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Algebraic study of a certain type of curvature tensor. Petrov's case 3

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Algebraic study of the curvature tensors that admit a vector $\boldsymbol{l}$ such that: $R_{\alpha\beta,\mu\nu}\, l^\alpha\, l^\beta = 0$, $*R_{\alpha\beta,\mu\nu}\, l^\alpha\, l^\beta = 0$ when $R_{\alpha\beta} = 0$. Petrov's case 3.

1. Let $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ (1) be the normal hyperbolic metric that is defined on V_4, let $R_{\alpha\beta,\mu\nu}$ be the curvature tensor, and let $*R_{\alpha\beta,\mu\nu}$ be the tensor that is defined as follows:

$$*R_{\alpha\beta,\mu\nu} = \tfrac{1}{2}\, \eta_{\alpha\beta\gamma\delta}\, R^{\gamma\delta}{}_{,\lambda\mu} \qquad (\eta\text{, the volume element form}).$$

If the tangent vector space T_x at x is referred to the basis $(\mathbf{e}_\alpha)$ then H_{IJ} and $*H_{\mathrm{IJ}}$ (1) denote the components of those tensors referred to the basis $\mathrm{E_I} = \mathbf{e}_\alpha \wedge \mathbf{e}_\beta$ (2) that is induced by $(\mathbf{e}_\alpha)$ in $T_x^{2\wedge}$ (subspace of the antisymmetric tensors of order 2). From now on, we shall suppose that $R_{\alpha\beta} = 0$ ($R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu,\beta\nu}$) and that is always an orthonormal frame $(\mathbf{e}_\alpha)$ ($\mathbf{e}_0^2 = +1$). The matrix (H_{IJ}) is written (1):

$$(H_{\mathrm{IJ}}) = \begin{pmatrix} X_{ij} & Z_{ij} \\ Z_{ij} & -X_{ij} \end{pmatrix} \tag{1}$$

$$\sum_l X_{ll} = 0, \qquad \sum_l Z_{ll} = 0$$

with respect to an arbitrary $(\mathbf{e}_\alpha)$, in which (X_{ij}) and (Z_{ij}) are the matrices of spatial components of the two tensors that are associated with $\mathbf{e}_0$, on the one hand, and H_{IJ} and $*H_{\mathrm{IJ}}$, on the other, respectively.

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(1) $\alpha, \beta, \ldots = 0, 1, 2, 3$; $i, j, \ldots = 1, 2, 3$; $\mathrm{I, J} = 1, 2, 3, 4, 5, 6$.

(2) The correspondence between indices α and I conforms to:

$$\begin{pmatrix} 23 & 31 & 12 & 10 & 20 & 30 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

2. If there exists $\boldsymbol{l}$ such that:

(2) $$R_{\alpha\beta,\lambda\mu}\, l^\alpha l^\lambda = 0, \qquad {}^*R_{\alpha\beta,\lambda\mu}\, l^\alpha l^\lambda = 0$$

then $\boldsymbol{l}$ will necessarily be isotropic. Suppose that $(\mathbf{e}_\alpha)$ is such that $\boldsymbol{l} = \mathbf{e}_0 + \mathbf{e}_1$. (From now on, $\boldsymbol{l}$ will be defined in that way.) It follows from (2) that:

(3) $$\begin{cases} X_{21} + X_{31} = 0, \quad X_{31} - X_{21} = 0, \\ X_{22} + X_{32} = 0, \quad X_{32} - X_{22} = 0, \quad X_{11} = Z_{11} = 0, \\ X_{23} + X_{33} = 0, \quad X_{33} - X_{23} = 0, \end{cases}$$

and by a direct calculation, one will see that:

(4) $$A = B = \det(H_{\mathrm{IJ}}) = 0 \qquad \left(A = \tfrac{1}{2} H_{\mathrm{IJ}}\, H^{\mathrm{IJ}},\ B = \tfrac{1}{2} H_{\mathrm{IJ}}\, {}^*H^{\mathrm{IJ}}\right).$$

Conversely, suppose that (4) is satisfied. Since $\det(H_{\mathrm{IJ}}) = 0$, there exists an antisymmetric tensor $F^{\alpha\beta}$ (F^{I}) such that $H_{\mathrm{IJ}}\, F^{\mathrm{I}} = 0$, or rather:

(5) $$X_{ij}\, u^i + Z_{ij}\, v^i = 0, \qquad X_{ij}\, u^i - Z_{ij}\, v^i = 0$$

$(u^\alpha = {}^*F^{\alpha 0},\ v^\alpha = F^{\alpha 0})$. If $F^{\alpha\beta}$ is singular then $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}^2 = \mathbf{v}^2$, in which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{e}_0$, and one can choose a $(\mathbf{e}_\alpha)$ such that $\mathbf{u} = k\,\mathbf{e}_2$, $\mathbf{v} = k\,\mathbf{e}_3$. (3) are satisfied from (5) and (1), and $\boldsymbol{l}$ satisfies (2). If $F^{\alpha\beta}$ is regular then we can choose a $(\mathbf{e}_\alpha)$ that is a principal frame. $\mathbf{u}$ and $\mathbf{v}$ are the collinear: $\mathbf{u} = k\,\mathbf{e}$, $\mathbf{v} = k'\mathbf{e}$ $(\mathbf{e}^2 = -1)$. It follows from (5): $X_{ij}\, e^i = Z_{ij}\, e^i = 0$. If $\mathbf{e} = \mathbf{e}_1$ (that is always possible), (X_{ij}) and (Z_{ij}) take the form:

$$(X_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & -\alpha \end{pmatrix}, \qquad (Z_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & \Phi \\ 0 & \Phi & -\sigma \end{pmatrix},$$

and the scalars A and B will become:

$$A = 2\,(\alpha^2 + \beta^2 - \sigma^2 - \Phi^2), \quad B = 4\,(\alpha\sigma + \beta\Phi).$$

