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On the general theory of differential parameters

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LAMÉ gave the name of *differential parameters* to certain expressions that are defined by the partial derivatives of a function of three variables that one encounters frequently in various doctrines of pure and applied analysis.

Those expressions first present themselves in the theory of the attraction of spheroids, and LAPLACE dealt with performing a transformation (which became quite celebrated) of the potential equation, which is an equation that results precisely from equating to zero what LAMÉ later called the *second-order differential parameter* of the attractive potential. It is intimately connected with some important transformations in the theory of *spherical functions*, which has been developed a great deal in this century and has led to many useful applications.

The LAPLACE transformation requires somewhat long-winded calculations with the usual methods. With his ingenious theory of curvilinear coordinates, LAMÉ achieved results that he reduced to supremely simple and elegant formulas in a much more extended category of transformations. His proofs of them, however, seemed to be somewhat artificial, and they were, on the other hand, subordinate to the hypothesis that the curvilinear coordinates were orthogonal. The first person to lift that restriction and to point out a much briefer path to achieving what LAMÉ had proposed to do was JACOBI in his beautiful paper "Sopra una soluzione particolare dell'equazione del potenziale." (See v. 2 of *Opuscula Mathematica*.) In that work, one will find the property that seems to me to be truly the most important in the study of differential parameters written down expressly, namely, that the transformation does not require anything except for knowledge of the form that line element assumes in the new system of variables. That property was indeed also revealed in the formulas of LAMÉ, but with the restriction that he imposed upon the nature of the coordinates, it does not appear to be clearly necessary.

Nevertheless, the JACOBI process was also not used by that author in all of the extent to which it is susceptible, and therefore, without a doubt, for the single reason that the question that he treated did not demand a great degree of generality, since the method would have lent itself to the extension that was alluded to with no difficulty. I mean that in the JACOBI paper, the original line element is always assumed to have the form $\sqrt{dx^2 + dy^2 + dz^2}$, while the theory of differential parameters likewise persists when the given element is not reducible to that form, but rather the laws of composition of its parameters keep all of them unaltered under that more general hypothesis. On the other hand, it is true that in the ordinary geometry of space, that hypothesis offers little interest. However, in order to convince oneself of the inopportune character of the restriction that was imposed, it is enough to consider that it tacitly assumes something that has been ignored for a long time, even in the case of just two variables, namely, the existence of the parameters of the surface, the nature and utility of which (especially the ones of order two) has been manifested in some recent studies of mine ["Ricerche d'analisi applicata alla geometria" in Giornale mathematico di Napoli, t. 2 and 3 (1864-65); "Delle variabili complesse, ecc." in Annali di Matematica (2) **1**; "Teoria generale delle superficie d'area minima," in Memorie dell' Accademia di Bologna (2) **7**].

In the present article, I propose to establish the general theory of differential parameters on a purely analytical basis, free from any unnecessary restriction, either on the number of variables or their significance. I hope that the simplicity of the method that is used, which does not differ from that JACOBI in its principal features (apart from the greater scope in which it is applied), will persuade one that the path that it follows is the most natural and direct one for achieving the goal.

The theory that is discussed here is contained substantially in § 3 of the present work. The first two §§ expose the principles upon which the adopted method is founded, which is an exposition that I wished to perform while reconciling brevity with clarity and addressing the fact that some readers might not have had a preliminary exposure to those principles yet. § 4 is dedicated to the search for some formulas of integral calculus that give a perfect confirmation and that are known already in some particular cases and that also give one an opportunity to understand the notion of differential parameters with all of the generality that I have tried to invest in it. In § 5 and the last one, one finds the proof, which is founded upon one of the general propositions, of a theory that was stated simply by (Carlo) NEUMANN [Schlömilch's Journal **12** (1867)] and which he proposed as an extension of that of GREEN.

I am obliged to mention, in addition to that of NEUMANN, some later writings in JACOBI's papers, in which the theory of differential parameters is recalled, in various aspects, and in greater generality than one finds in the work of LAMÉ.

CHELINI has defined the general expressions for the corresponding conventional expressions in ordinary rectangular coordinates in the elegant paper "Sulle formole fondamentali risguardanti la curvatura delle superficie e delle linee" Annali di Scienze Matematiche e Fisiche of Prof. TORTOLINI, Rome (1853). For that purpose, he took advantage of some very spontaneous and simple analytico-geometric considerations that often served to shed some light on it, as well as other arguments, and he referred to the beautiful paper "Sulla teoria delle coordinate curvilinee," which the same author has presented recently to the Academy, and in which he summarized the essence of his research on that interesting subject.

In *Teorica dei determinanti* [Pavia 1854, § X, eq. (114)], BRIOSCHI gave a general transformation of the sum of the second derivatives of a function of *n* variables, which is a transformation that the illustrious author had obtained with great elegance and simplicity, and from which we deduced (by means of the special variables that he also made use of in § 5 of that paper) a formula that reduces to that of LAPLACE in the case of three variables.

In volume 8, series VII, of Reports of St. Petersburg (1865), one can read a paper of SOMOFF that contains an interesting exposition of the theory of differential parameters in the case of three arbitrary curvilinear coordinates. The basis for the SOMOFF method is essentially the same as that of JACOBI, but the author gave it a dynamical context by regarding the variables as the coordinates of a moving point and considering the *vis viva* in place of the line element. Without detracting from the merit of the research, which is carried out with elegance and originality, for the most part, it seems to me that the viewpoint is not, perhaps, the most preferable one for a purely analytical question.

Finally, in the first paper "Sulle coordinate curvilinee d'una superficie e dello spazio," Annali di Matematica (2) **1** (1868) (Milan), CODAZZI has calculated the expressions for the differential parameters in arbitrary coordinates, by starting from their normal form in the system of orthogonal, rectilinear coordinates and performing all of the necessary transformations in detail.

It emerges from these brief hints that the more general results are, up to now, the ones that BRIOSCHI achieved in the cited classic reference. However, it is worth pointing out that the process of proving them that was employed by that author assumes essentially that the quadratic differential expression upon whose coefficients the definition of the parameters depend is deducible from the normal form $\sqrt{dx_1^2 + dx_2^2 + \cdots + dx_n^2}$. The purpose of the present work is precisely to exclude the necessity of that supposition without resorting to transformations that would be too laborious, which however would appear to be simple verifications under those more general hypotheses that are ill-suited to create an opportunity to consider the expressions in question. Nonetheless, I shall not neglect to confirm the result upon which the analytical definition of the second-order parameter is founded with a calculation of that type (which will be done with the greatest possible speed).

Before entering into that material, I beg your permission to use geometric language sometimes, notwithstanding the fact that the number of coordinates can be greater than three. The present study, like all of the ones that are connected with multiple integration, belongs essentially (as GAUSS said in regard to some other analytic investigations) "to a higher field of the abstract study of quantities that are independent of any concept of space, and that has as its objective the combinations of quantities that proceed continuously, which is a field that has been cultivated very little in our time, and in which *one cannot take a step without invoking the phraseology that is appropriate to the figures that exist in space.*" [Göttinger Berichte **4** (1850)].

§ 1.

ALGEBRAIC THEOREMS ON QUADRATIC FORMS

Suppose that one has the quadratic form in n variables:

$$\phi = \Sigma_{rs}\, a_{rs}\, x_r\, x_s \qquad (a_{rs} = a_{sr}), \tag{1}$$

in which the Σ sign is extended over all terms that arise when one writes all of the values 1, 2, …, n for each of the two indices r, s.

If one sets:

$$\frac{1}{2}\frac{d\phi}{dx_r} = X_r \qquad (r = 1, 2, \ldots, n) \tag{2}$$

then from EULER's theorem in homogeneous functions one will have:

$$x_1 X_1 + x_2 X_2 + \dots + x_n X_n = \phi\,. \tag{3}$$

Moreover, if one solves equations (2) for the x then one will have:

$$x_r = A_{1r} X_1 + A_{2r} X_2 + \dots + A_{nr} X_n\,, \tag{4}$$

in which A_{rs} are the quotients of the complements of a_{rs} in the discriminant:

$$a = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

over a, so that (if one considers a_{rs} to not be distinct from a_{sr}) one can write:

$$A_{rs} = \frac{d \log a}{d a_{rs}}\,,$$

and one will have $A_{rs} = A_{sr}$.

Now, if one considers the quadratic form:

$$\Phi = \Sigma_{rs} A_{rs} X_r X_s \tag{1′}$$

then it will be clear that formula (4) can be written:

$$\frac{1}{2}\frac{d\Phi}{dX_r} = x_r\,, \tag{2′}$$

so, by the cited theorem of EULER:

$$x_1 X_1 + x_2 X_2 + \dots + x_n X_n = \Phi,$$

and therefore, from (3):

$$\Phi = \phi.$$

Hence, the new form Φ is nothing but the old one ϕ when the variables x are transformed into the X by means of equations (2).

It is clear that if one operates on Φ as one would on ϕ then one must revert back to ϕ itself. For that reason, the two quadratic forms (1) and (1′) are called *reciprocal*. Formulas (2), (2′) serve to transform the one into the other. The reciprocity of the two forms certainly shows that a_{rs} is the quotient of the complement of A_{rs} in the discriminant:

$$A = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}$$

over a, which also results from the theory of determinants, so one can write (if one considers A_{rs} to not be distinct from A_{sr}):

$$a_{rs} = \frac{d \log A}{dA_{rs}}.$$

It results from the rules for the multiplication of determinants that $Aa = 1$.

Now, suppose that the variables x are substituted for other ones y by means of the linear equations:

(5) $$x_r = p_{1r}\, y_1 + p_{2r}\, y_2 + \ldots + p_{nr}\, y_n \qquad (r = 1, 2, \ldots, n)$$

from which, one infers that, reciprocally:

(6) $$y_r = q_{r1}\, x_1 + q_{r2}\, x_2 + \ldots + q_{rn}\, x_n\,,$$

so if p, q, denote the determinants that are formed by the coefficients p_{rs}, q_{rs}, respectively, then one will have:

(7) $$q_{rs} = \frac{d \log p}{dp_{rs}}, \qquad p_{rs} = \frac{d \log q}{dq_{rs}}, \qquad pq = 1.$$

Assume that when the form ϕ is transformed by means of (5) it will become:

(8) $$\psi = \Sigma_{rs}\, b_{rs}\, y_r\, y_s\,,$$

so that one will have:

(9) $$b_{rs} = \Sigma_{uv}\, a_{uv}\, p_{ru}\, p_{sv} \qquad (b_{rs} = b_{sr})$$

and therefore if one substitutes the values (6) in the new functions (8), while recalling (1), then one will have, reciprocally:

(10) $$a_{rs} = \Sigma_{uv}\, b_{uv}\, q_{ru}\, q_{sv}\,.$$

The form ψ possesses the reciprocal:

(8′) $$\Psi = \Sigma_{rs}\, B_{rs}\, Y_r\, Y_s\,,$$

which one gets by putting:

(11) $$\frac{1}{2}\frac{d\psi}{dy_r} = Y_r$$

in it, and which will give the inverse formula:

(11′) $$\frac{1}{2}\frac{d\Psi}{dY_r} = y_r\,.$$

The B_{rs} are given (in the usual way) by the formula:

$$B_{rs} = \frac{d\log b}{db_{rs}},$$

in which b is the discriminant of the form ψ.

Write (5), (6) in the following way:

$$y_u = \Sigma_r\, q_{ur}\, x_r\,, \qquad x_u = \Sigma_r\, p_{ru}\, y_r\,,$$

and apply the double sum Σ_{uv} to both sides, after multiplying the left-hand side by $b_{uv}\, q_{vs}$ and the right-hand side by $a_{uv}\, p_{sv}$. While taking (9), (10) into account, one will find in that way that:

$$\Sigma_r\, a_{ur}\, x_r = \Sigma_v\, q_{vs}\,(\Sigma_u\, b_{uv}\, y_u), \qquad \Sigma_r\, b_{rs}\, y_r = \Sigma_v\, p_{sv}\,(\Sigma_u\, a_{uv}\, x_u),$$

or, from (1), (8):

$$\frac{1}{2}\frac{d\phi}{dx_s} = \Sigma_v\, q_{vs}\,\frac{1}{2}\frac{d\psi}{dy_v}, \qquad \frac{1}{2}\frac{d\psi}{dy_s} = \Sigma_v\, p_{sv}\,\frac{1}{2}\frac{d\phi}{dx_v},$$

or finally, from (2), (11):

(5′) $$X_s = q_{1s}\, Y_1 + q_{2s}\, Y_2 + \ldots + q_{ns}\, Y_n\,,$$

(6′) $$Y_s = p_{1s}\, X_1 + p_{2s}\, X_2 + \ldots + p_{ns}\, X_n\,,$$

which are formulas in which the coefficients are obviously the same as in the original substitutions (5), (6), except that p_{rs} and q_{rs} are found to be exchanged with each other. Thanks to that simple exchange, the substitutions that serve to transform the one form ϕ into the other one ψ turn into the ones that serve to transform the reciprocal form Φ into the other on Ψ.

