Excerpted from Henri Beghin, *Étude théorique des compas gyrostatiques Anschutz et Sperry*, Théses de l'entre-deux-guerres, 1922. Translated by D. H. Delphenich.

FIRST CHAPTER

EXTENDING THE LAGRANGE AND APPELL EQUATIONS TO SYSTEMS THAT INCLUDE A SERVO

___

3. The general theorems of mechanics (in particular, the kinetic moment theorem) are convenient for an elementary study of the compass, but their application to a more precise study presents some difficulties of a practical nature that one can avoid by the use of the Lagrange equation or the Appell equations.

I would like to recall the conditions under which those equations apply. I will show that those conditions are not realized in systems like the Sperry compass that include a servo (†), but I shall indicate the modifications that must be made in order to permit that application.

4. Constraints. – If the parameters $q_1, q_2, \ldots, q_n$ that the configuration and position of a system depend upon are coupled by a certain number of finite equations of the form:

$$\varphi(q_1, q_2, \ldots, q_n, t) = 0$$

then one says that the system is *holonomic*. That is the case, for example, for a system of invariable solid bodies that are in contact with each other and some fixed or moving foreign obstacles that are known in advance as functions of t.

If certain constraints are expressed by non-integrable equations between the parameters and their derivatives with respect to time then the system is called *non-holonomic*. That is true, for example, for solid bodies that are subject to rolling over each other without slipping. The equations that translate those conditions are linear with respect to the first derivatives of the parameters: They are *first-order linear constraints* (1).

In regard to that, I would like to recall that the constraints are considered from a purely-*analytical* viewpoint that is independent of the particular manner by which they are realized.

However, *can one abstract from the manner by which a constraint is realized?* In other words, is the motion of a system determined entirely by the constraint equations and the initial values of the parameters $q_1, q_2, \ldots, q_n$ and their first derivatives?

(†) Translator: The French word *asservissement* means "servitude" or "slavery," but the modern terminology for these constraints is "servo-constraints," so I have chosen to translate the word as "servo."

(1) On this topic, see P. APPELL [**1-5**]; J. HADAMARD [**1**]; E. DELASSUS [**1**, **2**]; P. APPELL gave numerous bibliographic references in [**2**].

That question has been the object of numerous studies ([1]). I will summarize some of the results that relate to holonomic systems and non-holonomic systems with first-order linear constraints.

A constraint L on a system Σ can be realized with or without recourse to an auxiliary system Σ_1 . In the first case, the realization of the constraint is called *perfect* when the introduction of that system imposes no restriction on the infinitesimal displacements of the system Σ that are, in turn, all of the displacements that are compatible with the constraint L. It is *imperfect* if there is no restriction on the infinitely-small displacements.

Hence, one has, for example, the following situation that was cited by DELASSUS and is realized imperfectly by the constraint $z = a$ that is imposed upon a material point whose coordinates are x, y, z : The fork of a unicycle of radius a is kept vertical by means of a tripod that rests on the plane P ($z = 0$). The unicycle touches the plane P, and the friction is assumed to be sufficient to make any slipping impossible. The material point is attached to the center of the unicycle.

The system Σ is composed of that point. The unicycle, fork, and tripod constitute the auxiliary system Σ_1 .

That arrangement obviously permits the material point to occupy all of the positions in the plane Q ($z = a$), and is realized, in turn, by the given constraint. However, it is realized *imperfectly* because under any infinitely-small displacement of the system $\Sigma\Sigma_1$, the displacement of the material point will obviously be in the plane of the wheel, so it cannot have an arbitrary direction in the plane Q : viz., there is a restriction.

On the contrary, if one attaches the point to the center of a sphere of radius a that rolls without slipping on the plane P then one will get a *perfect* realization of that constraint.

One easily verifies that an imperfect realization will necessarily provide non-holonomic constraints.

4. Applying d'Alembert's principle. – If the constraints L that are imposed upon a system Σ are realized by means of an auxiliary system Σ_1 then I shall apply d'Alembert's principle to the system $\Sigma\Sigma_1$: The virtual will done by inertial forces, given forces, and constraint forces is zero for any displacement, and in particular, any displacement that is compatible with the constraints on the system $\Sigma\Sigma_1$ that might exist at the instant t.

If the constraint forces – i.e., the forces whose purpose is to insure that various constraints – *do zero work for each other those displacements*, and if on the other hand, the inertial forces in the system Σ_1 are negligible (the mass of Σ_1 is negligible), as well as the given forces that are applied to Σ_1 , then the inertial forces on the system Σ and the given forces that are applied to it are the only ones that enter into d'Alembert's equations.

If the system Σ_1 realizes the constraints *perfectly* then the displacements of the system Σ that are compatible with the constraints on the system $\Sigma\Sigma_1$ will be the same as the displacements that are compatible with the constraints L, in such a way that the d'Alembert equation is the same and applies to the same displacements as if the constraints L were realized without the help of Σ_1 . *The motion of the system Σ is determined by the equations that express the constraints L and the initial values of the parameters and their first derivatives (initial positions and velocity). It is independent of* *the manner by which the constraints L are realized, and in particular, the auxiliary system* Σ_1 .

On the contrary, if Σ_1 is realized *imperfectly* then the constraints *L* and the application of d'Alembert's principle to the $\Sigma\Sigma_1$ will give only one part of the equations that are capable of determining the motion, and which *will*, *in turn, depend upon the way that the constraints are realized.*

However, those results suppose in an essential way that the work done by the constraint forces is zero under all displacements that are compatible with the constraints that might exist at the instant *t*.

Now, despite the very general character that is most often left to the nature of the constraint force, it does not seem that the authors have considered (to confine myself to first-order constraints) any constraints that expressed anything beyond the contact condition, or rolling without slipping or pivoting or any forces beyond the corresponding contact forces. The foreign obstacles are assumed to be fixed or moving as a function of *t* that is known in advance.

Under those conditions, the fundamental hypothesis that was stated above on the zero value of the work done by constraint forces is equivalent to the following one on the nature of the body considered:

The resistance to rolling is neglected for all types of contact. For the ones where there is pivoting, the resistance to pivoting is also neglected. For the ones where there is sliding, the reaction is, in addition, supposed to be normal. In other words, *any source of the dissipation of energy is neglected.*

Those propositions apply to systems of invariable solid bodies and extend to incompressible liquids and perfectly flexible and inextensible strings and membranes on the condition that there is no viscosity or stiffness, and more generally, to any system that is not capable of contraction or dilatation and exhibits no phenomena that involve the dissipation of energy.

