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On the flexion of ruled surfaces ([*])

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Translated by D. H. Delphenich

The admirable research that GAUSS established on the general theory of surfaces, which he committed to two justly-celebrated papers ([**]), opened up a path to the solution of some problems in which those surfaces are considered from a viewpoint that is essentially different from that of the geometers that preceded him, among which, we cite EULER and MONGE, without mentioning everyone. Indeed, whereas they had regarded the surface as the limit of a body, and therefore as entities that cannot be subjected to any other spatial displacements than the ones that are common to the solids that determined them, GAUSS naturally treated them as solids in which one of the dimensions vanishes, as well, by his method of applying analysis to the study of surfaces. In that way, if one supposes that these new geometric entities are susceptible to being flexed, but not extended or contracted, then it will be clear that the displacements of their parts are, within certain limits, mutually independent, and the study of those relative displacements no longer has anyhing in common with the study of the absolute displacements of a rigid surface, which will always be supposed in the research that follows.

That research has therefore provided an indeed specialized case that relates to the theory of flexible and inextensible surfaces: We would like to discuss that of the *developable* surface, or the ones that are exactly mappable onto a plane, which is well-known to have many uses in multiple questions in pure, as well as applied, mathematics. It is also strange that before GAUSS, no one (that I know of) had thought to generalize the new concept that the surface had introduced spontaneously into geometry when one regarded it as flexible. Be that as it may, it is certain that when one abstracts from the simplest case that I mentioned right now, the theory of flexible surfaces presents grave difficulties, which the geometers are invited to make the object of diligent study. The importance and beauty of the fundamental theorem by which GAUSS inaugurated that new branch of analysis leaves no doubt that other theorems of equal or greater fecundity will be the reward for those who penetrate more deeply into that thorny question.

([*]) The preface to this paper is excepted from the note by the author "Intorno all flessione delle superficie rigata" that was read to the Venice Athenaeum on 10 August 1865, which is a note that will not be reproduced here, since it contains only a summary of the more extended paper that is printed in the text. [Ed. note].

([**]) "Allgemeine Auflösung der Aufgabe die Theile einer gegebener Fläche…" This paper was awarded a prize by the Copenhagen Academy in 1822 and published for the first time by SCHUMACHER in the *Astronomsiche Abhandlungen.* – "Disquisitiones generales circa superficie curvas." This paper was published in v. VI of the *Commentationes recentiores* of the Göttingen Academy (1828).

The difficulty in that (as far as I know) proceeds chiefly from the fact that we do not possess a clear idea of the way by which we can effect the flexion of a curved surface in the general case, even in a less-extensive treatment, so one is obliged to trust completely in the bare analysis when one starts from formulas that characterize the inextensibility. We can almost never make good use of the auxiliary considerations – whether direct or indirect – that lead promptly and elegantly to the final objective for the better part of the ordinary problems of analytic geometry.

The truth in that observation seems to me to be confirmed by the developments into which I am about to enter, and which will take into consideration, more specifically, the surfaces that can be generated by the motion of a line. For those surfaces, when considered to be flexible and inextensible, the difficulty in the question is, for the most part, derived from the fact that if one overlooks those flexions whose effect is to make the original generators cease to be rectilinear then it will be possible to get a very clear and simple idea of the manner by which the flexion can be produced. Indeed, any surface of that class can be mentally decomposed into an infinite number of infinitely-thin zones, each of which is found between two contiguous generators, and one can imagine that the flexion of the surface comes about by means of an infinitesimal rotation that is performed on each of those zones around that generator that it has in common with the preceding zone. In that way, it will become clear that the new surface that is produced by a well-defined flexion is defined completely by the elements that characterize the original surface and the series of successive infinitesimal rotations that were just mentioned. Indeed, until 1838, the talented geometer MINDING had expressed the values of the three coordinates of the transformed surface by quadrature by introducing an arbitrary function that represented precisely the law by which those rotations proceeded.

MINDING's solution, which was subsequently discussed once more by BONNET (1848) and BOUR (1860), was undoubtedly invested with all of the desirable analytic generality. However, the presence of an arbitrary function in the final formulas means that one will encounter a series of analytical difficulties when one wishes to determine the nature of the transformed surface that depends upon special conditions that are prescribed *a priori*. Thus, it seems that in those cases, it would be far from advantageous to introduce the conditions that are prescribed for the transformation from the beginning, in such a way that one would arrive at a special solution for it. That process is not unlike the one that has been followed for a long time in many branches of analysis, and thanks to it, the solutions to many problems would become possible, while they would present very great difficulties when treated by the general methods.

§ 1.

Let ξ, η, ζ be the orthogonal coordinates of an arbitrary line that is traced on a ruled surface, which will be regarded as the *director* of that surface and will be subject to the single condition that it should not coincide with one of the rectilinear generators. Let λ, μ, ν be the cosines of the angles that the generator that passes through the point (ξ, η, ζ) makes with the three axes, and let v be the length of the portion of the generator that is found between that point and another arbitrary point of that generator. The surface can be represented by the equations:

(1) $$x = \xi + \upsilon\, l, \qquad y = \eta + \upsilon\, m, \qquad x = \zeta + \upsilon\, n,$$

in which l, m, n are coupled by the usual relation:

(2) $$l^2 + m^2 + n^2 = 1.$$

We suppose that the l, m, n, as well as the ξ, η, ζ, are functions of the arc length u of the director, and we will likewise have the relation:

(3) $$l'^2 + m'^2 + n'^2 = 1,$$

in which the prime denotes the derivative with respect to u.

Take:

(4) $$\begin{cases} l'\xi' + m'\eta' + n'\zeta' = \kappa, \\ \quad l'^2 + m'^2 + n'^2 = \varepsilon'^2, \end{cases}$$

for brevity, and represent the angle that the generator forms with the director by θ; i.e., set:

(5) $$l\,\xi' + m\,\eta' + n\,\zeta' = \cos\theta.$$

That angle, like the other ones that will present themselves in what follows, will be measured in the sense in which one proceeds from the positive direction of the director (i.e., the one in which u increases) to the positive direction of the generator (i.e., that of increasing υ).

If one regards u, υ as the curvilinear coordinates and adopts the known GAUSS nomenclature then one will find that:

(6) $$E = 1 + 2\,\kappa\,\upsilon + \varepsilon'^2\,\upsilon^2, \qquad F = \cos\theta, \qquad G = 1.$$

Now suppose that the surface, which is considered to be flexible and inextensible, changes form in such a way that its rectilinear generators will be converted into other ones. It is clear the director transforms into a certain other curve for which ξ_1, η_1, ζ_1 will denote the coordinates of the point that corresponds to the point (ξ, η, ζ), while the l, m, n will change into l_1, m_1, n_1 . The variables u, υ have the same value at the corresponding points of the two surfaces, so in order for the line element to be identical for the two surfaces, as it must (i.e., in order for the E, F, G to be the same for one surface and the other), it is obviously necessary and sufficient that the three equations must be valid:

(7) $$\begin{cases} \qquad l_1'^2 + m_1'^2 + n_1'^2 = \varepsilon'^2, \\ l_1\,\xi_1' + m_1\,\eta_1' + n_1\,\zeta_1' = \cos\theta, \\ l_1'\,\xi_1' + m_1'\,\eta_1' + n_1'\,\zeta_1' = \kappa, \end{cases}$$

to which one must add the following two:

$$l_1^2 + m_1^2 + n_1^2 = 1, \qquad \xi_1'^2 + \eta_1'^2 + \zeta_1'^2 = 1. \tag{8}$$

Note that in order to have:

$$(l\,\xi' + m\,\eta' + n\,\zeta')' = \kappa + l\,\xi'' + m\,\eta'' + n\,\zeta'',$$

one must have:

$$l\,\xi'' + m\,\eta'' + n\,\zeta'' = -(\kappa + \theta' \sin\theta). \tag{9}$$

Now, let ρ denote the radius of curvature of the original director at the point (u), and let ω be the angle that this radius makes with the tangent to the surface, so one will have:

$$l\,\xi'' + m\,\eta'' + n\,\zeta'' = \sin\theta \frac{\cos\omega}{\rho}, \tag{10}$$

so

$$\sin\theta \frac{\cos\omega}{\rho} = -(\kappa + \theta' \sin\theta). \tag{11}$$

Therefore, by virtue of the preceding, one can substitute the following equation for the last of equations (7):

$$\frac{\cos\omega_1}{\rho_1} = \frac{\cos\omega}{\rho}, \tag{12}$$

which expresses the idea that the geodetic curvatures of the two directors are the same at the corresponding points. That result can be established *a priori* as a consequence of the known fundamental property of that curvature. However, the process keeps teaching us (and this is very important for our purposes) that the three properties that are expressed by the first two equations (7) and (12), which are obviously *necessary* conditions for the identity of the line elements of the two surface, will also be *sufficient* to determine that identity.

Let δw denote the minimum distance between the infinitely-close generators that correspond to the values u and $u + \delta u$, so one has:

$$\delta w = \frac{\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}}{\varepsilon'}\,\delta u, \qquad \varepsilon'^2 \sin^2\theta - \kappa^2 = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ \xi' & \eta' & \zeta' \end{vmatrix}^2, \tag{13}$$

in which the quantity $\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}$, which is always real, can be zero only when the surface is developable. That formula will not be true, or it would be better to say, it will become indeterminate for the cylindrical surface, for which one will have, at the same, $\varepsilon' = \kappa = 0$. In that particular case, one will obviously have:

$$\delta w = \delta u \sin\theta. \tag{13'}$$

The functions ξ_1, η_1, ζ_1, l_1, m_1, n_1, which determine the transformed surface, must satisfy five equations that are equivalent to (7), (8). It is then clear that one of them must be capable of being chosen arbitrarily; that is, the general expression for the aforementioned six quantities must involve an arbitrary function. Such general expressions were posed by MINDING (*), who was the first to address that argument, and by the other authors that followed in his footsteps (**). However, if one can say that the problem is solved analytically in full generality with the explicit introduction of an arbitrary function then things will not be the same when one has the geometric question as one's target. Indeed, it is clear that in order to completely determine the nature of the transformed surface, one can prescribe a new condition that is expressible by means of one finite or differential equation between the ξ_1, η_1, ζ_1, l_1, m_1, n_1, u, v. If we would like to determine MINDING's arbitrary function by means of that equation then most of the time we would encounter very grave analytical difficulties that would have to be overcome in order to treat the six equations between the aforementioned quantities directly.