If one multiplies both sides of equation (9) by q_{ri} and sums over the index r then one will find that:

$$\Sigma_r\, b_{rs}\, q_{ri} = \Sigma_v\, a_{iv}\, p_{sv}\,,$$

because $\Sigma_r\, p_{ru}\, q_{ri} = 1$ or 0 according to whether $u = i$ or not, resp. If one suitably alters the indices then one will have:

(12) $$\Sigma_m\,(a_{rm}\, p_{sm} - b_{ms}\, q_{mr}) = 0,$$

which is an equation that will persist for any pair of values for the indices r and s. One easily obtains the formulas that express the coefficients p as functions of the coefficients q, and vice versa. Indeed, if one multiplies the preceding equation, first by A_{ri} and then by B_{is} and sums, the first time with respect to r and the second time with respect to s, then one will find that:

$$p_{si} - \Sigma_{mr}\, b_{ms}\, A_{ri}\, q_{mr} = 0, \qquad \Sigma_{ms}\, a_{rm}\, B_{is}\, p_{sm} - q_{ir} = 0,$$

so, upon altering the indices, one will deduce the formulas:

$$(13) \qquad p_{rs} = \Sigma_{uv}\, b_{ru}\, A_{sv}\, q_{uv} = 0, \qquad q_{rs} = \Sigma_{uv}\, a_{sv}\, B_{ru}\, p_{uv}\,,$$

which are precisely the ones that we shall treat. Finally, if one multiplies the first of these by B_{ri} and sums over r then one will get a result that can be written as follows:

$$(14) \qquad \Sigma_m\, (A_{rm}\, q_{sn} - B_{ms}\, p_{mr}) = 0,$$

and which is the reciprocal of the one that is contained in equation (12), which could have been established without any further proof.

Along with the form (1), it is often necessary to consider the bilinear expression:

$$\xi = \Sigma\, a_{rs}\, x_r\, x'_s\,,$$

which is defined by not one, but two, series of n variables:

$$x_1\,,\, x_2\,,\, \ldots,\, x_n\,,$$

$$x'_1\,,\, x'_2\,,\, \ldots,\, x'_n\,,$$

and it will be necessary to know some of their properties.

In the first place, one observes that if the variables x' are transformed linearly with the same substitutions (5), i.e., if one sets:

$$x'_r = \Sigma_u\, p_{ur}\, y'_u\,,$$

then obviously, together with:

$$\Sigma\, a_{rs}\, x_r\, x_s = \Sigma\, b_{rs}\, y_r\, y_s\,,$$

one will also have:

$$\Sigma\, a_{rs}\, (x_r + \lambda x'_r)\, (x_s + \lambda x'_s) = \Sigma\, a_{rs}\, (y_r + \lambda y'_r)\, (y_s + \lambda y'_s)$$

for any value of λ, i.e.:

$$\phi + 2\lambda\, \xi + \lambda^2 \phi' = \psi + 2\lambda\, \eta + \lambda^2 \psi',$$

in which $\eta = \sum b_{rs}\, y_r\, y'_s$. Therefore, if one lets $\phi = \psi$, $\phi' = \psi'$ then, by virtue of the substitution (5), one must likewise have $\xi = \eta$, by virtue of that, such that the substitutions that transform either of the two quadratic expressions:

$$\sum a_{rs}\, x_r\, x_s\,, \qquad \sum b_{rs}\, y_r\, y_s$$

into the other one will also transform either of the two bilinear expressions:

$$\sum a_{rs}\, x_r\, x'_s\,, \qquad \sum b_{rs}\, y_r\, y'_s$$

into the other. For the same reason, the inverse substitutions (5′), (6′) will transform one of the two functions:

(16) $$\sum A_{rs}\, X_r\, X'_s\,, \quad \sum B_{rs}\, Y_r\, Y'_s$$

into the other one.

It is known that the coefficients of a quadratic form can be such that they remain positive for all real values of the variables. It is not necessary to write down the conditions (of inequality) under which that result is true, and which can be given in many different forms. It is enough to know that when those conditions are fulfilled, any transform that contains only squares of the variables (a reduction that is well-known to follow in an infinitude of ways) will necessarily have all of its coefficients positive. Therefore, if one supposes that the form ϕ stays positive for all real values of the variables then one can always reduce it to the form:

$$\phi = \sum_r x_r^2 \qquad (r = 1, 2, \ldots, n)$$

by a suitable real linear substitution. Now, from what we saw earlier, the same linear substitution will also make:

$$\phi' = \sum_r x_r'^2\,, \quad \xi = \sum_r x_r\, x'_r\,.$$

Hence, one will have:

$$\phi\phi' - \xi^2 = \sum_r x_r \cdot \sum_r x'_r - \left(\sum_r x_r\, x'_r\right)^2,$$

or, by a well-known algebraic theorem:

$$\phi\phi' - \xi^2 = \sum_{r,s} (x_r x'_s - x_s\, x'_r)^2\,.$$

An important property emerges from this that the function $\phi\phi' - \xi^2$, namely:

$$\sum_{r,s} a_{rs}\, x_r x_s \cdot \sum_{r,s} a_{rs}\, x'_s\, x'_r - \left(\sum_{r,s} a_{rs}\, x_r x'_s\right)^2,$$

stays positive for any system of real values of the variables x, x' when that property is true for $\sum a_{rs} x_r x_s$.

In general, one has (as is easily proved):

(17)
$$\sum_{r,s} a_{rs} x_r x_s \cdot \sum_{r,s} a_{rs} x'_s x'_r - \left(\sum_{r,s} a_{rs} x_r x'_s\right)^2$$
$$= \sum (a_{rs} a_{tu} - a_{ru} a_{ts})(x_r x'_t - x_t x'_r)(x_s x'_u - x_u x'_s),$$

in which the four indices r, s, t, u in the right-hand side must separately take on all of the values 1, 2, …, n, in such a way that any square of one of the binomials that are formed from the variables will appear just once, and any product of two binomials will appear twice.

One must observe that since two reciprocal forms will be made identical by the transformation formulas, and since those formulas are linear with respect to one and the other variable, it is obvious that if one form is kept positive for any system of real values of its own variables then the same property will also be true for the other one.

The algebraic theory of reciprocal quadratic forms is susceptible to an elegant application of the method of rectilinear coordinates, which is an application that deserves to find its place in the literature of analytic geometry. To that end, it is enough to set:

$$\phi = x^2 + y^2 + z^2 + 2yz \cos \alpha + 2zx \cos \beta + 2xy \cos \gamma,$$

in which α, β, γ are the angles between the three oblique axes Ox, Oy, Oz, when taken two at a time. In that way, the quantity ϕ will express the square of the distance r from the point (x, y, z) to the origin O, and the variables X, Y, Z in the reciprocal form to ϕ are nothing but the orthogonal projections of r onto the three axes Ox, Oy, Oz. I shall briefly deviate from the present subject to consider where one might encounter some elegant relations that were found by CHELINI amongst what he called *component* coordinates and *projection* coordinates, along with their analytical origin. We make only the general observation that the quadratic expression for the distance from a point to the origin has the same status in respect to finite geometry in rectilinear coordinates as that of linear element in respect to the infinitesimal geometry of curvilinear coordinates. The most important, and most essential, formulas of the one and the other geometry do not depend upon the coefficients of the quadratic forms that represent that two aforementioned geometric elements.

§ 2.

PROPERTIES OF THE QUADRATIC DIFFERENTIAL EXPRESSIONS

Let:

(1)
$$ds^2 = \sum_{r,s} a_{rs}\, dx_r\, dx_s \qquad (a_{rs} = a_{sr})$$

be a quadratic differential expression in n variables $x_1, x_2, \dots, x_n$ such that the coefficients a_{rs} are functions of those variables. Many theorems are true with respect to that expression (for which ds^2 must be regarded as simply its representative symbol, for now) that are perfectly analogous to the ones in the preceding §. That analogy is based upon the fact that if all of the variables $x_1, x_2, \dots, x_n$ are replaced with n new variables $y_1, y_2, \dots, y_n$ that are coupled to the latter by n independent equations then the differentials of x will be coupled to those of the y by two (equivalent) systems of linear equations. If one also sets:

$$p_{rs} = \frac{dx_s}{dy_r}, \qquad q_{sr} = \frac{dy_r}{dx_s} \tag{2}$$

then those equations will be the same as (5), (6) in § 1, as long as one writes the differentials dx, dy in place of x, y, respectively. As a consequence, if one represents the transform of (1) by:

$$ds^2 = \sum_{r,s} b_{rs}\, dy_r\, dy_s \qquad (b_{rs} = b_{sr}) \tag{3}$$

then one will get immediately the following relations from (9), (10) of § 1:

$$b_{rs} = \sum_{u,v} a_{uv} \frac{dx_u}{dy_r}\frac{dx_v}{dy_s}, \qquad a_{rs} = \sum_{u,v} b_{uv} \frac{dy_u}{dx_r}\frac{dy_v}{dx_s}. \tag{4}$$

(12) of § 1 will then become:

$$\sum_m \left(a_{rm} \frac{dx_m}{dy_s} - b_{sm} \frac{dy_m}{dx_r} \right) = 0, \tag{5}$$

which is an equation that will be valid for any pair of values of the indices r, s. (13) will then become:

$$\frac{dx_r}{dy_s} = \sum_{u,v} b_{sv} A_{ru} \frac{dy_v}{dx_u}, \qquad \frac{dy_r}{dx_s} = \sum_{u,v} a_{sv} B_{ru} \frac{dx_v}{dy_u}, \tag{6}$$

in which A_{rs}, B_{rs} are the coefficients of the reciprocal forms to (1), (3), and are therefore functions of x and y, respectively. Finally, (14) is converted into:

$$\sum_m \left(A_{rm} \frac{dy_s}{dx_m} - B_{sm} \frac{dx_r}{dy_m} \right) = 0. \tag{5′}$$

Under the hypothesis that the expression (1) is simply:

$$ds^2 = dx_1^2 + dx_2^2 + \dots + dx_n^2,$$

the preceding equations will reduce to the ones that serve as the basis for the theory of curvilinear coordinates.

One also encounters the doctrine of reciprocity of quadratic algebraic forms when one considers quadratic differential expressions. In fact, in the preceding §, we found that the two forms:

$$\sum_{r,s} A_{rs} X_r X_s\,, \qquad \sum_{r,s} B_{rs}\, Y_r\, Y_s\,,$$

which are reciprocal to the ϕ, ψ, are the transforms of the respective formulas (5′), (6′) of that §, and due to (2), they will become:

$$X_r = \frac{dy_1}{dx_r} Y_1 + \frac{dy_2}{dx_r} Y_2 + \cdots + \frac{dy_n}{dx_r} Y_n\,,$$
$$Y_r = \frac{dx_1}{dy_r} X_1 + \frac{dx_2}{dy_r} X_2 + \cdots + \frac{dx_n}{dy_r} X_n\,,$$

in the present case. Now, these will obviously be satisfied when one sets:

$$X_r = \frac{dU}{dx_r}\,, \qquad Y_r = \frac{dU}{dy_r}\,,$$

in which U is any function of $x_1, x_2, \ldots, x_n$ or $y_1, y_2, \ldots, y_n$, respectively. An interesting property then results from this (which was pointed out by JACOBI in the case of three variables, and which was proved in general in my article in the *Giornale matematico* of Naples, t. 5, pp. 24), namely, that the transformations of variables that render the equation:

$$\sum a_{rs}\, dx_r\, dx_s = \sum b_{rs}\, dy_r\, dy_s$$

an identity will also render the other equation:

$$\sum A_{rs} \frac{dU}{dx_r}\frac{dU}{dx_s} = \sum B_{rs} \frac{dU}{dy_r}\frac{dU}{dy_s}$$

an identity, and conversely, so the nature of the expression:

$$\sum A_{rs} \frac{dU}{dx_r}\frac{dU}{dx_s} \tag{7}$$

is that one can assign the results of its transformation without actually knowing the n relations that exist between the original variables x and the new variables y, since it is enough to know only the results that are obtained by transformations that are analogous to the quadratic differential expressions (1). By virtue of what was proved in § 1 in regard to the functions (16), that property belongs to all of the expressions:

$$\sum_{r,s} A_{rs} \frac{dU}{dx_r}\frac{dV}{dx_s}, \tag{8}$$

no matter what the two functions U, V are.