Leaving aside the systems that are capable of contraction or dilatation, I return to the question that was posed above (§ **4**):

Can one abstract from the manner by which a constraint is realized?

From what was just recalled, it seems that the answer must be in the affirmative whenever one is dealing with systems that involve no dissipation of energy and perfect realizations. In particular, it seems that this will be the case for holonomic constraints without friction.

I would like to show that this is not true: On the contrary, there exists an important category that realizes the constraints by a method that is completely different from the ones that were just examined. For those mechanisms, the answer to the preceding question is in the negative: *One cannot abstract from the way that the constraints are realized.*

6. Mechanisms that include a servo. – The constraints that are realized by these mechanisms can be arbitrary; most often, they are holonomic. However, instead of those realizations being – so to speak – passive ones that are obtained by simple contact, they are ones that use arbitrary forces (e.g., electromagnetic forces, pressure from compressed air, etc.); in a word: *auxiliary energy sources that come into play automatically and are* *automatically designed to realize this or that constraint at each instant.* One can even imagine a living being that acts by contact and regulates its actions in such a way as to realize this or that constraint.

Let a solid body Σ (a disc, for example) move around a diameter Δ under the influence of certain given forces. A solid body Σ_1 (a concentric ring of diameter Δ, for example) moves around Δ without having any contact with Σ. The ring Σ_1 carries a toothed wheel a whose axis is Δ and which meshes with a pinion b that is press-fitted (*calé sur*) onto the shaft of a motor M. It is easy to imagine an arrangement ([2]) that does not act directly on either Σ or Σ_1, but brings the motor M into play, in one sense or the other, whenever Σ and Σ_1 are not in the same plane. If α and α_1 are the azimuths of Σ and Σ_1 then the constraint:

$$\alpha = \alpha_1$$

will be found to be realized in such a way that *the ring Σ_1 follows the disc Σ in all of its motions around Δ without being driven by it*. It is obvious that the manner in which that system behaves has nothing in common with the manner in which it would behave if Σ drove Σ_1 by direct contact. For example, if a small spring is fixed to Σ_1 and acts upon Σ then the system will take on a uniformly-accelerated motion in the case of servitude, while it will obviously remain immobile under the second hypothesis.

What are the forces of constraint on the system in the preceding example?

If I consider the system $\Sigma\Sigma_1$ then those forces will be, on the one hand, the reactions along the axis Δ, which will be the ordinary constraint forces, and the reactions of the pinion b on the wheel a. Those reactions, which play a major role in the problem, have an entirely special character, because the pinion b (viz., the foreign obstacle) that exerts them is not fixed, nor is its motion known in advance as a function of t : It is *an obstacle whose position is known in advance as a function of the parameters (which are α, α_1 here) upon which the system considered $\Sigma\Sigma_1$ depends.*

If I surround the rotor R of the motor in the system considered then the constraint forces will be, in addition to the actions due to contact between the fixed obstacles and the ones due to the contact $R\Sigma_1$, which are ordinary constraint forces, the electromagnetic actions to which the rotor is subject on the part of the stator. Indeed, those forces have the character of constraint forces: *They are unknown, but one knows that they have the value that is necessary to insure the constraint considered.*

Under any elementary displacement that is compatible with the constraint $\alpha = \alpha_1$, the ordinary constraint forces will do zero work. On the contrary, the other constraint forces (which amount to the reactions of the foreign obstacles whose position depends upon the parameters α, α_1, or those electromagnetic actions that are exerted on the rotor from a distance) will do non-zero work, and that is why the mechanisms that include a servo are distinguished from the other ones.

([2]) See the description of the Sperry compass in *The Sperry gyrocompass and navigation equipment,* publ. by Sperry Gyro. Co., NY, 1913.

7. General study of mechanisms that include a servo. D'Alembert's principle. – Let Σ be a material system that involves no cause for the dissipation of energy. I suppose, in addition, that no part of that system is capable of contraction or dilatation, with the exception of what will be assumed below.

Upon taking into account the contacts that are imposed upon it, that system will be supposed to depend upon a limited number h of parameters $q_1, q_2, \ldots, q_h$ in such a manner that the coordinates x, y, z of each element of Σ will functions of those parameters that are known in advance, and also possibly functions of time:

$$(1) \qquad x = f(q_1, q_2, \ldots, q_h, t), \qquad y = \ldots, \qquad z = \ldots$$

Some of the foreign obstacles that Σ is in contact with are fixed or depend upon t. Others, as a result of the contacts imposed, are supposed to depend upon a certain number k of the previous parameters, namely, $q_1, q_2, \ldots, q_k$, and also possibly t.

Those contact conditions are *holonomic contact constraints.*

I suppose, in addition, that the system is subject to certain non-holonomic relations – i.e., that the parameters $q_1, q_2, \ldots, q_h$ are coupled by a certain number p of differential relations that express the conditions of rolling without slipping or pivoting for some of the contacts. Those relations will permit one to express the p elementary variations:

$$dq_{n+1}, dq_{n+2}, \ldots, dq_{n+p} \qquad (n + p = h)$$

as functions of $dq_1, dq_2, \ldots, dq_n$, and dt; they have the form:

$$(2) \qquad (p \text{ relations}) \quad \begin{cases} A_1\, dq_1 + \cdots + A_h\, dq_h + A\, dt = 0, \\ B_1\, dq_1 + \cdots + B_h\, dq_h + B\, dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Those conditions are *non-holonomic contact constraints.* These are the only two types of constraints that one encounters in the modern problems.

Under any elementary displacement that is compatible with the constraints that might exist at the instant t (i.e., ones for which dt is zero and $\delta q_1, \delta q_2, \ldots, \delta q_n$ are arbitrary), the mutual reactions between the bodies of the system do zero work, as well as the reactions of the fixed obstacles or the ones that depend upon t. I will say that those reactions are *constraint forces of the first kind.*

In addition, the system Σ is assumed to be subject to some other constraints that I shall call *servo-constraints*, which are also expressed by finite equations or linear differential equations, but are realized by means of forces that are entirely different. Those forces, which I shall call *generalized constraint forces*, or ones *of the second kind*, are applied to the bodies of the system. They can be *external* or *internal*.