We propose to show how the second method can be applied to some cases that are chosen from the more interesting ones, with the hope that the simplicity of the calculations and results can tempt other to proceed with that research, which seems to promise an abundant harvest of new and elegant theorems.

§ 2.

If one considers the first conditions (7) to be satisfied and supposes that the geodetic curvature of the first director is zero then it will obviously result from (12) that a necessary and sufficient condition for this is that the geodetic curvature of the transformed director must be zero: That is to say, if the director if the first surface is a geodetic line then it will be necessary and sufficient that it is also a geodetic of the second one, as long as the other two conditions are satisfied. Having said that, one examines whether the transformed director can be a line. Assuming that the line is the z-axis and measuring its length u from the origin, one will have:

$$\xi_1 = 0, \qquad \eta_1 = 0, \qquad \zeta_1 = u, \qquad \frac{\cos\omega_1}{\rho_1} = 0.$$

In order to satisfy the second of the conditions (7), it is obviously enough to suppose that θ is equal to the angle that generator of the transformed surface makes with the z-axis. We therefore set:

$$l_1 = \sin\theta\cos\varphi, \qquad m_1 = \sin\theta\sin\varphi, \qquad n_1 = \cos\theta, \tag{14}$$

from which:

$$l_1'^2 + m_1'^2 + n_1'^2 = \theta'^2 + \varphi'^2\sin^2\theta.$$

(*) Journal für reine und angewandte Mathematik **18** (1838), pp. 297, 365.

(**) O. BONNET, Journal de l'École Polytechnique **19**, cahier 32 (1848), pp. 1; BOUR, *ibid.*, **22**, cahier 39 (1862), pp. 1.

If one then assumes that the original director is a geodetic line of the given surface then the second of those equations will be satisfied identically, and the first one will likewise determine φ by the formula:

$$\varphi = \int \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin\theta}\,du, \tag{15}$$

with which, (14) will give the values of l_1, m_1, n_1. Hence:

Any ruled surface can always be transformed by means of simple flexion in such a way that one of its geodetic lines will become a straight line, and that transformation will depend upon just one quadrature.

Consider, for example, the hyperboloid of rotation:

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1,$$

for which, if one assumes that the circumference of the throat is the director and is one of its geodetic lines then one can set:

$$\xi = a\cos\frac{u}{a}, \qquad \eta = a\sin\frac{u}{a}, \qquad \zeta = 0,$$

$$l = -\cos\theta\sin\frac{u}{a}, \qquad m = \cos\theta\cos\frac{u}{a}, \qquad n = \sin\theta, \qquad \tan\theta = \frac{b}{a}.$$

In that case, formula (15) will give:

$$\varphi = \frac{u\cot\theta}{a} = \frac{u}{b},$$

and therefore:

$$l_1 = \sin\theta\cos\frac{u}{b}, \qquad m_1 = \sin\theta\sin\frac{u}{b}, \qquad n_1 = \cos\theta.$$

The coordinates of the transformed surface, which is a helicoid with a rectilinear director, are then:

$$x = \frac{b\upsilon}{\sqrt{a^2+b^2}}\cos\frac{u}{b}, \qquad y = \frac{b\upsilon}{\sqrt{a^2+b^2}}\sin\frac{u}{b}, \qquad z = u + \frac{a\upsilon}{\sqrt{a^2+b^2}},$$

and when one eliminates u, υ from this, one can deduce that:

$$z = \frac{a}{b}\sqrt{x^2+y^2} + b \arctan\frac{y}{x},$$

which is the equation for a surface that can be mapped to the hyperboloid of rotation whose semi-axes are *a* and *b*.

It is good to observe that when the director is a geodetic line, one will have from (11) that $\kappa = -\theta' \sin\theta$, and therefore, (13):

$$\delta w = \frac{\sin\theta\sqrt{\varepsilon'^2 - \theta'^2}}{\varepsilon'}\,\delta u.$$

It will then result from (15) that the original surface is developable if one has φ = const., and therefore the transformed surface will be a plane. That fact is a necessary consequence of the hypothesis that we assumed, namely, that the generators stay rectilinear under the transformation.

It is worthwhile to point out the following two applications of the general theorem that was proved at the beginning of this §.

1. Any curved surface on which there exists a geodetic line that is normal to all of the generators can obviously be considered to have been generated by the lines that are perpendicular to the osculating planes of a line of double curvature. Now, when that geodetic line transforms into a straight line, the generators of the surface will arrange themselves to all be normal to that line. One can then say that:

Any curved surface that is generated by the perpendiculars to the osculating planes of a line of double curvature can be mapped into a conoidal surface (*).

The formulas that relate to that transformation are quite simple. Indeed, if *l*, *m*, *n* are the cosines of the angles that the three axes make with the normal to the osculating plane of a line of double curvature whose arc length is *u*, while *r* is its radius of torsion, then one will have:

$$\varepsilon' = \frac{1}{r},$$

and then:

$$\xi_1 = 0, \qquad \eta_1 = 0, \qquad \zeta_1 = 0, \qquad \varphi = \int\frac{du}{r},$$

$$l_1 = \cos\varphi, \qquad m_1 = \sin\varphi, \qquad n_1 = 0.$$

2. By virtue of the general theorem that was proved in this §, one will see that the number of ruled surfaces that *rectify* an arbitrary line, whether planar or of double curvature, is *unlimited.* In fact, it is clear that if one draws a line from any point of that line that perpendicular to the principal normal and inclined with respect to the tangent by

(*) ENNEPER, Zeit. Math. Phys. **9** (1864), pp. 398.

an angle that varies by an arbitrary law then it will generate a ruled surface for which the given line will be a geodetic line. One can then always transform that surface in such a way that the line will be converted into a line.

Let a_1, b_1, c_1 ; a_2, b_2, c_2 ; a_3, b_3, c_3 denote the cosines of the angles that the three axes make with the tangent, the principal normal, and the perpendicular to the osculating plane of the given line, resp., and let ρ, r be the radii of first and second curvature, resp. If one lets l, m, n denote the cosines of the angles that the three axes make with the generators of a rectifying ruled surface then one can set:

$$l = a_1 \cos\theta + a_3 \sin\theta, \qquad m = b_1 \cos\theta + b_3 \sin\theta, \qquad n = c_1 \cos\theta + c_3 \sin\theta,$$

from which, by the known formulas of SERRET (*), one will deduce:

$$l' = \left(\frac{\cos\theta}{\rho} + \frac{\sin\theta}{r}\right) a_2 - \theta'(a_1 \sin\theta - a_3 \cos\theta),$$

$$m' = \left(\frac{\cos\theta}{\rho} + \frac{\sin\theta}{r}\right) b_2 - \theta'(b_1 \sin\theta - b_3 \cos\theta),$$

$$n' = \left(\frac{\cos\theta}{\rho} + \frac{\sin\theta}{r}\right) c_2 - \theta'(c_1 \sin\theta - c_3 \cos\theta),$$

and therefore:

(16) $$\varepsilon'^2 = \left(\frac{\cos\theta}{\rho} + \frac{\sin\theta}{r}\right)^2 + \theta'^2.$$

With that value, (15) will become:

$$\varphi = \int \left(\frac{\cot\theta}{\rho} + \frac{1}{r}\right) du .$$

When one determines θ from the equation:

$$\frac{\cot\theta}{\rho} + \frac{1}{r} = 0,$$

the transformed surface will be a plane, and one will then get the *rectifying developable*, which is the only rectifying surface to which the geometers have directed their attention so far. Indeed, the known value of θ that relates to the generators of that surface coincides with the preceding one.

Equation (16) is a special case of a more general formula that will be established elsewhere.

(*) See BERTRAND, *Traité de calcul différentiel*, §§ 590, 591.

§ 3.

The theorem that was proved in the preceding § can be made more intuitive by means of some very simple geometric considerations that will also lead to a more general property.

Indeed, any ruled surface can be considered to be composed of an infinitude of strips, each of which lies between two contiguous generators. Imagine an arbitrary line that is traced on that surface. The tangent plane to the surface at a point on that line will be determined by the direction of the generator that passes through that point and the direction of the element of the curve that terminates at that point. Now it is clear that one can rotate the strip that contains the successive element around the generator that contains the point that is common to it and that element until the other end of the second element, and therefore all of its elements, come to lie in the tangent plane that contains the first element. In the same way, one can rotate the third strip until the element that is contained in it is arranged in the plane that is determined by the preceding element and the generator that is common to the second and third strip, and so on. In that way, the original ruled surface will be transformed into another one on which the transformed curve is found to be traced in such a way that each pair of consecutive elements exists in the tangent plane to that surface. That is to say, the transformed curve has all of its osculating planes tangent to the transformed surface and is therefore an *asymptotic* line of that surface. Hence:

Any ruled surface can always be transformed in such a way that an arbitrary line that is traced on it will become an asymptotic line of the transformed surface.

When the original line is a geodetic, no matter how one bends the surface on which it is traced, it must always continue to be a geodetic on the transformed surface. It cannot become an asymptotic line without transforming into a straight line then, since in any case, it will be impossible that its osculating planes are, at the same time, normals and tangent to the transformed surfaces. One will then arrive at the theorem of the preceding §.