In the applications that one makes as a result of this notion, one always supposes that the coefficients a_{rs} satisfy the conditions that are necessary for the differential expressions (1) to be positive for any system of values for the ratios:

$$dx_1 : dx_2 : \ldots : dx_n$$

(in particular, one consequence of this is that the discriminant a is never negative), so there will always exist that positive infinitesimal quantity ds whose square is equal to the value of the differential expression (at least, if one does not consider some domain of values for the variables in which that property is not verified). With that hypothesis, if $\delta x_1, \delta x_2, \ldots, \delta x_n$ is a second system of infinitesimal increments of $x_1, x_2, \ldots, x_n$, and if one sets:

$$\delta s^2 = \sum a_{rs}\, dx_r\, dx_s$$

then from what was proved at the end of § 1, it will result that the expression:

$$ds^2 \cdot \delta s^2 - (\sum a_{rs}\, dx_r\, \delta x_s)^2$$

is positive, and therefore that the expression:

$$\frac{\sum a_{rs} dx_r \delta x_s}{ds \cdot \delta s}$$

is not greater than unity, such that one can always assign a real angle θ for which one has:

$$\sum a_{rs}\, dx_r\, dx_s = ds \cdot \delta s \cos \theta. \tag{9}$$

By virtue of equation (17) in § 1, the sign of the angle θ will be given by the formula:

$$\sum (a_{rs}\, a_{lu} - a_{ru}\, a_{st})\, (dx_r\, \delta x_l - dx_l\, \delta x_r)\, (dx_s\, \delta x_u - dx_u\, \delta x_s) = ds^2 \cdot \delta s^2 \sin^2 \theta. \tag{10}$$

The possibility of satisfying equation (9) with a real value of θ as soon as the conditions are satisfied for the quadratic differential expression to stay positive for any system of values for dx leads to the important consequence that the ds that is given by the expression (1) can be considered to be a *line element* that is analogous to the one that bears that name in the theory of surfaces and in the analytic geometry of space. Thus, if one calculates the three values of ds that arise from the following three systems of values for the variables, when considered two at a time:

$$(x_1, x_2, \ldots, x_n),$$

$$(x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n),$$
$$(x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n),$$

then one will find three numbers that serve to express the length of the three sides of a rectilinear triangle. Indeed, let M, M', M'' denote the aforementioned three systems of values, and let MM' represent ds, while MM'' represents δs. The values of the system M can be deduced from those of the system M by means of the increments that are given to the latter:

$$\delta x_1 - dx_1, \qquad \delta x_2 - dx_2, \qquad \ldots, \qquad \delta x_n - dx_n,$$

respectively. Thus, if one neglects infinitesimals of order higher than two then one can set:

$$\overline{M'M''}^2 = \sum a_{rs}\,(\delta x_r - dx_r)(\delta x_s - dx_s) = ds^2 + \delta s^2 - 2\sum a_{rs}\,dx_r\,\delta x_s,$$

or, from (9):

(11) $$\overline{M'M''}^2 = \overline{M\,M'}^2 + \overline{M\,M''}^2 - 2\overline{M\,M'}\cdot\overline{M\,M''}\cdot\cos\theta,$$

in which θ is a real angle. That equation proves the stated property, and one can understand how it is possible to associate any system of values for the variables $x_1, x_2, \ldots, x_n$ to a definite *point* with those coordinates. It is along that same line of thinking that the two line elements ds, δs can be considered to be *orthogonal* when one has $\theta = \pi / 2$, i.e., (9), when the increments d, δ that relate to them satisfy the condition:

(12) $$\sum_{r,s} a_{rs}\,dx_r\,\delta x_s = 0,$$

which one can call an *orthogonality condition*, for ease of expression. If one pursues the same analogy then one can say that the left-hand side of equation (10) expresses the square of the area of a parallelogram whose sides are ds, δs.

It is useful to observe that, by virtue of what was said at the end of the preceding §, it will result from the conditions that were given there in regard to the sign of the expression for ds^2 that the expression (7) will always stay positive for any real function U.

When the quadratic expression (1) is kept constantly positive, with the introduction of a determinant that depends upon the n variables $x_1, x_2, \ldots, x_n$ of a single independent variables t, one can define a (generally continuous) series of systems of values for the n variables, which is a series that be conceived to be a *line* for which ds is the element arc. If one writes, for brevity, s' and x'_r, in place of $\frac{ds}{dt}$, $\frac{dx_r}{dt}$, resp., then the differential equations that characterize the *minimal line* will be the following ones:

(13) $$\frac{ds'}{dx_r} = \frac{d}{dt}\left(\frac{ds'}{dx'_r}\right) \qquad (r = 1, 2, \ldots, n),$$

in which one intends that s' should have the expression:

$$s' = \sqrt{\sum a_{rs}\, x'_r\, x'_s}\,,$$

and in which the indicated derivation in the left-hand side refers to the fact that x_r are contained explicitly in the coefficients a_{rs} .

If the line along which only x_1 varies – or, as one says more briefly, if the line (x_1) – is that minimal line then the preceding equations must be satisfied by:

$$x'_2 = x'_3 = \ldots = x'_n = 0,$$

and in that case, if one takes $t = x_1$ then one will have the following $n - 1$ equations, in place of (13):

(14) $$\frac{d\sqrt{a_{11}}}{dx_r} = \frac{d\left(a_{r1}/\sqrt{a_{11}}\right)}{dx_r} \qquad (r = 2, 3, \ldots, n).$$

If, moreover, the parameter of the line (x_1) – i.e., the variable x_1 – depends upon only the arc length s then one will have $ds = dx_1\sqrt{a_{11}}$ for that arc length, and it is clear that a_{11} must be a function of only the variable x_1, such that from the preceding equations, one must have:

$$\frac{d\left(a_{r1}/\sqrt{a_{11}}\right)}{dx_r} = 0,$$

so:

(15) $$a_{r1} = f_r\,(x_1, x_2, \ldots, x_n)\cdot\sqrt{a_{11}}\,,$$

in which f_r is the symbol for an arbitrary function. Suppose that the lines (x_1) are, in addition, orthogonal to the domain $x_1 = c$ (c is a specific constant) – i.e., they are orthogonals to all of the line elements that exist in it and emanate from the point of intersection with each of those lines. Since if one sets:

$$dx_2 = dx_3 = \ldots = dx_n = 0, \qquad \delta x_1 = 0$$

in (12) then that equation will reduce to the following one:

$$a_{12}\,\delta x_2 + a_{13}\,\delta x_3 + \ldots + a_{1n}\,\delta x_n = 0,$$

which cannot be satisfied by any element δs that belongs to $x_1 = c$ unless one has:

$$a_{12} = a_{13} = \ldots = a_{1n} = 0$$

for $x_1 = c$, given the form of the expressions (13), one will see that if a_{11} is non-zero for $x_1 = c$ then the $n - 1$ coefficients a_{12} , a_{13} , … a_{1n} will necessarily always be equal to zero when they are zero for just the value $x_1 = c$. Hence, if a_{11} is non-zero for $x_1 = c$ then it will result from the preceding that the lines (x_1) will all be orthogonal to the domain x_1 = *const.* when just one of them is. That property, in conjunction with the other one that any two of those domains will cut out equal arcs along the lines (x_1) (since a_{11} is a function of only x_1), constitute the obvious generalization of a known theorem of GAUSS regarding systems of geodetic lines on a surface. Conversely, if the domains x_1 = *const.* are all orthogonal to the lines (x_1), which are assumed to be *minimal*, then from (13), one will have:

$$a_{12} = a_{13} = \ldots = a_{1n} = 0$$

for any value of x_1, and then, from (14):

$$\frac{d\sqrt{a_{11}}}{dx_r} = 0 \qquad (r = 2, 3, \ldots, n),$$

from which, it will emerge that a_{11} is a function of only x_1 and therefore the arcs that are cut out by the two regions will all be equal.

From the conditions that were just assumed, it is legitimate to assume for the variable x_1 that the distance between two regions x_1 = *const.* is constant – i.e., that $a_{11} = 1$ – and in that case, with $x_0, x_1, \ldots, x_{n-1}$ in place of $x_1, x_2, \ldots, x_n$, one will obtain the line elements in the noteworthy form:

(16) $$ds^2 = dx_0^2 + \sum_{r,s} a_{rs}\, dx_r\, dx_s \qquad (r, s = 1, 2, \ldots, n-1).$$

If the lines x_0 all emanate from the same point ($x_0 = 0$) then the coefficients a_{rs} will all contain the factor x_0^2

In the case of just two variables x_0, x_1, (16) will reproduce the known reduction that was pointed out and used by GAUSS for the formula of the line element of a surface. As for the general case, it is good to observe that one can introduce n arbitrary functions of just as many new variables in place of the n original variables, so one can, in general, satisfy n conditions with those new variables that can consist of n relations that are prescribed for the coefficients of the new line element. When the form (16) is compared with (1), it will offer an example of precisely that determination. In fact, the n conditions:

$$a_{00} = 1, \qquad a_{01} = a_{02} = \ldots = a_{0,\,n-1} = 0$$

will be satisfied for the form (16).

We conclude this § with an important observation. Let W be any function of the n variables $x_1, x_2, \ldots, x_n$, and form the n-fold integral:

$$\int^{(n)} W\sqrt{a}\; dx_1\, dx_2 \ldots dx_n\,,$$

which is extended over a certain continuous region of those variables, and which we denote by S_n. In order to perform the transformation of that integral with respect to n

new variables $y_1, y_2, \ldots, y_n$, according to the known rule, one needs to replace the product:

$$dx_1\, dx_2 \ldots dx_n$$

with

$$p\, dy_1\, dy_2 \ldots dy_n ,$$

in which p is the determinant (2) that is defined by the derivatives p_{rs}. However, from the known theorem on the discriminant of the form (1), one will have $b = ap^2$, so the transformation in question will be expressed by the equation:

$$\text{(17)} \qquad \int^{(n)} W\sqrt{a} \cdot dx_1\, dx_2 \ldots dx_n = \int^{(n)} W\sqrt{b} \cdot dy_1\, dy_2 \ldots dy_n ,$$

in which the roots $\sqrt{a}$, $\sqrt{b}$ are taken to be positive. (As is known already, a and b are necessarily positive quantities as long as ds^2 is a positive quantity.) The form of the preceding quantity allows one to regard the quantities:

$$\sqrt{a} \cdot dx_1\, dx_2 \ldots dx_n , \qquad \sqrt{b} \cdot dy_1\, dy_2 \ldots dy_n$$

as two different expressions for the element dS_n of the region S_n over which the two integrals are extended, not only in the sense that the numerical values of the two expressions are the same, by in the sense that first form for the element is the one that corresponds to the decomposition of S_n in terms of the variables x, and the second one is the one that corresponds to the decomposition in terms of the y. That is obvious on the basis of the analogy with what one has in the case of surfaces and ordinary three-dimensional space.

As a consequence of that, either one or the other form for the integral will be denoted by the notation:

$$\text{(18)} \qquad \int W dS_n ,$$

which is very useful as an abbreviation for the writing down the integral formulas.

§ 3.

DEFINITION AND PROPERTIES OF THE DIFFERENTIAL PARAMETERS

We saw in the preceding § that the expression:

$$\text{(1)} \qquad \Delta_1 U = \sum_{r,s} A_{rs} \frac{dU}{dx_r} \frac{dU}{dx_s}$$

has the property of transforming into another one of the same form when one replaces the original variables x with the new variables y; that is to say, in order to perform that transformation, it is enough to replace the derivatives of U with respect to x with the homologous derivatives with respect to y, and to replace the coefficients A_{rs} (viz., the reciprocal of a_{rs}) with the homologous coefficients B_{rs} (which are reciprocal to b_{rs}). The expression will be called the *first* (or *first-order*) *differential parameter* of the function U, and will be denoted by the symbol $\Delta_1 U$.

Since that term was already applied by LAMÉ to an expression that is used often in the geometry of space and in much research in mechanics and physics, and it is already used in that accepted sense by many writers, it will be necessary to show that the extension of that term to the more general expression (1) is legitimate – i.e., it is founded upon an essential analogy.

