"Sulle equazioni dinamiche di Lagrange," Rend. Ist. Lombardo, Accad. di sci. (2) **28** (1895), 744-752.

On Lagrange's dynamical equations

Note by EUGENIO BELTRAMI

Translated by D. H. Delphenich

For some time, the direction into which the mathematical investigations of numerous classes of physical phenomena has entered has bestowed great honor upon LAGRANGE's dynamical equations, the first of which MAXWELL did not really consider to be a consequence of pure analysis, due to the more expeditious and more elegant treatment of the problems that belong to classical mechanics.

That is despite the fact that although much use is made of those equations now, which is undoubtedly an advantage to theoretical physics, perhaps too often they leave one in the dark as to what the fundamental principle would be that would subsume all of them in a single simple formula. Ever since LAGRANGE, that principle was always considered to be the true foundation of analytical mechanics, and as BOLTZMANN's example showed, it can be effectively invoked for the direct deduction of the important physico-mathematical laws.

As a proof of the preceding assertion, one can cite the celebrated *Treatise on natural Philosophy* by W. THOMSON and TAIT, in the first edition of which, the equations in question were obtained from that fundamental principle (following the example of LAGRANGE himself), whereas in the second edition, they were, however, established by means of the transformation of the ordinary Cartesian equations (in order to avoid "*an unnecessary complication*," as those illustrious authors said). Other writers refer back to the stated principle, but only in order to pass from it to those of HAMILTON and to then obtain the Lagrangian equations from the latter. With that, one must introduce an extraneous loop into the line of reasoning for that purpose whose power is, nonetheless, unquestionable in other aspects.

The question of the method to which we allude is certainly not of great significance in itself, but it would be appropriate to stress it its importance. However, the argument that is of interest to us is treated more often in physics than in mathematics, and that is why it would be desirable to remove any algorithmic artifice from the discussion without sacrificing any useful element, so the following brief review of the direct use of LAGRANGE's principle might not seem superfluous.

Let m be the mass, let x, y, z be the orthogonal Cartesian coordinates of any one of the material points that constitute the system under study, and let X, Y, Z be the components of the applied force that the point is subject to at the instant t. Represent the total derivative with respect to time by a prime and let u collectively denote the three coordinates x, y, z, and let U denote the homologous components of the force X, Y, Z. LAGRANGE's fundamental principle is expressed by the formula:

$$\sum (U - m\,u'')\ \delta u = 0,$$

in which the sum extends over all masses and all coordinates $u = x, y, z$ of each mass, and in which δu is a *virtual* variation of the coordinate u. As usual, let:

$$T = \tfrac{1}{2}\sum m u'^2$$

denote the total *vis viva*, and let:

$$\delta L = \sum U\ \delta u$$

denote the total virtual work. As is known, the preceding formula can be put into the equivalent form:

$$\delta L = \left(\sum \frac{\partial T}{\partial u'}\delta u\right)' - \delta T.$$

Having done that, let $(q_1, q_2, \ldots)$ represent a group of independent variables – or *general coordinates* – by means of which one can express the Cartesian coordinates of all points in question when one takes into account the constraints on the system. The number of those new coordinates (which is supposed to be finite) is the *degree of mobility* in the system. One should not exclude the case in which the expressions for the coordinates u in terms of the q might also contain time t explicitly (when one has moving constraints) in such a way that one will have:

$$u' = \frac{\partial u}{\partial t} + \sum \frac{\partial u}{\partial q} q'.$$

When expressed in terms of the new variables q, the *vis viva* T will become a (generally inhomogeneous) function of degree two in the derivatives q' that contains the variables q and time t in its coefficients. If one observes that when one gives the increments δq to the q' in the preceding equation, the u' will receive the increments δu (because the virtual variations of the coordinates must be taken with constant t) then one will immediately have:

$$\sum \frac{\partial T_u}{\partial u'}\delta u = \sum \frac{\partial T}{\partial q'}\delta q,$$