In the first case, they will be either actions at a distance, such as electromagnetic actions or others, which are governed *automatically* in such a manner as to assure the finite of differential constraint that they are charged with realizing, or contact actions from foreign obstacles whose position is supposed to depend upon $q_1, q_2, \ldots, q_h$, t, and whose motion must be governed *automatically* in such a manner that certain finite or differential equations are verified by the parameters p at each instant ([3]).

If the constraint forces of the second kind are internal then they will be either actions at a distance, such as electromagnetic ones, or internal stresses in the bodies that are capable of contraction or dilatation (e.g., compressed air, muscles in a living being), which are stresses that are governed *automatically* (for example, by the will of the living being) in such a manner as to realize this or that constraint ([4]).

The system Σ can be composed of an electric motor whose velocity ω is independent of the load, which might be, for example, a bypass motor (*moteur-dérivation*), within certain limits. The servo-constraint thus-realized will have the form:

$$d\theta = \omega dt.$$

The system can consist of a cyclist and his machine. The cyclist can contract his muscles, not by a given quantity, but by a quantity that is sufficient for certain constraints to be realized: It governs the action of his legs in such a manner that it realizes a constant angular velocity, or rather, he contracts the muscles of his body in such a way that it will realize an inclination of the frame that is a function of t, etc. The methods that will be described below will permit one to study the variation of the unknown parameters.

One can also imagine, as an application, a ship Σ such that one part σ of its cargo is put into motion automatically by a motor in such a manner as to realize certain constraints: As an example of a servo-constraint, one can have that the ship must remain constantly vertical, which is realized by a roll stabilizer. A small gyrostatic apparatus that is based upon the principle of the Schlick stabilizer might indicate the true vertical on board. The servomotor will come into action when that vertical is not in the symmetry plane of the ship. One can also govern the motion of σ in such a manner as to realize that relation between its position and the inclination of the ship. One can then change the period of oscillation of the ship at will and avoid the synchronism of the hull, where appropriate. One can govern the motion of σ in such a manner as to realize that relation between its position and the angular velocity of the ship, which will permit one to absorb the oscillations, etc. The constraint forces of the second kind will be the mutual actions of Σ and σ here.

A material system that presents constraint forces of the second kind will be said to *include a servo*. It is obvious that the *virtual work done by constraint forces of the second kind will be generally non-zero*.

Having posed those definitions, I will suppose that there are r servo-constraints, some of which are finite, while others are differentials, so they will have the form:

$$\text{(3)} \qquad (r \text{ relations}) \quad \begin{cases} g(q_1, \ldots, q_h, t) = 0, \ldots \\ \varepsilon_1\, dq_1 + \varepsilon_2\, dq_2 + \cdots + \varepsilon_h\, dq_h + \varepsilon\, dt = 0, \ldots \end{cases}$$

([3]) It is interesting to remark that these contact actions have a mixed character, since they are, on the one hand, associated with contact constraints, and on the other hand, with servo-constraints.

([4]) Except for that exception, as was assumed at the beginning of this paragraph, the system will not be assumed to be compressible.

The virtual displacements of the system that are compatible with the *contact* constraints that might exist at the instant t ($\delta t = 0$) are obtained by taking $h - p$ of the elementary variations $\delta q_1, \ldots, \delta q_h$ arbitrarily. The other p are defined by the relations (2), which reduce to:

$$(2') \qquad (p \text{ relations}) \qquad \begin{cases} A_1\,\delta q_1 + \cdots + A_h\,d\delta_h + A\,\delta t = 0, \\ B_1\,\delta q_1 + \cdots + B_h\,\delta q_h + B\,\delta t = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

here.

Among those displacements, there will exist ones for which one can confirm *a priori* that the force done by constraint forces of the second kind is zero without knowing anything but their mode of action ([5]). I suppose that they are the ones that simultaneously verify the j relations:

$$(4) \qquad (j \text{ relations}) \qquad \begin{cases} a_1\,\delta q_1 + \cdots + a_h\,d\delta_h = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1\,\delta q_1 + \cdots + l_h\,\delta q_h = 0. \end{cases}$$

D'Alembert's principle, when applied to any of those displacements, can be expressed by the equation:

$$(5) \qquad \sum m\,(x''\delta x + y''\delta y + z''\delta z) = \sum (X\,\delta x + Y\delta y + Z\delta z)\,,$$

The Σ sign on the left-hand side extends over all elements of the system, m denotes the mass of one of those elements, and x'', y'', z'' are the projections of its acceleration, while the Σ sign on the right-hand side extends over all of the given forces X, Y, Z . Indeed, it is obvious that for those displacements, the constraint forces, which are of the first or second kind, do zero work.

That equation decomposes into h-p-j equations, since only h-p-j of those variations are arbitrary, because the h elementary variations $q_1, \ldots, q_h, t$ are subject to the p relations (2′) and the j relations (4).

([5]) If two solid bodies S and S' in the system exert actions F, F', resp., on each other that are constraint forces of the second kind then the virtual work that is done by those two forces will be independent of the reference frames, since those two forces form a system that is equivalent to zero. I shall refer to the solid body S'. By itself, the force F that is applied to S will do work. The virtual displacement is supposed to take place during the fictitious time interval δt, so that work will have the expression:

$$(U_x\,X + U_y\,Y + U_z\,Z + pL + qM + rN)\,dt,$$

in which X, Y, Z, L, M, N denote the coordinates of F with respect to arbitrary axes, while p, q, r, U_x, U_y, U_z are those of the system of vectors that characterize the virtual velocities of the various points of S with respect to the solid body S' (H. BEGHIN [**1**], t. 1, pp. 131). That expression is linear with respect to δq. When one is dealing with a cyclist, one will immobilize the articulations that are commanded by the muscles whose internal stresses are constraint forces of the second kind, while leaving free the ones that are commanded by muscles whose action is zero or given in advance.

