"Über uneigentliche Somen," Math. Ann. **87** (1920), 152-156.

On improper somas

By

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1. In a paper "Über lineare Somenmannigfaltigkeiten" ([1]), the author was concerned with (*inter alia*) the concept of real, improper somas in Euclidian space. Whereas a proper soma can be associated with a coordinate system – i.e., an ordered triple of three spears through a point – an improper soma, when interpreted as an ordered triple of (oriented) *directions*, will be a concept that one already finds in Study ([2]).

Now, E. Kosiol has recently given an interesting interpretation of the real, improper somas in hyperbolic space ([3]). They can be associated with *oriented threads* in a completely gapless, invertible, single-valued way. (A thread will be called oriented when its principal axis is.)

One now asks whether traces of that connection between improper somas and threads have been preserved in Euclidian space. We will answer that question in the affirmative, and then expose the pathological element that robs that connection of any practical use. The fact that the argument that is presented nonetheless has considerable importance will be suggested at the conclusion.

2. A soma has eight homogeneous coordinates:

$$\mathfrak{X}_0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{123} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12}$$

that satisfy the quadratic relation:

$$\tfrac{1}{2}(\mathfrak{X}\mathfrak{X}) = \mathfrak{X}_0\,\mathfrak{X}_{123} + \mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12} = 0. \tag{1}$$

It will be called *proper* when the expression $(\mathfrak{X}/\mathfrak{X}) \neq 0$.

That expression is hyperbolic when:

([1]) Math. Annalen **81** (1920).

([2]) Sitzungsberichte der Berl. Math. Ges. **12** (1913), 50.

([3]) "Grundlagen der Kinematik in hyperbolischen Raum," Diss., Bonn, 1922.

(2*h*) $(\mathfrak{X}/\mathfrak{X}) = \mathfrak{X}_0^2 + \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 - \kappa^2\mathfrak{X}_{123}^2 - \kappa^2\mathfrak{X}_{23}^2 - \kappa^2\mathfrak{X}_{31}^2 - \kappa^2\mathfrak{X}_{12}^2$

or Euclidian:

(2*E*) $(\mathfrak{X}/\mathfrak{X}) = \mathfrak{X}_0^2 + \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2$,

when the absolute polar system is given by the system:

Point coordinates: $x_0\, y_0 - \kappa^2\, x_1\, y_1 - \kappa^2\, x_2\, y_2 - \kappa^2\, x_3\, y_3 = 0.$

Plane coordinates: $-\kappa^2\, \mathfrak{x}_0\, \mathfrak{y}_0 + \mathfrak{x}_1\, \mathfrak{y}_1 + \mathfrak{x}_2\, \mathfrak{y}_2 + \mathfrak{x}_3\, \mathfrak{y}_3 = 0.$

(*h*) $\kappa > 0.$ (*E*) $\kappa = 0.$

The thread $\bar{\mathfrak{X}}$:

$$\mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \,,$$

will arise from the soma $\mathfrak{X}$ (in general) by "abbreviation" ([4]) – i.e., by dropping the two coordinates $\mathfrak{X}_0$ and $\mathfrak{X}_{123}$ – and the overbar on $\mathfrak{X}$ shall suggest the abbreviation process. Corresponding statements are true for the two symbols $(\bar{\mathfrak{X}}\bar{\mathfrak{X}})$ and $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}})$, in which, for example, only three terms are abbreviated in the first one.

In particular, the *improper* soma $\mathfrak{X}$ will determine the thread $\bar{\mathfrak{X}}$:

always uniquely.	with the single exception of the soma $0 : 0 : 0 : 0 : 1 : 0 : 0 : 0.$

3. The thread $\bar{\mathfrak{X}}$ will be mapped to a point of R_5 in a known way. In it, one finds the singularity-free M_4^2 whose equations is $(\bar{\mathfrak{X}}\bar{\mathfrak{X}}) = 0$, as well as the $\mathfrak{M}_4^2$ whose equation is $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) = 0$. The latter is:

Singularity-free: There is an interior $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) > 0$ and an exterior $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) < 0$.	Endowed with a singular plane. The form $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}})$ is positive-definite

A point $\bar{\mathfrak{X}}$ in R_5 that does not lie on $\mathfrak{M}_4^2$ determines a polar R_4 with respect to it, and it once more determines a pole relative to M_4^2 that can be connected to $\bar{\mathfrak{X}}$ with a line. It meets M_4^2 in two points:

One is outside $\mathfrak{M}_4^2$, and the other one is inside it.	One lies on $\mathfrak{M}_4^2$, while the other does not.

