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On the mechanical interpretation of Maxwell's formulas

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The study that forms the subject of the present paper refers essentially to those formulas that I studied, from a different viewpoint, in the paper that I had the honor of presenting last year to this illustrious Academy ["Sull'uso delle coordinate curvilinee nelle theorie del potenziale e dell'elasticità," (4), t. VI]. I allude to the formulas by which MAXWELL defined the systems of stresses that are generated in an elastic medium by a force field that one ordinarily considers to be represented by a Newtonian potential function.

Those formulas, which are reproduced here with the same symbols that were used in the preceding paper, are the following ones:

$$
(1) \qquad \left\{
\begin{aligned}
X_x &= -\frac{1}{4\pi}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{8\pi}\Delta_1 V, \qquad Y_z = Z_y = -\frac{1}{4\pi}\frac{\partial V}{\partial y}\frac{\partial V}{\partial z}, \\
Y_y &= -\frac{1}{4\pi}\left(\frac{\partial V}{\partial y}\right)^2 + \frac{1}{8\pi}\Delta_1 V, \qquad Z_x = X_z = -\frac{1}{4\pi}\frac{\partial V}{\partial z}\frac{\partial V}{\partial x}, \\
Z_z &= -\frac{1}{4\pi}\left(\frac{\partial V}{\partial z}\right)^2 + \frac{1}{8\pi}\Delta_1 V, \qquad X_y = Y_x = -\frac{1}{4\pi}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}, \\
\Delta_1 V &= \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2,
\end{aligned}
\right.
$$

in which V is the Newtonian potential function that one considers, and the components of the stress X_x, X_y, … are just the symbols that were used in KIRCHHOFF's notation, in which one agrees to let X_n, Y_n, Z_n represent the components along three orthogonal axes x, y, z of the unit pressure that is exerted on a planar element with normal n.

Now, here is the question that I propose to treat and that, and which I believe presents itself naturally to anyone who reflects upon the significance that the theory of elasticity attributes to the stresses or tensions that are present in an elastic medium. One leaves aside any preconceived notion in regard to any possible physical correlation between the so-called actions-at-a-distance and the stresses or tensions that that are defined in MAXWELL's formulas, and considers those stresses or tensions to be simply generated in an elastic medium by a slight deformation of it; i.e., by a slight displacement of any of its points (relative to an initial equilibrium state). When regarding matters from that aspect, it is natural to demand to know: Does there truly exist a deformation that is capable of generating that stress, and if it does exist, what are the components of the displacement of any point of the medium?

In order to respond to that demand, one first needs to establish a few things in regard to the nature of the elastic medium in which one supposes that the stresses and tensions that are defined by MAXWELL's formulas are generated. From the purely-mathematical viewpoint, one can require that the constitution of the medium is not constrained by any condition other than that of continuity. However, in order to avoid the prolixity of a very general and abstract study, I shall suppose that I am dealing with simply a homogeneous and isotropic medium, while assuming that the constants of isotropy can be different in the regions of space that are found to be in different conditions with respect to the distribution of the potentiating masses; i.e., the masses that contribute to the formation the potential function. Nonetheless, in order to avoid a degree of generality that is theoretically justifiable, but unused in practice, one correspondingly assumes that in the formation of that potential function in terms of contributing masses that are extended in three dimensions, those masses have constant density, just as the density in an ordinary isotropic medium is constant. Moreover, that restriction has little importance with respect to the physical theory in which MAXWELL's formulas are presented, since surface potential functions do not appear in them, as a rule.

Having thus circumscribed the problem, it is possible to undertake its mathematical treatment, and it is precisely that treatment that forms the subject of the present work. Its conclusions are almost-entirely negative, since it will be established that in a truly isotropic medium, it is not possible to perform displacements that are capable of reproducing the system of stresses that are defined by MAXWELL's formulas, unless the potential function is linear in the coordinates. Not only is that particular case devoid of all interest, but it cannot be verified when the space considered is infinite, either. If we would like to avoid the necessity of attributing that particular form to the potential function, which is not always admissible, then we will need to concede the existence of a medium that is *sui generis* isotropic, which is a property that does not correspond to any reality that is known so far. In order to envision the nature of such a medium, it is convenient to recall that the elementary potential of elasticity for a given properly-isotropic medium that is assumed to have the form that was so opportunely adopted by GREEN consists of two very distinct parts, which can be regarded in that way as the distinct elastic potentials of two essentially-different, irreducible media whose combinations with varying ratios will result in the isotropic medium that one ordinarily considers (*). The first of those media corresponds to a well-known real entity, since its properties tally precisely with those of ordinary elastic fluids in which only longitudinal waves are transmitted. However, the second one, in which only transversal waves are transmitted, does not correspond (and probably cannot correspond) to any known entity, since, among other things, the conditions of stability of equilibrium are not generally verified in them. Now, it is precisely when one assumes the existence of the second medium in the analysis that will be presented below that it will first seem possible, among other things, to pursue the investigation of the deformations in spaces that are

(*) According to molecular theory, that ratio will always be 3 to 1, and according to WERTHEIM, 4 to 1.

devoid of potentiating masses, except that the ultimate progress in that analysis will show that (at least in the infinite space that surrounds the mass) the deformations will again be possible only when the potential function has the simple form that it would take if all of the masses could be concentrated into a fixed, but arbitrary, point. It is then legitimate to assert that if one is given a potential function arbitrarily then it will not be generally possible to reproduce the system of stresses that are defined by MAXWELL's formulas by means of the deformations of an isotropic medium.

I have induced myself (not without some hesitation) to make those results public, not only to raise some objections against MAXWELL's doctrine, but to show the necessity of investigating the mechanical interpretation along a different path.

§ 1.

As usual, let u, v, w denote the components of the displacement of an arbitrary point (x, y, z) of the elastic medium and set:

$$(2)\qquad \begin{cases} \alpha = \dfrac{\partial u}{\partial x}, \quad \lambda = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}, \\ \beta = \dfrac{\partial v}{\partial y}, \quad \mu = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}, \quad \vartheta = \alpha + \beta + \gamma, \\ \gamma = \dfrac{\partial w}{\partial z}, \quad \nu = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}; \end{cases}$$

i.e., α, β, γ, λ, μ, ν denote the known components of the deformation of the medium at the point (x, y, z), and ϑ denotes the cubic dilatation. After that (while admitting the isotropy hypothesis), assume that the elastic potential has the form:

$$(2.a)\qquad \Phi = \tfrac{1}{2}\{A\vartheta^2 + B(\lambda^2 + \mu^2 + \nu^2 - 4\beta\gamma - 4\gamma\alpha - 4\alpha\beta)\},$$

in which A and B are the isotropy constants that were introduced in the manner of GREEN used and are proportional to the squares of the velocities of propagation of the longitudinal and transverse waves, respectively. In order for the initial state of the medium (i.e., the one in which the three functions u, v, w are zero) to be a true state of stable equilibrium, it is necessary and sufficient for those two constants A and B to be subject to the limitations:

$$(2.b)\qquad B > 0, \qquad 3A - 4B > 0,$$

which will be satisfied by those functions Φ that are not only kept constantly *positive* for any given proper deformation, but will be annulled only when *all* of the components of deformation are separately zero.

The components of stress:

$$X_x, \quad Y_y, \quad Z_z, \quad Y_z, \quad Z_x, \quad Y_y,$$

that are due to the deformation components:

$$\alpha, \quad \beta, \quad \gamma, \quad \lambda, \quad \mu, \quad \nu$$

are, as is known, minus the derivatives of the potential Φ with respect to the latter components, so they will be given by:

$$\begin{aligned} X_x &= 2B(\beta + \gamma) - A\vartheta, & Y_z &= -B\lambda, \\ Y_z &= 2B(\gamma + \alpha) - A\vartheta, & Z_x &= -B\mu, \\ Z_z &= 2B(\alpha + \beta) - A\vartheta, & X_y &= -B\nu. \end{aligned}$$

If we set:

$$P = X_x + Y_y + Z_z,$$

for the moment, then we will get:

$$P = -(3A - 4B)\,\vartheta,$$

and then we will easily have the reciprocal formulas:

$$\alpha = \frac{1}{2B}\left(\frac{A-2B}{3A-4B}P - X_x\right), \qquad \lambda = -\frac{1}{B}Y_z,$$

$$\beta = \frac{1}{2B}\left(\frac{A-2B}{3A-4B}P - Y_y\right), \qquad \mu = -\frac{1}{B}Z_x,$$

$$\gamma = \frac{1}{2B}\left(\frac{A-2B}{3A-4B}P - Z_z\right), \qquad \nu = -\frac{1}{B}X_y.$$