If the line that one considers is an orthogonal trajectory of the generators then it will be clear that when it is transformed into an asymptotic line, its principal normals will be directed along the generators of the transformed surface. One concludes from this that *one can always turn all of the generators of a ruled surface into principal normals to one of its orthogonal trajectories by an opportune flexion* (*).

It is easy to find the formulas that relate to the transformation under discussion.

Indeed, (12) will initially give:

$$\frac{1}{\rho_1} = \frac{\cos \omega}{\rho},$$

since $\omega_1 = 0$ for the asymptotic line. That confirms the observation that if the first director is a geodetic then the transformed one will be a straight line. By virtue of (11), the preceding equation can be written:

(*) BOUR, J. École Poly. **22**, cahier 39 (1862), pp. 52.

(17) $$\rho_1 = -\frac{\sin\theta}{\kappa + \theta' \sin\theta}.$$

When the curve that one considers is the line of striction, one will have $\kappa = 0$, and therefore:

$$\rho_1 = -\frac{1}{\theta'} = -\frac{du}{d\theta},$$

or, if one calls the contingency angle of the transformed line θ then:

$$d\eta + d\theta = 0,$$

which is a formula that one deduces from a known theorem (*).

Now let $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$ denote the cosines of the angles that the three axes make with the tangent, the principal normal, and the perpendicular to the osculating plane, resp., of the transformed directors. If one observes that the generator of the transformed surface is in the osculating plane of that curve and makes an angle of θ with it then one will see that:

(18) $$l_1 = \alpha_1 \cos\theta + \alpha_2 \sin\theta, \qquad m_1 = \beta_1 \cos\theta + \beta_2 \sin\theta, \qquad n_1 = \gamma_1 \cos\theta + \gamma_2 \sin\theta,$$

from which, if one recalls the SERRET relations that were cited above, one can deduce that:

$$l_1'^2 + m_1'^2 + n_1'^2 = \frac{\sin^2\theta}{r_1^2} + \left(\frac{1}{\rho_1} + \theta'\right)^2,$$

in which r_1 is the radius of torsion of the transformed curve. One gets from this [(7), (17)] that:

(19) $$r_1 = \frac{\sin^2\theta}{\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}}.$$

The two equations (17), (19), along with (3), define the three functions ξ_1, η_1, ζ_1 completely, and when one knows them, (18) will give l_1, m_1, n_1 .

The value of r_1 will become infinite only when $\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2} = 0$; i.e., the transformed curve can be planar only when the original surface is developable, which is also clear in its own right. We then need to emphasize the case in which the director coincides with the edge of regression of the developable surface, since one will then have $\theta = 0$, $\kappa = 0$, and the preceding formula will become indeterminate, as it must. Indeed, no matter how one folds a developable surface, while its original generators stay rectilinear, it will be clear that the edge of regression must always preserve the characteristic

(*) PAUL SERRET, *Théorie nouvelle géométrique et mécanique des lignes à double courbure*, Paris, 1860, pp. 150.

property of the asymptotic lines, and whereas its curvature of the first type will be remain invariant at each point, the torsion can take on values that vary according to arbitrary law.

When the director coincides with the line of striction of the surface (which is not supposed to be developable), (19) will yield the simple formula:

$$r_1 = \frac{\sin\theta}{\varepsilon'},$$

which will coincide with one that we encountered before in the penultimate application of § **2**.

The transformed director will prove to be defined, under the present transformation, by the expressions of the two radii of first and second curvature as functions of the arc length. The problem of determining a curve under those conditions was treated recently by HOPPE (*). It is clear, moreover, that the complete integration of the three equations (17), (19), (3), which have orders 2, 3, and 1, resp., must introduce six arbitrary constants that can be determined by fixing the absolute position of the new direction in space. All of the surfaces that have been transformed in that way can therefore differ from each other only in position, which would also emerge from the preceding geometric considerations.

§ 4.

If one takes:

$$A = mn' - m'n, \qquad B = nl' - n'l, \quad C = lm' - l'm,$$

for brevity, then one will deduce from the three equations:

$$l\,\xi' + m\,\eta' + n\,\zeta' = \cos\theta,$$

$$l'\xi' + m'\eta' + n'\zeta' = \kappa,$$

$$\xi'^2 + \eta'^2 + \zeta'^2 = 1,$$

the following values:

$$(20)\qquad \left\{\begin{aligned} \xi' &= l\cos\theta + \frac{\kappa l' \pm A\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}}{\varepsilon'^2}, \\ \eta' &= m\cos\theta + \frac{\kappa m' \pm B\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}}{\varepsilon'^2}, \\ \zeta' &= n\cos\theta + \frac{\kappa n' \pm C\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}}{\varepsilon'^2}, \end{aligned}\right.$$

(*) Jour. f. d. reine u. angew. Math. **60** (1862), pp. 182; *ibid.* **63** (1864), pp. 122.

which are formulas in which the radical must be taken with the same sign in all three.

If one excludes the case of developable surfaces then the two systems of values for ξ', η', ζ' will always be essentially different, since the quantity $\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}$ cannot be zero independently of u. It will then be clear that only one of those systems considered will coincide with the values of ξ', η', ζ' that were previously known from the derivatives; in other words, in order for the given substitution of value for ξ, η, ζ, l, m, n to make the preceding three formulas identities, one will need to give a well-defined sign to the radicals.

Having said that, observe that if one sets:

(21) $$l_1 = l, \qquad m_1 = m, \qquad n_1 = n$$

then the five equations (7), (8) will reduce to just the three equations:

$$l\,\xi_1' + m\,\eta_1' + n\,\zeta_1' = \cos\theta,$$

$$l'\,\xi_1' + m'\,\eta_1' + n'\,\zeta_1' = \kappa;$$

$$\xi_1'^2 + \eta_1'^2 + \zeta_1'^2 = 1,$$

which differ by only the substitution of ξ_1', η_1', ζ_1' for ξ', η', ζ' from the ones that provided the values (20), and which must therefore give values to the first three quantities are identical to them. Hence, if one takes the radicals in those formulas to have the opposite sign to the one that makes their right-hand sides equal to ξ', η', ζ' identically then one will have values for ξ_1', η_1', ζ_1' that correspond to a curved surface that is distinct from the given one such that its line element will be identical to that of the first surface and its generators will be parallel to the corresponding generators of the first surface. Therefore:

It is always possible to transform a curved surface in such a way that each generator of the transformed surface will be parallel to the corresponding generator of the original one.

We shall now make some applications of that theorem.

1. First of all, consider a surface that is endowed with a rectilinear generator and set:

$$\xi = 0, \qquad \eta = 0, \qquad \zeta = u,$$

$$l = \sin\theta\cos\varphi, \qquad m = \sin\theta\sin\varphi, \qquad n = \cos\theta,$$

so

$$\varepsilon' = \sqrt{\theta'^2 + \varphi'^2\sin^2\theta}, \qquad \kappa = -\,\theta'\sin\theta, \qquad \sqrt{\varepsilon'^2\sin^2\theta - \kappa^2} = \varphi'\sin^2\theta.$$

Substituting those values in the formulas (20) and taking the convenient sign, one will find that:

$$\xi_1' = \frac{\varphi' \sin^2\theta}{\theta'^2 + \varphi'^2 \sin^2\theta} (\varphi' \sin 2\theta \cos\varphi + 2\theta' \sin\varphi),$$

$$\eta_1' = \frac{\varphi' \sin^2\theta}{\theta'^2 + \varphi'^2 \sin^2\theta} (\varphi' \sin 2\theta \sin\varphi - 2\theta' \cos\varphi),$$

$$\zeta_1' = \frac{\theta'^2 + \varphi'^2 \cos 2\theta \sin^2\theta}{\theta'^2 + \varphi'^2 \sin^2\theta},$$

which are formulas that resolve to simply:

$$\xi_1 = \sin 2\theta \int \cos\varphi \, du, \qquad \eta_1 = \sin 2\theta \int \sin\varphi \, du, \qquad \zeta_1 = u \cos 2\theta,$$

in which θ is constant.

2. Suppose that the original surface is composed of the principal normals to a line of double curvature, and then set:

$$l = a_2, \qquad m = b_2, \qquad n = c_2,$$

from which, one will deduce that:

$$l' = -\left(\frac{a_1}{\rho} + \frac{a_3}{r}\right), \qquad m' = -\left(\frac{b_1}{\rho} + \frac{b_3}{r}\right), \qquad n' = -\left(\frac{c_1}{\rho} + \frac{c_3}{r}\right).$$

One will find that:

$$\theta = \frac{\pi}{2}, \qquad \varepsilon'^2 = \frac{1}{\rho^2} + \frac{1}{r^2}, \qquad \kappa = -\frac{1}{\rho}, \qquad \sqrt{\varepsilon'^2 - \kappa^2} = \frac{1}{r},$$

$$A = \frac{a_3}{\rho} - \frac{a_1}{r}, \qquad B = \frac{b_3}{\rho} - \frac{b_1}{r}, \qquad C = \frac{c_3}{\rho} - \frac{c_1}{r}.$$

If one substitutes those values in formulas (20), while taking a convenient sign, and takes:

$$\frac{\rho}{r} = \tan \tfrac{1}{2}\Theta$$

then one will find the following formulas:

$$\xi_1' = a_1 \cos \Theta + a_3 \sin \Theta, \qquad \eta_1' = b_1 \cos \Theta + b_3 \sin \Theta, \qquad \zeta_1' = c_1 \cos \Theta + c_3 \sin \Theta,$$

from which, one can deduce the coordinates of the transformed director by integration in special cases.

One gets the relation:

$$\xi_1' a_1 + \eta_1' b_1 + \zeta_1' c_1 = \cos \Theta$$

from that, which says that the angle between the tangents to the two directors at the corresponding points will be equal to Θ.