To that end, note that in order for the quantities A_{rs} to be coefficients of the reciprocal quadratic form to the one whose homologous coefficients are a_{rs}, one must (with the same definition of the reciprocal form) pass directly from the expression:

$$ds^2 = \sum a_{rs}\, dx_r\, dx_s \tag{2}$$

to (1), by equating one half of the derivative of ds^2 with respect to dx_r to the partial derivative dU / dx_r, and replacing the values of $dx_1, dx_2, \ldots, dx_n$ in ds^2 with the values of the n linear equations that are thus established. However, in order to maintain the differential homogeneity, one should form the n equations:

$$\frac{1}{2}\frac{d\left(\sum a_{rs} dx_r dx_s\right)}{d(dx_r)} = dk \cdot \frac{dU}{dx_r} \qquad (r = 1, 2, \ldots, n), \tag{3}$$

and in that way, by means of the aforementioned substitution, one will get:

$$ds^2 = dk \cdot \Delta_1 U. \tag{4}$$

Now, equations (3), when multiplied by $\delta x_1, \delta x_2, \ldots, \delta x_n$ and summed, will give:

$$a_{rs}\, dx_r\, \delta x_s = dk \cdot \delta U, \tag{5}$$

so if the increments δ leave the value of U unaltered, or if they make $\delta U = 0$, then it will be clear that any element δs that corresponds to it will be orthogonal [by virtue of equation (12) of § 2] to the element ds for which it will satisfy equations (3). Conversely, the variations d that refer to (3) are directly orthogonal to the region $U = const.$ On the other hand, when one supposes that the increments δ are identical to d that were defined just now, (5) will give:

$$ds^2 = dk \cdot dU. \tag{6}$$

Therefore, if one eliminates dk from that equations and (4) then one will have:

$$\Delta_1 U = \frac{dU^2}{ds^2}. \tag{7}$$

That formula expresses the idea that the first differential parameter of the function U is equal to the square of the ratio of the increment dU that is due to a variation ds that is normal to $U = const.$ to the one that is due to the normal variation ds. Now, that property concurs precisely with the one that is characteristic of the parameters that were considered by LAMÉ in ordinary three-dimensional space, and one cannot avoid the fact that such concurrence (which is manifested by all of the geometric evidence in the parameters of surfaces) is not just contingent upon the identity of the analytical relations, but is in fact founded upon it.

Formula (7) confirms the property that was pointed out already in the preceding § that the first differential parameter of any real function is always a positive quantity when that is true for ds^2.

One infers from (7) that:

(7, cont.) $$\frac{dU}{ds} = \sqrt{\Delta_1 U}\,,$$

which is an equation in which (as in any other formula into which $\sqrt{\Delta_1 U}$ enters) one supposes that the radical is given the *positive* sign, which one intends to mean that the normal element ds points towards the direction in which U increases.

From the theory of reciprocal quadratic forms, when equations (3) are solved with respect to $dx_1, dx_2, \ldots, dx_n$, they will give:

(3′) $$dx_r = dk \cdot U_r \qquad (r = 1, 2, \ldots, n),$$

in which we have set:

(8) $$U_r = \frac{1}{2}\frac{d(\Delta_1 U)}{d(dU/dx_r)}\,,$$

for brevity. These new equations (3′), when multiplied by $\frac{dU}{dx_1}, \frac{dU}{dx_2}, \ldots, \frac{dU}{dx_r}$, and summed, give:

$$dV = dk \sum_r U_r \frac{dV}{dx_r}$$

or

$$dV = dk \sum_{r,s} A_{rs} \frac{dU}{dx_r}\frac{dV}{dx_s}\,,$$

so when one eliminates dk using (6), one will infer that:

$$\sum_{r,s} A_{rs} \frac{dU}{dx_r}\frac{dV}{dx_s} = \frac{dU\,dV}{ds^2}\,,$$

which is an equation in which the variations d are, as in (7), normal to the region $U = const.$ We already saw that the left-hand side of this equation possesses the same character as the differential parameters – i.e., that it transforms into an expression of the same nature when one changes the variables. It will be denoted by the symbol:

$$\Delta_1 UV = \sum_{r,s} A_{rs} \frac{dU}{dx_r} \frac{dV}{dx_s}, \tag{9}$$

and for that reason, one can call it the *intermediate (or mixed) parameter* of the two functions U, V. That expression will be converted into a first differential parameter when the two functions U, V are equal and by virtue of the foregoing will satisfy the relation:

$$\Delta_1 UV = \frac{dU\,dV}{ds^2}, \tag{10}$$

which is equivalent to the other ones (7), (7, cont.):

$$\Delta_1 UV = \frac{dV}{ds}\sqrt{\Delta_1 U}, \qquad \Delta_1 UV = \frac{dV}{dU}\Delta_1 U, \tag{10, cont.}$$

in which ds is the normal element to $U = const.$ that points in the direction of increasing U, and dU, dV are the increments of U, V along that element.

If one had $dV = 0$ then one would have to say that any variation ds that was normal to $U = const.$ would make $V = const.$ In that case, the two regions $U = const.$, $V = const.$ would be considered to be mutually orthogonal, and the necessary and sufficient condition for that is consequently:

$$\Delta_1 UV = 0.$$

One can observe that (9) gives:

$$\Delta_1 x_r\, x_s = A_{rs}, \tag{11}$$

such that (9) can be written:

$$\Delta_1 UV = \sum_{r,s} \frac{dU}{dx_r} \frac{dV}{dx_s} \Delta_1 x_r\, x_s,$$

which is an equation that obviously also persists when the $x_1, x_2, \ldots, x_n$, are not the independent variables, by n arbitrary functions of them.

In my article "Ricerche di analisi applicata alla geometria" (art. IV), I proved that the equation:

$$\Delta_1 U = 1 \tag{12}$$

[in place of which one can consider, with no greater generality, $\Delta_1 U = f(U)$] defines a certain relation on the surface that I called *geodetic parallelism*, which consists of saying that the system of orthogonal lines to $U = const.$ is formed from *geodetic* (or minimal) lines on which (from GAUSS's theorem) the $U = const.$ cut out constant lengths. That property, whose justification one already sees in formula (7), is also preserved (analytically speaking, if one prefers) in the general case of n variables, as we shall now proceed to prove.

In order to do that, one now represents the quantities $\frac{ds}{dt}$, $\frac{dx_r}{dt}$, by s', x'_r (as in the preceding §) and observes that, in light of (4), equations (3), (3′) of the present § can be written as:

$$\frac{ds'}{dx'_r} = \frac{1}{\sqrt{\Delta_1 U}}\frac{dU}{dx_r}, \qquad \frac{dx_r}{ds} = \frac{1}{\sqrt{\Delta_1 U}}U_r. \tag{13}$$

Assuming that, one recalls the equations of the preceding §:

$$\frac{ds'}{dx_r} = \frac{d}{dt}\left(\frac{ds'}{dx'_r}\right) \qquad (r = 1, 2, \ldots, n), \tag{14}$$

which characterize the minimal line, and supposes that they have the following n first integrals:

$$x'_r = \text{func.}\ (x_1, x_2, \ldots, x_n) \qquad (r = 1, 2, \ldots, n) \tag{15}$$

Imagine that one replaces the values (15) of $x'_1, x'_2, \ldots, x'_n$ as functions of $x_1, x_2, \ldots, x_n$ in the expressions:

$$s' = \sqrt{\sum a_{rs}\, x'_r\, x'_s}, \qquad \frac{ds'}{dx'_r}$$

and if one represents the derivative that is taken with respect to x_r (with those hypotheses) by $\mathrm{d} / \mathrm{d}x_r$ then one will have two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{ds'}{dx'_r}\right) = \sum_m x'_m \frac{\mathrm{d}}{\mathrm{d}x_m}\left(\frac{ds'}{dx'_r}\right), \qquad \frac{\mathrm{d}s'}{\mathrm{d}x_r} = \frac{ds'}{dx_r} + \sum_m \frac{ds'}{dx'_m}\frac{\mathrm{d}x'_m}{\mathrm{d}x_r}. \tag{16}$$

In the same way, the identity equation:

$$s = \sum_m \frac{ds'}{dx'_m} x'_m$$

will give:

$$\frac{\mathrm{d}s'}{\mathrm{d}x_r} = \sum_m x'_m \frac{\mathrm{d}}{\mathrm{d}x_m}\left(\frac{ds'}{dx'_r}\right) + \sum_m \frac{ds'}{dx'_m}\frac{\mathrm{d}x'_m}{\mathrm{d}x_r},$$

and when one compares this with the second equation in (16), one will get:

$$\frac{ds'}{dx'_m} = \sum_m x'_m \frac{\mathrm{d}}{\mathrm{d}x_m}\left(\frac{ds'}{dx'_r}\right).$$

From this equation and the first of (16), the system of equations (14) will transform into the following one:

$$\text{(14, cont.)} \qquad \sum_m \left\{ \frac{\mathrm{d}}{\mathrm{d}x_m}\left(\frac{ds'}{dx'_r}\right) - \frac{\mathrm{d}}{\mathrm{d}x_r}\left(\frac{ds'}{dx'_m}\right) \right\} x'_m = 0 \qquad (r = 1, 2, \ldots, n),$$

which is notable for its *Pfaffian* form. For $n = 2$, one will obtain from it the transformation that was discussed in my note "Sulla teoria delle linee geodetiche," Atti dell'Istituto Lombardo, t. 1 of series II.

The preceding equations (14, cont.), and thus (14), can obviously be satisfied when one can assign a function U such that one has:

$$\text{(17)} \qquad \frac{ds'}{dx'_r} = \frac{dU}{dx_r} \qquad (r = 1, 2, \ldots, n).$$

Now, if one observes the first of equations (13) then one will see that this condition is verified by any function U that satisfies the partial differential equation (12). Hence, the lines that cross the region $U = const.$ orthogonally when U is a solution of equation (12) all be minimal lines, and their differential equations will be (17) and the equivalent ones:

$$\frac{\mathrm{d}x_r}{\mathrm{d}s} = U_r .$$

These equations, or (15), can be integrated in the following way:

Imagine that the expressions (15) for x'_r as functions of x_r contain an arbitrary constant α. Obviously, that constant will also enter into the function U, and since one has, from (17), that:

$$\mathrm{d}U = \sum_r \frac{ds'}{dx'_r} dx_r ,$$

differentiating with respect to α will give:

$$\text{(18)} \qquad \mathrm{d}\frac{dU}{d\alpha} = \sum_r \frac{d}{d\alpha}\left(\frac{ds'}{dx'_r}\right) \cdot dx_r .$$

One likewise has:

$$\frac{ds'}{d\alpha} = \sum_r \frac{ds'}{dx'_r}\frac{dx'_r}{d\alpha} ;$$

however, from the identity equation:

$$s' = \sum_r \frac{ds'}{dx'_r} x'_r ,$$

one will infer that:

$$\frac{ds'}{d\alpha} = \sum_r \frac{d}{d\alpha}\left(\frac{ds'}{dx'_r}\right)\cdot x'_r + \sum_r \frac{ds'}{dx'_r}\frac{dx'_r}{d\alpha},$$

so

$$\sum_r \frac{d}{d\alpha}\left(\frac{ds'}{dx'_r}\right)\cdot x'_r = 0. \tag{19}$$

By virtue of that equation, it is clear that in place of:

$$dx_1 : dx_2 : \ldots : dx_n = x'_1 : x'_2 : \ldots : x'_n \tag{20}$$

in (18), one can equivalently set:

$$\mathrm{d}\frac{dU}{d\alpha} = 0, \qquad \text{i.e.,} \qquad \frac{dU}{d\alpha} = \beta,$$

in which β is a new constant. Now, observe that if U is a *complete* solution of equation (12) then it will contain $n - 1$ arbitrary constants $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$, in addition to an additive constant. Having assumed that, one assumes that one has $n - 1$ new arbitrary constants $\beta_1, \beta_2, \ldots \beta_{n-1}$, and establishes the equations:

$$\frac{dU}{d\alpha_1} = \beta_1, \qquad \frac{dU}{d\alpha_2} = \beta_2, \qquad \ldots, \qquad \frac{dU}{d\alpha_{n-1}} = \beta_{n-1}. \tag{21}$$

If one compares the two systems of equations that one deduces from (18), (19) when one sets α equal to $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ in succession then one will easily see that the system of equations that results from differentiating (21) (without varying $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$; $\beta_1, \beta_2, \ldots, \beta_{n-1}$) is equivalent in substance to the system (20), or the system (15) of first integrals of which it is part. If one concludes that equations (21) are nothing by the finite integrals with 2 $(n - 1)$ arbitrary constants of the minimal lines that constitute the system that is orthogonal to $U = const.$

The correspondence between that process and the HAMILTON-JACOBI method of integration is obvious.