The solutions of the system of equations $A = B = 0$ are:

(S_1) $$\alpha = -\Phi, \qquad \beta = \sigma,$$

(S_2) $$\alpha = \Phi, \qquad \beta = -\sigma.$$

For (S_1), $\boldsymbol{l}$ satisfies:

(6) $$R_{\alpha\beta,\lambda\mu}\, l^\alpha = 0, \qquad {}^*R_{\alpha\beta,\lambda\mu}\, l^\alpha = 0,$$

and (2) *a fortiori*. The result will remain valid for S_2 when one switches $\mathbf{e}_1$ with $-\mathbf{e}_1$. Therefore: $A = B = \det(H_{\mathrm{IJ}}) = 0$ *are the necessary and sufficient conditions for there to exist a vector* $\boldsymbol{l}$ *such that* (2) *are satisfied.*

3. There always exists a $(\mathbf{e}_\alpha)$ such that (X_{ij}) and (Z_{ij}) take one of the following three forms (3):

$$(C_1) \qquad (X_{ij}) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad (Z_{ij}) = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

$$(\textstyle\sum_i \alpha_i = 0,\ \sum_i \beta_i = 0);$$

$$(C_2) \qquad (X_{ij}) = \begin{pmatrix} -2\alpha & 0 & 0 \\ 0 & \alpha-\sigma & 0 \\ 0 & 0 & \alpha+\sigma \end{pmatrix}, \qquad (Z_{ij}) = \begin{pmatrix} -2\beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\beta \end{pmatrix},$$

$$(C_3) \qquad (X_{ij}) = \begin{pmatrix} 0 & -\sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (Z_{ij}) = \begin{pmatrix} 0 & 0 & \sigma \\ 0 & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix}.$$

For (C_1), (4) will imply that $R_{\alpha\beta,\,\lambda\mu} = 0$. For (C_2), it will imply that $\alpha = \beta = 0$. $\boldsymbol{l}$ will then satisfy (6) (special case 2). (4) are satisfied for (C_3). Therefore: *if* $A = B = \det(H_{\mathrm{IJ}}) = 0$ *then* H_{IJ} *will belong to either special case* 2 *or special case* 3.

4. Suppose that H_{IJ} belongs to case 3 at each point a domain D of V_4, and let $(\mathbf{e}_\alpha)$ be the frame for which (X_{ij}) and (Z_{ij}) take the form that was indicated above. The tensor (4):

$$T_{\beta\gamma,\mu\nu} = g_{\beta\gamma}\, g_{\mu\nu}\, A - R^{\alpha}{}_{\beta,}{}^{\lambda}{}_{\mu}\, R_{\alpha\gamma,\lambda\nu} - R^{\alpha}{}_{\beta,}{}^{\lambda}{}_{\nu}\, R_{\alpha\gamma,\lambda\mu}$$

admits the reduction:

$$T_{\beta\gamma,\mu\nu} = -\,2\sigma^2\,(P_{\mu\nu}\, l_\beta\, l_\gamma + P_{\beta\gamma}\, l_\mu\, l_\nu + Q_{\beta\gamma}Q_{\mu\nu} + G_{\beta\gamma}G_{\mu\nu}),$$

in which:

$$P_{\mu\nu} = e_{(0)\mu}\, e_{(0)\nu} - e_{(1)\mu}\, e_{(1)\nu} + e_{(2)\mu}\, e_{(2)\nu} + e_{(3)\mu}\, e_{(3)\nu}\,,$$

$$Q_{\mu\nu} = e_{(2)\mu}\, l_\nu + e_{(2)\nu}\, l_\mu\,, \qquad G_{\mu\nu} = e_{(3)\mu}\, l_\nu + e_{(3)\nu}\, l_\mu\ .$$

It follows from $\nabla_\beta\, T^{\beta}{}_{\gamma,\mu\nu} = 0$ (4) that:

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(3) PETROV, Sci. Not. Kazan. St. Univ. **114** (1954), pp. 55.
(4) BEL, C. R. Acad. Sci. Paris **247** (1958), pp. 1094.

$$- l^{\mu}\; e^{\nu}_{(2)}\, \nabla_{\beta} T^{\;\beta}{}_{\gamma,\mu\nu} = -\, 4\sigma^2\; l_{\gamma}\, e^{\nu}_{(2)}\, l^{\beta}\, \nabla_{\beta}\, l_{\nu} = 0,$$
$$- l^{\mu}\; e^{\nu}_{(3)}\, \nabla_{\beta} T^{\;\beta}{}_{\gamma,\mu\nu} = -\, 4\sigma^2\; l_{\gamma}\, e^{\nu}_{(3)}\, l^{\beta}\, \nabla_{\beta}\, l_{\nu} = 0,$$

namely:

$$e^{\nu}_{(2)}\, l^{\beta}\, \nabla_{\beta}\, l_{\nu} = 0, \qquad\qquad e^{\nu}_{(3)}\, l^{\beta}\, \nabla_{\beta}\, l_{\nu} = 0.$$

However, since $l^{\beta}\, \nabla_{\beta}\, l_{\nu}$ is also orthogonal to ***l***, one must have:

$$l^{\beta}\, \nabla_{\beta}\, l_{\nu} = a\; l_{\nu}\,.$$

Therefore: *The trajectories of the vector field* ***l*** *are isotropic geodesics of the metric.*