If one substitutes the values (1) in these formulas, after observing that from the result that:

$$P = \frac{\Delta_1 V}{8\pi},$$

then one will get the following expressions for the $\alpha, \beta, \gamma, \lambda, \mu, \nu$ in terms of V:

$$(3) \qquad \left\{ \begin{aligned} \alpha &= \frac{1}{8\pi B}\left\{\left(\frac{\partial V}{\partial x}\right)^2 - \frac{A-B}{3A-4B}\Delta_1 V\right\}, & \mu &= \frac{1}{4\pi B}\frac{\partial V}{\partial y}\frac{\partial V}{\partial z}, \\ \beta &= \frac{1}{8\pi B}\left\{\left(\frac{\partial V}{\partial y}\right)^2 - \frac{A-B}{3A-4B}\Delta_1 V\right\}, & \lambda &= \frac{1}{4\pi B}\frac{\partial V}{\partial z}\frac{\partial V}{\partial x}, \\ \gamma &= \frac{1}{8\pi B}\left\{\left(\frac{\partial V}{\partial z}\right)^2 - \frac{A-B}{3A-4B}\Delta_1 V\right\}, & \nu &= \frac{1}{4\pi B}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}. \end{aligned} \right.$$

These are then the values of the components of deformation of an isotropic medium that correspond to the six components of stress that result from Maxwell's theory.

One deduces from (3) that:

$$\vartheta = -\frac{\Delta_1 \vartheta}{8\pi(3A - 4B)}, \tag{3.a}$$

$$\lambda^2 + \mu^2 + \nu^2 - 4\beta\gamma - 4\gamma\alpha - 4\alpha\beta = (A - B)(3A - 5B)\left\{\frac{\Delta_1 V}{4\pi B(3A - 4B)}\right\}^2, \tag{3.b}$$

and then, if one substitutes this in the expression (2.*a*), one will find that:

$$\Phi = \frac{4A - 5B}{2B(3A - 4B)}\left(\frac{\Delta_1 V}{8\pi}\right)^2. \tag{3.c}$$

That is then the particular value that the elastic potential assumes when the stresses that exist in the isotropic medium are the ones that were just pointed out.

However, in order for a *given* system of components of deformation $\alpha, \beta, \gamma, \lambda, \mu, \nu$ to effectively correspond to a system of components of displacement u, v, w, or in other words, in order for a *given* system of functions $\alpha, \beta, \gamma, \lambda, \mu, \nu$ to represent a *possible* deformation of the elastic medium, it will be necessary and sufficient (*) that it should satisfy the following equations identically:

$$\frac{\partial^2 \beta}{\partial z^2} + \frac{\partial^2 \gamma}{\partial y^2} = \frac{\partial^2 \lambda}{\partial y\,\partial z}, \qquad 2\frac{\partial^2 \alpha}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \mu}{\partial y} + \frac{\partial \nu}{\partial z} - \frac{\partial \lambda}{\partial x}\right),$$

$$\frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \alpha}{\partial z^2} = \frac{\partial^2 \lambda}{\partial z\,\partial x}, \qquad 2\frac{\partial^2 \beta}{\partial z\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \nu}{\partial z} + \frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial y}\right),$$

$$\frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \beta}{\partial x^2} = \frac{\partial^2 \nu}{\partial x\,\partial y}, \qquad 2\frac{\partial^2 \gamma}{\partial x\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \lambda}{\partial x} + \frac{\partial \mu}{\partial y} - \frac{\partial \nu}{\partial z}\right).$$

Therefore, in order for there to truly exist a deformation of the medium that is defined by the six components (3), it is necessary and sufficient that the function V should satisfy the six following conditions, which result from substituting the expressions (3) for $\alpha, \beta, \gamma, \lambda, \mu, \nu$ in the preceding six equations:

(*) For the proof that those equations are *sufficient*, see the *Note* at the end of the present article.

$$(4)\qquad \begin{cases} 2\left\{\dfrac{\partial^2 V}{\partial y^2}\dfrac{\partial^2 V}{\partial z^2}-\left(\dfrac{\partial^2 V}{\partial y\,\partial z}\right)^2\right\}+C\left(\dfrac{\partial^2\Delta_1 V}{\partial y^2}+\dfrac{\partial^2\Delta_1 V}{\partial z^2}\right)=0,\\[2mm] 2\left\{\dfrac{\partial^2 V}{\partial y^2}\dfrac{\partial^2 V}{\partial x^2}-\left(\dfrac{\partial^2 V}{\partial z\,\partial z}\right)^2\right\}+C\left(\dfrac{\partial^2\Delta_1 V}{\partial z^2}+\dfrac{\partial^2\Delta_1 V}{\partial x^2}\right)=0,\\[2mm] 2\left\{\dfrac{\partial^2 V}{\partial x^2}\dfrac{\partial^2 V}{\partial y^2}-\left(\dfrac{\partial^2 V}{\partial x\,\partial y}\right)^2\right\}+C\left(\dfrac{\partial^2\Delta_1 V}{\partial x^2}+\dfrac{\partial^2\Delta_1 V}{\partial y^2}\right)=0,\\[2mm] 2\left\{\dfrac{\partial^2 V}{\partial z\,\partial x}\dfrac{\partial^2 V}{\partial x\,\partial y}-\dfrac{\partial^2 V}{\partial x^2}\dfrac{\partial^2 V}{\partial x\,\partial y}\right\}-C\dfrac{\partial^2\Delta_1 V}{\partial y\,\partial z}=0,\\[2mm] 2\left\{\dfrac{\partial^2 V}{\partial x\,\partial y}\dfrac{\partial^2 V}{\partial y\,\partial z}-\dfrac{\partial^2 V}{\partial y^2}\dfrac{\partial^2 V}{\partial z\,\partial x}\right\}-C\dfrac{\partial^2\Delta_1 V}{\partial z\,\partial x}=0,\\[2mm] 2\left\{\dfrac{\partial^2 V}{\partial y\,\partial x}\dfrac{\partial^2 V}{\partial z\,\partial x}-\dfrac{\partial^2 V}{\partial z^2}\dfrac{\partial^2 V}{\partial x\,\partial y}\right\}-C\dfrac{\partial^2\Delta_1 V}{\partial x\,\partial y}=0, \end{cases}$$

in which we have set:

$$(4.a)\qquad \frac{A-B}{3A-4B}=C,$$

for brevity. That constant C can be expressed, more briefly, as:

$$(4.b)\qquad C=\frac{B}{E},$$

in which E is the modulus of elasticity of the isotropic medium.

The solution of the problem that was posed is contained completely in the preceding equations (4), which we shall now study.

§ 2.

In order to simplify the following calculations, we shall first adopt the following notations:

$$(5)\qquad \begin{cases} \dfrac{\partial V}{\partial x}=a, & \dfrac{\partial^2 V}{\partial x^2}=e, & \dfrac{\partial^2 V}{\partial y\,\partial z}=e',\\[2mm] \dfrac{\partial V}{\partial y}=b, & \dfrac{\partial^2 V}{\partial y^2}=f, & \dfrac{\partial^2 V}{\partial z\,\partial x}=f',\\[2mm] \dfrac{\partial V}{\partial z}=c, & \dfrac{\partial^2 V}{\partial z^2}=g, & \dfrac{\partial^2 V}{\partial x\,\partial y}=g', \end{cases}$$

If one performs the differentiations in the expression $\Delta_1 V$ then one will find the following values in that way:

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial x^2} = e^2 + f'^2 + g'^2 + a\frac{\partial e}{\partial x} + b\frac{\partial e}{\partial y} + c\frac{\partial e}{\partial z},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial y^2} = f^2 + g'^2 + e'^2 + a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial z^2} = g^2 + e'^2 + f'^2 + a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y} + c\frac{\partial g}{\partial z},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial y\,\partial z} = f'g' + e'(f + g) + a\frac{\partial e'}{\partial x} + b\frac{\partial e'}{\partial y} + c\frac{\partial e'}{\partial z},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial z\,\partial x} = g'e' + f'(g + e) + a\frac{\partial f'}{\partial x} + b\frac{\partial f'}{\partial y} + c\frac{\partial f'}{\partial z},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial x\,\partial y} = e'f' + g'(e + f) + a\frac{\partial g'}{\partial x} + b\frac{\partial g'}{\partial y} + c\frac{\partial g'}{\partial z}.$$