From the preceding formulas [or from (7)], one has $U = s$, from which it results (and this is a conformation of what was proved in the preceding §) that the portions of the minimal lines that are cut out between $U = c_1$ and $U = c_2$ are all equal, no matter what the constants c_1, c_2 are.

I shall pass over the consequences that one deduces from these formulas in the case of ordinary three-dimensional space, as they are too obvious.

If one now considers the n-fold integral that is extended over the region S_n :

$$\int \Delta_1 U \cdot dS_n = \int^{(n)} \Delta_1 U \cdot \sqrt{a} \cdot dx_1\, dx_2 \ldots dx_n\,,$$

and if one recalls that by virtue of equation (17) of the preceding §, one will have:

$$\int^{(n)} \Delta_1 U \cdot \sqrt{a} \cdot dx_1\, dx_2 \,\ldots\, dx_n = \int^{(n)} \Delta_1 U \cdot \sqrt{b} \cdot dy_1\, dy_2 \,\ldots\, dy_n\,,$$

in which one intends that the parameter $\Delta_1 U$ should be expressed by the formula:

$$\sum A_{rs} \frac{dU}{dx_r} \frac{dU}{dx_s}$$

on the left-hand side and by:

$$\sum B_{rs} \frac{dU}{dy_r} \frac{dU}{dy_s}$$

on the right. If one varies the function U and denotes its variation by δU (which is assumed to be zero on the boundary of the region S_n) then one will get, from known rules [and using the notation in (8)], that:

$$\int^{(n)} \delta U \cdot \sum_r \frac{d(U_r \sqrt{a})}{dx_r} \cdot dx_1\, dx_2 \,\ldots\, dx_n = \int^{(n)} \delta U \cdot \sum_r \frac{d(U_r' \sqrt{b})}{dy_r}\; dy_1\, dy_2 \,\ldots\, dy_n\,,$$

or

$$\int \delta U \left\{ \frac{1}{\sqrt{a}} \sum_r \frac{d(U_r \sqrt{a})}{dx_r} \right\} dS_n = \int \delta U \left\{ \frac{1}{\sqrt{b}} \sum_r \frac{d(U_r' \sqrt{b})}{dy_r} \right\} dS_n\,,$$

in which U_r' expresses the quantity that is analogous to U_r when the variables are the y, instead of the x. One obviously concludes from this that if there is always just one region (which is otherwise arbitrary) over which one or the other integral is extended then one will have:

$$(\dagger) \qquad \frac{1}{\sqrt{a}} \sum_r \frac{d(U_r \sqrt{a})}{dx_r} = \frac{1}{\sqrt{b}} \sum_r \frac{d(U_r' \sqrt{b})}{dy_r}\,,$$

which is an equality that must be true by virtue of the relations that are established between the x and y. However, the two sides of this equality are constructed in completely analogous ways, the one, from only the coefficients of the expression:

$$\sum a_{rs}\, dx_r\, dx_s\,,$$

and the other, from only those of the expression:

$$\sum b_{rs}\, dy_r\, dy_s\,,$$

and it is clear that in order to perform the transformation of the one side into the other one, it would be not necessary to know all of the relations that were established between the x and y, but it would be enough to know the form that the line elements assume in the one system of variables or the other. That property, which we encountered before in the first differential parameter, confers great importance upon the equivalent expressions that we just encountered, and contains the first and second derivatives of the function U. The expression:

$$\Delta_2 U = \frac{1}{\sqrt{a}} \sum_r \frac{d(U_r \sqrt{a})}{dx_r}, \tag{22}$$

will be called the *second* (or *second-order*) *differential parameter* of the function U, and will be denoted by the symbol $\Delta_2 U$. It is then appropriate to justify the suitability of that term, since the device by which one gets that second parameter from the first-order parameter is precisely the one that was used already by JACOBI in order to arrive at the same objective with respect to the ordinary LAMÉ parameters. One should note only that for $n = 2$, formula (22) will yield the expression that I have often used by the same name in the theory of surfaces, and that CHELINI has recently recovered by his own methods in his excellent memoir on the curvilinear coordinates.

One infers from (22) that:

$$\Delta_2 x_r = \frac{1}{\sqrt{a}} \sum_s \frac{d(A_{rs} \sqrt{a})}{dx_s},$$

and from this, if one recalls formula (11) then one can easily conclude the following development of the second differential parameter:

$$\Delta_2 U = \sum_r \frac{dU}{dx_r} \Delta_2 x_r + \sum_{r,s} \frac{d^2 U}{dx_r \, dx_s} \Delta_1 x_r x_s, \tag{22, cont.}$$

in which one can obviously suppose that $x_1, x_2, \ldots, x_n$ are n arbitrary functions of the independent variables. This development also includes the one that CAUCHY made for three variables in t. 2 of his *Exercises d'analyse et de physique mathématique*, pp. 347, as a particular case.

The fundamental property of the expression (22) can be easily verified *a posteriori* in the following way:

By virtue of formula (13, 2a), the equation:

$$dx_r = \sum_m \frac{dx_r}{dy_m} dy_m$$

will give rise to this one:

$$U_r = \sum_m \frac{dx_r}{dy_m} U'_m.$$

[This can be easily proved by a direct route when one includes equations (6) of § 2.] After one multiplies both sides of this by $\sqrt{a}$ [and recalling the notation (2) of § 2], one will deduce:

$$\frac{d(U_r\sqrt{a})}{dx_r} = \sum_m \left\{\frac{dx_r}{dy_m}\left(\sum_\nu \frac{d(U'_m\sqrt{a})}{dy_\nu}\frac{dy_\nu}{dx_r}\right) + \frac{dp_{mr}}{dx_r}U'_m\sqrt{a}\right\}.$$

However:

$$\frac{dp_{mr}}{dx_r} = \sum_\nu \frac{dp_{mr}}{dy_\nu}q_{\nu r} = \sum_\nu \frac{dp_{\nu r}}{dy_m}q_{\nu r}\ ,$$

or, from (7) in § 1:

$$\frac{dp_{mr}}{dx_r} = \sum_\nu \frac{d\ln p}{dp_{\nu r}}\frac{dp_{\nu r}}{dy_m},$$

so:

$$\frac{d(U_r\sqrt{a})}{dx_r} = \sum_{m,\nu}\left(\frac{dy_\nu}{dx_r}\frac{dx_r}{dy_m}\right)\frac{d(U'_m\sqrt{a})}{dy_\nu} + \sum_{m,\nu}\left(\frac{d\ln p}{dp_{\nu r}}\frac{dp_{\nu r}}{dy_m}\right)U'_m\sqrt{a}\ .$$

One now takes the sums of both sides over the index *r*. Since the expression:

$$\sum_r \frac{dy_\nu}{dx_r}\frac{dx_r}{dy_m}$$

is equal to 1 or 0 according to whether the indices *m*, *n* are equal or unequal, respectively, the first group of terms in the left-hand side will reduce to:

$$\frac{d(U'_m\sqrt{a})}{dy_\nu}.$$

The second group can be written:

$$\sum_m \left\{\sum_{\nu,r}\frac{d\ln p}{dp_{\nu r}}\frac{dp_{\nu r}}{dy_m}\right\}U'_m\sqrt{a} = \sum_m \frac{d\ln p}{dy_m}U'_m\sqrt{a}\ ,$$

so one will have:

$$\frac{d(U'_m\sqrt{a})}{dy_r} = \sum_r \left\{\frac{d(U'_r\sqrt{a})}{dy_r} + \frac{d\ln p}{dy_r}U'_r\sqrt{a}\right\}\ ,$$

or

$$\sum_r \frac{d(U'_r\sqrt{a})}{dy_r} = \frac{1}{p}\sum_r \frac{d(U'_r\,p\sqrt{a})}{dy_r},$$

in which if one recalls that $p\sqrt{a} = \sqrt{b}$ then one will finally deduce that:

$$\frac{1}{\sqrt{a}}\sum_r \frac{d(U_r\sqrt{a})}{dx_r} = \frac{1}{\sqrt{b}}\sum_r \frac{d\left(U'_r\sqrt{b}\right)}{dy_r},$$

which is an equation that is identical to (†), which was found directly by the calculus of variations, and which served to define the second differential parameter analytically.

If one keeps (13, 2a) in mind then (22) can be written:

$$\Delta_2 U = \frac{1}{\sqrt{a}} \sum_r \frac{d\left(\frac{dx_r}{ds}\sqrt{\Delta_1 U}\cdot\sqrt{a}\right)}{dx_r}, \tag{23}$$

or

$$\Delta_2 U = \frac{d\sqrt{\Delta_1 U}}{ds}\left\{\frac{d\ln\sqrt{a}}{ds} + \sum_r \frac{d(dx_r/ds)}{dx_r}\right\}\sqrt{\Delta_1 U},$$

so, from (7, cont.):

$$\frac{\Delta_2 U}{\sqrt{\Delta_1 U}} - \frac{d\sqrt{\Delta_1 U}}{dU} = \frac{d\ln\sqrt{a}}{ds} + \sum_r \frac{d(dx_r/ds)}{dx_r}, \tag{24}$$

which is a symbolic equation in which the quantities:

$$\frac{d\sqrt{\Delta_1 U}}{dU}, \qquad \frac{d\ln\sqrt{a}}{ds}, \qquad \frac{dx_r}{ds}$$

are not (in general) true derivatives, but simply quotients of the simultaneous variations of the quantities:

$$\sqrt{\Delta_1 U}, \qquad U, \qquad \sqrt{a}, \qquad x_r$$

by a displacement ds that is normal to $U = const.$

For $n = 2$, the left-hand side of equation (24) will become the expression for the *tangential curvature* of the line $U = const.$ at the point (x_1, x_2), which would emerge from formulas (55) in the cited *Ricerche d'analisi*, etc. (pp. 71). That observation will become interesting as a result of the significance that one assumes for its left-hand side in the case of ordinary three-dimensional space. In fact, suppose that one has:

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad \text{and thus } a = 1,$$

so the quantities $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ will be nothing but the cosines X, Y, Z of the angles that the normal to the point (x, y, z) of the surface $U = const.$ makes with the axes, so that (24) will give:

$$\frac{\Delta_2 U}{\sqrt{\Delta_1 U}} - \frac{d\sqrt{\Delta_1 U}}{dU} = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}.$$

However, from the identity $X^2 + Y^2 + Z^2 = 1$, one will have:

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=\left(\frac{dX}{dx}-\frac{X}{Z}\frac{dX}{ds}\right)+\left(\frac{dY}{dy}-\frac{Y}{Z}\frac{dY}{ds}\right),$$

or, from the fact that $-\frac{X}{Z}=\frac{dz}{dx}, -\frac{Y}{Z}=\frac{dz}{dy}$:

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=\left(\frac{dX}{dx}\right)+\left(\frac{dY}{dy}\right),$$

in which the derivatives in parentheses are taken with respect to the x, y, which are considered to be principal variables in which the z is a function by virtue of the equation $U = const.$ Due to a well-known theorem (cf., Correspondance sur l'École Polytechnique, t. 3, pp. 168):

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=\frac{1}{R_1}+\frac{1}{R_2},$$

in which R_1, R_2 denote the principal radii of curvature of the surface $U = const.$ at the point (x, y, z). By virtue of that relation, which one can also establish directly (see, e.g., BORCHARDT in t. 19 of LIOUVILLE's Journal, pp. 374), one will have:

$$\frac{\Delta_2 U}{\sqrt{\Delta_1 U}}-\frac{d\sqrt{\Delta_1 U}}{dU}=\frac{1}{R_1}+\frac{1}{R_2}. \tag{25}$$

This result (which was found already in LAMÉ, *Leçons sur les coord. curv.*, pp. 42), in conjunction with the one that we recalled earlier in the case of $n = 2$, reveals a perfect analogy between the *tangential (or geodetic) curvature* of a line that is traced on a surface and the *sum of the principal curvatures* of a surface that exists in space, insofar as either quantity is represented (abstracting from the number of variables) by just one and the same analytic expression. That analogy is the true origin of two properties that have been known for quite some time, namely, that the sum of the principal curvatures is constant for the surfaces whose *area* is a minimum for the same volume enclosed, and that it is zero for the ones whose *area* is an absolute minimum between given limits. Indeed, those properties offer an exact counterpoint to the other two, namely, that the tangential curvature (of a line that is traced on a surface) is constant for those lines whose *length* is a minimum for the same *area* enclosed, and it is zero for the ones whose *length* between two given points is an absolute minimum.