In order to effectively write these equations, I shall employ the method of Lagrange multipliers: If x, y, z are expressed as functions of q_1 , …, q_h , t by equations (1) then the left-hand side of equation (5) will be the sum of h terms of the form:

$$\delta q \sum m\left(x'' \frac{\partial x}{\partial q} + y'' \frac{\partial y}{\partial q} + z'' \frac{\partial z}{\partial q} \right) = P\ \delta q, \tag{6}$$

in which q denotes any of the h parameters. The right-hand side is the sum of h terms of the form:

$$\delta q \sum m\left(X \frac{\partial x}{\partial q} + Y \frac{\partial y}{\partial q} + Z \frac{\partial z}{\partial q} \right) = Q\ \delta q\ . \tag{7}$$

The d'Alembert equation is written:

$$(P_1 - Q_1)\ \delta q_1 + (P_2 - Q_2)\ \delta q_2 + \ldots + (P_h - Q_h)\ \delta q_h = 0. \tag{8}$$

I append to that equation the p relations (2), multiplied by the coefficients Λ, M, …, respectively, and the j relations (4), multiplied by λ, μ, …, respectively, where those coefficients Λ, M, …, λ, μ, … constitute $p + j$ auxiliary unknowns. I get the equation:

$$\sum\ (P_i - Q_i + \Lambda\, A_i + M\, B_i + \ldots + \lambda\, a_i + \mu\, b_i + \ldots)\ \delta q_i = 0, \tag{9}$$

in which i represents the indices 1, 2, …, h. The multipliers Λ, M, …, λ, μ, … can be chosen in such a way that the coefficients of $p + j$ of the variations δq_i will be zero, because, of course, the relations (2′) and (4) are independent in the preceding. Equations (9) must be verified for any h-p-j of the other variations δq_i , in such a way that the coefficients of those h-p-j variations in equation (9) also be zero.

In summary, the problem comes down to solving the h equations:

$$\begin{cases} P_1 - Q_1 - \Lambda\, A_1 + MB_1 + \cdots + \lambda a_1 + \mu b_1 + \cdots = 0, \\ P_2 - Q_2 - \Lambda\, A_2 + MB_2 + \cdots + \lambda a_2 + \mu b_2 + \cdots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases} \tag{10}$$

to which, there is good reason to append the p equations (2) that express the non-holonomic contact constraints and the r servitude equations (3), so in all $h + p + r$ equations in $h + p + j$ unknowns (q_1 , …, q_h , Λ, M, …, λ, μ, …).

If it happens that r of them are greater than j then the problem will generally be impossible to solve; i.e., it is not possible to realize a number of the servo-constraints that is greater than the number of restrictive conditions that one must impose upon the q parameters in order to annul the virtual work done by the constraint forces of the second kind.

If r is equal to j then the problem will be solved by the equations (2), (3), and (10).

If r is less than j then the motion will be indeterminate: One imagines, moreover, that if the functions that must replace the forces of the second kind are not sufficiently well-defined then their elimination will become impossible, and that the motion cannot be studied without one giving at least part of them.

8. Special cases:

1. I suppose that equations (2′), which express the idea that the virtual displacements are compatible with the non-holonomic contact constraints, and that equations (4), which one is led to introduce in order to annul the work done by constraint forces of the second kind, are solved with respect to the $p + j = m$ variations $\delta q_1, \dots, \delta q_m$:

$$(11) \qquad \begin{cases} \delta q_1 = \mathcal{A}_{m+1}\delta q_{m+1} + \cdots + \mathcal{A}_h \delta q_h, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \delta q_m = \mathcal{L}_{m+1}\delta q_{m+1} + \cdots + \mathcal{L}_h \delta q_h. \end{cases}$$

The Lagrange multipliers then become superfluous. If I replace $\delta q_1, \dots, \delta q_m$ in equation (8) with those expression then I will get an equation that is linear in $\delta q_{m+1}, \dots, \delta q_h$, which must be verified for any of those variations, so for $h - m$ equations of the form:

$$(12) \qquad P_{m+i} - Q_{m+i} + \mathcal{A}_{m+i}\,(P_1 - Q_1) + \dots + \mathcal{L}_{m+i}\,(P_m - Q_m) = 0,$$

in which i denotes one of the numbers 1, 2, …, $h - m$.

There is good reason to append the p equations (2) and the r servitude equations (3) to these equations.

9. – 2. If equations (11) reduce to:

$$(13) \qquad \delta q_1 = 0, \dots, \delta q_m = 0$$

then the equations of motion will reduce to the simple form:

$$(14) \qquad P_{m+1} = Q_{m+1}, \dots, \qquad P_h = Q_h .$$

10. – 3. I suppose that the constraint forces of the second kind are solely the contact actions of one auxiliary system Σ_1 of moving obstacles whose positions depend upon a certain number $q_1, \dots, q_k$ of the parameters $q_1, \dots, q_h$. In that case, the relations (4) will be:

$$(15) \qquad \delta q_1 = 0, \dots, \delta q_h = 0,$$

because it is by keeping those obstacles fixed that one will annul the work done by their actions on the given system Σ. The multipliers λ, μ, … will become superfluous, because equation (8) will no longer contain δq_{k+1}, …, δq_h . Equations (10) will reduce to the following ones, which are $h - k$ in number:

(16)
$$\begin{cases} P_{k+1} - Q_{k+1} + \Lambda\, A_{k+1} + M\, B_{k+1} + \cdots = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P_h - Q_h + \Lambda\, A_h + M\, B_h + \cdots = 0, \end{cases}$$

and as in the general case, there is good reason to append the p equations (2) and the r relations (3) to them, namely, $h - k + p + r$ relations in $h + p$ unknowns. The problem is determinate when the number of servitude equation is equal to the number k of parameters upon which the auxiliary system Σ_1 depends.

11. – 4. If the hypotheses are the ones in the preceding paragraph (10. – 3) then I will suppose, in addition, that the contact constraints on the system are all holonomic ($p = 0$). The multipliers Λ, M, … will also become superfluous, and equations (10) will reduce to the following $h - k$ equations:

(17)
$$P_{m+1} = Q_{m+1}\,, \ \ldots, \qquad P_h = Q_h\,,$$

to which there is good reason to append the r equations (3) that express the servitude. The unknowns are solely the q_1 , …, q_h .

12. Remarks:

1. In the systems without servitude, the virtual displacements to which one applies d'Alembert's equations are the ones that are compatible with all of the constraints. In the systems that include servitude, they are completely different displacements: One has then exhibited the analytical reasons for the difference that exists between those two categories of systems, and one can understand all of the interest that is attached to the mechanisms that include servitude from the industrial viewpoint.

13. – 2. In the case where the constraint forces of the second kind are solely the reactions from moving obstacles whose position is a function of certain parameters q (§ **10**, § **11**), the solution to the problem is independent of the inertia of those bodies and the given forces that are applied to them.