If one differentiates these formulas then one will find that:

$$\xi_1'' = \frac{a_2}{\rho} + (a_3 \cos \Theta - a_1 \sin \Theta)\, \Theta',$$

$$\eta_1'' = \frac{b_2}{\rho} + (b_3 \cos \Theta - b_1 \sin \Theta)\, \Theta',$$

$$\zeta_1'' = \frac{c_2}{\rho} + (c_3 \cos \Theta - c_1 \sin \Theta)\, \Theta',$$

from which, one infers:

$$\frac{1}{\rho_1^2} = \frac{1}{\rho^2} + \Theta'^2 .$$

In addition:

$$\xi_1'' a_2 + \eta_1'' b_2 + \zeta_1'' c_2 = \frac{1}{\rho},$$

and therefore, if one calls the angle that the principal normal to the transformed director makes with the original one ψ then:

$$\cos \psi = \frac{\rho_1}{\rho} = \frac{1}{\sqrt{1+\rho^2\Theta'^2}} .$$

In order for the principal normals to the two curves to be parallel (and therefore, for the principal normals of the transformed curve to coincide with the generators of the second surface, as well), one will need to have $\cos \psi = 1$ and therefore $\rho_1 = \rho$, $\Theta = \text{const.}$, $\rho / r = \text{const.}$ As is known, the last equation is true for only the cylindrical helices. One likewise sees that in that special case the transformed director will be a helix that is traced upon that cylinder, and will be equal and symmetric to the first one with respect to plane that is normal to the generators of the cylinder. Two corresponding points can be found on that generator.

3. Consider surfaces that have their lines of striction orthogonal to the generators; i.e., ones that are composed of the perpendiculars to the osculating plane to a line with double curvature. In that case, one will have:

$$\cos\theta = 0, \qquad \kappa = 0, \qquad l = a_3, \qquad m = b_3, \qquad n = c_3,$$

from which:

$$\varepsilon' = \frac{1}{r}, \qquad A = -\frac{a_1}{r}, \qquad B = -\frac{b_1}{r}, \qquad C = -\frac{c_1}{r},$$

and therefore:

$$\xi_1' = -a_1, \qquad \eta_1' = -b_1, \qquad \zeta_1' = -c_1.$$

If one integrates then one will get:

$$\xi_1 = x_0 - \xi, \qquad \eta_1 = y_0 - \eta, \qquad \zeta_1 = z_0 - \zeta,$$

from which, one will see that the transformed director is simply symmetric to the original one with respect to the point that has the coordinates $\frac{1}{2}x_0$, $\frac{1}{2}y_0$, $\frac{1}{2}z_0$; i.e., that this (arbitrary) point will bisect all of the lines that join two corresponding points of the two directors.

In § **6**, we will find another example of parallelism of the corresponding generators for two surfaces that transform into each other.

§ 5.

Now consider the case in which the director must transform into a plane curve. If one assumes that the plane of the transformed curve is the *xy*-plane then one will have $\zeta_1 = 0$, and the equations of the transformation will be the following ones:

$$\text{(22)} \qquad \left\{ \begin{array}{c} l_1\,\xi_1' + m_1\,\eta_1' = \cos\theta, \quad l_1'\,\xi_1' + m_1'\,\eta_1' = \kappa, \quad \xi_1'^2 + \eta_1'^2 = 1, \\ l_1^2 + m_1^2 + n_1^2 = 1, \quad l_1'^2 + m_1'^2 + n_1'^2 = \varepsilon'^2, \end{array} \right.$$

and their number will be equal to the number of functions to be determined.

We begin by excluding the case in which we have:

$$\cos\theta = 0, \qquad \kappa = 0,$$

simultaneously; i.e., the case in which the line of striction is an orthogonal trajectory to the generators, which is a case that was consider many times before. Under that hypothesis, the first two equations (22) can be written:

$$l_1\,\xi_1' + m_1\,\eta_1' = 0, \qquad l_1\,\xi_1'' + m_1\,\eta_1'' = 0,$$

and in order to satisfy them, we need to supposes that:

$$l_1 = m_1 = 0, \qquad n_1 = 1,$$

or

$$\xi_1' \eta_1'' - \xi_1'' \eta_1' = 0.$$

The first solution is admissible only when $\varepsilon' = 0$, and that will make the given surface be cylindrical. In that case, the transformation will remain indeterminate, which should be obvious *a priori*. In the second case, one will have $1 / \rho_1 = 0$, and then the transformed director will be a straight line; i.e., one would get back to ENNEPER's theorem, which was proved already in § **2**.

Hence, if one excludes those two cases then the first two equations in (22) will give:

$$(l_1 m_1' - l_1' m_1) \xi_1' = m_1' \cos\theta - m_1 \kappa,$$

$$(l_1 m_1' - l_1' m_1) \eta_1' = l_1 \kappa - l_1' \cos\theta,$$

from which, when one squares and sums, one will deduce that:

$$(l_1 m_1' - l_1' m_1)^2 = (\varepsilon'^2 - \eta_1'^2) \cos^2\theta + (1 - n_1^2)\,\kappa^2 + 2\kappa n_1 n_1' \cos\theta.$$

However:

$$(l_1 m_1' - l_1' m_1)^2 = (l_1^2 + m_1^2)(l_1'^2 + m_1'^2) - (l_1 l_1' + m_1 m_1')^2 = \varepsilon'^2 (1 - n_1^2) - n_1'^2,$$

so one substitutes:

$$(\varepsilon'^2 - \eta_1'^2) \sin^2\theta = \kappa^2 + (\varepsilon'^2 - \kappa^2) n_1^2 + 2\kappa n_1 n_1' \cos\theta,$$

in which:

(23) $$n_1' \sin^2\theta + \kappa n_1 \cos\theta = \sqrt{\sin^2\theta - n_1^2}\,\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}\,.$$

That differential equation serves to determine n_1 ; if one knows that quantity then the l_1, m_1, ξ_1, η_1 will be give by simple quadratures. Indeed, if one sets:

(24) $$l_1 = \sin\varphi \cos\psi, \qquad m_1 = \sin\varphi \sin\psi, \qquad n_1 = \cos\psi$$

then one will have:

$$\varepsilon'^2 = \varphi'^2 + \psi'^2 \sin^2\varphi,$$

and thus:

(25) $$\psi = \int \frac{\sqrt{\varepsilon'^2 - \varphi'^2}}{\sin\varphi}\, du\,,$$

which is an equation that will give ψ, and therefore l_1, m_1 . The preceding values of ξ_1', η_1' then assume the form:

$$\text{(26)} \qquad \xi_1' = \frac{m_1' \cos\theta - \kappa m_1}{\psi' \sin^2\varphi}, \qquad \eta_1' = -\frac{l_1' \cos\theta - \kappa l_1}{\psi' \sin^2\varphi},$$

and will provide the coordinates of the transformed director upon integration.

Since none of the operations that are necessary to solve the present problem imply any impossibility, one can generally state the theorem:

Any ruled surface can always be transformed in such a way that any one if its lines will become planar.

Let us consider some special cases.

1. The director is an orthogonal trajectory of the generators.

If one has $\theta = \pi / 2$ then (23) will become:

$$n_1' = \sqrt{1 - n_1^2}\sqrt{\varepsilon'^2 - \kappa^2},$$

from which, when one compares this with (24), (25), one will infer that:

$$\varphi = -\int \sqrt{\varepsilon'^2 - \kappa^2}\, du, \qquad \psi = \int \frac{\kappa\, du}{\sin\varphi}.$$

If one substitutes the values of l_1, m_1, ψ in (26) then one will have:

$$\xi_1' = -\sin\psi, \qquad \eta_1' = \cos\psi,$$

from which:

$$\xi_1'' = -\cos\psi \cdot \psi', \qquad \eta_1'' = -\sin\psi \cdot \psi',$$

and therefore:

$$\rho_1 = \pm\frac{\sin\varphi}{\kappa},$$

which is an expression that could have been deduced from (11), (12), which will give:

$$\frac{\cos\omega_1}{\rho_1} = -\kappa$$

for $\theta = \pi / 2$, and one observes that φ is obviously the complement of the angle that the tangent plane to the transformed surface makes with the new director; i.e., $\varphi = \pi / 2 - \omega_1$.

If one supposes that the surface is composed of principal normals to a line of double curvature, and one therefore has:

$$l = a_2\,, \qquad m = b_2\,, \qquad n = c_2\,, \qquad \varepsilon'^2 = \frac{1}{\rho^2} + \frac{1}{r^2}\,, \qquad \kappa = -\frac{1}{\rho}\,,$$

then one will find that:

$$\varphi = -\int \frac{du}{r}\,, \qquad \psi = -\int \frac{du}{\rho \sin\varphi}\,,$$

which will give:

$$\xi_1 = -\int \sin\psi\, du\,, \quad \eta_1 = \int \cos\psi\, du\,, \qquad \rho_1 = \rho \sin\varphi,$$

$$l_1 = \sin\varphi \cos\psi, \quad m_1 = \sin\varphi \sin\psi, \qquad n_1 = \cos\varphi\,.$$

In that case, one sees that ψ coincides in absolute value with all of the angles of torsion of the line considered.

Of the four arbitrary constants that enter into that formula, three of them correspond to a simple displacement of the transformed director in the *xy*-plane, but the fourth one corresponds to transformations that are truly distinct from each other, in general.

2. The director is a geodetic line.

Since one has, from (11), $\kappa = -\theta' \sin\theta$ in that case, equations (23) will become:

$$n_1' \sin\theta - n_1 \cos\theta \cdot \theta' = \sqrt{\sin^2\theta - n_1^2}\,\sqrt{\varepsilon'^2 - \theta'^2}\,.$$

That equation has the general integral:

$$n_1 = \sin\theta \sin\left[\int \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin\theta}\, du\right],$$

and possesses a singular integral, in addition, which is:

$$n_1 = \sin\theta\,.$$

A simple consideration will show that the solution to our problem is contained in that singular integral. Indeed, when a geodetic line transforms into a planar line, it will continue to be a geodetic line on the transformed surfaces, and then the lines normal to that surface at its points must be found in its plane and be normal to that curve. Consequently, the generators of the transformed surface must project onto the plane of the new director tangentially to that director, and must therefore make an angle θ with the *xy*-plane on which it is traced. One must then have $n_1 = \sin\theta$, which is precisely what the singular integral expresses.