([4]) Lineare Somenmannigfaltigkeiten, pp. 189.

The second of these points will be interpreted as the *principal axis* $\bar{\mathfrak{A}}$ of the thread $\bar{\mathfrak{X}}$ and the second, as the *auxiliary axis*. This derivation enjoys, at the same time, a very elegant advantage (which probably goes back to Study), namely, that it is also applicable to spaces of other dimensions. It was first found by W. Kniebes ([5]) (n = 8), and also found a surprising application recently by H. Lücking ([6]) for n = 4.

The principal axis $\bar{\mathfrak{A}}$ of the thread $\bar{\mathfrak{X}}$ then proves to be:

$$\mathfrak{A}_{0i} = \sigma\mathfrak{X}_{0i} - \kappa^2\tau\mathfrak{X}_{kl}, \qquad \mathfrak{A}_{kl} = \sigma\mathfrak{X}_{kl} + \tau\mathfrak{X}_{0i}, \qquad (i, k, l = 1, 2, 3; 2, 3, 1; 3, 1, 2) \tag{3}$$

in which σ and τ are determined uniquely, up to a proportionality factor, from the two conditions:

$$(\bar{\mathfrak{A}}\bar{\mathfrak{A}}) = (\sigma^2 - \kappa^2\tau^2)\,(\bar{\mathfrak{X}}\bar{\mathfrak{X}}) + 2\,\sigma\tau\,(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) = 0, \tag{4}$$

$$(\bar{\mathfrak{A}}/\bar{\mathfrak{A}}) = (\sigma^2 - \kappa^2\tau^2)\,(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) - 2\,\kappa^2\sigma\tau\,(\bar{\mathfrak{X}}\bar{\mathfrak{X}}) > 0, \tag{5}$$

which is even rational in Euclidian space. [viz., $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) \neq 0$].

4. For the principal axis of the thread $\bar{\mathfrak{X}}$ that belongs to the improper soma $\mathfrak{X}$, one has, in *hyperbolic space:*

$$\mathfrak{A}_{0i} = \kappa^2\,\mathfrak{X}_{123}\,\mathfrak{X}_{0i} - \kappa^2\,\mathfrak{X}_0\,\mathfrak{X}_{kl}, \qquad \mathfrak{A}_{kl} = \kappa^2\,\mathfrak{X}_{123}\,\mathfrak{X}_{kl} + \mathfrak{X}_0\,\mathfrak{X}_{0i}, \tag{6h}$$

so $\sigma = \kappa^2\,\mathfrak{X}_{123}$, $\tau = \mathfrak{X}_0$. However, since σ and τ are determined from (4) and (5), up to a proportionality factor, we will have a first linear equation for $\mathfrak{X}_0$ and $\mathfrak{X}_{123}$:

$$\sigma\mathfrak{X}_0 - \kappa^2\tau\mathfrak{X}_{123} = 0. \tag{7}$$

One has, moreover:

$$(\bar{\mathfrak{A}}/\bar{\mathfrak{A}}) = \kappa^2(\mathfrak{X}_0^2 + \kappa^2\mathfrak{X}_{123}^2)^2 = \kappa^2(\mathfrak{X}_{123}\sigma + \mathfrak{X}_{01}\tau)^2 .$$

By arbitrarily setting:

$$\sqrt{\bar{\mathfrak{A}}/\bar{\mathfrak{A}}} = \kappa(\tau\mathfrak{X}_0 + \sigma\mathfrak{X}_{123}), \tag{8}$$

one will have a second linear equation for $\mathfrak{X}_0$ and $\mathfrak{X}_{123}$. Now, since the principal axis for the thread $\bar{\mathfrak{X}}$ is oriented, one can also say:

([5]) "Über das Hypersoma," Diss., Bonn, 1921.

([6]) "Ein zweidimensionales Analogon zur Plückerschen Liniengeometrie," Diss., Bonn., 1922.

An improper soma $\mathfrak{X}$ is determined uniquely by the oriented thread $\bar{\mathfrak{X}}$. In the case where the thread $\bar{\mathfrak{X}}$ itself is improper – viz., $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) = 0$, $(\bar{\mathfrak{X}}\,\bar{\mathfrak{X}}) = 0$ – equations (4) and (5) will break down. Nevertheless, one will have $\mathfrak{X}_0 = \mathfrak{X}_{123} = 0$, uniquely. The thread is oriented oppositely to itself and consists of the line of intersection of a tangent to the absolute surface.

