' + qq' = 0, \end{cases}$$

which, together with the given ones, can define a central surface (any surface). Those equations then say only that the points of the two surfaces $z = f(x, y)$, $z' = \varphi(x', y')$ can be so arranged that they will be contact points of a common tangent to the two surfaces, and that the tangent planes of the two surfaces at any point will be perpendicular to each other. Any two associated points will be the two centers of principal curvature of a point of a surface that has the two surfaces $z = f(x, y)$, $z' = \varphi(x', y')$ for its central surface.

Equations (20) represent a surface transformation (**). If we add the equation:

$$(21) \qquad (x - x')^2 + (y - y')^2 + (z - z')^2 = \frac{1}{\alpha^2}$$

to them then (20), (21) will express the condition for the surface pair $z = f(x, y)$, $z' = \varphi(x', y')$ to define the central surface of a surface for which the difference of the principal radii of curvature at a point will be constant and equal to $1 / \alpha$ [$\rho - \rho' = 1 / \alpha$]. The four equations (20), (21) will be of the form (11). In order to recognize for which surfaces in $r = (x, y, z)$ they will define a surface transformation, we must first present the form of the involution equation that was written down at the beginning of no. 18 that is valid for the present case. One has to set:

$$[F_2, F_2]_{z'x'p'} = 0, \qquad [F_1, F_2]_{z'x'p'} = 1 + p^2 + q^2, \qquad [F_1, F_4]_{z'x'p'} = 0,$$

(*) Lie: "Zur Theorie der Flächen constanter Krümmung," Archiv for Mathematik og Naturvidenskab, Bdd. **4**, **5** (Hefte 3). Christiania, 1879, 1880. I mostly know only of Bianchi's investigations from the work of Lie. The following transformation (20), (21) is cited by Lie as the analytical expression of Bianchi's transformation of the surfaces of constant curvature.

(**) The equations that express the connection between a surface and its central surface determine a surface transformation of the kind that I discussed in M. A. , Bd. XI, , pp. 199, § 3.

$$[F_1, F_3]_{z'x'p'} = 0, \qquad [F_2, F_4]_{z'x'p'} = -\frac{2}{a^2}, \quad [F_3, F_4]_{z'x'p'} = 0,$$

$$(2, 4) = 2\,(1 + p^2 + q^2)(1 + p'^{\,2} + q'^{\,2})\,\frac{y' - y}{p' - p},$$

$$(1, 3) = -\,(rt - s^2)\,(1 + p'^{\,2} + q'^{\,2})\,\frac{y' - y}{p' - p},$$

and the condition equation in question will then assume the following form:

$$rt - s^2 + a^2\,(1 + p^2 + q^2)^2 = 0. \tag{22}$$

It is independent of z', x', y', p', q'. All of its integral surfaces will therefore be converted into families of ∞^1 surfaces by the transformations (20), (21), and since the system of transformation equations is symmetric in the primed and unprimed symbols, the transformed surfaces must fulfill the same equation (22) in r'. However, they represent the most general surfaces of constant curvature $\frac{1}{R'R} = -a^2$. For that reason, the surfaces of this constant curvature go to each other under the transformation (20), (21), which was to be concluded from the theorem that was cited at the outset, exactly.

Therefore, it follows (no. 17) *that one has to determine the ∞^1 surfaces that correspond to a given surface of constant curvature – a^2 in such a way that each of them defines a central surface of a surface $\rho - \rho' = 1 / a$ with the given surface by integrating an ordinary differential equation in two variables (and one arbitrary parameter).* In the same way, one obtains new surfaces of constant curvature, etc., from each of these ∞^1 surfaces, *such that one must be able to derive infinitely many surfaces of constant curvature besides the original ∞^1 surfaces from a surface of constant curvature by repeated integration of differential equations in two variables.* (*)

A strip in r will be converted by any transformation (11) into ∞^1 strips in r' that are common integrals of the those three first-order partial differential equations that arise by eliminating x, y, z, p, q from (11) and the equations of the given strip. A strip of a surface of constant curvature – a^2 will thus be taken to ∞^1 strips by means of the transformation (20), (21) that lie on one of the ∞^1 surfaces that correspond to the given surface. As a result, any *characteristic* of (22) will be converted into ∞^1 other *characteristics* of the same equation. These others are obtained from the first one by mere elimination in the event that these ∞^1 surfaces are known ones that correspond to one of the former characteristics that lie on a surface of constant curvature.

Now, the characteristics of (22) are the principal tangent curves to the integral surfaces of the equation. Consequently: *From a surface of constant curvature whose principal tangent curves are known, one derives ∞ other surfaces of the same constant curvature with their principal tangent curves by the aforementioned operations.*

(*) According to Lie (see the papers cited above), one obtains ∞^∞ surfaces of constant curvature from a given surface in this way. If the geodetic curves of the given surface are known then, according to Bianchi and Lie, one will obtain the new surfaces by mere quadratures.