That argument will cease to be correct only when the director is transformed into a straight line. That case, which was treated already in § **2**, corresponds precisely to the general integral, so we shall not address it.

Since one then has $\varphi = \pi / 2 - \theta$, one can infer from (25) that:

$$\psi = \int \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\cos\theta} du ,$$

so

$$l_1 = \cos\theta \cos\psi, \qquad m_1 = \cos\theta \sin\psi, \qquad n_1 = \sin\theta ;$$

after that, one will deduce from (26) that:

$$\xi_1 = \int \cos\psi \, du , \qquad \eta_1 = \int \sin\psi \, du .$$

One will then have:

$$\xi_1'' = -\sin\psi \cdot \psi', \qquad \eta_1'' = \cos\psi \cdot \psi',$$

from which:

$$\frac{1}{\rho_1} = \pm \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\cos\theta} .$$

All of the elements of the transformed surface are thus determined.

To avoid any misunderstanding, recall that the present transformation cannot be applied to the case in which the geodetic meets all of the generators orthogonally, or it would be better to say that the transformed planar line could not be anything but a straight line in that case.

Furthermore, observe that the transformation that was considered in this § can be treated more directly in the case of the geodetic line by deducing the values of l_1, m_1, n_1 from the equations:

$$l_1 \xi_1' + m_1 \eta_1' = \cos\theta,$$

$$l_1 \xi_1'' + m_1 \eta_1'' = 0,$$

$$l_1^2 + m_1^2 + n_1^2 = 1$$

and substituting them in:

$$l_1'^2 + m_1'^2 + n_1'^2 = \varepsilon'^2,$$

and in that way one will get an equation in the second derivatives in ξ_1 , η_1 that is equivalent to:

$$\frac{1}{\rho_1} = \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\cos\theta} ,$$

which can also be obtained by a different method, and then, when combined with:

$$\xi_1'^2 + \eta_1'^2 = 1,$$

we would be given the same values that were found just recently. If one applies that process to the proof of the theorem that will define the objective of the following § then that will serve to establish a general formula later on.

§ 6.

Any ruled surface can always be transformed in such a way that any of its geodetic lines transforms into a cylindrical helix.

Indeed, suppose that the cylinder on which the helix must be traced has its generators parallel to the *z*-axis, and call the constant angle that is formed between the helix and those generators μ_1 . One needs to set:

$$\zeta_1' = \cos \mu_1 , \tag{27}$$

and that equation, when combined with the following five:

$$\left\{ \begin{array}{c} l_1 \xi_1' + m_1 \eta_1' = \cos\theta - n_1 \cos\mu_1, \quad l_1 \xi_1'' + m_1 \eta_1'' = 0, \quad \xi_1'^2 + \eta_1'^2 = \sin^2 \mu_1, \\ l_1^2 + m_1^2 + n_1^2 = 1, \quad l_1'^2 + m_1'^2 + n_1'^2 = \varepsilon'^2, \end{array} \right. \tag{28}$$

will determine completely the six quantities that relate to the transformed surface, which will prove the possibility of the transformation in which it is obvious that μ_1 can assume arbitrary values. We need to emphasize the case in which one has $\cos \mu_1 = 0$, $\cos \theta = 0$ simultaneously, which is a case that was excluded before in the preceding § and which was spoken of before; or more generally, in which one has $\mu_1 = 0$ if θ is constant.

If one first observes that one has, in general:

$$\begin{aligned} \frac{1}{\rho_1^2} &= (\eta_1' \zeta_1'' - \eta_1'' \zeta_1')^2 + (\zeta_1' \xi_1'' - \zeta_1'' \xi_1')^2 + (\xi_1' \eta_1'' - \xi_1'' \eta_1')^2 \\ &= \xi_1''^2 + \eta_1''^2 + \zeta_1''^2 \end{aligned}$$

then one will have in the present case that:

$$\frac{1}{\rho_1^2} = (\xi_1''^2 + \eta_1''^2) \cos^2 \mu_1 + (\xi_1' \eta_1'' - \xi_1'' \eta_1')^2 = \xi_1''^2 + \eta_1''^2 ,$$

so

$$\frac{1}{\rho_1^2} = \xi_1''^2 + \eta_1''^2 , \qquad \xi_1' \eta_1'' - \xi_1'' \eta' = \frac{\sin \mu_1}{\rho_1} . \tag{29}$$

From the second of these equations, the third of (28), and:

$$\xi_1' \xi_1'' - \eta' \eta_1'' = 0,$$

one can deduce the following values, in addition:

(30) $$\xi_1'' = -\frac{\eta_1'}{\rho_1 \sin \mu_1}, \qquad \eta_1'' = \frac{\xi_1'}{\rho_1 \sin \mu_1}.$$

Having said that, when one recalls (30), the first two equations in (28) will give:

$$l_1 \sin^2 \mu_1 = (\cos \theta - n_1 \cos \mu_2)\ \xi_1',$$

$$m_1 \sin^2 \mu_1 = (\cos \theta - n_1 \cos \mu_2)\ \eta_1',$$

from which, upon squaring and summing, one will get:

$$n_1^2 - 2n_1 \cos \mu_1 \cos \theta + \cos^2 \theta - \sin^2 \mu_1 = 0,$$

and therefore:

$$n_1 = \cos (\mu_1 \pm \theta), \qquad \cos \theta - n_1 \cos \mu_1 = \sin \mu_1 \sin (\mu_1 \pm \theta).$$

Consequently, if one takes just the lower sign in order to remain in agreement with the convention that was established in § **1** then one will have:

(31) $$l_1 = \frac{\xi_1' \sin(\mu_1 - \theta)}{\sin \mu_1}, \qquad m_1 = \frac{\eta_1' \sin(\mu_1 - \theta)}{\sin \mu_1}, \quad n_1 = \cos(\mu_1 - \theta).$$

If one substitutes these values in the last of (28) then one will find that:

$$\varepsilon'^2 = \frac{\sin^2(\mu_1 - \theta)}{\rho_1^2 \sin^2 \mu_1} + \theta'^2,$$

from which:

(32) $$\frac{1}{\rho_1} = \frac{\sin \mu_1 \sqrt{\varepsilon'^2 - \theta'^2}}{\sin(\mu_1 - \theta)}.$$

If one lets R' denote the curvature of the cross-section of the cylinder on which the helix is traced then one will have, as is known, $R' = \rho_1 \sin^2 \mu_1$, and therefore:

(33) $$\frac{1}{R'} = \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin \mu_1 \sin(\mu_1 - \theta)}.$$

If one substitutes the value of ρ_1 that is given in (32) in (30) and observes the third of equations (28) then one will find the formula:

$$\frac{\xi_1''}{\sqrt{\sin^2\mu_1 - \xi_1'^2}} = -\frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin(\mu_1 - \theta)}, \qquad \frac{\eta_1''}{\sqrt{\sin^2\mu_1 - \eta_1'^2}} = \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin(\mu_1 - \theta)},$$

from which, when one sets:

$$(34) \qquad \varphi = \int \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin(\mu_1 - \theta)} du,$$

one will infer that:

$$(35) \qquad \xi_1 = \sin\mu_1 \int \cos\varphi\, du, \qquad \eta_1 = \sin\mu_1 \int \sin\varphi\, du, \qquad \zeta_1 = u\cos\mu_1,$$

and therefore:

$$(36) \qquad l_1 = \sin(\mu_1 - \theta)\cos\varphi, \qquad m_1 = \sin(\mu_1 - \theta)\sin\varphi, \qquad n_1 = \cos(\mu_1 - \theta).$$

We have then arrived at all of the formulas that relate to our question by means of only quadratures.

Let us make the following two applications:

1. The formulas:

$$\xi = a\cos\frac{u\sin\mu}{a}, \qquad \eta = a\sin\frac{u\sin\mu}{a}, \qquad \zeta = u\cos\mu,$$

$$l = -\sin\nu\sin\frac{u\sin\mu}{a}, \qquad m = \sin\nu\cos\frac{u\sin\mu}{a}, \qquad n = \cos\nu$$

represent a ruled helicoid whose helix of striction (whose arc length is u) is traced on a cylinder of radius a, and makes an angle of μ with its generators, while ν is the angle that the generators of the helicoid make with those of the cylinder.

If one takes:

$$\frac{a\sin(\nu + \mu_1 - \mu)\sin\mu_1}{\sin\nu\sin\mu} = a_1,$$

for brevity, in which μ is an arbitrary constant, then one will find that φ can be taken to be equal to $\frac{u\sin\mu_1}{a_1} + \frac{\pi}{2}$, since the values of the constants that are added to the integral (34) will not influence its absolute position in the transformed surface; if one substitutes that value of φ in formulas (35), (36) then one will have:

$$\xi_1 = a_1\cos\frac{u\sin\mu_1}{a_1}, \quad \eta_1 = a_1\sin\frac{u\sin\mu_1}{a_1}, \quad \zeta_1 = u\cos\mu_1,$$

$$l_1 = -\sin(\nu + \mu_1 - \mu)\sin\frac{u\sin\mu_1}{a_1}, \qquad m_1 = \sin(\nu + \mu_1 - \mu)\cos\frac{u\sin\mu_1}{a_1},$$

$$n_1 = \cos(\nu + \mu_1 - \mu),$$

which is a formula that is perfectly analogous to the one that represented the first helicoid, and that shows that a_1, μ_1, $\nu + \mu_1 - \mu$ are quantities with the same significance as a, μ, ν with respect to the transformed helicoid.