One should note that the curvature $1 / r$ of a plane curve $U = const.$ can be expressed, on the basis of (24), by the formula:

$$\frac{1}{r}=\frac{dX}{dx}+\frac{dY}{dy} \qquad (X^2+Y^2=1),$$

in which X, Y are the cosines of the angles that the normal to $U = const.$ that points in the direction of increasing U makes with two orthogonal axes.

The differential parameters assume a noteworthy form when the line element has the form (16) of § 2. Indeed, from equations (1), (9), (22) of the present §, one will have, with that hypothesis:

$$(26)\qquad \begin{cases} \Delta_1 U = \left(\dfrac{dU}{dx_0}\right)^2 + \Delta_1' U, \\ \Delta_1 UV = \dfrac{dU}{dx_0}\dfrac{dV}{dx_0} + \Delta_1' UV, \\ \Delta_2 U = \dfrac{1}{\sqrt{a}}\dfrac{d}{dx_0}\left(\dfrac{dU}{dx_0}\sqrt{a}\right) + \Delta_2' U, \end{cases}$$

in which Δ_1', Δ_2' denote the parameters that relate to the element:

$$\sum_{r,s} a_{rs}\, dx_r\, dx_s \qquad (r, s = 1, 2, \ldots, n-1).$$

The first formula says that if U is a function of only x_0 then the same thing will be true for $\Delta_1 U$. Thus, in general, if the orthogonal trajectories of the region $U = const.$ are all minimal lines then one will have:

$$\Delta_1 U = f(U),$$

which reproduces, in a different way, the theorem that was proved already in this § as a consequence of equation (12), to which the last equation that was written will reduce immediately.

The last of formulas (26) says that the equation:

$$(27)\qquad \Delta_2 U = 0$$

cannot be satisfied by a function of only x_0, unless the discriminant a of the differential expression in $x_1, x_2, \ldots, x_{n-1}$ is the product of a function of only x_0 with a function of the other $n - 1$ variables. In fact, in that case, if X_0 is the factor that is a function of only x_0 then it will be enough to set:

$$(28)\qquad U = k\int \frac{dx_0}{\sqrt{X_0}}.$$

For example, if one sets:

$$ds^2 = dy_1^2 + dy_2^2 + \cdots + dy_n^2$$

and

$$y_1 = \lambda_1\, x_0, \qquad y_2 = \lambda_2\, x_0, \qquad \ldots, \qquad y_n = \lambda_n\, x_0,$$

with the condition that:

$$\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = 1,$$

then one will obtain:

$$ds^2 = dx_0^2 + x_0^2\, d\Lambda^2, \quad d\Lambda^2 = d\lambda_1^2 + d\lambda_2^2 + \cdots + d\lambda_n^2.$$

The coefficients of the element $d\Lambda^2$ are obviously reducible to functions of $n - 1$ independent variables of x_0, so the discriminant of that element will be equal to the product of $x_0^{2(n-1)}$ with a function of the $n - 1$ variables, and from (28) one will have the following solution for equation (27):

$$U = k \int \frac{dx_0}{x_0^{n-1}};$$

i.e.:

$$U = \frac{1}{x_0^{n-2}} \qquad \text{when } n > 2,$$

$$U = \ln \frac{1}{x_0} \qquad \text{when } n = 2,$$

in which:

$$x_0 = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}\,.$$

§ 4.

PROOFS OF SOME INTEGRAL FORMULAS

In what follows, we shall suppose that the domain of values for the variables $x_1, x_2, \ldots, x_n$ is always limited in such a way that the functions a_{rs} all stay monodromic, finite, and continuous, along with their first derivatives. In addition, one supposes the equation $ds^2 = 0$ cannot satisfy any real relations that inside of that domain unless one sets $dx_1 = 0$, $dx_2 = 0, \ldots, dx_n = 0$.

One lets $U_1, U_2, \ldots, U_n$ denote n of the variables $x_1, x_2, \ldots, x_n$ that are monodromic, continuous, and finite in all of the interior of a domain S_n, within which, one has the aforementioned assumptions, and one considers the n-fold integral:

$$W_r = \int \frac{d\left(U_r \sqrt{a}\right)}{dx_r}\, dx_1\, dx_2 \ldots dx_n,$$

or

$$\text{(1)} \qquad W_r = \int \frac{1}{\sqrt{a}} \frac{d\left(U_r \sqrt{a}\right)}{dx_r}\, dS_n,$$

which is extended over all systems of values of the values that are found in S_n .

The boundary of S_n is an n – 1-dimensional region that one will denote by S_{n-1}, and which one assumes to be composed of the complex of systems of values of the variables that satisfy the equation:

$$y_0 = h, \tag{2}$$

in which y_0 is a given function of $x_1, x_2, \ldots, x_n$, and h is a constant. From the level of precision of the considerations and what was just said, one also supposes that the function y_0 will increase in value when one passes from an internal point of S_n to an external one (which are both close to the boundary S_{n-1}).

Having said that, one will have:

$$\int \frac{d\left(U_r\sqrt{a}\right)}{dx_r}\,dx_r = \sum \left(U_r\sqrt{a}\right)_{s'} - \sum \left(U_r\sqrt{a}\right)_{s}, \tag{3}$$

in which the notations:

$$\left(U_r\sqrt{a}\right)_{s}, \qquad \left(U_r\sqrt{a}\right)_{s'}$$

denote the values that the expression $U_r\sqrt{a}$ will assume when, having first assigned well-defined values (included in the domain S_n) to the n – 1 variables:

$$x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n,$$

one attributes values to the x_r (at least two in number and always an even number) that satisfy equation (2), along with the preceding ones; i.e., values that define systems of values that belong to the boundary region S_{n-1} with the preceding ones. If one supposes that x_r varies continuously from the smallest to the largest of those values then the corresponding values of the places where the expression $U_r\sqrt{a}$ is odd will be denoted by just one prime, and the places where it is even will be denoted by two primes. In figurative language, one can say that those values of x_r correspond to points at which a line (x_r) enters or exits the region S_n ; i.e., the points at which it crosses the boundary region S_{n-1} .

If one multiplies both sides of equation (3) by:

$$dx_1\,dx_2 \ldots dx_{r-1}\,dx_{r+1} \ldots dx_n$$

and integrates over the entire region S_n then one will get:

$$W_r = \int \left\{ \sum \left(U_r\sqrt{a}\right)_{s'} - \sum \left(U_r\sqrt{a}\right)_{s} \right\} dx_1\,dx_2 \ldots dx_{r-1}\,dx_{r+1} \ldots dx_n\,. \tag{4}$$

The integral on the right-hand side must obviously be extended over just the region S_{n-1} . It is composed of several partial integrals, each of which refers to a portion of the region S_{n-1} that is bounded by systems of values of the variables in which the value of x_r (while keeping those of the other variables constant) is a double root of equation (2); i.e., it is bounded by the points at which the aforementioned line does not cross, but only touches, the boundary region S_{n-1} .

We now agree to replace the original variables with n new variables $y_1, y_2, \ldots, y_{n-1}$ (which are analogous to the $y_1, y_2, \ldots, y_n$ of the preceding §§), the first of which is precisely the function y_0, and is therefore constant in all of the boundary region S_{n-1} . If one sets:

$$p = \sum \left(\pm \frac{dx_1}{dy_0} \frac{dx_2}{dy_1} \cdots \frac{dx_n}{dy_{n-1}} \right)$$

then it will be clear, from the rule for the transformation of multiple integrals, that in place of:

$$dx_1\, dx_2 \ldots dx_{r-1}\, dx_{r+1} \ldots dx_n ,$$

one will need to set:

$$\pm \frac{dp}{d(dx_r / dy_0)}\, dy_1\, dy_2 \ldots dy_{n-1} ,$$

in the $(n - 1)$-fold integral, or, from a known relation:

$$\pm\, p \frac{dy_0}{dx_r}\, dy_1\, dy_2 \ldots dy_{n-1} .$$

The sign of that quantity must be chosen in such a way that it proves to be positive as long as one supposes that the determinant p (which cannot be annulled) is kept positive. Now, since, by hypothesis, y_0 increases from the inside to the outside of the region S_n, the derivative dy_0 / dx_r will be negative when the line (x_r) enters the given region and positive when the line leaves it. One then needs to take the – sign in the former case and the + sign in the latter, from which it will result that $(n - 1)$-fold integral can be written in the following way:

$$\int \sum \left(U_r \frac{dy_0}{dx_r}\, p \sqrt{a} \right) dy_1\, dy_2 \ldots dy_{n-1} .$$

In the expression in parentheses, it is intended that one must replace the $x_1, x_2, \ldots, x_n$ with the $y_1, y_2, \ldots, y_n$ and give the value of h to y_0 . One can suppress the Σ sign, provided that one understands that the integration extends over all of the region S_{n-1} . In addition, one can write $\sqrt{b}$, in place of $p \sqrt{a}$, where b is the discriminant of the quadratic expression that defined by the y, and finally, one can suppress the parentheses, as long as one understands that the integration is taken over just the boundary region. In that way, the $(n - 1)$-fold integral can be denoted in the following way:

$$\int U_r \frac{dy_0}{dx_r} \sqrt{b}\, dy_1\, dy_2 \ldots dy_{n-1} .$$

If one substitutes that integral in equation (4) and takes the sum over r on both sides then one will get:

$$(5)\qquad \int U_r \frac{dy_0}{dx_r}\sqrt{b}\; dy_1\, dy_2 \ldots dy_{n-1} = \int \frac{1}{\sqrt{a}}\sum_r \frac{d\left(U_r\sqrt{a}\right)}{dx_r}\, dS_n .$$

The first integral must be taken over the entire region S_{n-1}, while the second one must be taken over the entire region S_n .

In order to make a first application of that general formula, take the product U_rV, in place of U_r, in which U_r is the expression that is deduced from U by using formula (8) of the preceding §. That requires that the function U must be monodromic, continuous, and finite in the region S_n , along with all of its first-order derivatives, which are conditions that one also assumes to be satisfied by the function V. From that substitution, one will have:

$$\sum_r U_r \frac{dy_0}{dx_r} = V \cdot \Delta_1\, U\, y_0 ,$$

and therefore, by virtue of equation (10, cont.) of § 3 (with the hypotheses that were made there regarding the way that y_0 varies):

$$\sum_r U_r \frac{dy_0}{dx_r} = - V\, \frac{dU}{dv}\sqrt{\Delta_1 y_0} ,$$

in which dv is the line element that is normal to the region S_{n-1} and *internal* to the region S_n, and dU is the increment that U takes on along dv . Equation (5) can then be written:

$$(6)\qquad \int V\frac{dU}{dv}\sqrt{b\,\Delta_1 y_0}\; dy_1\, dy_2 \ldots dy_{n-1} + \int \frac{1}{\sqrt{a}}\sum_r \frac{d\left(U_r\sqrt{a}\right)}{dx_r}\, dS_n = 0.$$

Now, if one takes $y_0 = h$, $dy_0 = 0$ in the expression:

$$ds^2 = \sum_{r,s} b_{rs}\, dy_r\, dy_s \qquad (r, s = 0, 1, 2, \ldots, n-1)$$

then the resulting value for ds, which one can denote by ds_0, and which is given by:

$$ds_0^2 = \sum_{r,s} b_{rs}\, dy_r\, dy_s \qquad (r, s = 0, 1, 2, \ldots, n-1),$$

expresses the generic line element of the region S_{n-1}, such that, from what was established at the end of § 2, one needs to set:

$$dS_{n-1} = \sqrt{\frac{db}{db_{00}}} \cdot dy_1\, dy_2 \ldots dy_{n-1}\,.$$

However, the quantity B_{00}, which is the inverse of b_{00}, is given by:

$$B_{00} = \frac{d \ln b}{db_{00}},$$

and on the other hand, from equation (11) of § 3, one will have:

$$B_{00} = \Delta_1 y_0\,,$$

so:

$$\frac{db}{db_{00}} = b\,\Delta_1 y_0\,,$$

and therefore:

(†) $$dS_{n-1} = \sqrt{b\,\Delta_1 y_0} \cdot dy_1\, dy_2 \ldots dy_{n-1}\,.$$

On the basis of this, the equation (6) can be written more briefly as:

(7) $$\int \frac{1}{\sqrt{a}} \sum_r \frac{d\left(U_r \sqrt{a}\right)}{dx_r} dS_n + \int V \frac{dU}{dv} dS_{n-1} = 0,$$

or also:

(8) $$\int (\Delta_1 UV + V \Delta_2 U)\, dS_n + \int V \frac{dU}{dv} dS_{n-1} = 0,$$

so:

$$\frac{1}{\sqrt{a}} \sum_r \frac{d\left(U_r V \sqrt{a}\right)}{dx_r} = \sum_r U_r \frac{dV}{dx_r} + V \frac{1}{\sqrt{a}} \sum_r \frac{d\left(U_r \sqrt{a}\right)}{dx_r}$$
$$= \Delta_1 UV + V\,\Delta_2 U.$$

If one compares equation (8) with the one that one deduces by permuting U and V then one will get:

(9) $$\int (U\Delta_2 V - V\Delta_2 U)\, dS_n + \int \left(U \frac{dV}{dv} - V \frac{dU}{dv}\right) dS_{n-1} = 0.$$

This last equation (9) contains the generalization (which, it would seem, increases with its amplitude) of a known, useful theorem from integral calculus. For the case of $n = 2$, it was established for the first time (with no unnecessary restriction) in my paper "Sulla variabili complesse in una superficie," Annali di matematica, (2), t. **1**.

