"Sur les conditions d'application des équations de Lagrange à un système non holonome," Bull. Soc. Math. France **57** (1929), 118-124.

On the conditions for the application of the Lagrange equations to a nonholonomic system.

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Translated by D. H. Delphenich

J. Hadamard ([1]) has studied the case in which the Lagrange equations apply to the *minimum* number of parameters for a material system that is subject to nonholonomic constraints in a general manner.

That will be true in the particular case where the equations of constraint in terms of total differentials form a completely integrable system. Indeed establishing the Lagrange equations without the introduction of multipliers supposes two things:

1. Before any dynamical considerations, the coordinates of any mass element must be representable by expressions in finite terms of the form $f(q_1, q_2, \ldots, q_h, t)$, in which $q_1, q_2, \ldots, q_h$ denote a finite number of parameters, and t denotes time.

2. For any displacement, $dq_1, dq_2, \ldots, dq_h$ ($dt = 0$), the elementary work done by forces of constraint is zero.

In the particular case considered, the system is supposed to depend upon k parameters $q_1, q_2, \ldots, q_k$, and t that are coupled by $k - h$ nonholonomic equations. Those $k - h$ equations are supposed to be integrable, so that will permit one calculate $k - h$ as functions of the h other ones $q_1, q_2, \ldots, q_h$, and t in such a way that the first stated condition is fulfilled: The coordinates of any element that is endowed with mass can be expressed as functions of the $q_1, q_2, \ldots, q_h, t$.

On the other hand, since the displacement $dq_1, dq_2, \ldots, dq_h$ ($dt = 0$) is compatible with the constraints that exist at the instant t, the work done by forces of constraint will be zero under that displacement if the constraints are realized without passive resistances (e.g., reactions that are normal at all contact points that involve slipping, oblique reactions without couples of resistance to rolling or to the pivoting at all contacts where the slipping is supposed to be zero).

Suppose that we are given a system of solid bodies in contact with each other and with fixed foreign obstacles or with a motion that is known in advance as a function of time, for which the nonholonomic equations are expressed uniquely as conditions of no slipping. I propose to

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([1]) *Mémoire de la Société des Sciences physiques et naturelles de Bordeaux*, 1895.

investigate whether one can recognize the case in which those nonholonomic equations form a completely integrable system with a simple geometric pictures.

Thus, for example, for two invariable profiles S and S_1 that move in the same plane and are subject to rolling without slipping on each other, when the contact condition is written out, no-slip constraint can be integrated in the form:

$$s - s_1 = C ,$$

in which s and s_1 denote the abscissas of the contact point on the two profiles, and C is a constant. That condition does not constitute an obstacle to the application of the Lagrange equations.

More generally, the no-slip conditions are completely integrable, and as a result, they will permit one to apply the Lagrange equations without multipliers whenever one knows in advance, *before any dynamical consideration*, the trajectories of the points of contact on the bodies that roll without slipping.

For example, I suppose that I am dealing with two solid bodies S and S_1 that are bounded by surfaces that are tangent to each other. Their relative positions can be determined by u, v and u_1, v_1, which are the curvilinear coordinates of the point of contact I on those two surfaces, and by θ, which is the angle at I between the lines u = const. and u_1 = const.

Knowing the trajectories of I on the two surfaces entails knowing relations of the form:

$$u = \varphi(v) ,$$
$$u_1 = \varphi_1(v_1) ,$$
$$\theta = \psi(u, u_1) ,$$
$$s(u) - s_1(u_1) = C .$$

The five parameters u, v, u_1, v_1, and θ will then reduce to just one, and it will be expressed in finite terms as a function in such a way that the two conditions for applying the Lagrange equations will be fulfilled.

Thus, for example, the Lagrange equations can be applied without multipliers to a solid body that rolls without slipping on a fixed solid body that it touches at two points, because in that case, the trajectories of the two contact points can be determined by *purely-kinematical* considerations.