If one sets $\mu_1 = 0$ then the helicoid will have a rectilinear director, and if one sets $\mu_1 = \pi / 2 + \mu - \nu$ then the helicoid will have a director plane. Finally, if one sets $\mu_1 = \pi / 2$ then one will have a hyperboloid of rotation, which is a surface to which all of the helicoids that are contained in the preceding equations can be mapped, as is known. The formulas that relate to this case are:

$$\xi_1 = \frac{a\cos(\mu-\nu)}{\sin\mu\sin\nu}\cos\frac{u\sin\mu\sin\nu}{a\cos(\mu-\nu)}, \quad \eta_1 = \frac{a\cos(\mu-\nu)}{\sin\mu\sin\nu}\sin\frac{u\sin\mu\sin\nu}{a\cos(\mu-\nu)}, \quad \zeta_1 = 0,$$

$$l_1 = -\cos(\mu-\nu)\sin\frac{u\sin\mu\sin\nu}{a\cos(\mu-\nu)}, \quad m_1 = \cos(\mu-\nu)\cos\frac{u\sin\mu\sin\nu}{a\cos(\mu-\nu)},$$

$$n_1 = \sin(\mu-\nu),$$

which will give:

$$\frac{x^2+y^2}{\cos^2(\mu-\nu)} - \frac{z^2}{\sin^2(\mu-\nu)} = \frac{a^2}{\sin^2\mu\sin^2\nu}$$

when they are substituted in (1) and one eliminates u, v.

In the exceptional case that was mentioned to begin with, this is presently verified for $\mu_1 = \mu - \nu$, as is easy to see *a posteriori*. That value of the constant μ_1 must therefore be considered to have been excluded.

2. Suppose that the director of the first surface is the z-axis, and therefore take:

$$\xi = 0, \quad \eta = 0, \quad \zeta = u,$$

$$l = \sin\theta\cos\psi, \qquad m = \sin\theta\sin\psi, \qquad n = \cos\theta.$$

In that case, one will have from (34):

$$\varphi = \int \frac{\psi'\sin\theta}{\sin(\mu_1-\theta)}\,du,$$

which is a value that will yield the desired transformation when it is substituted in formulas (35), (36).

If one supposes that θ is constant and equal to $\mu_1 / 2$ then one will have simply $\varphi = \psi$, and therefore:

$$l_1 = l, \qquad m_1 = m, \qquad n_1 = n\ ;$$

i.e., the generators of the transformed surface will be parallel to those of the original one, which is a particular case of the question that was treated in § **4**. We will then get a theorem that can be stated as follows:

Trace a helix on a cylinder with an arbitrary base that forms the arbitrary angle μ *with the generators of the cylinder, and slide a line that is inclined with respect to the generators by the angle* $\mu / 2$ *along that helix. The curved surface that is obtained in that way can be superimposed with that generator by a line that one keeps constantly parallel to the generators of the preceding surface, while one if its points moves along an axis that is parallel to the generators of the cylinder and sweeps out lengths along that axis that are constantly equal to the corresponding arc lengths of the helix.*

§ 7.

The geometric considerations that we made use of in § **3** can serve to easily prove that one can always transform a ruled surface in such a way that its generators become parallel to those of a director cone that is given arbitrarily (which can therefore never be reduced to a simple line, except when the surface is a cylinder). That property was established before by MINDING (*loc. cit.*) and more recently by BOUR (*), who based it upon an ingenious classification of the ruled surfaces. Thus, we shall not return to that argument and will limit ourselves to presenting some considerations that relate to the case in which the cone that is assumed to be the director is a line.

Under that hypothesis, let λ denote the angle that the generators of the cone make with its axis, which is supposed to be parallel to Oz, so one can set:

$$l_1 = \sin\lambda\cos\varphi, \qquad m_1 = \sin\lambda\sin\varphi, \qquad n_1 = \cos\lambda,$$

from which:

$$\varepsilon' = \varphi' \sin\lambda\ .$$

The equations that must be satisfied by the functions ξ_1, η_1, ζ_1 are then the following ones:

$$(37) \qquad \left\{ \begin{aligned} &\xi_1'^2 + \eta_1'^2 + \zeta_1'^2 = 1, \\ &(\xi_1'\cos\varphi + \eta_1'\sin\varphi)\sin\lambda + \zeta_1'\cos\lambda = \cos\theta, \\ &-(\xi_1'\sin\varphi - \eta_1'\cos\varphi)\,\varepsilon' = \kappa. \end{aligned} \right.$$

The projection of the transformed generator onto the xy-plane is represented by:

$$(x - \xi_1)\sin\varphi - (y - \eta_1)\cos\varphi = 0,$$

(*) Cited paper, page 43.

and therefore the envelope of all the analogous projection must satisfy the equation:

$$[(x - \xi_1)\cos\varphi + (y - \eta_1)\sin\varphi]\,\varphi' = \xi_1'\sin\varphi - \eta_1'\cos\varphi,$$

or, from (1):

(38) $$\xi_1'\sin\varphi - \eta_1'\cos\varphi = \upsilon\,\varphi'\sin\lambda\,.$$

This equation between u and υ (which one would likewise arrive at if one had supposed that the angle λ was variable) represents the curve on the transformed surface along which it is enveloped by a cylindrical surface that has its generators parallel to the z-axis. The corresponding curve on the original surface is represented by an equation between u and υ that one can write as:

$$\kappa + \upsilon\,\varepsilon'^2 = 0,$$

by virtue of the last equation (37), and under the hypothesis that λ is constant and therefore $\varphi'\sin\lambda = \varepsilon'$. If one substitutes the value of υ that one infers from that equation in (1) then one will get the rectangular coordinates of the curve in question, which will be:

$$x = \xi - \frac{\kappa l}{\varepsilon'^2}, \qquad y = \eta - \frac{\kappa m}{\varepsilon'^2}, \qquad z = \zeta - \frac{\kappa n}{\varepsilon'^2}.$$

One deduces from this that:

$$x' = \xi' - \frac{\kappa l'}{\varepsilon'^2} - l\left(\frac{\kappa}{\varepsilon'^2}\right)',$$

$$y' = \eta' - \frac{\kappa m'}{\varepsilon'^2} - m\left(\frac{\kappa}{\varepsilon'^2}\right)',$$

$$z' = \zeta' - \frac{\kappa l'}{\varepsilon'^2} - n\left(\frac{\kappa}{\varepsilon'^2}\right)',$$

and therefore:

$$l'x' + m'y' + n'z' = 0,$$

which is a result that says that the line in question is nothing but the line of striction. Hence:

The line of striction of a ruled surface that has all of its generators inclined the same with respect to a fixed plane is the line of contact between that surface and the cylinder that is normal to the plane that involves (involvente) the given surface.

Conversely:

If the line of contact between a curved surface and an involved cylindrical surface is the line of striction of the first surface then its generators are all inclined the same with respect to those of the second one.

Indeed, if one supposes that λ is variable and considers the line of striction to be the director (i.e., one sets $\kappa = 0$) then one will get the following equation in place of the third equation in (37):

$$[(\xi_1' \cos\varphi + \eta_1' \sin\varphi)\cos\lambda - \zeta_1' \sin\lambda]\,\lambda' = (\xi_1' \sin\varphi - \eta_1' \cos\varphi)\,\varphi' \sin\lambda\,,$$

and then, by virtue of the second equation in (37), which remains invariant, and the fact that ζ_1' = cos ($\lambda + \theta$), equation (38) will become:

$$\lambda' \sin\theta = \upsilon\,\varphi'^2 \sin^2\lambda.$$

Now the line of contact between the transformed surface and the cylindrical surface must, by hypothesis, coincide with the director; i.e., with $u = 0$, so λ' sin $\theta = 0$. If the surface is not developable then sin θ cannot be zero, so one will need to have $\lambda' = 0$; i.e., λ = const.

This property can be made to appear obvious by means of some easy geometric considerations, moreover.

Indeed, when two lines that are concurrent at a point in space are inclined the same with respect to a fixed plane, it will be clear that if one projects onto that plane the common normal that goes through it from the point of intersection of the two lines then one will get a line that bisects the angle that is defined by the projections of the two lines. It results from this that if two lines that are infinitely-close and not situated in the same plane make the same angle with a fixed plane then the direction of their minimum separation distance must project onto that plane parallel to the bisector of the infinitesimal angle that is defined by the projections of the two lines. Now, the length of the minimum distance is infinitely small, so its projection must be that way, as well, and therefore if one takes into account the direction that the projection takes then it will be clear that the projections of the feet of the aforementioned minimum distances must fall on the projections of the two lines at points that are infinitely-close to the intersection of those two projections.

If one applies that observation to the successive pairs of infinitely-close generators of a curved surface all have generators that are equally-inclined with respect to a fixed plane then one will conclude immediately that: *The line of striction of such a curved surface projects onto that plane along the envelope of the projections of the generators.*

Conversely, if two contiguous generators *do not* make the same angle with a fixed plane then the direction of their minimum distance, when projected onto that plane, will define a finite angle with the projections of the generators. Hence, in order for the projection of the minimum distance to be infinitely small of the same order as the angle that is found between two generators (and therefore their projections, as well,) one will require that the projections of the feet of that minimum distance must fall at a *finite* distance from the intersection of the two projections, from which, it will obviously result that the projection of the line of striction *cannot* coincide with the envelope of the projections of the generators. Hence, along with the preceding theorem, one will have the converse theorem, and therefore its reciprocal, as well.

The preceding considerations make some of the properties that were pointed out by BONNET (*) seem obvious, since they are all immediate consequences of formula (11), moreover. If the generators of a curved surface are all inclined the same with respect to a fixed plane then they will likewise make a constant angle with the generators of the cylindrical surface that is normal to that plane and involves the curve surface along its line of striction. Therefore, if that line makes a constant angle with the generators of the curved surface then they will likewise make a constant angle with those of the cylindrical surface – viz., it will be a cylindrical helix – and therefore a geodetic line on the cylindrical surface, as well as on the tangent curved surface. Conversely, if the line of striction is a geodetic on the curved surface then the same thing will also be true with respect to the cylindrical surface, and that line of striction will therefore make a constant angle with the generators of the cylindrical surface, as well as with those of the curved surface. Now any curved surface can be transformed into another one that has all of its generators inclined the same above a fixed plane, and the characteristic property of the geodetic lines and lines of striction will remain unaltered by the transformation; it is then clear that the preceding observations will lead to this theorem:

If the line of striction of a curved surface meets all the generators at a constant angle then it will be, at the same time, a geodetic line, and conversely, if it is a geodetic line then it will cross all the generators at a constant angle.