Therefore, if one can divide a system that is subject to r servitude relations into two parts Σ, Σ_1, such that the partial system Σ is not subject to any constraint force of the second kind beyond the reactions of the system Σ_1, and if on the other hand, the number of parameters upon which the system Σ_1 depends is equal to the number of servitude relations then the inertia and given forces that are applied to Σ_1 will not influence the motion of Σ . The method that was indicated in paragraphs **10** and **11** will permit one to present the problem in terms of equations without introducing either inertial forces or given forces. The partial system then plays an auxiliary role. That special case frequently presents itself in the applications.

14. Equilibrium in systems that include a servo. – D'Alembert's principle gives conditions of equilibrium when one suppresses the P in them, which are the terms that are due to the inertial forces in the system considered. Equations (10) relate to the general case, and equations (12), (14), (16), or (17), which relate to the special cases that were studied, will then give the equations of equilibrium when one replaces the P in them with zero. There is good reason to append the equations of servitude, which are finite, to those equations. The differential expressions will express non-holonomic constraints, which are due to either contact or servitude, and must obviously not be appended; they are verified independently.

15. Extension of Lagrange's equations. – The conditions are the general conditions that were defined in paragraph **7**. The coordinates x, y, z of the various elements of the system considered Σ are expressed in terms of finite expressions [§ **7**, eq. (1)] as functions of time t and the parameters q_1 , …, q_h upon which the system will depend when one takes into account only the holonomic contact constraints. In the treatises on rational mechanics ([6]), one establishes that the expression:

$$P = \sum m\left(x''\frac{\partial x}{\partial q} + y''\frac{\partial y}{\partial q} + z''\frac{\partial z}{\partial q}\right)$$

will have the value:

$$P = \frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q},$$

in which q denotes any of the parameters q_1 , …, q_k , q' is its derivative with respect to time, and $2T$ denotes the expression for the *vis viva* of the system Σ as a function of the q_1 , …, q_h ; q_1' , …, q_h' , t.

One will then extend the Lagrange equations to systems that involve servitude by replacing P_1 , …, P_h with those expressions in equations (10) in paragraph **7** ([7]).

([6]) P. APPELL [**1**], t. II, pp. 309.

([7]) With an eye towards applications, I recall that the *vis viva* of an invariable solid body that moves around a fixed point O is (P. APPELL [**1**], t. II, pp. 147):

$$2T = f(p, q, r) = A\,p^2 + B\,q^2 + C\,r^2 - 2D\,qr - 2E\,rp - 2F\,pq\,,$$

in which p, q, r denote the projections of the instantaneous rotations around three rectangular axes Ox, Oy, Oz, whether fixed or not, where:

$$f(x, y, z) = A\,x^2 + B\,y^2 + C\,z^2 - 2D\,yz - 2E\,zx - 2F\,xy = 1$$

is the equation of the ellipsoid of inertia of the center O when referred to the same axes.

In the special cases that were defined in paragraphs **9** and **11**, the Lagrange equations take the simple well-known form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right) - \frac{\partial T}{\partial q} = Q,$$

in which q denotes any of the parameters q_{m+1} , …, q_h in the case of paragraph **9** and any of the parameters q_{k+1} , …, q_h in the case of paragraph **11**.

It is essential to remark that the *vis viva* must be calculated as a function of the q_1 , …, q_h , q_1', …, q_h', t *without taking into account the servitude constraints.* The same thing is true for the elementary work $Q_1\ \delta q_1 + \ldots + Q_h\ \delta q_h$ done by given forces. If the forces admit a force function (i.e., if Q_1 , …, Q_h are the derivatives $\frac{\partial U}{\partial q_1}$, …, $\frac{\partial U}{\partial q_h}$ of a function U of q_1 , …, q_h , t) then that function U will be calculated without appealing to servitude. It is only in the equations themselves – i.e., in the expressions, Q, $\frac{\partial T}{\partial q}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial q'}\right)$ – that it can be taken into account. Meanwhile, since the derivative of $\frac{\partial T}{\partial q'}$ with respect to t is taken along the real motion, which is compatible with the servo-constraints, one can perform all of the simplifications on $\frac{\partial T}{\partial q'}$ that result from those constraints before differentiating with respect to t. In summary: *One can take the servitude into account after having concludes the calculation of the three categories of expressions* Q, $\frac{\partial T}{\partial q}$, $\frac{\partial T}{\partial q'}$.

16. Equation of *vis viva*. – Since the contact constraints are not supposed to depend upon t, in particular, equations (2), which represent the non-holonomic constraints, have no terms in dt (so $A = B = \ldots = 0$) . Since the given forces are supposed to admit the force function U (q_1 , …, q_h), I shall multiply equations (10), which give the motion in the general case, by dq_1 , …, dq_h , resp., which are the elementary variations of the parameters for the real displacements, so the expression:

$$P_1\,dq_1 + \ldots + P_h\,dq_h$$

will give the work done by the inertial forces (with the sign changed)

$$\sum m\,(x''dx + y''dy + z''dz)\,;$$

If one is dealing with a solid body that is animated with an arbitrary motion then one must add to the *vis viva* of a solid body that is due to its motion around the center of gravity (which has the form that I just indicated) the *vis viva* $M V_0^2$ that the mass M of the solid body would have if it were concentrated at the center of gravity (Koenig's theorem; P. APPELL [**1**], t. II, pp. 56).

i.e., the differential dT of the semi-*vis viva*.

The expression:

$$Q_1\, dq_1 + \ldots + Q_h\, dq_h$$

is equal to dU. The multiplier Λ has the coefficient:

$$A_1\, dq_1 + \ldots + A_h\, dq_h$$

which is zero, since the displacement verifies equations (2); the same thing will be true for the analogous coefficients M, …

One will then have the equation:

$$d\,(T - U) + \lambda\,(a_1\, dq_1 + \ldots + a_h\, dq_h) + \mu\,(b_1\, dq_1 + \ldots + b_h\, dq_h) + \ldots = 0.$$

One sees that $T - U$ is not constant. The terms in λ, μ, … represent the elementary work done by constraint forces of the second kind, which is not zero, in general, since the conditions (4) are not imposed upon the real displacement. According to its sign, that work done will correspond to an *input* or an *output of mechanical energy* for the system Σ considered.

The same thing is true in each special case that was defined in paragraphs **9**, **10**, **11**. The combination of *vis vivas* will not give the expression $d\,(T - U)$, because only some of the expressions $P_1, \ldots, P_h$, $Q_1, \ldots, Q_h$ will enter into the equations of motion.