We agree to make a caveat regarding the preceding results: In this §, it was assumed, in principle, that from the nature of the functions a_{rs}, the expression for ds^2 could not be annulled for $dx_1 = dx_2 = \ldots = dx_n = 0$. That condition was necessary for validating the proof that was adopted, but is not indispensible in itself, since equations (8), (9) no longer contain any trace of the special system of variables that served to deduce them. Those equations can then apply in any case, provided that the integrations are suitably adjusted according to the circumstances, with rigorous attention paid to the nature of the variables with which one works. One will see a simple example of that in the special study that defines the subject of the following §.

By virtue of the formula that was found before, equation (9) can also be put into the form:

$$(10)\qquad \int (U\Delta_2 V - V\Delta_2 U)\,dS_n = \int \left(U\Delta_1 V\, y_0 - V\Delta_1 V\, y_0\right)\sqrt{b}\,dy_1\,dy_2\cdots dy_{n-1}\ ,$$

in which the integral in the right-hand side is extended over the entire region S_{n-1}.

We now pass on to another application of formula (5).

Imagine that the position of each point $(x_1, x_2, \ldots, x_n)$ varies with time t. With that hypothesis, the derivatives $x_1', x_2', \ldots, x_n'$ of the coordinates with respect to time will become (generally speaking) functions of those coordinates and time, and, properly speaking, functions that one can assume to be monodromic, continuous, and finite. Assuming that, one sets:

$$U_1 = V x_1', \qquad U_2 = V x_2', \ldots, U_n = V\, x_n'$$

in (5), where V is another function of $x_1, x_2, \ldots, x_n$, and t that is monodromic, continuous, and finite. Since y_0 does not contain t, one will have:

$$\sum_r U_r \frac{dy_0}{dx_r} = V \sum_r \frac{dy_0}{dx_r} x_r' = V\, y_0',$$

and then equation (5) will become:

$$\int V\sqrt{b}\cdot y_0'\ dy_1\ dy_2\ \ldots\ dy_{n-1} = \int \frac{1}{\sqrt{a}} \sum_r \frac{d\left(V\, x_r' \sqrt{a}\right)}{dx_r} dS_n\ .$$

One now observes that during the infinitesimal time interval dt, the region S_n, which is limited by the boundary region S_{n-1}, will change into another region S_n' that is limited by a boundary region S_{n-1}' and is infinitely close to S_{n-1}. (One regards S_n' as being composed of points that were first in S_n.) Under that change, the integral:

$$V = \int V_t\, dS_n\ ,$$

which is taken over the region S_n, will change into the integral:

$$V' = \int V_{t+dt}\, dS_n'\ ,$$

which is taken over all of the region S'_n, and will take on an increment $dV = V' - V$ that must be calculated.

That increment is composed of two parts.

In fact, the two regions S_n and S'_n have a third region S''_n in common, in which the variation of V depends upon only the increment dt that is given at time t that figures *explicitly* in the function V. The part of dV that relate to that common region is then:

$$dt \int \frac{dV}{dt} dS''_n ,$$

which is, however, a quantity in which one can correctly assume that the other one:

$$dt \int \frac{dV}{dt} dS_n , \tag{12}$$

which differs only to second order, in order for $S_n - S''_n$ to be obviously an infinitesimal quantity.

The other part of dV is provided by the aggregate of elements $V\,dS$ that are found between the bounding regions S_{n-1} and S'_{n-1}, which are elements that appear as increments of decrements in V according to whether the corresponding dS are external or internal to S_n. Moreover, these elements can correctly be considered in the state that relates to the instant t, instead of the instant $t + dt$, as they properly should be. Now, the function y_0, which is constant in the entire region S_{n-1}, takes on the increment $y'_0\,dt$ while passing to S'_{n-1}, so that the general expression (in the variables y) for an element dS that is found between S_{n-1} and S'_{n-1} will be:

$$dS = \sqrt{b} \cdot y'_0 \; dt \, dy_1 \, dy_2 \ldots dy_{n-1} .$$

Keeping in mind the hypotheses that were made about the function y_0, the dS that is given by that expression will be positive at the places where it is external to S_{n-1} (with respect to S_n) and negative at the ones where is internal to S_{n-1}. It results from this that the quantity:

$$dt \int V \sqrt{b} \cdot y'_0 \; dt \, dy_1 \, dy_2 \ldots dy_{n-1} ,$$

or (11):

$$dt \int \frac{1}{\sqrt{a}} \sum_r \frac{d\left(V\,x'_r \sqrt{a}\right)}{dx_r} dS_n , \tag{13}$$

expresses precisely the second part of the increment dV in both numerical value and sign.

If one combines the two parts (12), (13) then one will obtain:

(14) $$\frac{\mathrm{d}V}{\mathrm{d}t} = \int\left(\frac{dV}{dt} + \frac{1}{\sqrt{a}}\sum_r \frac{d\left(V\,x_r'\sqrt{a}\right)}{dx_r}\right) dS_n\,,$$

or:

(14′) $$\frac{\mathrm{d}V}{\mathrm{d}t} = \int\left(V' + \frac{V}{\sqrt{a}}\sum_r \frac{d\left(\,x_r'\sqrt{a}\right)}{dx_r}\right) dS_n\,,$$

in which the integral is extended over the entire initial region S_n . That is the formula that gives one the variation of the integral V, which depends upon the motion of the points that fill up the region, when one supposes that this (moving) region is always composed of the same (moving) points.

When V expresses the value of an entity that is (or is assumed to be) invariable in time (for any S_n), one will have $dV = 0$, and therefore:

$$\frac{dV}{dt} + \frac{1}{\sqrt{a}}\sum_r \frac{d\left(V\,x_r'\sqrt{a}\right)}{dx_r} \qquad \text{or} \qquad (\ln V)' + \frac{1}{\sqrt{a}}\sum_r \frac{d\left(V\,x_r'\sqrt{a}\right)}{dx_r} = 0.$$

In ordinary three-dimensional space, the preceding equation will coincide with the one that is called the *equation of continuity* in hydrodynamics when V is the density of the fluid in question.

If $V = 1$ then (14′) will become:

$$\frac{\mathrm{d}S_n}{\mathrm{d}t} = \int \frac{1}{\sqrt{a}}\sum_r \frac{d\left(x_r'\sqrt{a}\right)}{dx_r}\,,$$

so

(15) $$\lim \frac{\mathrm{d}\ln S_n}{\mathrm{d}t} = \frac{1}{\sqrt{a}}\sum_r \frac{d\left(x_r'\sqrt{a}\right)}{dx_r} \qquad \text{for } S_n = 0.$$

One can deduce a definition of the second-order differential parameter of an arbitrary function U (x_1, x_2, …, x_n) from this, which is a definition that inherently includes the one that was given by SOMOFF (cf., cited paper) in the case of ordinary three-dimensional space. In fact, suppose that the trajectories of the various points are everywhere normal to the region U = *const.* and that their velocities are everywhere governed by the equation:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\Delta_1 U}\,,$$

so formula (23) of the preceding § will give:

$$\Delta_2 U = \frac{1}{\sqrt{a}} \sum_r \frac{d\left(x'_r \sqrt{a}\right)}{dx_r},$$

and then (15) will give:

$$\lim \frac{\mathrm{d} \ln S_n}{\mathrm{d}t} = \Delta_2 U \qquad \text{for } S_n = 0.$$

One can then say that the second differential parameter of a function U is the limit, for $S_n = 0$, of the derivative:

$$\frac{\mathrm{d} \ln S_n}{\mathrm{d}t} \tag{16}$$

under the hypotheses that any point of S_n is displaced normally to U = *const.* with a velocity = $\sqrt{\Delta_1 U}$.

The quantity (16) was called *the mean cubical dilatation* of the volume S_3 by SOMOFF (in the case of ordinary space).

From the general equation (5), which can be written in the following way:

$$\int \frac{\sum_r U_r \dfrac{dy_0}{dx_r}}{\sqrt{\Delta_1 y_0}} \, dS_{n-1} = \int \frac{1}{\sqrt{a}} \sum_r \frac{d(U_r \sqrt{a})}{dx_r} \, dS_n ,$$

by virtue of (†), when one makes:

$$U_r = \frac{V\,Y_r}{\sqrt{\Delta_1 y_0}}, \qquad \text{in which} \qquad Y_r = \frac{1}{2} \frac{d(\Delta_1 y_0)}{d\left(\dfrac{dy_0}{dx_r}\right)}$$

(one supposes that y_0 is a function that is monodromic, continuous, and finite in the entire region S_n, along with its derivatives), one will deduce that:

$$\int V dS_{n-1} = \int \frac{1}{\sqrt{a}} \sum_r \frac{d}{dx_r}\left(\frac{VY_r \sqrt{a}}{\sqrt{\Delta_1 y_0}}\right) dS_n , \tag{17}$$

which is a formula that includes the one that BORCHARDT (Journal de Liouville, t. 19) and SOMOFF (*loc. cit.*) gave for the quadrature on the surface as a special case. By virtue of formula [13, (2a)] and (24) of § 3, it can be further transformed into the following one:

$$\int V dS_{n-1} = \int \left\{\frac{\mathrm{d}V}{\mathrm{d}p} + V\left(\frac{\Delta_2 y_0}{\sqrt{\Delta_1 y_0}} - \frac{\mathrm{d}\sqrt{\Delta_1 y_0}}{\mathrm{d}y_0}\right)\right\} dS_n , \tag{18}$$

in which dp is the normal element to S_{n-1} that is external to S_n .

In the case of ordinary three-dimensional space, by virtue of equation (25) of § 3, one infers from this that:

$$\int V d\Omega = \int \left\{ \frac{\mathrm{d}V}{\mathrm{d}p} + V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right\} dS , \tag{19}$$

and for the case of an arbitrary surface, by virtue of the expression for the tangential curvature 1 / r that was given in § 3:

$$\int V \mathrm{d}s = \int \left\{ \frac{\mathrm{d}V}{\mathrm{d}p} + \frac{V}{r}\right\} d\Omega . \tag{20}$$

The second integral in equation (19) is extended over the entire volume S that is contained within the surface Ω, over which the first integral is extended, and the principal radii R_1, R_2 refer to the surface y_0 = *const.*, which constitutes part of the boundary surface Ω. The second integral in equation (28) is extended over the entire area Ω that is enclosed by the contour s, over which the first integral is extended, and the tangent curvature 1 / r refers to the line y_0 = *const.*, which constitutes part of the contour s.

For $V = 1$, equation (19) reproduces the known formula:

$$\Omega = \int \left(\frac{1}{R_1} + \frac{1}{R_2}\right) dS.$$

§ 5.

APPLICATIONS OF THE PRECEDING FORMULAS

In my cited paper "Sulla variabili complesse, etc.," I showed that for the case of n = 2, equation (9) of the preceding § can be deduced from another formula that could be considered to be the analogue of GREEN's theorem. The deduction of a formula of that nature in the case of arbitrary n presents very appreciable difficulties, unless one introduces special hypotheses about the expression for the line element. I shall then confine myself to presenting that deduction in a special case that was considered already by (Carlo) NEUMANN in his excellent work on spherical and ultra-spherical functions (Schlömilch's Journal, Bd. 12, 1867).