Another property that was stated before by BONNET (*ibid.*) will also persist; namely: *If a geodetic line meets all the generators at a constant angle then it will be the line of striction.* Indeed, transform the surface in such a way that the geodetic in question becomes a straight line. All of the generators of the transformed surface will prove to be inclined the same with that line, and therefore, from a previous theorem, that line will be the line of striction of the transformed surface. (Therefore, etc.)

§ 8.

If one lets a, b, c denote the cosines of the angles that the three axes make with the normal to the ruled surface at the point (u) on the director then one will have:

$$a = \frac{\eta' n - \zeta' m}{\sin\theta}, \qquad b = \frac{\zeta' l - \xi' n}{\sin\theta}, \qquad c = \frac{\xi' m - \eta' l}{\sin\theta}. \tag{39}$$

In order for the transformed surface to be tangent at all points of the new director to a cylinder that is normal to the xy-plane, one needs to have:

(*) J. École Poly. **19**, cahier 32 (1848), pp. 71. A much-more-general theorem was given by BRIOSCHI in the Giornale dell'Istituto Lombardo e Biblioteca Italiana **9** (1856), pp. 400.

BONNET's theorems were proved geometrically by PAUL SERRET; see *Théorie nouvelle*, pp. 149.

(40) $$\xi_1' m_1 - \eta_1' l_1 = 0,$$

and that equation, when combined with the usual five (7) and (8), can serve to determine the six functions ξ_1, η_1, ζ_1, l_1, m_1, n_1. One can proceed with that search in the following way:

One gets from (20) that:

$$\xi' m - \eta' l = \frac{1}{\varepsilon'^2}\left[-\kappa(l\,m' - l'\,m) \pm n'\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}\,\right],$$

and therefore the condition (40) can be replaced with the following one:

$$\kappa^2\,(l_1\,m_1' - l_1'\,m_1)^2 = n_1'^2\,(\varepsilon'^2 \sin^2\theta - \kappa^2),$$

or

$$n_1'^2 \sin^2\theta = \kappa^2\,(1 - n_1^2),$$

from which, one gets:

(41) $$n_1 = \cos\varphi,$$

by setting:

$$\varphi = \int \frac{\kappa\,du}{\sin\theta}.$$

Hence, if one takes:

(42) $$\psi = \int \frac{\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}}{\sin\theta \sin\varphi}\,du$$

once more, then one will get the values of l_1, m_1, n_1 from the formulas:

(43) $$l_1 = \sin\varphi\cos\psi, \qquad m_1 = \sin\varphi\sin\psi, \qquad n_1 = \cos\varphi.$$

If one sets κ equal to 0 then one will have φ = const., and one will get back to a theorem that was proved in the preceding §. If the original surface is developable then one will have:

$$\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2} = 0,$$

and therefore:

$$\psi = \text{const.};$$

i.e., the transformed surface will be planar. However, if the director makes the same edge of regression then one will simultaneously have $\kappa = 0$, $\theta = 0$, and the formulas of the transformation will become indeterminate, which is obvious *a priori*.

It is clear that the angle that the transformed director makes with the generators of the involved cylinder is equal to $\varphi + \theta$, and that the angle that is made by the tangent to the cross-section of that cylinder and the x-axis is equal to ψ. It will then result that:

(44) $$\xi_1' = \sin(\varphi + \theta)\cos\psi, \qquad \eta_1' = \sin(\varphi + \theta)\sin\psi, \qquad \zeta_1' = \cos(\varphi + \theta),$$

which is a value that can be deduced from (20) by changing l, m, n into l_1, m_1, n_1 .

One deduces from this formula that:

$$\frac{1}{\rho_1^2} = \frac{(\kappa + \theta' \sin\theta)^2}{\sin^2\theta} + \frac{(\varepsilon'^2 \sin^2\theta - \kappa^2)\sin^2(\varphi+\theta)}{\sin^2\varphi \sin^2\theta},$$

Or also, from (11):

$$\frac{1}{\rho_1^2} = \left(\frac{\cos\omega}{\rho}\right)^2 + \frac{(\varepsilon'^2 \sin^2\theta - \kappa^2)\sin^2(\varphi+\theta)}{\sin^2\varphi \sin^2\theta}. \tag{45}$$

One also easily determines the curvature $1 / R'$ of the cross-section of the cylinder. Indeed, if one lets $1 / R_1$ denote the curvature of the normal section that is made in the transformed surface tangentially to the new director then one will get from (45) that:

$$\frac{1}{R_1} = \frac{\sin(\varphi+\theta) \cdot \sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}}{\sin\varphi \sin\theta},$$

and therefore:

$$\frac{1}{R'} = \frac{\sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2}}{\sin\varphi \sin\theta \sin(\varphi+\theta)}. \tag{46}$$

The preceding developments permit us to state the following theorem:

It is always possible to transform a ruled surface in such a way that any one of its lines will become a line of contact between the transformed surface and a cylindrical surface.

When the director is a geodetic line, one will have $\kappa = -\theta' \sin\theta$, so:

$$\varphi = \theta_0 - \theta, \qquad \zeta_1' = \cos\theta_0,$$

from which, it will emerge that the new director is a helix, as it must be. In that case, one will also have:

$$\frac{1}{\rho_1} = \frac{\sin\theta_0 \sqrt{\varepsilon'^2 - \theta'^2}}{\sin(\theta_0 - \theta)}, \qquad \frac{1}{R_1} = \frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\sin\theta_0 \sin(\theta_0 - \theta)},$$

which are formulas that agree with (32), (33).

When the director is the line of striction, one will find that:

$$\frac{1}{\rho_1^2} = \theta'^2 + \frac{\varepsilon'^2 \sin(\varphi+\theta)}{\sin^2\varphi}, \qquad \frac{1}{R'} = \frac{\varepsilon'}{\sin\varphi \sin(\varphi+\theta)},$$

in which φ is a constant angle.

When the director is an orthogonal trajectory of the generators, one will have:

$$\frac{1}{\rho_1^2} = \kappa^2 + (\varepsilon'^2 - \kappa^2)\cot^2\varphi, \qquad \frac{1}{R'} = \frac{\sqrt{\varepsilon'^2 - \kappa^2}}{\sin\varphi\cos\varphi}.$$

§ 9.

Last, but not least, we shall mention a condition of a very general nature by which one can prescribe the transformation.

If one sets:

$$M = (m\,n' - m'\,n)\,\xi' + (n\,l' - n'\,l)\,\eta' + (l\,m' - l'\,m)\,\zeta',$$

$$N = (\eta'\zeta'' - \eta''\zeta')\,l + (\zeta'\xi'' - \zeta''\xi')\,m + (\xi'\eta'' - \xi''\eta')\,n,$$

for brevity, then one will easily see that at any point (u) of the director $\upsilon = 0$, the equation:

$$M\,d\upsilon - N\,du = 0$$

will define the direction $d\upsilon / du$ that is conjugate with respect to that director at that point. Wherefore, if one substitutes a well-defined function of u for the ratio $d\upsilon / du$ then the resulting equation will express the condition that must be satisfied in order for the director to have its tangent conjugate to the direction that is defined by that function at that point. Hence, e.g., the equation $N = 0$ expresses the condition for the director to be an asymptotic line (which will follow with no further discussion from the geometric significance of the expression for N). However, the equation $M = 0$ expresses the condition for the director to be the conjugate line with respect to the rectilinear generators. It is clear enough *per se* that this last situation can be verified only for developable surfaces. That will emerge immediately from our formulas, moreover, when one observes that (20) will give:

$$M = A\,\xi' + B\,\eta' + C\,\zeta' = \pm\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}\,,$$

and that the quantity $\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}$ can be annulled only on developable surfaces, as was said in § **1**.

In order for the direction $d\upsilon / du$ to be normal to that of the director, one needs to set:

$$\frac{d\upsilon}{du} = -\frac{1}{\cos\theta},$$

and therefore, the equation:

$$M + N\cos\theta = 0$$

will express the condition for the director $\upsilon = 0$ to be a line of curvature of the ruled surface. When the surface is not developable, one will see that the preceding condition can never be satisfied for $\theta = \pi / 2$, which recalls the preceding observation (*).

If one lets N_1 denote what N will become when one passes from the original surface to the transformed one then it will be clear that if one adds the relation:

$$N_1 \cos\theta \pm \sqrt{\varepsilon'^2 \sin^2\theta - \kappa^2} = 0 \tag{47}$$

to the five relations (7), (8) then one can, in general, determine the six functions ξ_1, η_1, ζ_1; l_1, m_1, n_1 , which is equivalent to saying that:

In general, one can transform a curved surface in such a way that a line that is traced on it and is neither a generator not an orthogonal trajectory to the generators will become a line of curvature on the transformed surface.