21. I would not like to leave a consequence of the foregoing untouched that concerns the determination of the geodetic curves of a surface of constant curvature. As we showed above, one can derive ∞^1 other surfaces of the same constant curvature from a surface of constant curvature. I shall choose one of them arbitrarily. Along with the first surface, it defines the central surface of a family of parallel surfaces. It will be determined by two involutory, first-order partial differential equations. However, an infinitesimal contact transformation is known for the integrals of these equations, namely, the parallel transformation. For that reason, any integral surface will be obtained by mere quadratures. Corresponding to the ∞^1 surfaces that will be derived from the original surface of constant curvature by the transformation (20), (21), one obtains ∞^2 surfaces ($\rho - \rho' = 1 / a$) in this way that possess the first surface as a shell for its central surface. They define a complete solution of the first-order partial differential equation that defines the most general surface for which any surface of constant curvature will define a part of the central surface. Now, however, the following theorem is true: If the normals to a doubly-infinite family of surfaces define a line complex, and if $f(x, y, z, \lambda) = 0$ represents ∞^1 of these surfaces, and they are not parallel surfaces, then the characteristics of the first-order partial differential equation that has the assumed doubly-infinite family of surface for its integrals that lie on these surfaces will be determined by the equations:

$$f(x, y, z, \lambda) = 0, \qquad [f'(x)]^2 + [f'(y)]^2 + [f'(z)]^2 = C[f'(x)]^2,$$

where C is an arbitrary constant (*).

(*) I have borrowed this theorem from my treatise on sphere complexes in the Jahresschrift der Universität Lund, v. IX. However, since it was only given provisionally without proof there, here I will write down the proof. Let $f(x, y, z, \lambda, \mu) = 0$ be the equation of a family of ∞^2 surfaces whose normals belong to a given line complex, and let λ, μ be chosen such that for constant λ any equation will represent a family of parallel surfaces. The parallel surfaces that are infinitely close to a surface $f(x, y, z, \lambda, \mu) = 0$ will be obtained by eliminating x, y, z from $f(x, y, z, \lambda, \mu) = 0$ and using the equations:

$$x' = x + \varepsilon \frac{f'(x)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}},$$

$$y' = y + \varepsilon \frac{f'(y)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}},$$

$$z' = z + \varepsilon \frac{f'(z)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}},$$

or

$$x = x' - \varepsilon \frac{f'(x)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}},$$

$$y = y' - \varepsilon \frac{f'(y)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}},$$

$$z = z' - \varepsilon \frac{f'(z)}{\sqrt{[f(x)]^2 + [f(y)]^2 + [f(z)]^2}}.$$

Moreover, we further know that the characteristics of the first-order partial differential equation whose integral surfaces have a given surface as part of their central surfaces, such that the normals to any surfaces all contact the given surfaces and will then define a special line complex, will be curvature curves on the integral surfaces. They will thus be obtained in the aforementioned way from the integral surfaces of the first-order partial differential equations by differentiation and elimination.

As a result, one derives from the initially-obtained family of ∞^1 surfaces $\rho - \rho' = 1 / a$, the one family of curvature curves on them, by purely algebraic operations. One further obtains the ∞^2 geodetic curves of the initially-chosen surface of constant curvature from it, and by likewise algebraic operations. *The geodetic curves of a surface of constant negative curvature can thus be found by integrating a differential equation* $F\left(x, y, \frac{dy}{dx}\right)$ *= an arb. constant (and subsequent quadratures).*

[I would like to thank Lie for a casual remark on the geodetic curves of the surfaces of constant curvature that I have been able to overlook due to a misconception that permeates this manuscript.]

One therefore expresses any parallel surface by the equation:

$$f - \varepsilon \sqrt{[f'(x)]^2 + [f'(y)]^2 + [f'(z)]^2} = 0,$$

if the prime on the symbols x, y, z is omitted. On the other hand, the equation of this parallel surface is of the form:

$$f + d\mu\, f'(\mu) = 0.$$

Therefore:

$$d\mu\, f'(\mu) = -\,\varepsilon \sqrt{[f'(x)]^2 + [f'(y)]^2 + [f'(z)]^2}\ .$$

The characteristics of the first-order partial differential equation that have our doubly-infinite family of surfaces for a complete system of integrals will be determined by the equation:

$$f'(\lambda) + \varepsilon f'(\mu) = 0.$$

C is an arbitrary constant, here. Now, if $f'(\mu)$ were replaced with its value above then this equation would assume the form:

$$\sqrt{[f'(x)]^2 + [f'(y)]^2 + [f'(z)]^2} = C f'(\lambda),$$

with which the theorem in question is proved.