If one of the solid bodies S is bounded by a curvilinear edge that is tangent to the surface of the solid S_1 then their relative positions can be determined by u, which is a parameter that fixes the position of the contact point I on that edge, while u_1 and v_1 are the curvilinear coordinates of the point I on the surface S_1, θ is the angle between the tangent to the edge and the line u_1 = const., and α_1 is the angle between the osculating plane to the edge and the tangent plane to the surface S_1 .

Knowing the trajectory of the point I on S_1 implies knowing relations of the form:

$$u_1 = \varphi_1(v_1) , \qquad \theta = \psi(u, u_1) , \qquad s(u) - s_1(u_1) = C ,$$

so the five parameters will reduce to two – for example, u and α – that are coupled to each other by equations in finite terms, and the Lagrange equations will apply to the minimum number of parameters.

Here are even more cases in which the equations that express the no-slip condition are completely integrable:

I imagine two solid bodies S and S_1 whose surfaces can be mapped to each other: I shall suppose that they are subject to touch at a point that occupies homologous positions on those surfaces whose homologous directions agree. Those conditions, which are supposed to be realized at each instant, imply that the surfaces S and S_1 do not slip on each other.

Now, those conditions can be expressed in holonomic form:

$$u = u_1 , \qquad v = v_1 , \qquad \theta = 0 .$$

Two of those conditions can obviously be replaced with the two no-slip conditions in such a way that the latter will obviously be integrable, and the Lagrange equations will apply to the minimum number of parameters.

Thus, they apply to two identical solids of revolution whose axes intersect and that initially touch each other at a point that occupies homologous positions on the two bodies, and which roll without slipping one each other at that point, in addition. Those two bodies will remain constantly symmetric with respect to their common tangent plane.

I would like to establish that those are the only cases in which the equations that express the no-slip condition between solid bodies are completely integrable:

Therefore, let there be a system of invariable solid bodies that are in contact with each other and with foreign obstacles that are either fixed or have a motion that is known in advance as a function of t. The slipping is supposed to be zero for some of those contacts. I would like to establish that the equations that express those no-slip conditions are completely integrable only if the contacts fall within one or the other of the following two categories:

a. Contacts such that the trajectories of the contact point on the two bodies that touch are known in advance, before any dynamical consideration.

b. Contacts between solid bodies that are bounded by mappable surfaces such that the contact point will occupy homologous positions on the two surfaces with agreement between the homologous directions.

I shall then suppose that a system depends upon some parameters $q_1, \ldots, q_k, t$ that are coupled by r equations of non-slippage:

$$a_1^1 q_1' + \cdots + a_1^k q_k' + a_1 = 0 ,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_r^1 q_1' + \cdots + a_r^k q_k' + a_r = 0 .$$

That system is supposed to be completely integrable, i.e., equivalent to r holonomic equations that are obtained by integration, and because of that, they will contain arbitrary constants that I shall not specify. Those equations can be solved for the r parameters $q_{h+1}, \ldots, q_k$ (one sets $k = h +$

r) in such a way that the coordinates x, y, z of the contact point I of the two bodies S and S', when referred to the body S, and the coordinates x', y', z' with respect to axes that are coupled with S' are functions of q_1, …, q_h, and t, one will have:

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \cdots + \frac{\partial x}{\partial q_h} dq_h + \frac{\partial x}{\partial t} dt \;,$$
$$dy = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$dz = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$dx' = \frac{\partial x'}{\partial q_1} dq_1 + \cdots + \frac{\partial x'}{\partial q_h} dq_h + \frac{\partial x'}{\partial t} dt$$
$$dy' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$dz' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The displacements dq_1, …, dq_h imply that none of the contact have slippage, so the two displacements dx, dy, dz and dx', dy', dz' agree in magnitude and direction. In particular, the same thing will be true for the $h + 1$ displacements that are obtained by varying q_1, …, q_h, t *separately*, which are displacements that are pair-wise identical in the common tangent plane to the two bodies S and S' considered.