However, if one sets:

(5.*a*)
$$\begin{cases} fg - e'^2 = E, & f'g' - ee' = E', \\ ge - f'^2 = F, & g'e' - ff' = F', \\ ef - g'^2 = G & e'f' - gg' = G', \\ e + f + g = \theta, & E + F + G = \Theta \end{cases}$$

then one will easily find that:

(5.*b*)
$$\begin{cases} e^2 + f'^2 + g'^2 = E + \theta e - \Theta, & f'g' + e'(f+g) = E' + \theta e', \\ f^2 + g'^2 + e'^2 = F + \theta f - \Theta, & g'e' + f'(g+e) = F' + \theta f', \\ g^2 + e'^2 + f'^2 = G + \theta g - \Theta, & e'f' + g'(e+f) = G' + \theta g'. \end{cases}$$

Finally, if one sets:

(5.*c*)
$$a^2 + b^2 + c^2 = H^2$$

then one can write:

(5.*d*)
$$a = H\frac{\partial x}{\partial n}, \qquad b = H\frac{\partial y}{\partial n}, \qquad c = H\frac{\partial z}{\partial n},$$

in which n is the normal to the level surface V = const. that passes through the point (x, y, z), and it is not necessary to fix a sense for its normal, since it will depend upon the sign that one attributes to the quantity:

(5.e)
$$H = \sqrt{\Delta_1 V} = \frac{\partial V}{\partial n}.$$

On the basis of these varied formulas, one will have:

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial x^2} = E + \theta e - \Theta + H\frac{\partial e}{\partial n},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial y^2} = F + \theta f - \Theta + H\frac{\partial f}{\partial n},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial z^2} = G + \theta g - \Theta + H\frac{\partial g}{\partial n},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial y\,\partial z} = E' + \theta e' + H\frac{\partial e'}{\partial n},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial z\,\partial x} = F' + \theta f' + H\frac{\partial f'}{\partial n},$$

$$\frac{1}{2}\frac{\partial^2 \Delta_1 V}{\partial x\,\partial y} = G' + \theta g' + H\frac{\partial g'}{\partial n},$$

and if one substitutes these expressions and the symbols in (5.a) in equations (4) then one will find that they can be given the following form:

(6)
$$\begin{cases} (1-C)E = C\left(\theta e + H\dfrac{\partial e}{\partial n} + \Theta - \theta^2 - H\dfrac{\partial \theta}{\partial n}\right), \\ (1-C)F = C\left(\theta f + H\dfrac{\partial f}{\partial n} + \Theta - \theta^2 - H\dfrac{\partial \theta}{\partial n}\right), \\ (1-C)G = C\left(\theta g + H\dfrac{\partial g}{\partial n} + \Theta - \theta^2 - H\dfrac{\partial \theta}{\partial n}\right), \end{cases}$$

$$\begin{cases} (1-C)E' = C\left(\theta e' + H\dfrac{\partial e'}{\partial n}\right), \\ (1-C)F' = C\left(\theta f' + H\dfrac{\partial f'}{\partial n}\right), \\ (1-C)G' = C\left(\theta g' + H\dfrac{\partial g'}{\partial n}\right). \end{cases}$$

If one sums corresponding sides of the first three of these equations then one will get a result that can be written in the following way:

$$(1 - 4C)\,\Theta + 2C\,\theta^2 + 2CH\,\frac{\partial \theta}{\partial n} = 0. \tag{6.a}$$

Now observe first of all that from the hypotheses that were made, the quantity θ, which is nothing but $\Delta_1 V$, can be equal to only zero or a constant; therefore, in any case:

$$\frac{\partial \theta}{\partial n} = 0. \tag{6.b}$$

In addition, from the significance (5.*a*) of E, F, G, Θ, one has:

$$2\Theta = (e + f + g)^2 - (e^2 - f^2 + g^2 + 2e'^2 + 2f'^2 + 2g'^2),$$

or

$$2\Theta = \theta^2 - (e^2 - f^2 + g^2 + 2e'^2 + 2f'^2 + 2g'^2). \tag{6.c}$$

By virtue of those equations (6.*b*, *c*) and the significance (4.*a*) of the constant C, the equation (6.*a*) will be equivalent to this other one:

$$(3A - 4B)\,\theta^2 + A\,(e^2 - f^2 + g^2 + 2e'^2 + 2f'^2 + 2g'^2) = 0. \tag{6.d}$$

Now, in order for the constants:

$$3A - 4B, \qquad A$$

to be both greater than zero, as is required by the two conditions (2.*b*), that equation cannot be satisfied by a *real* function V unless one has:

$$e = f = g = e' = f' = g' = 0$$

at every point of the space considered – i.e., unless all second derivatives of that function are zero – which means that one can only have the linear form:

$$V = ax + by + cz + d$$

in this case, in which *a*, *b*, *c*, *d* are four constants.

If one therefore overlooks the special cases in which *V* can assume this form (which can never happen in an infinite space, in particular, and which, on the other hand, present nothing of interest in regard to the problems of elasticity that lead to constant values for the components of stress and deformation) then one will necessarily need to attribute values to the isotropy constants that are irreconcilable with the conditions (2.*b*) for the stable equilibrium of the medium. In particular, consider a space that is devoid of potentiating masses – i.e., one in which the function θ satisfies the equation $\theta = 0$. One will then necessarily need to suppose that:

$$A = 0, \qquad B > 0 \tag{6.e}$$

for the isotropic medium that permeates that space. The second of those conditions is imposed by the form that that the expression (3.*c*) for the elastic potential (or the energy of deformation per unit volume) assumes for $A = 0$, which will become:

$$\Phi = \frac{5}{8B}\left(\frac{\Delta_1 V}{8\pi}\right)^2 . \tag{6.f}$$

The conditions (6.*e*) are in patent contradiction to the ones in (2.*b*). The medium that is defined by them cannot be in stable equilibrium with respect to *all* possible deformations: However, the nature of the expression (6.*f*) is such that the equilibrium is still stable with respect to the special deformations that we present for consideration – i.e., to the ones that are defined by formulas of the type (3) in a space in which the function *V* satisfies the equation $\Delta_2 V = 0$. Such a medium, if it exists, will have the property that it transmits only transverse vibrations. The known coefficient of contraction or dilatation:

$$\eta = \frac{A - 2B}{2(A - B)} \tag{6.g}$$

will to be equal to unity for it, so a prismatic portion of that medium that is subject to a simple longitudinal extension will always contract in the transverse in an equal ratio.

§ 3.

We shall now see, while still assuming the existence of the medium that was just considered, that the determination of the deformation that corresponds to a given potential function *V* will follow in any case, and in order to avoid most difficulties, we shall deal solely in the infinite space that is excluded by the masses that belong to that potential function, especially since it is that space that the supposition of the existence of an isotropic medium is made largely possible.

If one introduces the conditions:

$$q = 0, \quad A = 0 \quad \text{or} \qquad C = \tfrac{1}{4}$$

into equations (6) then those equations will reduce to the following simpler form:

(7)
$$\begin{cases} H\dfrac{\partial e}{\partial n} = 3E - \Theta, & H\dfrac{\partial e'}{\partial n} = 3E', \\ H\dfrac{\partial f}{\partial n} = 3F - \Theta, & H\dfrac{\partial f'}{\partial n} = 3F', \\ H\dfrac{\partial g}{\partial n} = 3G - \Theta, & H\dfrac{\partial g'}{\partial n} = 3G'. \end{cases}$$

Represent the determinant of the second derivatives of V by Δ; i.e., set:

$$\Delta = \begin{vmatrix} e & g' & f' \\ g' & f & e' \\ f' & e' & g \end{vmatrix},$$

and observe that this determinant can be developed in various ways. Indeed, one has:

$$\begin{aligned} \Delta &= Ee + F'f' + G'g', \\ &= Ff + G'g' + E'e', \\ &= Gg + E'e' + F'f', \end{aligned}$$

and therefore also:

$$3\Delta = Ee + Ff + Gg + 2E'e' + 2F'f' + 2G'g'.$$

If one combines these various expressions opportunely then one will find these other ones:

(7.*a*)
$$\begin{aligned} \Delta &= Ff + Gg + 2E'e' - Ee, \\ &= Gg + Ee + 2F'f' - Ff, \\ &= Ee + Ff + 2G'g' - Gg. \end{aligned}$$