It is interesting to conclude that *servitude can permit one to increase or decrease the mechanical energy in the system at will, and in particular, to damp out the oscillations of a system that presents no source of energy dissipation.*

17. Application. – Let a plate Σ in a fixed plane articulate at a point C with a circular plate Σ_1 that moves around its center O . A constant force F that is parallel to a fixed line Ox is exerted on the plate Σ at a point A that is situated on the line that joins C to the center of gravity G. A servomotor M acts on the plate Σ_1 by means of gears in such a manner as to constantly realize the constraint:

(1) $$\alpha - \beta = \frac{\pi}{2},$$

$$[a = (Ox, OC), \qquad b = (Ox, CA), \qquad OC = R, \qquad CA = a, \qquad CG = b]$$

Since the servitude constraint is unique, and on the other hand, the plate Σ_1 depends upon only one parameter α, the system Σ, when taken in isolation, will fall into the special case that was defined in paragraph **11**: One can then apply the Lagrange equations to just the plate Σ . One sees that the mass of the plate Σ_1 will have no influence on the motion. The *vis viva* of Σ is:

$$2T = M\,(R^2\,\alpha'^2 + b^2\,\beta'^2 + 2R\,b\alpha'\beta'\cos(\alpha - \beta) + k^2\,\beta'^2),$$

where $M k^2$ denotes the moment of inertia of Σ about G.

The virtual work done by the force F is:

$$d\,\mathcal{T} = F\,\delta(R\cos\alpha + a\cos\beta).$$

By itself, the equation that relates to β is written:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \beta'}\right) - \frac{\partial T}{\partial \beta} = -F\,a\sin\beta. \tag{4}$$

However:

$$\frac{\partial T}{\partial \beta'} = M\,[b^2\,\beta' + R\,b\alpha'\cos(\alpha - \beta) + k^2\,\beta'] = M\,(b^2 + k^2)\,\beta',$$

if one takes the servitude constraint into account (§ **15**). On the other hand:

$$\frac{\partial T}{\partial \beta} = M\,R\,b\alpha'\beta'\sin(\alpha - \beta) = M\,R\,b\,\beta'^2.$$

The equation of motion is then:

$$M\,(b^2 + k^2)\,\beta'' - MRb\beta'^2 + Fc\sin\beta = 0. \tag{5}$$

If the constraint $\alpha - \beta = \pi / 2$ is realized by direct contact between Σ and Σ_1 then the motion will be entirely different: It will be governed by the equation:

$$[M\,(R^2 + b^2 + k^2) + I_1]\,\beta'' + F\,(a\sin\beta + R\cos\beta) = 0, \tag{6}$$

in which I_1 denotes the moment of inertia of the plate at O. Equation (5) easily gives the motion: β'^2 is obtained by adding an exponential term to a term that is sinusoidal in β : β varies between two limits, one of which can be pushed out to infinity. On the contrary, equation (6) will give a pendulum motion.

The equilibrium positions are obtained by annulling the right-hand side of equation (4). One then finds the two positions for which the force F passes through C. On the contrary, equation (6) will give the positions for which the force F passes through O.

18. Extending Appell's equations. – As the author has remarked before, the Appell equations ([8]) present the following advantages:

1. They can be applied to systems that are subject to non-holonomic constraints without one having to introduce a system of multipliers as auxiliary unknowns.

([8]) P. APPELL [**1**], t. II, pp. 374, [**3**].

2. They permit one employ auxiliary parameters that are coupled to the true coordinates $q_1, \ldots, q_h$ by differential relations.

For example, in the motion of a solid body around a fixed point, if p, q, r denote the projections onto the three axes of the instantaneous rotation then it can be advantageous to use the auxiliary parameters λ, μ, ν that are coupled to p, q, r by the relations:

$$d\lambda = p\,dt, \qquad d\mu = q\,dt, \qquad d\nu = r\,dt\,.$$

$d\lambda$, $d\mu$, $d\nu$ are then the elementary angles that the solid body must turn through around those three axes in order to pass from the position that it occupied at the instant t to the one that it occupies at the instant $t + dt$.

Therefore, let Σ be a system that fulfills the conditions that were indicated in paragraph **7**. Upon taking the *holonomic contact constraints* into account that were imposed, its position will depend upon h parameters $q_1, \ldots, q_h$, and maybe t, in such a way that the coordinates of each element of matter will be finite functions of the form:

(1) $$x = f(q_1, \ldots, q_h, t), \qquad y = \ldots, \qquad z = \ldots$$

I suppose that s auxiliary parameters $q_{h+1}, \ldots, q_{h+s}$ are appended to these parameters, which are coupled with the preceding one by some differential relations that serve to define them, *and which are relations that do not, in turn, correspond to any constraint force.* I shall count them with the relations that express the non-holonomic contact constraints, because they are used in the same way when exhibiting the equations.

I will then have p differential relations ($p \geq s$) of the form:

(2) (p relations) $$\begin{cases} A_1\,dq_1 + \cdots + A_{h+s}\,dq_{h+s} + A\,dt = 0, \\ B_1\,dq_1 + \cdots + B_{h+s}\,dq_{h+s} + B\,dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

I suppose that the servo-constraints are represented by r finite or differential relations:

(3) (r relations) $$\begin{cases} g(q_1, \cdots q_{h+s}, t) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varepsilon_1\,dq_1 + \cdots + \varepsilon_{h+s}\,dq_{h+s} + \varepsilon\,dt = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Finally, the virtual displacements that annul the work done by constraint forces of the second kind are the ones that verify the j relations:

(4) (j relations) $$\begin{cases} a_1\,\delta q_1 + \cdots + a_{h+s}\,\delta q_{h+s} = 0, \\ b_1\,\delta q_1 + \cdots + b_{h+s}\,\delta q_{h+s} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

Having said that, I form the expression:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2)\,,$$

which is called the *energy of acceleration.* If I express *x*, *y*, *z* in terms of the parameters , *t*, and the first and second derivatives of the parameters *q* with respect to *t* then a calculation that is analogous to the one that served to establish the Lagrange equations will show () that the terms *P* in d'Alembert's equation will have the expressions:

$$P_1 = \frac{\partial S}{\partial q_1''},\ \ldots,\ P_k = \frac{\partial S}{\partial q_k''},$$

which then establish the Appell equations.