Two cases must be distinguished:

1. *The* $h + 1$ *elementary displacements of the contact point I between the bodies S and* S' *have the same support.* In this case, the partial derivatives of x, y, z with respect to q_1, …, q_h, t will be proportional, or what amounts to the same thing, the functional determinants:

$$\frac{D(y,z)}{D(q_i,q_j)}, \qquad \ldots, \qquad \frac{D(y,z)}{D(q_i,t)}$$

are zero. One knows that in this case, x and y are coupled by one independent relation in q_1, …, q_h, t, and similarly y and z. In other words, the locus of I on the surface S is determined in the absence of any dynamical consideration: The same thing is true for the locus of I on the surface S'.

2. *The* $h + 1$ *elementary displacements of the contact point I between the bodies S and* S' *do not have the same support.* I suppose, for example, that the displacements that correspond to dq_1 and dq_2 have different supports. If I would like to vary q_1 and q_2 while leaving q_1, …, q_h, t fixed then I can choose dq_1 / dq_2 in such a manner as to give any direction that I would like to the displacement of I in the common tangent plane to S and S', and as a result, choose the trajectory of the point I on one of the two bodies S or S' arbitrarily. Those trajectories are not known in advance. In addition, one sees that the fact that the positions of the point I on the two parameters q_1 and q_2 will establish a point-wise correspondence between those two surfaces.

However, from the fundamental property of no slipping, that correspondence will preserve lengths in such a way that the surfaces S and S' will be mappable, so they will touch at homologous points and the homologous curves (i.e., possible trajectories of the point I) will agree at that point.

It is thus established that those are the only cases in which nonholonomic equations that express no-slip contact will be integrable.

For example, the integrability condition is not fulfilled by the rolling of a circle on a table when the trajectory of the contact point on the table is unknown. It is fulfilled by the rolling of a sphere on a plane and on a fixed cylinder whose generators are perpendicular to the plane. However, it will not be fulfilled when the cylinder, which is supposed to be one of revolution, turns around its axis with motion that is a given function of t. Indeed, in that case, the trajectories of the contact points are known on the plane and the cylinder, but unknown on the sphere, and one cannot apply the Lagrange equations without multipliers.

It is interesting to point out that the condition for the two mappable surfaces to constantly touch at homologous points with agreement of the homologous lines is that the condition must be fulfilled at the initial instant and that as a result, the pivoting will be zero.

Indeed, let C and C' be those trajectories of the point of contact I on the surface S and S', and let Σ and Σ' be the developables that are circumscribed by S and S' along C and C'. They are tangent at I. Let P be a plane that rolls without slipping on Σ in such a manner that it constantly coincides with the tangent plane at I. Likewise, let P' be a plane that rolls on Σ'.

Since the slipping at I is zero, the motion P' / P will be a rotation around the normal at I.

Having said that, the motion $\dfrac{S'}{S}$ will decompose into $\dfrac{S'}{P'}$; $\dfrac{P'}{P}$; $\dfrac{P}{S}$. The first and third of those motions are rolling motions around the generators of Σ and Σ' that pass through I. The second one is a rotation around the normal. It will then result that the rotation is nothing but the pivoting ω of the motion $\dfrac{S'}{S}$.

Now, I describes a curve γ in P, and a curve γ' in P'. γ and C are homologous under the rolling motion P / Σ, so they will have the same center of geodesic curvature K, since the deformation of the surfaces will preserve the geodesic curvature. Similarly, γ' and C' have the same center of geodesic curvature K'.

The Euler-Savary formula, when applied to the motion P' / P, is written:

$$\frac{1}{\overline{\overline{I K}}} - \frac{1}{\overline{\overline{I K'}}} = -\frac{\omega}{V} \, ,$$

in which V denotes the velocity of the point I. It will exhibit the relationship between the difference of the geodesic curvatures of the curves C and C' and the pivoting. In order for the curves C and C' to be homologous under the transformation that maps the surfaces S and S' to each other, it is necessary and sufficient that they must have the same geodesic curvature. Therefore, the pivoting ω will be zero.