In addition to the known property of that determinant Δ, three pairs of identities will spring from it that have the type:

$$G'f' + Fe' + E'g = 0, \qquad F'g' + E'f + Ge' = 0,$$

and if one sums the corresponding sides of the identities of each type then one will have the three formulas:

(7.*b*)
$$\begin{cases} F'g' + G'f' - E\,e' - E'e + e'\,\Theta = 0, \\ G'e' + E'g' - F\,f' + F'f + f'\,\Theta = 0, \\ E'f' + F'e' - G'\,g - G'g + g'\,\Theta = 0. \end{cases}$$

Having said that, differentiate the expressions for E, F, G, E', F', G' that are given by formulas (5.*a*) with respect to the normal n and substitute the values of the normal derivatives of *e*, *f*, *g*, *e′*, *f′*, *g′* that are given by the six equations (7) in the equations thus-obtained. With the help of the preceding identity (7.*a*, *b*), one will then find these other six equations:

$$\text{(7.}e\text{)}\qquad \left\{\begin{aligned} H\frac{\partial E}{\partial n} &= -2\Theta\, e - 3\Delta, & H\frac{\partial E'}{\partial n} &= -2\Theta\, e', \\ H\frac{\partial F}{\partial n} &= -2\Theta\, f - 3\Delta, & H\frac{\partial F'}{\partial n} &= -2\Theta\, f', \\ H\frac{\partial G}{\partial n} &= -2\Theta\, g - 3\Delta, & H\frac{\partial G'}{\partial n} &= -2\Theta\, g', \end{aligned}\right.$$

which form a system that is equivalent to the one in equations (7). The sum of the first three of those equations (7) will yield:

$$\text{(7.}d\text{)}\qquad H\frac{\partial \Theta}{\partial n} = -\,9\Delta.$$

Finally, when one differentiates any of the preceding expression for the determinant with respect to the normal and substitutes the values that equations (7), (7.*c*) provide for the normal derivatives of *e*, *f*, *g*, *e′*, *f′*, *g′*, *E*, *F*, *G*, *E′*, *F′*, *G′*, one will deduce that:

$$H\frac{\partial \Delta}{\partial n} = 2\Theta^2.$$

If one eliminates *H* from the two equations (7.*d*, *e*) then one will find that:

$$\text{(7.}e\text{)}\qquad \frac{\partial}{\partial n}(4\,\Theta^2 + 27\,\Delta^2) = 0,$$

and therefore one concludes that the expression:

$$4\,\Theta^2 + 27\,\Delta^2$$

is constant along each line of force. However, that expression is annulled at any point at an infinite distance, so one must always have:

$$\text{(7.}f\text{)}\qquad 4\,\Theta^2 + 27\,\Delta^2 = 0.$$

That conclusion will also persist when only one part of the line of force extends to an infinite distance, since in order for *V* to be a function that is continuous and finite, along with all of its derivatives, in the infinite space where $\Delta_2 V = 0$, any expression that is formed from the derivatives of that function cannot be zero in one region of that space without it also being zero in the rest of that space.

The left-hand side of equation (7.*f*) is the well-known condition for the equality of two roots of the third-degree equation with no second-order term:

$$s^3 + s\,\Theta - \Delta = 0,$$

in which *s* is the unknown. If one writes that equation in the form:

$$\Delta - (E + F + G)\,s + (e + f + g)\,s^2 - s^3 = 0$$

then one will immediately recognize that it is equivalent to the following one:

$$\begin{vmatrix} e-s & g' & f' \\ g' & f-s & e' \\ f' & e' & g-s \end{vmatrix} = 0.$$

It is known that under the reality hypothesis, the condition (7.*f*) for the equality of the two roots of the last equations will split into some other ones, which will be, as one sees, the ones that will lead to the solution of our problem. However, we prefer to get those individual conditions directly by taking advantage of the one noteworthy property of the system of equations (7) that we shall now discuss.

§ 4.

Differentiate each of equations (7) with respect to normal *n* and substitute the values of the normal derivatives of *E*, *F*, *G*, *E′*, *F′*, *G′*, Θ that are provided by equations (7.*c*, *d*) in the left-hand sides of the equations that one obtains. One will then find the following six new equations:

$$(8)\qquad\left\{\begin{aligned} &H\frac{\partial\left(H\dfrac{\partial e}{\partial n}\right)}{\partial n}+6\Theta e=0, &\quad &H\frac{\partial\left(H\dfrac{\partial e'}{\partial n}\right)}{\partial n}+6\Theta e'=0,\\ &H\frac{\partial\left(H\dfrac{\partial f}{\partial n}\right)}{\partial n}+6\Theta f=0, &\quad &H\frac{\partial\left(H\dfrac{\partial f'}{\partial n}\right)}{\partial n}+6\Theta f'=0,\\ &H\frac{\partial\left(H\dfrac{\partial g}{\partial n}\right)}{\partial n}+6\Theta g=0, &\quad &H\frac{\partial\left(H\dfrac{\partial g'}{\partial n}\right)}{\partial n}+6\Theta g'=0. \end{aligned}\right.$$

It results from this that the six second derivatives of *V* all satisfy one and the same differential equation of the form:

$$(8.a)\qquad H\frac{\partial\left(H\dfrac{\partial \omega}{\partial n}\right)}{\partial n} + 6\Theta\,\omega = 0,$$

in which:

$$\omega = e,\ f,\ g,\ e',\ f',\ g'.$$

Now let ω and ω' be any two of those second derivatives, such that, along with equation (8.*a*), one will also have the following one:

$$H\frac{\partial\left(H\dfrac{\partial\omega'}{\partial n}\right)}{\partial n}+6\Theta\,\omega'=0.$$

One deduces from that equation and (8.*a*) that:

$$\omega\frac{\partial\left(H\dfrac{\partial\omega'}{\partial n}\right)}{\partial n}-\omega'\frac{\partial\left(H\dfrac{\partial\omega}{\partial n}\right)}{\partial n}=0,$$

or

$$\frac{\partial}{\partial n}\left\{H\left(\omega\frac{\partial\omega'}{\partial n}-\omega'\frac{\partial\omega}{\partial n}\right)\right\}=0.$$

It results from this that the expression:

$$H\left(\omega\frac{\partial\omega'}{\partial n}-\omega'\frac{\partial\omega}{\partial n}\right)$$

is constant along each line of force. However, as was observed before in regard to the left-hand side of equation (7.*f*), that expression will vanish at any point at an infinite distance, so one must always have:

(8.*b*) $$\omega\frac{\partial\omega'}{\partial n}-\omega'\frac{\partial\omega}{\partial n}=0.$$

In particular, one deduces the following two equalities from that equation, which persists for any two of the second derivatives e, f, g, e', f', g':

(9) $$\frac{\dfrac{\partial e'}{\partial n}}{e'}=\frac{\dfrac{\partial f'}{\partial n}}{f'}=\frac{\dfrac{\partial g'}{\partial n}}{g'},$$

and therefore, by virtue of the last three equations (7), these other two, as well:

(9.*a*) $$\frac{E'}{e'}=\frac{F'}{f'}=\frac{G'}{g'}.$$

Those relations contain the essential elements of the solution to our problem.

While putting the preceding equality into the indicated form, it was supposed that none of the second derivatives e', f', g' were identically zero. Now, that assumption is legitimate, since no special hypotheses were made on the directions of the coordinate axes, so it is always possible to fix them in such a way as to avoid the aforementioned exception. The only case in which that proves to be impossible will be the one in which *all* of the second derivatives are identically zero, or in which V is a linear function of the coordinates. However, that case, which was previously overlooked for other reasons, is presently excluded by the infinitude of the space to which the function V refers.