19. Case in which the differential equations of the contact constraints and the definitions (2) **are solved for the** ***p*** **variations** δq. – In order to give the Appell equations their full simplicity, it is useful to solve those *p* equations (2) for *p* of the *h* + *s* = *n* + *p* variations δq . I thus express, on the one hand, the *p* derivatives $q'_{n+1},\ \ldots,\ q'_{n+p}$ as functions of the $q'_1,\ \ldots,\ q'_n$ by means of relations of the form:

(5)
$$\left\{\begin{array}{l} q'_{n+1} = \alpha_1\, q'_1 + \cdots + \alpha_n\, q'_n + \alpha, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ q'_{n+p} = \gamma_1\, q'_1 + \cdots + \gamma_n\, q'_n + \gamma, \end{array}\right.$$

and on the other hand, the *p* virtual displacements δq_{n+p} , …, δq_{n+1} as functions of the δq_1, …, δq_n :

(6)
$$\left\{\begin{array}{l} \delta q_{n+1} = \alpha_1\, \delta q_1 + \cdots + \alpha_n\, \delta q_n + \alpha, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_{n+p} = \gamma_1\, \delta q_1 + \cdots + \gamma_n\, \delta q_n + \gamma, \end{array}\right.$$

in which the coefficients α_1, …, γ_1 are functions of the q_1 , …, q_{n+p} , *t* . Of course, those parameters q_1 , …, q_n can just as well be chosen from among the true coordinates as from among the auxiliary parameters q_{n+p} , …, q_{n+s} .

Having said that, instead of expressing *S* as a function of the parameters q_1 , …, q_h , and their first and second derivatives, as I supposed in the preceding paragraph, it is interesting to utilize equations (5), which replace equations (2): Upon differentiating them with respect to *t*, one expresses the second derivatives $q''_{n+1},\ \ldots,\ q''_{n+p}$ as functions of the $q''_1,\ \ldots,\ q''_{h+s}$, and first derivatives of the parameters *q* . One can also make the *p* second derivatives $q''_{n+1},\ \ldots,\ q''_{n+p}$ disappear from *S*. *S* will become a function of q_1 , …, q_{h+s} , *t*, $q'_1,\ \ldots,\ q'_{h+s}$, and the *n* second derivatives $q''_1,\ \ldots,\ q''_n$. Appell [note (8), pp. 14] shows that under those conditions, the virtual work done by inertial forces (with the sign changed) will be:

$$\left(\frac{\partial S}{\partial q_1''}\right)\delta q_1 + \cdots + \left(\frac{\partial S}{\partial q_n''}\right)\delta q_n . \tag{7}$$

On the other hand, if one expresses the virtual work done by the given forces by means of only δq_1 , …, δq_n then upon using the relations (6), one will get an expression of the form:

$$Q_1\ \delta q_1 + \ldots + Q_n\ \delta q_n \tag{8}$$

for that work.

Those two expressions must be equal for any displacement that annuls the work done by constraint forces of the second kind – i.e., ones that verify the j relations (4). Here again, it is interesting to take the relations (6) into account, which will permit one to make the δq_1 , …, δq_{n+p} disappear from equations (4) . When those equations are solved for j of the remaining variations δq_1 , …, δq_n , they can be written:

$$\left\{\begin{array}{l} \delta q_1 = \mathcal{A}_{j+1}\,\delta q_{j+1} + \cdots + \mathcal{A}_n\,\delta q_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta q_j = \mathcal{L}_{j+1}\,\delta q_{j+1} + \cdots + \mathcal{L}_n\,\delta q_n. \end{array}\right. \tag{6}$$

If I replace the δq_1 , …, δq_j in the expressions (7) and (8) with these values and express their equality for any of the remaining variations δq_{j+1} , …, δq_n then I will get the Appell equations in the form

$$\left\{\begin{array}{l} \left(\dfrac{\partial S}{\partial q_{j+1}''} - Q_{j+1}\right) + \mathcal{A}_{j+1}\left(\dfrac{\partial S}{\partial q_1''} - Q_1\right) + \cdots + \mathcal{L}_{j+1}\left(\dfrac{\partial S}{\partial q_j''} - Q_j\right) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \left(\dfrac{\partial S}{\partial q_n''} - Q_n\right) + \mathcal{A}_n\left(\dfrac{\partial S}{\partial q_1''} - Q_1\right) + \cdots + \mathcal{L}_n\left(\dfrac{\partial S}{\partial q_j''} - Q_j\right) = 0, \end{array}\right. \tag{10}$$

These equations are simpler than the Lagrange equations that one can write for the same problem [see § **8**, eq. (12)] because the number of terms in each of the Appell equations is $j + 1$, instead of $m + 1 = p + j + 1$, in the case of the Lagrange equations. The complication that is introduced in that way by the presence of the coefficients $\mathcal{A}$ and $\mathcal{L}$ provides solely the relations that express the idea that the work done by constraint forces of the second kind is zero and provide none of the non-holonomic constraints.

There is good reason to append the p equations (5) to equations (10), along with the r equations (3) that express the servo-constraints.

20. Case in which the displacements that annul the virtual work done by constraint forces of the second kind are defined by *j* relations of the form:

$$\delta q_1 = 0, \ldots, \delta q_j = 0 . \tag{11}$$

Under the same hypotheses as in the preceding paragraph, I shall suppose that the conditions that a displacement must replace in order to annul the virtual work done by constraint forces of the second kind have the simple form (11). Furthermore, one can always put oneself in this case by introducing conveniently-chosen auxiliary parameters, if needed, as was indicated in paragraph **18**.

In that case (which is, in summary, the general case), if one performs the calculations in the manner that was just described then equations (10) will simplify and take the same form as in the case of a system without servitude:

$$\frac{\partial S}{\partial q''_{j+1}} = Q_{j+1} , \ldots, \frac{\partial S}{\partial q''_n} = Q_n . \tag{12}$$

One sees that the Appell equations give a general solution to the question in a simpler form than the Lagrange equations. There is good reason to append the p equations (5) and the r servitude equations (3) to these $n - j$ equations. If $r = j$ then the number of equations will be equal to the number of unknowns ([9]).