in which for clarity one momentarily lets T_u and T_q denote the two equivalent expressions for the *vis viva* in the original and new coordinate systems, respectively. It then results from this, with no further discussion, that:

$$\delta L = \left(\sum \frac{\partial T}{\partial q'}\delta q\right)' - \delta T \tag{1}$$

is the expression for LAGRANGE's principle in the general coordinates q. If, at the same time, one sets:

$$\delta L = \sum Q\ \delta q, \tag{1.a}$$

in which the quantities Q, which are easy to calculate, are (in the general sense) the components of the applied force along the homologous coordinates q, then it will suffice to perform the two indicated operations on the right-hand side of (1) in order to obtain equations of the type:

$$Q = \left(\frac{\partial T}{\partial q'}\right)' - \frac{\partial T}{\partial q}, \tag{1.b}$$

which are LAGRANGE's dynamical equations.

In particular, if one supposes that $\delta = d$ in (1), which will not be legitimate when the expressions for the coordinates u in terms of q are independent of time, and consequently if T is a homogeneous quadratic function of the q' then one will get:

$$dL = d\left(\sum q' \frac{\partial T}{\partial q'} - T\right), \tag{1.c}$$

which is an equation that expresses the *vis viva* theorem, because the quantity in parentheses will be equal to T in that case.

Now suppose that for reasons that are based upon the nature of the problem being treated it is appropriate (as often happens) to divide the variables q into two distinct groups. For convenience, one preserves the collective notation q for the variables of the first group and denotes the second group by r, and also collectively. One can correspondingly give the form:

$$\delta L = \left(\sum \frac{\partial T}{\partial q'}\delta q + \sum \frac{\partial T}{\partial r'}\delta r\right)' - \delta T \tag{2}$$

to equation (1), in which the left-hand side has the significance of:

$$\delta L = \sum Q\,\delta q + \sum R\,\delta r. \tag{2.a}$$

One introduces the new quantities ρ (which are as many in number as the r) by means of equalities of the type:

$$\rho = \frac{\partial T}{\partial r'}, \tag{2.b}$$

which are linear in the ρ and the r'. One deduces from this that:

$$\sum \rho\, dr' = \sum \frac{\partial T}{\partial r'} dr'$$

or:

$$d \sum \rho\, r' = dT - \sum \frac{\partial T}{\partial \alpha} d\alpha ,$$

in which, for the moment, α indicates any of the quantities (which are considered to be independent) that T is a function of, except for the r': i.e., any of the quantities q, q', r, and also any of the other parameters that can possibly enter into T – for example, time t. If the preceding equality, which is a simple consequence of (2.b), is written in the form:

$$\sum r' d\rho - \sum \frac{\partial T}{\partial \alpha} d\alpha = dU = \sum \frac{\partial U}{\partial \rho} d\rho + \sum \frac{\partial U}{\partial \alpha} d\alpha ,$$

in which one sets:

$$U = \sum \rho\, r' - T, \tag{2.c}$$

then one will immediately get:

$$r' = \frac{\partial U}{\partial \rho}, \qquad \frac{\partial T}{\partial \alpha} = -\frac{\partial U}{\partial \alpha}, \tag{2.d}$$

and by virtue of the equality (2.b, c), as well as the second of (2.d) for $\alpha = q'$, the fundamental equation (2) will become:

$$\delta L = \left(\sum \rho\, \delta r - \frac{\partial U}{\partial q'} \delta q \right)' - \delta\left(\sum \rho\, r - U \right) ,$$

or rather:

$$\delta L = \delta U - \left(\sum \frac{\partial U}{\partial q'} \delta q \right)' + \sum (\rho'\, \delta r - r'\, \delta \rho) . \tag{3}$$

