"Sur la theorie de la deformation infiniment petite d'un milieu," C. R. Acad. Sci. **108** (1889), 502-505.

On the theory of infinitely-small deformations of a medium

Extract of a letter from **EUG. Beltrami** to Maurice Lévy

Pavia, 2 March 1889

Translated by D. H. Delphenich

I shall take the liberty of communicating some remarks to you that I had occasion to make in some research into the infinitely-small deformations of a continuous medium and which pertain to an entirely elementary point in that theory.

… as is customary, let u, v, w be the three components of the displacement of the medium at the point (x, y, z), and let a, b, c, f, g, h be the six components of deformation at the same point – i.e., the six quantities that are defined by the equalities:

$$u_x = a, \qquad v_y = b, \qquad w_z = c,$$

$$w_y + v_z = 2f, \qquad u_z + w_x = 2g, \qquad v_x + u_y = 2h,$$

in which u_x, u_y, u_z, … are the partial derivatives of u, v, w with respect to the coordinates. One knows that if one considers those six quantities to be given functions that are independent of the u, v, w then it will be necessary and sufficient that they satisfy six particular second-order partial differential equations if they are to represent the components of a possible deformation – i.e., in order for them to be expressible by means of three arbitrary functions u, v, w from the preceding equalities. Now, in the research that I just spoke of, I was led to demand to know whether there would be any possible advantage to deducing those six equations from the variation of a single triple integral of the type that is very useful in many cases, such as the classical potential equation.

Instead of immediately describing the triple integral that enjoys the property in question here, and which can take an infinitude of forms, moreover, I would prefer to present things to you in a somewhat-indirect manner that nonetheless gives a better accounting of the result to which one arrives, and above all if one approaches that result from the considerations by which I have further proved (addition to my paper "Sur l'interprétation mécanique des formulas de Maxwell") that the six equations that one addresses are sufficient.

Set:

$$2p = w_y - v_z, \quad 2q = u_z - w_x, \quad 2r = v_x - u_y;$$

i.e., let p, q, r denote the components of what one is led to call the *rotation of the medium* at the point (x, y, z). Those three equations establish relations between those components and the coordinates by virtue of which one can imagine that the latter are functions of the variables p, q, r, because there is no finite equation between those three quantities, in general, although there is one (viz., $p_x + q_y + r_z = 0$) between their derivatives. Upon adopting that viewpoint, let S denote the integral:

$$S = \tfrac{1}{2}\iiint\left(\frac{\partial x}{\partial p}+\frac{\partial y}{\partial q}+\frac{\partial z}{\partial r}\right) dp\, dq\, dr\ ,$$

whose bounded region is supposed to be arbitrary, but fixed. The determinant Δ of the equations:

$$\begin{aligned} dp &= p_x\, dx + p_y\, dy + p_z\, dz\ , \\ dq &= q_x\, dx + q_y\, dy + q_z\, dz\ , \\ dr &= r_x\, dx + r_y\, dy + r_z\, dz \end{aligned}$$

is not generally zero, so one can infer from these equations that:

$$\Delta\left(\frac{\partial x}{\partial p}+\frac{\partial y}{\partial q}+\frac{\partial y}{\partial r}\right) = q_y\, r_z - q_z\, r_y + r_z\, p_x - r_x\, p_z + p_x\, q_y - p_y\, q_x\ ,$$

and when the variables p, q, r in integral S are transformed into x, y, z, it will become:

$$S = \tfrac{1}{2}\iiint\ (q_y\, r_z - q_z\, r_y + r_z\, p_x - r_x\, p_z + p_x\, q_y - p_y\, q_x)\ dx\, dy\, dz.$$

In that form, one will see immediately that S is only a triple integral in appearance, because one can reduce it immediately to an integral over the bounding surface, and in several different ways. For example, one can reduce it to the form:

$$S = \tfrac{1}{4}\int\left(\frac{\partial p}{\partial s}\frac{\partial x}{\partial n}+\frac{\partial q}{\partial s}\frac{\partial y}{\partial n}+\frac{\partial r}{\partial s}\frac{\partial z}{\partial n}\right)\sqrt{p^2+q^2+r^2}\,d\sigma\ ,$$

in which s is the direction of the axis of rotation, $d\sigma$ is the element on the bounding surface, and n is the direction of the interior normal to that element. However, it is entirely pointless to account for that reduction. It is important to point out that if one varies the functions p, q, r in the triple integral then the indefinite part of the corresponding variation of S (I would like to say that it is similarly represented by a triple integral) will reduce to zero identically.

Having said that, the derivatives p_x , p_y , … of the components of rotation can all be expressed in terms of the six components of deformation, as one can easily assure oneself. Upon replacing those expressions for p_x , p_y , … with the derivatives of a, b, …, one will get:

$$\begin{aligned} S = \tfrac{1}{2}\iiint\ \ [&(h_z - f_x)(f_x - g_y) + (f_z - g_y)(g_y - h_z) \\ &+ (h_y - h_z)(h_z - f_x) - (g_z - c_x)(b_x - h_y) \\ &+ (h_x - a_y)(c_y - f_z) - (f_y - b_z)(a_z - g_x)\ ]\ dx\, dy\, dz\ , \end{aligned}$$

and the indefinite part δS of the variation of that integral, which responds to the arbitrary variation δa, δb, … of the components of deformation, will necessarily have the form:

$$\delta S = \iiint (A\,\delta a + B\,\delta b + C\,\delta c + F\,\delta f + G\,\delta g + H\,\delta h)\,dx\,dy\,dz\,.$$

From the preceding remark, that variation will or will not be zero for any variations δa, δb, … according to whether the six functions a, b, … are or are not suitable to represent the derivatives of the three functions p, q, r of x, y, z in terms of the expressions that one just exhibited in place of p_x, p_y, …; i.e. (in the spirit of the proof that I cited above), according to whether those six functions can or cannot present the components of a possible deformation. Indeed, if one calculates the coefficients A, B, … of the variations δa, δb, … by the known rule then one will find that:

$$A = \frac{\partial^2 f}{\partial y\,\partial z} - \frac{1}{2}\left(\frac{\partial^2 b}{\partial z^2} + \frac{\partial^2 c}{\partial y^2}\right),$$

$$B = \frac{\partial^2 f}{\partial z\,\partial x} - \frac{1}{2}\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 a}{\partial z^2}\right),$$

$$C = \frac{\partial^2 f}{\partial x\,\partial y} - \frac{1}{2}\left(\frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 b}{\partial x^2}\right),$$

$$F = \frac{\partial^2 a}{\partial y\,\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right),$$

$$G = \frac{\partial^2 b}{\partial z\,\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial z} + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right),$$

$$H = \frac{\partial^2 c}{\partial x\,\partial y} - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - \frac{\partial h}{\partial z}\right),$$

and upon setting:

$$A = B = C = F = G = H = 0,$$

one will recover precisely the six equations that one knows to express the conditions for any possible deformation.

The last expression for S, whose indefinite variation δS will give all of those equations, can be transformed in several ways and can be given a wide variety of interpretations; however, at the moment, I shall confine myself to the preceding simple indications.

___