Let:

$$\frac{\dfrac{\partial h'}{\partial n}}{h'}$$

denote the value of the three equal ratios in (9). One can set:

$$e' = \frac{h'}{\xi}, \qquad f' = \frac{h'}{\eta}, \qquad g' = \frac{h'}{\zeta},$$

in which ξ, η, ζ are three functions that remain constant along one line a force – i.e., that satisfy the conditions:

(10) $$\frac{\partial \xi}{\partial n} = \frac{\partial \eta}{\partial n} = \frac{\partial \zeta}{\partial n} = 0$$

and that can be additionally subject to the relation:

(10.*a*) $$\xi^2 + \eta^2 + \zeta^2 = 1,$$

for ease of calculation. The preceding values of e', f', g' will imply that:

$$E' = \frac{h'(h'\xi - e\eta\zeta)}{\xi\eta\zeta}, \quad F' = \frac{h'(h'\eta - e\zeta\xi)}{\xi\eta\zeta}, \quad G' = \frac{h'(h'\zeta - e\xi\eta)}{\xi\eta\zeta},$$

and therefore (9.*a*):

$$\frac{h'\xi}{\eta\zeta} - e = \frac{h'\eta}{\zeta\xi} - f = \frac{h'\zeta}{\xi\eta} - g.$$

Let h denote the common value of the three equal expressions, so one will have:

$$e = \frac{h'\xi}{\eta\zeta} - h, \quad f = \frac{h'\eta}{\zeta\xi} - h, \quad g = \frac{h'\zeta}{\xi\eta} - h,$$

and the condition $\theta = e + f + g = 0$ will give (10.*a*):

$$\frac{h'}{\xi\eta\zeta} - 3h = 0, \qquad \text{i.e.,} \qquad h' = 3h\,\xi\eta\zeta.$$

One will then finally have:

$$(10.b)\qquad \begin{cases} e = h(3\xi^2 - 1), & e' = 3h\eta\zeta, \\ f = h(3\eta^2 - 1), & f' = 3h\zeta\xi, \\ g = h(3\zeta^2 - 1), & g' = 3h\xi\eta. \end{cases}$$

That will imply:

$$E = h^2\,(3\xi^2 - 2), \qquad E' = 3h^2\,\eta\zeta,$$
$$F = h^2\,(3\eta^2 - 2), \qquad F' = 3h^2\,\zeta\xi,$$
$$G = h^2\,(3\zeta^2 - 2), \qquad G' = 3h^2\,\xi\eta,$$

$$\Theta = -\,3h^2, \qquad \Delta = 2h^2.$$

If one substitutes these values in (7), while recalling the conditions (10), then one will find that it reduces all of them to the single equation:

$$(10.c)\qquad H\frac{\partial h}{\partial n} = 3h^2.$$

§ 5.

Now return to the formulas (5), from which one has:

$$da = e\,dx + g'\,dy + f'\,dz,$$
$$db = g'\,dx + f\,dy + e'\,dz,$$
$$dc = f'\,dx + e'\,dy + g\,dz,$$

and therefore, with the preceding values (10.*b*):

$$(11)\qquad \begin{cases} da = 3h\xi\,(\xi\,dx + \eta\,dy + \zeta\,dz) - h\,dx, \\ db = 3h\eta\,(\xi\,dx + \eta\,dy + \zeta\,dz) - h\,dy, \\ dc = 3h\zeta\,(\xi\,dx + \eta\,dy + \zeta\,dz) - h\,dz. \end{cases}$$

The nine integrability conditions (which can be reduced to eight):

$$\frac{\partial g'}{\partial z} - \frac{\partial f'}{\partial y} = 0, \qquad \frac{\partial f'}{\partial x} - \frac{\partial e}{\partial z} = 0, \qquad \frac{\partial e}{\partial y} - \frac{\partial g'}{\partial x} = 0,$$

$$\frac{\partial f}{\partial z} - \frac{\partial e'}{\partial y} = 0, \qquad \frac{\partial e'}{\partial x} - \frac{\partial g'}{\partial z} = 0, \qquad \frac{\partial g'}{\partial y} - \frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial e'}{\partial z} - \frac{\partial g}{\partial y} = 0, \qquad \frac{\partial g}{\partial x} - \frac{\partial f'}{\partial z} = 0, \qquad \frac{\partial f'}{\partial y} - \frac{\partial e'}{\partial x} = 0$$

will translate into some other partial differential equations between the four functions ξ, η, ζ, h that can be written in the following way:

$$(11.\xi)\qquad \begin{cases} L\xi + h\left(\eta\dfrac{\partial\xi}{\partial z} - \zeta\dfrac{\partial\xi}{\partial y}\right) = 0, \\ M\xi + h\left(\zeta\dfrac{\partial\xi}{\partial x} - \xi\dfrac{\partial\xi}{\partial z}\right) = -\dfrac{1}{3}\dfrac{\partial h}{\partial z}, \\ N\xi + h\left(\xi\dfrac{\partial\xi}{\partial y} - \eta\dfrac{\partial\xi}{\partial y}\right) = +\dfrac{1}{3}\dfrac{\partial h}{\partial y}, \end{cases}$$

$$(11.\eta)\qquad \begin{cases} L\eta + h\left(\eta\dfrac{\partial\eta}{\partial z} - \zeta\dfrac{\partial\eta}{\partial y}\right) = +\dfrac{1}{3}\dfrac{\partial h}{\partial z}, \\ M\eta + h\left(\zeta\dfrac{\partial\eta}{\partial x} - \xi\dfrac{\partial\eta}{\partial z}\right) = 0, \\ N\eta + h\left(\xi\dfrac{\partial\eta}{\partial y} - \eta\dfrac{\partial\eta}{\partial y}\right) = -\dfrac{1}{3}\dfrac{\partial h}{\partial y}, \end{cases}$$

$$(11.\zeta)\qquad \begin{cases} L\zeta + h\left(\eta\dfrac{\partial\zeta}{\partial z} - \zeta\dfrac{\partial\zeta}{\partial y}\right) = -\dfrac{1}{3}\dfrac{\partial h}{\partial z}, \\ M\zeta + h\left(\zeta\dfrac{\partial\zeta}{\partial x} - \xi\dfrac{\partial\zeta}{\partial z}\right) = +\dfrac{1}{3}\dfrac{\partial h}{\partial y}, \\ N\zeta + h\left(\xi\dfrac{\partial\zeta}{\partial y} - \eta\dfrac{\partial\zeta}{\partial y}\right) = 0, \end{cases}$$

in which, we have set:

$$(11.a)\qquad L = \frac{\partial\cdot h\eta}{\partial z} - \frac{\partial\cdot h\zeta}{\partial y}, \qquad M = \frac{\partial\cdot h\zeta}{\partial x} - \frac{\partial\cdot h\xi}{\partial z}, \qquad N = \frac{\partial\cdot h\xi}{\partial y} - \frac{\partial\cdot h\eta}{\partial x},$$

for brevity. If one sums the equations of the each of the three triples (11.ξ, η, ζ), after having multiplied them by ξ, η, ζ, in the ordinary way, then one will have:

$$(11.b)\qquad \begin{cases} (L\xi + M\eta + N\zeta)\xi = \dfrac{1}{3}\left(\dfrac{\partial h}{\partial y}\zeta - \dfrac{\partial h}{\partial z}\eta\right), \\ (L\xi + M\eta + N\zeta)\eta = \dfrac{1}{3}\left(\dfrac{\partial h}{\partial z}\xi - \dfrac{\partial h}{\partial x}\zeta\right), \\ (L\xi + M\eta + N\zeta)\zeta = \dfrac{1}{3}\left(\dfrac{\partial h}{\partial x}\eta - \dfrac{\partial h}{\partial y}\xi\right). \end{cases}$$

However, if one sums the first equations in each of those triples, then the second ones, and then the last ones, after ordinary-multiplying each time by ξ, η, ζ, then when one recalls the relations (10.*a*), one will have:

$$\text{(10.}e\text{)}\qquad \begin{cases} L+\dfrac{1}{3}\left(\dfrac{\partial h}{\partial y}\zeta-\dfrac{\partial h}{\partial z}\eta\right)=0, \\ M+\dfrac{1}{3}\left(\dfrac{\partial h}{\partial z}\xi-\dfrac{\partial h}{\partial x}\zeta\right)=0, \\ N+\dfrac{1}{3}\left(\dfrac{\partial h}{\partial x}\eta-\dfrac{\partial h}{\partial y}\xi\right)=0. \end{cases}$$

Finally, if one sums either the three equations (11.*b*) or the three equations (11.*c*), after ordinary-multiplying them by ξ, η, ζ, then one will get:

$$L\,\xi+M\,\eta+N\,\zeta=0,$$

and therefore, due to equations (11.*b*, *c*), one will have:

$$\frac{\partial h}{\partial y}\zeta-\frac{\partial h}{\partial z}\eta=0, \qquad \frac{\partial h}{\partial z}\xi-\frac{\partial h}{\partial x}\zeta=0, \qquad \frac{\partial h}{\partial x}\eta-\frac{\partial h}{\partial y}\xi=0,$$

as well as:

$$L=0, \qquad M=0, \qquad N=0.$$

One can give the first three of these latter six equations the form:

$$\text{(11.}d\text{)}\qquad \frac{\frac{\partial h}{\partial x}}{\xi}=\frac{\frac{\partial h}{\partial y}}{\eta}=\frac{\frac{\partial h}{\partial z}}{\zeta}=\frac{dh}{\xi\,dx+\eta\,dy+\zeta\,dz};$$

however, if one takes the preceding equations into account then one can give the last three the form:

$$\text{(11.}e\text{)}\qquad \frac{\partial\eta}{\partial z}-\frac{\partial\zeta}{\partial y}=0, \qquad \frac{\partial\zeta}{\partial x}-\frac{\partial\xi}{\partial z}=0, \qquad \frac{\partial\xi}{\partial y}-\frac{\partial\eta}{\partial x}=0.$$

If follows immediately from those three equations (11.*e*) that there will exist a function r of the coordinates x, y, z whose total differential is given by:

$$\text{(12)}\qquad \xi\,dx+\eta\,dy+\zeta\,dz=dr,$$

and the preceding equations (11.*d*) show that the quantity h can depend upon only that function r.