Other than the quantities θ, κ, ε, the value of N_1 will contain only the radius of curvature ρ_1 of the transformed director. Indeed, from the three equations:

$$l_1 \xi_1' + m_1 \eta_1' + n_1 \zeta_1' = \cos\theta,$$

$$l_1 \xi_1'' + m_1 \eta_1'' + n_1 \zeta_1'' = \sin\theta \frac{\cos\omega}{\rho},$$

$$l_1^2 + m_1^2 + n_1^2 = 1,$$

one will deduce the following values of l_1, m_1, n_1 :

$$\left\{\begin{aligned} l_1 &= \alpha_1 \cos\theta + \varpi\,\rho_1\,\alpha_2 + \alpha_3\sqrt{\sin^2\theta - \varpi^2\rho_1^2}, \\ m_1 &= \beta_1 \cos\theta + \varpi\,\rho_1\,\beta_2 + \beta_3\sqrt{\sin^2\theta - \varpi^2\rho_1^2}, \\ n_1 &= \gamma_1 \cos\theta + \varpi\,\rho_1\,\gamma_2 + \gamma_3\sqrt{\sin^2\theta - \varpi^2\rho_1^2}, \end{aligned}\right. \tag{48}$$

in which:

$$\varpi = \sin\theta \frac{\cos\omega}{\rho},$$

and the quantities α, β, γ have the same significance that they had in § **3**. With those values, one will deduce that:

(*) The property that *any ruled surface that has an orthogonal trajectory to its generators for a line of curvature is necessarily a developable surface* can be regarded as a consequence of the other one that *the tangents that go through the points of a line to two different evolutes of it define a constant angle between them.*

(49)
$$N_1 = \frac{\sqrt{\sin^2\theta - \varpi^2\rho_1^2}}{\rho_1}.$$

If one substitutes that value in equation (47) then one will get:

(50)
$$\frac{1}{\rho_1} = \pm\frac{\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2 + \varpi^2\cos^2\theta}}{\sin\theta\cos\theta},$$

which will always be real since $\varepsilon'^2\sin^2\theta - \kappa^2$ is a positive quantity (§ **1**).

If one lets $1 / R_1$ denote the normal curvature of the transformed director then one will have:

(51)
$$\frac{1}{R_1} = \pm\frac{\sqrt{\varepsilon'^2\sin^2\theta - \kappa^2}}{\sin\theta\cos\theta}$$

for that value of ϖ, which is a value that will become indeterminate when the original director is an orthogonal trajectory of the generators of a developable surface, as is obvious *a priori*.

When the director is a geodetic, one will have $\varpi = 0$, $\kappa = -\theta'\sin\theta$, and therefore:

$$\frac{1}{\rho_1} = \frac{1}{R_1} = \pm\frac{\sqrt{\varepsilon'^2 - \theta'^2}}{\cos\theta},$$

which is a value that coincides with the one that was found in § **5**, application 2; it must indeed be true then, since a geodetic line cannot be, at the same time, a line of curvature without being planar, as is known.

If the director is the line of striction then one will have $\kappa = 0$, $\varpi = -\theta'\sin\theta$, and therefore:

$$\frac{1}{\rho_1} = \pm\frac{\sqrt{\varepsilon'^2 + \theta'^2\cos^2\theta}}{\cos\theta}, \qquad \frac{1}{R_1} = \pm\frac{\varepsilon'}{\cos\theta}.$$

§ 10.

Since we would not like to multiply these examples greatly, it will be sufficient to show one advantage of the method.

Everyone can see that many of the questions that we treated are susceptible to generalization. Hence, e.g., the problem of § **5** is a special case of this other one: *Transform a ruled surface in such a way that a given line will be arranged on another given surface.* The one in § **6** will come down to the following: *Transform a ruled surface in such a way that one of its geodetic lines will become a geodetic line on another surface.* The last one is, in turn, contained in this other one, which is a generalization of the one that resulted in § **8**: *Transform a ruled surface in such a way that a given line on* *it will become a line of contact between that surface and another given surface*, and so on. The solution to this and other questions would certainly be less simple than what we could achieve in the special cases that we treated, but that is precisely why it deserves to be made the object of later research.

We conclude with the observation that, in general, it is not possible to transform the director (which is an arbitrarily-traced line on the surface, moreover) into another line of that type, since that would imply *two* conditions on the transformation. Therefore, it could happen that this transformation would be possible in certain situations, and we had a very obvious example of that in the question that was treated in § **6**. In order to be able to judge the possibility that those transformations exist in any case, we establish an equation that must be regarded as fundamental in the theory of ruled surfaces, since it expresses a condition that must be necessarily satisfied by any transformed curve independently of the surface into which the original surface was transformed.

That equation is obtained by eliminating the three quantities l_1, m_1, n_1 from the three equations (48) and the first of (8).

If one sets:

$$k = \rho_1\,\varpi, \qquad k = \sqrt{\sin^2\theta - \rho_1^2\varpi^2}\,,$$

for the moment, then one will get from (48) that:

$$l_1' = \kappa\,\alpha_1 + \left(h' + \frac{\cos\theta}{\rho_1} + \frac{k}{r_1}\right)\alpha_2 + \left(k' - \frac{h}{r_1}\right)\alpha_3,$$

$$m_1' = \kappa\,\beta_1 + \left(h' + \frac{\cos\theta}{\rho_1} + \frac{k}{r_1}\right)\beta_2 + \left(k' - \frac{h}{r_1}\right)\beta_3,$$

$$n_1' = \kappa\,\gamma_1 + \left(h' + \frac{\cos\theta}{\rho_1} + \frac{k}{r_1}\right)\gamma_2 + \left(k' - \frac{h}{r_1}\right)\gamma_3$$

by means of SERRET's formula, from which, one will get:

$$\varepsilon'^2 = \kappa^2 + \left(h' + \frac{\cos\theta}{\rho_1} + \frac{k}{r_1}\right)^2 + \left(k' - \frac{h}{r_1}\right)^2,$$

upon summing and squaring. If one sets:

$$P = h' + \frac{k}{r_1} = \left(\rho_1 \sin\theta\,\frac{\cos\omega}{\rho}\right)' + \frac{\sin\theta\sqrt{1 - \rho_1^2\left(\dfrac{\cos\omega}{\rho}\right)^2}}{r_1}$$

then that formula can be written in the following way (*):

$$(52) \qquad \left(P+\frac{\cos\theta}{\rho_1}\right)^2+\frac{\left(\theta'\cos\theta-P\rho_1\dfrac{\cos\omega}{\rho}\right)^2}{1-\rho_1^2\left(\dfrac{\cos\omega}{\rho_1}\right)^2}+\kappa^2=\varepsilon'^2,$$

and constitutes a relation between the four quantities:

$$u, \qquad \rho_1, \qquad r_1, \qquad \frac{d\rho_1}{du},$$

which will always be kept the same, no matter what transformation is performed on the ruled surface (as long as one keeps its original generators rectilinear). In other words, it will be a differential equation that belongs to all of the curves into which one can transform the director of the ruled surface.

Hence, for example, when the director is a geodetic line, one will have:

$$\frac{\cos\omega}{\rho}=0, \qquad P=\frac{\sin\theta}{r_1}, \qquad \kappa=-\theta'\sin\theta,$$

and the preceding formula will reduce to this other very simple one:

$$(53) \qquad \frac{\cos\theta}{\rho_1}+\frac{\sin\theta}{r_1}=\sqrt{\varepsilon'^2-\theta'^2},$$

which has the same form as (16), § **2**, which should be obvious. If one takes $1 / r_1 = 0$ then the last formula will reduce to the one that gave the value of ρ_1 in application 2, § **5**.

However, if the direction is an orthogonal trajectory of the generators then one will have:

$$\frac{\cos\omega}{\rho}=-\kappa; \qquad \theta'=0, \qquad \cos\theta=0,$$

and (52) will be:

$$(54) \qquad \frac{1}{r_1}-\frac{(\rho_1\kappa)'}{\sqrt{1-\rho_1^2\kappa^2}}=\sqrt{\varepsilon'^2-\kappa^2},$$

which is an equation that will give the same value for ρ_1 that was obtained by a different method in the application 1, § **5** by setting $1 / r_1 = 0$.

(*) That transformation excludes only the case that was verified in § **3** in which one has $\frac{1}{\rho_1}=\frac{\cos\omega}{\rho}$. Under that hypothesis, one can therefore have $h = \sin\theta$, $k = 0$ in the preceding formula, and one will immediately get the value of r_1 that was given in equation (19) of § **3**.

If the director is a line of striction then one will have:

$$\kappa = 0, \qquad \frac{\cos\omega}{\rho} = -\,\theta',$$

and (52) will easily reduce to the following one:

$$\frac{\cos\theta - \rho_1(\rho_1\,\theta'\sin\theta)'}{\rho_1\sqrt{1-\rho_1^2\,\theta'^2}} + \frac{\sin\theta}{r_1} = \varepsilon'. \tag{55}$$

The preceding formulas can serve to determine one of the quantities ρ_1, r_1 when the other one is given or determinate under certain conditions.

Hence, in § **4**, application 2, we found the value:

$$\rho_1 = \frac{\rho}{\sqrt{1+\rho^2\,\Theta'^2}},$$

which depended upon the relations:

$$\theta = \frac{\pi}{2}, \qquad \kappa = -\frac{1}{\rho}, \qquad \sqrt{\varepsilon'^2 - \kappa^2} = \frac{1}{r}.$$

If one applies formula (54) then one will find that:

$$\frac{1}{r_1} = \frac{1}{r} + \frac{(\rho\,\Theta')'}{1+\rho^2\,\Theta'^2},$$

which is a value that can be deduced less rapidly from the formulas in the cited §.

Formula (32) in § **6** provides the value of ρ_1 that relates to a geodetic line that transforms into a cylindrical helix. If one substitutes that value in (53) then one will have:

$$\frac{1}{r_1} = -\frac{\cos\mu_1\sqrt{\varepsilon'^2-\theta'^2}}{\sin(\mu_1-\theta)},$$

which is a formula that will reproduce the known property of the cylindrical helix when it is combined with the aforementioned one.

We finally note that when one eliminates r_1 from equation (52) and:

$$\rho_1^2 + r_1^2\,\rho_1'^2 = g^2,$$

which characterizes the lines that are traced on a sphere of radius g, one will get a first-order differential equation in ρ_1 and u, and its integration will yield the spherical transform of the original director.

Pisa, May 1865