The function U that appears in those formulas and in (2.d) is meant to be composed from the quantities q, q', r, ρ by way of the substitution in (2.c) of the values of r' that is provided linearly by equations (2.b). It has degree two with respect to the quantities q' and ρ, and contains the quantities q, r, and possibly t in its coefficients. Conversely, one gets (2.d) from (2.c) that:

$$T = \sum \rho \frac{\partial U}{\partial \rho} - U . \tag{3.a}$$

When one performs the indicated operations in the right-hand side of (3), one will get the following three groups of dynamical equations:

$$\left.\begin{aligned} Q &= \frac{\partial U}{\partial q} - \left(\frac{\partial U}{\partial q'}\right)', \\ R &= \frac{\partial U}{\partial r} + \rho', \\ 0 &= \frac{\partial U}{\partial \rho} - r', \end{aligned}\right\} \tag{3.b}$$

whose number is equal to that of the quantities q, r, ρ. The equations of the last two groups are not first-order differential, while those of the first group are of second order, just as (1.b) are already.

When the constraints are independent of time, it is legitimate to set $\delta = d$ in (3), with which one will get:

$$dL = d\left(U - \sum q' \frac{\partial U}{\partial q'}\right),$$

or also, since U is a homogeneous quadratic function of the q' and the ρ in that case:

$$dL = d\left(\sum \rho \frac{\partial U}{\partial \rho} - U\right);$$

i.e., that reproduces (3.a), which is the *vis viva* theorem.

When the transformation (2.b, c, d) is applied to all of the coordinates, it will be nothing but the HAMILTON transformation. When it is applied to only some of the coordinates, it will lead to dynamical equations (3.b) that present themselves as being partially of the Lagrangian type and partially of the Hamiltonian type. The use of that intermediate transformation is particularly advantageous in the case where it is legitimate to carry out what the English call "ignoration of coordinates."

Suppose that T (and therefore also U) does not contain the coordinate r in its coefficients. Equations (3.b) of the second group will reduce to:

$$R = \rho'$$

in that case. Therefore, if *all* of the forces R are *zero* then that will say that the quantities ρ behave like just as many invariable parameters. Hence, based upon that, when one sets $\rho' = 0$, as well as $\delta\rho = 0$, equation (3) will reduce to simply:

$$\delta L = \delta U - \left(\sum \frac{\partial U}{\partial \rho} \delta q\right)'. \tag{4}$$

Having done that, one decomposes the *vis viva* T (in which the coefficients in that expression, which are of second degree in the q', r', are all functions of q, and possibly t, from what was said) into three parts:

$$T = T_q + T_r + \Lambda, \tag{4.a}$$

the first of which T_q includes all of the terms that do not contain any r' (and is therefore of second degree, but generally inhomogeneous, with respect to the q'), the second of which T_r is a homogeneous quadratic function of the r', and finally, the third of which:

$$\Lambda = \sum \lambda\, r' \tag{4.b}$$

is linear and homogeneous with respect to the r' with coefficients λ that are first-degree (generally inhomogeneous) functions of the q'. In that way, one will have (2.b):

$$\rho = \frac{\partial T_r}{\partial r'} + \lambda, \tag{4.c}$$

and the value of U in (2.c), (4.a) will be expressed by:

$$U = T_r - T_q ,$$

in which it is intended that one substitutes the values of r' that are given by (4.c) in T_r. One can represent the result of that substitution quite simply by introducing the quadratic form T that is reciprocal to T_r : Indeed, it is clear (4.c) that the desired expression is nothing but that of the reciprocal quadratic form T, which composed from the arguments $\rho - \lambda$; that is to say, one can write:

$$T_r = \mathsf{T}_\rho - \mathsf{T}_\lambda - \sum \frac{\partial \mathsf{T}_\lambda}{\partial \lambda} \rho , \tag{4.d}$$

in which T_ρ and T_λ represent the quadratic form T that is composed from the constant arguments ρ in one case and the variable arguments λ, in the other. By virtue of the expressions thus-obtained for U and T_r , if one sets:

$$\mathsf{U} = T_q - \mathsf{T}_\lambda - \mathsf{T}_\rho , \qquad \mathsf{V} = \sum \frac{\partial \mathsf{T}_\lambda}{\partial \lambda} \rho \tag{5}$$

then one will find that:

$$- U = \mathsf{U} + \mathsf{V}, \tag{5.a}$$

in whose right-hand side, the first part U has degree two with respect to the q', while the second part V has degree one. With that expression for U, one can deduce from the third group of equations (3.b) that:

$$r' = \frac{\partial \mathsf{T}_\rho}{\partial \rho} - \frac{\partial \mathsf{T}_\lambda}{\partial \lambda},$$

in which:

$$\sum \lambda\, r' = \sum \frac{\partial \mathsf{T}_\rho}{\partial \rho}\lambda - 2\mathsf{T}_\lambda,$$

or, from (4.b):

$$\Lambda = \sum \frac{\partial \mathsf{T}_\lambda}{\partial \lambda}\rho - 2\mathsf{T}_\lambda,$$

and therefore, from (4.d):

$$T_r + \Lambda = \mathsf{T}_\rho - \mathsf{T}_\lambda .$$

The total *vis viva* T (4.a) can then be put into the definitive form:

$$T = T_q - \mathsf{T}_\lambda - \mathsf{T}_\rho, \tag{5.b}$$

which is a form that can also result from applying the process (3.a) to the expression (5.a) for U.

Nothing remains now but to substitute that expression for U in (4), which will yield:

$$\left.\begin{aligned} \delta L = & \left(\sum \frac{\partial \mathsf{U}}{\partial q'}\delta q\right)' - \delta \mathsf{U} \\ & + \left(\sum \frac{\partial \mathsf{V}}{\partial q'}\delta q\right)' - \delta \mathsf{V}. \end{aligned}\right\} \tag{5.c}$$

Developing the operations that are indicated in the right-hand side will give results of many different forms for the part in U and the one in V. In regard to the former, the coefficients of δq have the usual form:

$$\left(\frac{\partial \mathsf{U}}{\partial q'}\right)' - \frac{\partial \mathsf{U}}{\partial q},$$

and as long as one remains in the general theory, they do not lend themselves to any useful reductions, as would be true essentially for second-order differentials. However, the other part does not effectively provide terms of first order, but in order to recognize the peculiar character of its development, one must arrange the expression (5) for V in terms of the q' and set:

$$\mathsf{V} = \sum \frac{\partial \mathsf{T}_\lambda}{\partial \lambda}\rho = \kappa_0 + \sum_i \kappa_i\, q_i' \qquad (i = 1, 2, \ldots), \tag{5.d}$$

in which the coefficients κ depend upon only the variables q, and possibly t. Upon performing the indicated operations in (5.c), one will find in that way that the coefficients of δq in the aforementioned second part will be:

$$\frac{\partial \kappa}{\partial t} - \frac{\partial \kappa_0}{\partial q} + \sum_i \left(\frac{\partial \kappa}{\partial q_i} - \frac{\partial \kappa_i}{\partial q} \right) q_i'$$

in which is implicit that κ and q have the *same* index (when it is not denoted). The definitive dynamical equations will then have the following type:

$$Q = \left(\frac{\partial \mathsf{U}}{\partial q'} \right)' - \frac{\partial \mathsf{U}}{\partial q} + \sum_i \left(\frac{\partial \kappa}{\partial q_i} - \frac{\partial \kappa_i}{\partial q} \right) q_i' + \frac{\partial \kappa}{\partial t} - \frac{\partial \kappa_0}{\partial q} . \tag{5.e}$$

The terms that are linear in the q' and are found inside the summation have the *Pfaffian* form, and are always zero for the ones among them for which $q_i = q$. The last two terms after the summation will be absent when one does not have moving constraints, in which case the Pfaffian terms will be the only ones that are linear in the q'.

In that case, the preceding equations will coincide, if one ignores the difference of signature, with what is given by THOMSON and TAIT, who used a different procedure, in *loc. cit.* (pp. 323 of the 2nd edition).

Equations that also belong to the type (5.e) were given by C. NEUMANN in *Hydrodynamische Untersuchungen* in 1883 and were reproduced more recently by that illustrious author, with a new analysis, in the seventh chapter of *Beiträge zu einzelnen Theilen der mathematischer Physik* in 1893.