The three of equations (11.ξ, η, ζ) whose right-hand sides are zero will reduce to the following ones:

(12.*a*) $$\eta\frac{\partial\zeta}{\partial x}-\zeta\frac{\partial\eta}{\partial x}=0,\qquad \zeta\frac{\partial\xi}{\partial y}-\xi\frac{\partial\zeta}{\partial y}=0,\qquad \xi\frac{\partial\eta}{\partial z}-\eta\frac{\partial\xi}{\partial z}=0,$$

by virtue of equations (11.*d*). The remaining six equations of the aforementioned system, when combined pair-wise in a sum in such a way as to eliminate the derivatives of *h*, will give:

(12.*b*) $$\begin{cases} \xi\left(\dfrac{\partial\eta}{\partial y}-\dfrac{\partial\zeta}{\partial z}\right)=\eta\dfrac{\partial\eta}{\partial x}-\zeta\dfrac{\partial\zeta}{\partial x}, \\ \eta\left(\dfrac{\partial\zeta}{\partial z}-\dfrac{\partial\xi}{\partial x}\right)=\zeta\dfrac{\partial\zeta}{\partial y}-\xi\dfrac{\partial\xi}{\partial y}, \\ \zeta\left(\dfrac{\partial\xi}{\partial x}-\dfrac{\partial\eta}{\partial y}\right)=\xi\dfrac{\partial\xi}{\partial z}-\eta\dfrac{\partial\eta}{\partial z}. \end{cases}$$

Finally, when those equations are combined pair-wise by subtraction, they will give:

$$h\xi\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)=-\frac{2}{3}\frac{\partial h}{\partial x},$$

$$h\eta\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)=-\frac{2}{3}\frac{\partial h}{\partial y},$$

$$h\zeta\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)=-\frac{2}{3}\frac{\partial h}{\partial z},$$

which are equations that summarize all of (12) in the single one:

$$\left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}\right)dr=-\frac{2}{3}\frac{dh}{h},$$

which one can put into the form:

(12.*c*) $$\frac{d\log h}{dr}=-\frac{3}{2}\Delta_2 r.$$

§ 6.

It is now easy to determine the two functions *r*, *h* upon which the solution to the problem presently depends.

When equations (12.*a*) are put into the form:

$$\frac{\partial}{\partial x}\left(\frac{\eta}{\zeta}\right) = 0, \quad \frac{\partial}{\partial y}\left(\frac{\zeta}{\xi}\right) = 0, \quad \frac{\partial}{\partial z}\left(\frac{\xi}{\eta}\right) = 0,$$

that will give:

(13) $$\eta = P\,\zeta, \qquad \zeta = P\,\xi, \qquad \xi = P\,\eta,$$

in which P is a function of y and z, Q is a function of z and x, and R is a function of x and y. Those three functions are constrained by the relation:

$$PQR = 1,$$

or

(13.*a*) $$\log P + \log Q + \log R = 0,$$

from which one deduces that:

$$\frac{\partial \log Q}{\partial x} = -\frac{\partial \log R}{\partial x}, \quad \frac{\partial \log R}{\partial y} = -\frac{\partial \log P}{\partial y}, \quad \frac{\partial \log P}{\partial z} = -\frac{\partial \log Q}{\partial z}.$$

Now, in the first of these latter equalities, the left-hand side is independent of y and the right-hand side is independent of z; hence, the two sides can be equal to only a function of just x. By an analogous argument, the two sides of the second equality can be equal to only a function of just y, and those of the third equality can be equal to just z. It follows quite easily from this and the necessity of satisfying the original relation (13.*a*) that the forms of $\log P$, $\log Q$, $\log R$ can be only the following ones:

$$\log P = Y - Z, \qquad \log Q = Z - X, \qquad \log R = X - Y,$$

in which X is a function of only x, Y is a function of only y, Z is a function of only z, and therefore equations (13) can be written in the following way, if one alters the notation for those arbitrary functions:

$$\frac{\xi}{X'} = \frac{\eta}{Y'} = \frac{\zeta}{Z'},$$

in which X' is the derivative of a function of only x, Y' is that of a function of only y, and Z' is that is a function of only z. One deduces from this and equation (12) that:

$$\frac{\xi}{X'} = \frac{\eta}{Y'} = \frac{\zeta}{Z'} = \frac{dr}{d(X+Y+Z)},$$

so, if one sets:

(13.*b*) $$X + Y + Z = t,$$

for the moment, then one will see that r is necessarily a function of only t and that one has:

(13.*c*) $$\xi = \frac{\partial r}{\partial x} = r' X', \qquad \eta = \frac{\partial r}{\partial y} = r' Y', \qquad \zeta = \frac{\partial r}{\partial z} = r' Z',$$

in which:

$$r' = \frac{dr}{dt};$$

equation (10.*a*) will then become:

(13.*d*) $$r'^2 (X'^2 + Y'^2 + Z'^2) = 1.$$

When the preceding values (13.*c*) of the functions ξ, η, ζ are substituted in equations (12.*b*), that will give:

$$r'^2 X' (Z'' - Y'') = 0, \qquad r'^2 Y' (X'' - Z'') = 0, \qquad r'^2 Z' (Y'' - X'') = 0,$$

and therefore one has:

(13.*e*) $$X'' = Y'' = Z'' = 2k,$$

in which $2k$ represents the common value of the three second derivatives of X, Y, Z, which can be only a *constant.*

That constant k cannot be zero. Indeed, if that were true then the quantities X', Y', Z' would be constants, and therefore (13.*d*) r', as well, so r would be a linear function of x, y, z, and one would have $\Delta_2 r = 0$. Consequently (12.*e*), h would also have a constant value, and that value could only be (10.*c*) zero, so (11) one would have:

$$da = 0, \qquad db = 0, \qquad dc = 0,$$

and the function V would prove to be linear, which is a form that was already excluded as inadmissible.

Therefore, since the constant k cannot be zero, one can assign (13.*e*) the following forms to the functions X, Y, Z :

$$X = k (x - x_0)^2 + l, \qquad Y = k (y - y_0)^2 + m, \qquad Z = k (z - z_0)^2 + n,$$

in which x_0 , y_0 , z_0 , l, m, n are new arbitrary constants. One deduces (13.*b*) from this that:

$$t = k \{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\} + l + m + n,$$

$$X'^2 + Y'^2 + Z'^2 = 4k (t - l - m - n),$$

and therefore (13.*d*):

$$dr = \frac{dt}{2\sqrt{k(t - l - m - n)}},$$

and if one drops the additive constant (which is obviously useless) then that will give:

$$r = \sqrt{\frac{t-l-m-n}{k}},$$

or

(14) $$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,.$$

It follows from that expression for r that:

$$\Delta_2\, r = \frac{2}{r},$$

such that equation (12.*c*) will become:

$$\frac{d\log h}{dr} = -\,\frac{3}{r},$$

and will give:

(14.*a*) $$h = \frac{M}{r^3},$$

in which M is a new constant.