The improper soma has a "midpoint," namely, one of the endpoints of the principal axis of the associated thread.

5. The essence of this derivation is that in hyperbolic geometry, one can succeed in solving the equations:

$$\text{(9}h\text{)} \qquad \mathfrak{X}_0^2 - \kappa^2 \mathfrak{X}_{123}^2 = -\,(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}), \qquad 2\,\mathfrak{X}_0\,\mathfrak{X}_{123} = -\,(\bar{\mathfrak{X}}\,\bar{\mathfrak{X}})$$

for $\mathfrak{X}_0$ and $\mathfrak{X}_{123}$ uniquely. In the Euclidian case, they reduce to ([7]):

$$\text{(9}E\text{)} \qquad \mathfrak{X}_0^2 = -\,\mathfrak{X}_{01}^2 - \mathfrak{X}_{02}^2 - \mathfrak{X}_{03}^2, \qquad \mathfrak{X}_0\,\mathfrak{X}_{123} = -\,\mathfrak{X}_{01}\,\mathfrak{X}_{23} - \mathfrak{X}_{02}\,\mathfrak{X}_{31} - \mathfrak{X}_{03}\,\mathfrak{X}_{12}\,.$$

If $(\bar{\mathfrak{X}}/\bar{\mathfrak{X}}) = \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 \neq 0$ then everything is in order. After disposing of $\sqrt{-(\bar{\mathfrak{X}}/\bar{\mathfrak{X}})}$, one gets $\mathfrak{X}_0$ uniquely, and it will follow that one gets $\mathfrak{X}_{123}$ uniquely, as well. However, the principal axis (6h) will not become the Euclidian principal axis of the thread in the limit, *but its absolute polar*, and thus, the auxiliary axis. It will also contain the midpoint of the improper thread (as one of its points of intersection with the absolute conic). The auxiliary axis will then be oriented by it, and *therefore the Euclidian principal axis, as well.* That arises by passing to the limit with the hyperbolic auxiliary axis:

$$\text{(10}h\text{)} \qquad \mathfrak{A}_{0i} = \mathfrak{X}_0\;\mathfrak{X}_{0i} + \kappa^2\,\mathfrak{X}_{123}\;\mathfrak{X}_{kl}\,, \quad \mathfrak{A}_{kl} = \mathfrak{X}_0\mathfrak{X}_{kl} + \mathfrak{X}_{123}\;\mathfrak{X}_{0i}\,.$$

For this, one has:

$$(\bar{\mathfrak{A}}/\bar{\mathfrak{A}}) = -(\mathfrak{X}_0^2 + \kappa^2 \mathfrak{X}_{123}^2)^2,$$

such that in the limit, after cancelling a common factor of $\mathfrak{X}_0$, one will get:

$$\sqrt{\bar{\mathfrak{A}}/\bar{\mathfrak{A}}} \to \sqrt{\bar{\mathfrak{X}}/\bar{\mathfrak{X}}}\,.$$

6. Two problems appear now. The improper soma $\mathfrak{X}$ cannot be real in Euclidian space when $\mathfrak{X}_0 \neq 0$. The connection by which an improper soma can also be associated with an oriented thread in Euclidian space still exists, *but it appears only in the imaginary domain.*

([7]) Except for the aforementioned exceptional case.

However, if $\mathfrak{X}_0 = 0$ then $\mathfrak{X}_{123}$ can no longer be calculated from the second equation (9*E*), and also not for real somas; both equations (9*E*) will now be fulfilled identically ($\mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$). In that case, one can associate one and the same (real, improper) thread $0 : 0 : 0 : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12}$ with ∞^1 improper, real somas $0 : 0 : 0 : 0 : \mathfrak{X}_{123} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12}$, and conversely, except for the exceptional case that has been mentioned several times already.

7. The issue is still not resolved with that; one can associate the ∞^3 improper somas in Euclidian space with the minimal lines of Euclidian space (viz., $\mathfrak{X}_0 = \mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$).

In fact, that is entirely possible when one represents the "minimal line" by ten homogeneous coordinates; it would then be better to call it a "minimal circle." That is connected with the fact that on a singularity-free M_3^2, the polar planes to the generators (and thus, the generators themselves) define a quaternary domain. (More on that soon…)

Thus, one's attention is directed to the oriented thread as a spatial element. It can be associated with well-known figures in an extremely simple way, which also do not pose the slightest difficulty for one's intuition, namely, the oriented line-elements (surface-elements, resp.), which are likewise images of improper somas. This situation, which is also quite fruitful in Lie's sphere geometry, shall be treated next.

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