([9]) With an eye towards applications, I recall that if a solid moves around a fixed point O, if p, q, r denote the projections of the instantaneous rotations onto the edges of a reference trihedron $Oxyz$, which is fixed or animated with an arbitrary motion that is given as a function of t or the parameters of the problem, and if P, Q, R are the projections onto the same axes of the instantaneous rotation of trihedron itself then the projections onto those axes of the acceleration of the point whose coordinates are x, y, z will be given by the following formula, and the two analogous ones that are deduced by cyclic permutation (P. APPELL [**1**], t. II, pp. 379; [**3**]):

$$J_x = - x\,(p + q + r) + p\,(p\,x + q\,y + r\,z) + (q - Q_1)\,z - (r - R_1)\,y ;$$

I have set:

$$P_1 = q\,R - r\,Q_1 = r\,P - p\,R, \qquad R_1 = q\,Q - q\,P .$$

If the axes are fixed or invariably coupled with the solid body then P_1, Q_1, R_1 will be zero.

The energy of acceleration of the solid body is defined by the relation (P. APPELL [**1**], t. II, pp. 393, ex. 16, [**3**]):

$$2S = f\,(p' - P_1, q' - Q_1, r' - R_1) + (p' - P_1)\left(q\frac{\partial f}{\partial r} - r\frac{\partial f}{\partial q}\right) + (q' - Q_1)\left(r\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial r}\right)$$
$$+ (r' - R_1)\left(p\frac{\partial f}{\partial q} - q\frac{\partial f}{\partial p}\right) + \ldots$$

the function $f\,(x, y, z)$ has the significance that was indicated above [note ([6]), pp. 11]. The unwritten terms, which do not contain the second derivatives of the parameters do not have to be calculated, because they do not enter into the Appell equations.

If one is dealing with a solid body that is animated with an arbitrary motion then the energy of acceleration will be calculated by means of a theorem that is analogous to Koenig's theorem (P. APPELL [**1**], t. II, pp. 381):

$$2S = M\,J_0^2 + 2\,S_1 .$$

(J_0 is the acceleration of the center of gravity G ; S_1 is the energy of acceleration in the motion around G.)

21. Application. – A material plane P can slide by translation along a fixed horizontal plane xOy. On that plane, a sphere Σ of radius R can roll without slipping. The motion of the plane P is governed automatically in such a manner that the center of the sphere that turns uniformly around Oz with a velocity ω with respect to the fixed axes Ox, Oy, Oz . Let us study the motion by means of the Appell equations.

Let u, v be the coordinates of a distinguished point A on the plane P with respect to the axes Ox, Oy, Oz . The position of that plane is defined by just those two parameters. The position of the sphere is defined by the coordinates ξ, η of its center and the Euler angles φ, θ, ψ that define its orientation, for example.

If p, q, r are the projections onto those axes of the instantaneous rotation of the sphere then the conditions that express rolling without slipping are obtained by writing that the material element of the sphere and the material element of the plane, which coincide at the instant t, will have the same velocity:

(1) $$\xi' - q\,R = u', \quad \eta' + p\,R = v'.$$

There are two servitude constraints:

(2) $$d\xi + \omega\,\eta\,dt = 0, \quad d\eta + \omega\,\xi\,dt = 0\ .$$

The number of those relations is equal to the number of parameters upon which the position in the plane P depends, so one can solve the problem by applying the Appell equations to only the sphere Σ.

Upon taking into account only the holonomic contact conditions, the sphere will be considered to depend upon the seven parameters u, v, ξ, η, φ, θ, ψ (h = 7). It is interesting to append three auxiliary parameters (s = 3) that are coupled with the preceding ones by the relations:

(3) $$d\lambda = p\,dt, \qquad d\mu = q\,dt, \qquad d\nu = r\,dt\ .$$

Those $h + s = 10$ parameters are coupled with those three relations and the two relations (1) that express the non-holonomic contact constraints. The relations (1) can be written:

(1) $$d\xi - R\,d_m = du, \qquad d\eta + R\,d\lambda = dv\ .$$

The relations (3) and (1) are the p differential relations [§ **18**, eq. (2)] of the general theory (p = 5).

I shall keep $h + s - p = n = 5$ of the $h + s = 10$ parameters; I shall choose u, v, ξ, η, ν . I express the energy of acceleration S of the sphere as a function of the second derivatives of those n parameters by utilizing the p = 5 relations (3) and (1′). Now [note ([9]), pp. 18], the value of S is defined by:

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5}MR^2\,(p'^2 + q'^2 + r'^2),$$

or, from (3) and (1′):

$$2S = M\,(\xi''^2 + \eta''^2) + \tfrac{2}{5} M\,[(v'' - \eta'')^2 + (\xi'' - u'')^2 + R^2\, \nu'')\,.$$

The virtual displacements that annul the work done by constraint forces of the second kind are defined by the $j = 2$ conditions:

(5) $$\delta u = 0, \qquad \delta v = 0,$$

since those forces are the reactions of the plane on the sphere. Those conditions have the form that was indicated in paragraph **20** [eq. (11)], in such a way that the equations of motion will have the form [§ **20**, eq. (12)].

References

P. APPELL:

1. *Traité de mécanique rationelle*, 2nd ed., (4 vol.), Paris, **1** (1902), **2** (1904).
2. *Les mouvements de roulement en dynamique*, Paris, 1899.
3. "Sur une forme générale des équations de la dynamique," J. reine u. angew. Math. **121** (1899), 310-319; J. math. pures appl. **6** (1900), 5-40; *ibid*. **7** (1901), 1-12; C. R. Acad. sci. Paris **129** (1899), 317-320; 423-427.
4. "Sur les liaisons exprimées par…" C. R. Acad. sci. Paris **152** (1911), 1197-1199; Rend. Palermo **32** (1911), 46-50.
5. "Sur les liaisons cachées…" C. R. Acad. sci. Paris **162** (1916), 27-29.

H. BEGHIN:

1. *Statique et Dynamique* (2 vol.), Paris, 1921.

E. DELASSUS:

1. *Leçons sur la dynamique des systèmes matériels*, Paris, 1913.
2. "Sur les liaisons et les mouvements…" Ann. Éc. norm. Paris **29** (1912), 305-370; *ibid.*, **30** (1913), 489-520; C. R Acad. sci. Paris **152** (1911), 1739-1743; *ibid.*, **156** (1913), 205-209; 677-679.

J. HADAMARD:

1. "Sur les mouvements de roulement," Mém. Soc. des sc. phys. et nat., Bordeaux (4) **5** (1895).