By virtue of formulas (14), (14.*a*), (13.*c*), equations (11) will become:

$$da = M\,\frac{3(x-x_0)\,dr - r\,dx}{r^4},$$

$$db = M\,\frac{3(y-y_0)\,dr - r\,dy}{r^4},$$

$$dc = M\,\frac{3(z-z_0)\,dr - r\,dz}{r^4},$$

and give:

$$a = -M\,\frac{x-x_0}{r^3} + a_0\,, \quad b = -M\,\frac{y-y_0}{r^3} + b_0\,, \quad c = -M\,\frac{z-z_0}{r^3} + c_0\,,$$

in which a_0, b_0, c_0 are three constants that can only be zero. That is already required by the property that the first derivative of V must have that it is zero at infinity, but will also necessarily follow from the equations of the problem. Indeed, the three quantities (13.*e*):

$$\xi = \frac{x-x_0}{r}, \qquad \eta = \frac{y-y_0}{r}, \qquad \zeta = \frac{z-z_0}{r}$$

must satisfy equations (10), which will be equivalent to:

$$\frac{\partial x}{\partial n} = \xi\,\frac{\partial r}{\partial n}, \qquad \frac{\partial y}{\partial n} = \eta\,\frac{\partial r}{\partial n}, \qquad \frac{\partial z}{\partial n} = \zeta\,\frac{\partial r}{\partial n},$$

for those values of ξ, η, ζ.

Those equations show that the quantity ξ, η, ζ must be proportional to the three first derivatives of V; i.e., to the quantities that were just determined:

$$a = -\frac{M\xi}{r^2} + a_0\,, \qquad b = -\frac{M\eta}{r^2} + b_0\,, \qquad c = -\frac{M\zeta}{r^2} + c_0\,,$$

and that proportionality obviously cannot happen unless $a_0 = b_0 = c_0 = 0$, as we said. We will then have:

$$a = \frac{\partial \frac{M}{r}}{\partial x}, \qquad b = \frac{\partial \frac{M}{r}}{\partial y}, \qquad c = \frac{\partial \frac{M}{r}}{\partial z},$$

and from which, we will finally get:

$$V = \frac{M}{r}. \tag{14.b}$$

The direction of the normal n can be made to coincide with that of the radius r in this case, so we will have (14.*a*, *b*):

$$\frac{\partial h}{\partial n} = -\frac{3M}{r^4}, \qquad H = \frac{\partial V}{\partial n} = -\frac{M}{r^2},$$

and equation (10.*e*) will then be satisfied, as well.

The value of V that was found will then satisfy all of equations (7). Moreover, if that value is satisfied in the original equations (4) then one will find that they reduce to the following ones:

$$\begin{aligned} &A\,(2r^2 - 3x^2) = 0, \qquad && Ayz = 0, \\ &A\,(2r^2 - 3y^2) = 0, && Azx = 0, \\ &A\,(2r^2 - 3z^2) = 0, && Axy = 0, \end{aligned}$$

and will become an identity when (and only when) $A = 0$, which agrees with the conclusions of § **2**.

§ 7.

It will result from the preceding that the number of cases is severely restricted in which it is possible to represent the MAXWELL system of stresses by the effective deformation of an infinite isotropic medium, which is a medium whose constitution does not, on the other hand, correspond to any known reality, due to the necessary condition that $A = 0$. Despite that fact, one will then see what the deformation of that medium will be that corresponds to the unique form for the function V that was just seen to be possible.

For simplicity, take:

$$x_0 = y_0 = z_0 = 0, \qquad r = \sqrt{x^2 + y^2 + z^2}\,,$$

and one will have:

$$\frac{\partial V}{\partial x} = -\frac{Mx}{r^3}, \quad \frac{\partial V}{\partial y} = -\frac{My}{r^3}, \quad \frac{\partial V}{\partial z} = -\frac{Mz}{r^3}, \quad \Delta_1 V = -\frac{Mx}{r^3},$$

and when one sets $A = 0$, formulas (3) will give:

$$\alpha = \frac{M^2(4x^2 - r^2)}{32\pi B r^6}, \qquad \lambda = \frac{8M^2 yz}{32\pi B r^6},$$

$$\beta = \frac{M^2(4y^2 - r^2)}{32\pi B r^6}, \qquad \mu = \frac{8M^2 zx}{32\pi B r^6},$$

$$\gamma = \frac{M^2(4z^2 - r^2)}{32\pi B r^6}, \qquad \nu = \frac{8M^2 xy}{32\pi B r^6},$$

or

$$\alpha = \frac{\partial^2\phi}{\partial x^2}, \qquad \lambda = 2\frac{\partial^2\phi}{\partial y\,\partial z},$$

$$\beta = \frac{\partial^2\phi}{\partial y^2}, \qquad \mu = 2\frac{\partial^2\phi}{\partial z\,\partial x},$$

$$\gamma = \frac{\partial^2\phi}{\partial z^2}, \qquad \nu = 2\frac{\partial^2\phi}{\partial x\,\partial y},$$

in which:

(15) $$\phi = \frac{V^2}{64\pi B}.$$

It will then result that the components u, v, w of the displacement of the point (x, y, z), which one supposes will vanish at infinity, are given by:

(15.*a*) $$u = \frac{\partial\phi}{\partial x}, \qquad v = \frac{\partial\phi}{\partial y}, \qquad w = \frac{\partial\phi}{\partial z},$$

or

$$u = -\frac{M^2}{32\pi B}\frac{x}{r^4}, \qquad v = -\frac{M^2}{32\pi B}\frac{y}{r^4}, \qquad w = -\frac{M^2}{32\pi B}\frac{z}{r^4}.$$

The displacement of any point of the medium is therefore *radial*, like the force, and will have the value:

(15.*b*) $$- \frac{M^2}{32\pi B}\frac{1}{r^2}.$$

The sign of that expression shows that it is always a *contraction* of the entire surrounding region towards the center $x = y = z = 0$, but the law by which that contraction decreases with distance from the center will be such that the cubic dilatation will always be *positive* and given by:

(15.*c*) $$\vartheta = \frac{M^2}{32\pi B}\frac{1}{r^4}.$$

One then finds that:

$$X_x = \frac{M^2}{8\pi}\cdot\frac{r^2 - 2x^2}{r^6}, \qquad Y_z = \frac{M^2}{8\pi}\cdot\frac{-2yz}{r^6},$$

$$Y_y = \frac{M^2}{8\pi}\cdot\frac{r^2 - 2y^2}{r^6}, \qquad Z_x = \frac{M^2}{8\pi}\cdot\frac{-2zx}{r^6},$$

$$Z_z = \frac{M^2}{8\pi}\cdot\frac{r^2 - 2z^2}{r^6}, \qquad X_y = \frac{M^2}{8\pi}\cdot\frac{-2xy}{r^6},$$

or also:

$$X_x = \frac{1}{r^2}\frac{\partial^2\psi}{\partial x^2}, \qquad Y_z = \frac{1}{r^2}\frac{\partial^2\psi}{\partial y\,\partial z},$$

$$Y_y = \frac{1}{r^2}\frac{\partial^2\psi}{\partial y^2}, \qquad Z_x = \frac{1}{r^2}\frac{\partial^2\psi}{\partial z\,\partial x},$$

$$Z_z = \frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2}, \qquad X_y = \frac{1}{r^2}\frac{\partial^2\psi}{\partial x\,\partial y},$$

in which:

$$\psi = \frac{M^2}{8\pi}\log r.$$

The deformation that was defined by formulas (15.*a*, *b*, *c*) can refer to the infinite space that is excluded, for example, from a sphere of finite radius and it can come about radially and uniformly around the center of that sphere. Now, the study of a deformation of that type presents itself in one of the more elementary applications of the theory of elasticity – i.e., in the problem of the equilibrium of an isotropic spherical shell that is subjected to constant pressures on its two surfaces, and in our analogous case, its external surface would be completely at infinity and would not support any pressure. One can then suppose that the preceding solution would be found to coincide with the one that is known already. However, that is not, nor can it be, for the reason that we shall now discuss.

§ 8.