The case in question is the one in which the line element has the form:

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2 , \tag{1}$$

and in which, as we saw already at the end of § 3, the equation $\Delta_2 V = 0$ will then be satisfied by the function:

(2) $$V = \frac{1}{\{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2\}^{n/2-1}},$$

in which $a_1, a_2, \ldots, a_n$ are constants.

If the system of values $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$ (or, as one can say more briefly, if the point a) is found within the region then one cannot apply formula (9) of the preceding § to the value (2) of V, because the function V will become infinite at that point. In order to remove that obstacle to the application of that formula, imagine that a small region S'_n is removed from the region S_n, which contains the point a and is bounded by another small region S'_{n-1}. In that way, formula (9) of § 4 will become applicable to the residual region $S_n - S'_n$, and will assume the form:

$$\int V\,\Delta_2 U \cdot dS'_n - \int V\,\Delta_2 U \cdot dS_n + \int \left(U\frac{\mathrm{d}V}{\mathrm{d}v} - V\frac{\mathrm{d}U}{\mathrm{d}v}\right) dS_{n-1} + \int \left(U\frac{\mathrm{d}V}{\mathrm{d}v'} - V\frac{\mathrm{d}U}{\mathrm{d}v'}\right) dS'_{n-1} = 0$$

for it, because when one subtracts S'_n from S_n, one must add S'_{n-1} to S_{n-1}, assuming that the element dv' is normal to S'_{n-1} and directed towards the inside of the residual space $S_n - S'_n$. However, if z_0 represents a function (that is analogous to y_0) that keeps the same value k at all points of S'_{n-1} and that increases from the inside to the outside of S'_n, if $z_1, z_2, \ldots, z_{n-1}$ represent $n - 1$ variables (analogous to $y_1, y_2, \ldots, y_{n-1}$), which, along with z_0, specify the points of S'_n, and if c represents the discriminant of the quadratic expression of $\mathrm{d}s^2$ that is defined by the variables $z_1, z_2, \ldots, z_{n-1}$, then one will have (from what we saw in the preceding § and if we observe that $\mathrm{d}v'$ is directed in the sense of increasing z_0):

$$\int \left(U\frac{\mathrm{d}V}{\mathrm{d}v'} - V\frac{\mathrm{d}U}{\mathrm{d}v'}\right) dS'_{n-1} = \int (U\,\Delta_1 V z_0 - V\,\Delta_1 U z_0)\sqrt{c} \cdot dz_1\, dz_2 \ldots dz_{n-1}.$$

The equation above can then be written:

(3) $$\int V\,\Delta_2 U \cdot dS'_n - \int V\,\Delta_2 U \cdot dS_n + \int \left(U\frac{\mathrm{d}V}{\mathrm{d}v} - V\frac{\mathrm{d}U}{\mathrm{d}v}\right) dS_{n-1} + \int (U\,\Delta_1 V z_0 - V\,\Delta_1 U z_0)\sqrt{c} \cdot dz_1\, dz_2 \ldots dz_{n-1} = 0.$$

The choice of the region S'_n is arbitrary, as long as the point a is contained in it. One can therefore define it by the condition:

(4) $$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2 \leq k^2,$$

in which k is a positive constant that is subject to only the condition that it must be small enough that the region S_n (which obviously includes the point a) does not come from the boundary of S_n. Afterwards, one can define the function z_0 by taking:

(5) $$z_0 = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}$$

and always attributing the positive value to the radical. In that way, z_0 will become $= k$ in the entire region S'_{n-1} and increase from the inside to the outside of S'_n, which is precisely what was assumed. That being the case, by virtue of the formulas of § 3, when they are applied to the present case, one will have:

$$\Delta_1\, V\, z_0 = \sum_r \frac{dV}{dx_r}\frac{dz_0}{dx_r} = \frac{2-n}{z_0^{n-1}},$$

$$\Delta_1 U\, z_0 = \frac{1}{z_0}\sum_r \frac{dU}{dx_r}(x_r - a_r) = \frac{2-n}{z_0^{n-1}},$$

so, for $z_0 = k$:

(6) $$U\,\Delta_1 V\, z_0 - V\,\Delta_1 U\, z_0 = -\frac{1}{k^{n-1}}\left\{(n-2)U + \sum_r \frac{dU}{dx_r}(x_r - a_r)\right\}.$$

It now remains for us to suitably fix the meaning of the new variables $z_1, z_2, \ldots, z_{n-1}$ over which the last expression must be integrated. Observe that, from the two relations:

$$\xi = \rho \cos \psi, \qquad \eta = \rho \sin \psi,$$

which give rise to the other two:

$$\xi^2 + \eta^2 = \rho^2, \qquad d\xi^2 + d\eta^2 = d\rho^2 + \rho^2\, d\psi^2,$$

one will see immediately that when one establishes the $n - 1$ pairs of formulas:

(7) $$\left\{\begin{aligned} x_r - a_r &= k_r \cos z_r, \\ k_{r+1} &= k_r \sin z_r, \end{aligned}\right. \qquad (r = 1, 2, \ldots, n-1)$$

in which k_n is intended to mean $x_n - a_n$, one will have, correspondingly, the following $n - 1$ pairs of relations:

$$\left\{\begin{aligned} (x_r - a_r)^2 + k_{r+1}^2 &= k_r^2, \\ dx_r^2 + dk_{r+1}^2 &= dk_r^2 + k_r^2 dz_r^2, \end{aligned}\right. \qquad (r = 1, 2, \ldots, n-1)$$

from which, one will infer, upon separately summing the first $n - 1$ and the second $n - 1$:

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2 = k_1^2,$$

(8) $$dx_1^2 + dx_2^2 + \ldots + dx_n^2 = dk_1^2 + k_1^2 dz_1^2 + k_2^2 dz_2^2 + \cdots + k_{n-1}^2 dz_{n-1}^2.$$

The first of these equations coincides with (5) when one sets $k_1 = z_0$, and with the same hypotheses, (7) will easily give:

$$(9)\qquad \begin{cases} x_r - a_r = z_0 \sin z_1 \sin z_2 \cdots \sin z_{r-1} \cos z_r, \\ x_n - a_n = z_0 \sin z_1 \sin z_2 \cdots \sin z_{r-2} \sin z_{r-1}, \end{cases} \qquad (r = 1, 2, \ldots, n-1),$$

$$(10)\qquad k_r = z_0 \sin z_1 \sin z_2 \ldots \sin z_{r-1} \qquad (r = 1, 2, \ldots, n-1).$$

Equations (9) show that in order to make $x_1, x_2, \ldots, x_n$ traverse all of the values that are found within the region S'_n just once, it is necessary and sufficient for one to vary z_0 from 0 to k, $z_1, z_2, \ldots, z_n$ from 0 to π, and z_{n-1} from 0 to 2π. However, in order to have all of the values that belong to the boundary region S'_{n-1}, it is necessary and sufficient for one to vary the $z_1, z_2, \ldots, z_n$ in the manner that was just given, while taking z_0 to be constant and $= k$. The quantity c, which is the discriminant of the quadratic differential expression that constitutes the right-hand side of (8), is given by:

$$c = k_1^2 \, k_2^2 \ldots k_{n-1}^2 ;$$

hence, for the values (10):

$$(11)\qquad \sqrt{c} = z_0^{n-1} (\sin z_1)^{n-2} (\sin z_2)^{n-3} \ldots \sin z_{r-2} .$$

We need to observe that in the right-hand side of equation (3), $\sqrt{c}$ refers to the boundary region S'_{n-1}, so that one must make $z_0 = k$, while in the left-hand side of that equation, one needs to set, more generally (11):

$$(12)\qquad dS'_n = z_0^{n-1} (\sin z_1)^{n-2} (\sin z_2)^{n-3} \ldots \sin z_{r-2}\, dz_0\, dz_1 \ldots dz_{n-1} .$$

If one substitutes the values (6), (11), (12), along with the value $V = 1 / z_0^{n-2}$ in equation (3) then one will find that:

$$\begin{aligned} &\int z_0\, \Delta_2 U \cdot (\sin z_1)^{n-2} (\sin z_2)^{n-3} \ldots \sin z_{r-2}\, dz_0\, dz_1 \ldots dz_{n-1} \\ &\qquad + \int \left(U \frac{\mathrm{d}V}{\mathrm{d}\nu} - V \frac{\mathrm{d}U}{\mathrm{d}\nu} \right) dS_{n-1} - \int V \Delta_2 U \, dS_n \\ &= \int \left\{ (n-2)U + \sum_r \frac{dU}{dx_r}(x_r - a_r) \right\} (\sin z_1)^{n-2} (\sin z_2)^{n-3} \ldots \sin z_{r-2}\, dz_0\, dz_1 \ldots dz_{n-1} . \end{aligned}$$

One now decreases the constant k indefinitely, while one recalls that the function U is kept finite, along with its derivatives, in the entire region S_n, and therefore, also at the point $z_0 = 0$. The first n-fold integral obviously converges to zero, from the factor z_0 that multiplies the element and is always found between 0 and k. In the last $(n-1)$-fold integral, the function U tends to assume the value U_n (i.e., the value that corresponds to $x_1 = a_1$, $x_2 = a_2$, …, $x_n = a_n$) over the entire course of integration, while the sum:

$$\sum_r \frac{dU}{dx_r}(x_r - a_r)$$

will obviously tend to zero for the values (9). It will then emerge that the equation that was found will reduce to the following one:

$$\int \left(U \frac{\mathrm{d}V}{\mathrm{d}v} - V \frac{\mathrm{d}U}{\mathrm{d}v} \right) dS_{n-1} - \int V \Delta_2 U \cdot dS_n = 2\pi (n-2)\, Z\, U_n$$

for $z_0 = 0$, in which:

$$Z = \int_0^\pi (\sin z)^{n-2} dz \cdot \int_0^\pi (\sin z)^{n-3} dz \cdots \int_0^\pi \sin z \, dz\,.$$

That last quantity is easily calculated by recalling that:

$$\int_0^\pi (\sin z)^m dz = \begin{cases} \pi \dfrac{1 \cdot 3 \cdots (m-1)}{2 \cdot 4 \cdots \quad m} & \text{when } m \text{ is even,} \\[2ex] 2 \dfrac{2 \cdot 4 \cdots (m-1)}{3 \cdot 5 \cdots \quad m} & \text{when } m \text{ is odd,} \end{cases}$$

from which one will deduce that:

$$Z = \begin{cases} \dfrac{(2\pi)^{\frac{n-2}{2}}}{2 \cdot 4 \cdots (n-2)} & \text{when } n \text{ is even,} \\[2ex] \dfrac{2(2\pi)^{\frac{n-3}{2}}}{3 \cdot 5 \cdots (n-2)} & \text{when } n \text{ is odd.} \end{cases}$$

If one, with NEUMANN, then sets:

$$N = \begin{cases} \dfrac{(2\pi)^{\frac{n}{2}}}{2 \cdot 4 \cdots n} & \text{when } n \text{ is even,} \\[2ex] \dfrac{2(2\pi)^{\frac{n-1}{2}}}{3 \cdot 5 \cdots n} & \text{when } n \text{ is odd} \end{cases}$$

then one will finally have:

$$n\,(n-2)\,N\,U_a = \int \left(U\frac{\mathrm{d}V}{\mathrm{d}v} - V\frac{\mathrm{d}U}{\mathrm{d}v} \right) dS_{n-1} \tag{13}$$

$$- \int \frac{\Delta_2 U \cdot dS_n}{\{(x_1-a_1)^2+(x_2-a_2)^2+\cdots+(x_n-a_n)^2\}^{\frac{n}{2}-1}},$$

in which, for brevity, we have neglected to substitute the value for V that was developed in (2) in the first integral on the right-hand side.

This equation will be valid as long as the point a is contained in S_n, because if that were not true then equation (9) of the preceding § would be true, and thus (13) would give $U_a = 0$.

If one supposes that the function U satisfies the equation $\Delta_1 U = 0$ in the entire region S_n then the preceding formula will contain the new theorem that was given without proof by NEUMANN as an extension of that of GREEN.

In the case of $n = 2$, equation (3) comes down to (9) in the preceding §, even when a is inside of S_n . In order to obtain the true equation that is analogous to (13) in this case, one needs to assume that:

$$V = \ln \frac{1}{\sqrt{(x_1-a_1)^2+(x_2-a_2)^2}},$$

as a result of the observation that concluded § 3, and as I did precisely in the paper that was cited already, in which the theorem under discussion was established without any restriction regarding the form of the line element.