Since the trinomial:

$$n\,dx + v\,dy + w\,dz$$

is an exact differential for one question, as well as the other, one can set:

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z},$$

from which, it will follow, as in the preceding §, that:

$$\alpha = \frac{\partial^2 \phi}{\partial x^2}, \qquad \lambda = 2\frac{\partial^2 \phi}{\partial y\,\partial z},$$

$$\beta = \frac{\partial^2 \phi}{\partial y^2}, \qquad \mu = 2\frac{\partial^2 \phi}{\partial z\,\partial x},$$

$$\gamma = \frac{\partial^2 \phi}{\partial z^2}, \qquad \nu = 2\frac{\partial^2 \phi}{\partial x\,\partial y}.$$

It results from this that the components of the stresses in an ordinary isotropic medium will have the values:

$$X_x = -(A - 2B)\,\Delta_2\phi - 2B\frac{\partial^2 \phi}{\partial x^2}, \qquad Y_z = -2B\frac{\partial^2 \phi}{\partial y\,\partial z},$$

$$Y_y = -(A - 2B)\,\Delta_2\phi - 2B\frac{\partial^2 \phi}{\partial y^2}, \qquad Z_x = -2B\frac{\partial^2 \phi}{\partial z\,\partial x},$$

$$Z_z = -(A - 2B)\,\Delta_2\phi - 2B\frac{\partial^2 \phi}{\partial z^2}, \qquad X_y = -2B\frac{\partial^2 \phi}{\partial x\,\partial y},$$

from which, one will deduce that:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = -A\frac{\partial \Delta_2 \phi}{\partial x},$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = -A\frac{\partial \Delta_2 \phi}{\partial y},$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = -A\frac{\partial \Delta_2 \phi}{\partial z}.$$

If X, Y, Z denote the components of the external force per unit volume then the indefinite equations of equilibrium will be:

$$A\frac{\partial \Delta_2 \phi}{\partial x} + X = 0, \qquad A\frac{\partial \Delta_2 \phi}{\partial y} + Y = 0, \qquad A\frac{\partial \Delta_2 \phi}{\partial z} + Z = 0,$$

and if one sets:

$$X = -k\frac{\partial V}{\partial x}, \quad Y = -k\frac{\partial V}{\partial y}, \quad Z = -k\frac{\partial V}{\partial z},$$

in which V is the potential function of the external force and k is the density, then those equations can be summarized in the single one:

$$A\,\Delta_2\phi - kV = \text{const.},$$

which will reduce to simply:

$$A\,\Delta_2\phi = \text{const.}$$

when there are no surface pressures.

Now, if A is non-zero, which would be necessarily true in an ordinary isotropic medium, and if, as would happen in the spherical problem that was cited above, the function ϕ depends upon only the distance r from the point (x, y, z) from the center of the shell then that equation will become:

$$\frac{1}{r}\frac{\partial^2(r\phi)}{\partial r^2} = \text{const.},$$

and when one drops an insignificant additive constant, that will give:

$$f = \frac{K}{r} + K'r^2,$$

in which K and K' are two constants. In the case of the infinite space that is excluded from a sphere, one needs to set $K' = 0$, and one will then get the expression:

$$\phi = \frac{K}{r},$$

which represents the ordinary solution, in which the constant K is then determined by the pressure that exists upon the internal spherical surface. However, no matter what that pressure is, the cubic dilatation of the medium, which is given by:

$$\vartheta = \Delta_2\,\phi,$$

will always be *zero*.

However, if, as one has in the singular case that one was led to consider in the preceding §§, the constant A is zero, then one will have:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0,$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0,$$

for any function ϕ. It will then result that from the very nature of the deformation, equilibrium can come about only in the absence of mass forces, but when that condition is satisfied, it will be consistent with any form for the function ϕ. That obvious contradiction with the fundamental theorem on the uniqueness of the equilibrium will clearly depend upon the fact that for $A = 0$, the elastic potential will no longer possess the character of an essentially-positive quadratic form. The precise determination that one has nonetheless obtained for the function ϕ in the preceding §, and the fact that its character will imply that the cubic dilatation (15.*c*) is *not* precisely zero, spring from another source, and therefore the components of the stresses X_x , X_y , … will assume *a priori* expressions of the *given form* – i.e., the expressions (1). Those expressions can very well contain an arbitrary function V, but in the infinite space in which those functions satisfy the equations $\Delta_2 V = 0$, as we proved before, it cannot relate to the stress components that one derives from a true deformation in the medium unless V has the form (14.*b*), and that special form of V will necessarily imply just the special determination of ϕ that was encountered in the preceding § and that was essentially different from the one that is suitable to an ordinary isotropic medium.

§ 9.

I would not wish to conclude this article without mentioning a case in which the problem of the determination of the functions u, v, w that correspond to a given form of the potential function V that is reducible to some very simple terms when one recalls the conditions (1). The case to which I allude is the one in which the aforementioned function depends upon only the distance r from the potentiating point to a fixed point, which one assumes to be the origin of the coordinates, for simplicity.

It is easy to understand (and also to prove) that the displacements must be radial and uniform in this case as well; i.e., that the components u, v, w must be the derivatives of a function of only the distance r. Having said that, for ease of calculation, consider V to be a function of the argument:

$$r = x^2 + y^2 + z^2 = r^2,$$

and set:

$$\frac{dV}{d\rho} = V';$$

one will have:

$$\frac{\partial V}{\partial x} = 2V'x, \quad \frac{\partial V}{\partial y} = 2V'y, \quad \frac{\partial V}{\partial z} = 2V'z, \quad \Delta_1 V = 4V'^2\, x,$$

and the formulas (3), (4.*a*) will give:

$$\alpha = \frac{V'^2}{2\pi B}(x^2 - C\rho), \qquad \lambda = \frac{V'^2}{\pi B} y\, z,$$

$$\beta = \frac{V'^2}{2\pi B}(y^2 - C\rho), \qquad \mu = \frac{V'^2}{\pi B} z\, x,$$

$$\gamma = \frac{V'^2}{2\pi B}(z^2 - C\rho), \qquad \nu = \frac{V'^2}{\pi B} x\, y.$$

After that, assume that there is another function ϕ of the same argument ρ and set:

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z},$$

and one will find that:

$$\begin{aligned} \alpha &= 2\phi' + 4\phi'' x^2, & \lambda &= 8\phi\, yz, \\ \beta &= 2\phi' + 4\phi'' y^2, & \mu &= 8\phi\, zx, \\ \gamma &= 2\phi' + 4\phi'' z^2, & \nu &= 8\phi\, xy. \end{aligned}$$

Now, these ultimate expressions cannot be identified with the preceding ones unless one simultaneously has the following *two* relations:

$$\phi'' = \frac{V'^2}{8\pi B}, \qquad \phi' = -\frac{CV'^2 \rho}{4\pi B},$$

from which it will result that:

$$\frac{\phi''}{\phi'} + \frac{1}{2C\rho} = 0.$$

The last equation already establishes that the form of the function ϕ (and consequently that of the function V, as well) depends uniquely upon the constitution of the isotropic medium that one considers, if one ignores a constant. That would be likewise true if the constitution of the medium varied with the distance from the center; i.e., if the isotropy parameters A and B were supposed to be functions of r, in such a way that the medium is envisioned to be formed of concentric, isotropic layers with isotropy parameters that vary continuously from one layer to another.

In the case of constant parameters, when one integrates the last equation, one will have:

$$\phi = K\, \rho^{-\eta},$$

in which K is an arbitrary constant and η already refers to the ratio of the transverse contraction to the longitudinal dilatation. It will result that:

$$V = 4\sqrt{\frac{\pi EK}{\eta}}\rho^{-\eta/2},$$

in which E is the modulus of elasticity. If one replaces ρ with r^2 again then one will conclude from this that the unique solution to the problem is given by:

$$V = \frac{M}{r^\eta}, \qquad \phi = \frac{\eta}{16\pi E}V^2,$$

in which M is a new constant.

The density of the distribution that produces that potential V is expressed by:

$$\frac{M\eta(1-\eta)}{4\pi r^{\eta+2}}.$$

In order for that density to be zero, one needs to have $\eta = 1$, and therefore:

$$A = 0, \qquad E = 4B.$$

One will then get the solution that was found before. However, if one desires that the aforementioned density should prove to be *constant* then one could reach some even more incongruous conclusions. Indeed, since one would need to have $\eta = -2$ in order for that to happen, one would have:

$$5A - 6B = 0, \quad \text{and therefore} \quad E = -2B,$$

so one would need to assume that either the modulus of elasticity of the medium was negative or that (3.*e*) the elastic potential:

$$\Phi = \frac{-1}{2E}\left(\frac{\Delta_1 V}{8\pi}\right)^2$$

was negative.

There is another case in which the problem that was treated in this paper can be solved in a direct way, although not as simply as when one makes the particular hypotheses that were just considered. It is the case in which the mass distribution that belongs to the potential function V is symmetric around a rectilinear axis, so V will become a function of only two coordinates that suffice to define the position of a point in a plane that passes through the symmetry axis. In that case, the six condition equations (4) will reduce to four, two of which will be the exact derivatives of the same second-order partial differential equation. That equation, when combined with $\Delta_2 V = 0$, will admit a new first integral, and from its form, one will easily deduce (when one benefits from the intervention of the *associated* function for that purpose) that if one excludes the linear form and one then sets $A = 0$ then the only possible form for V is the one that is represented by equation (14.*b*) for a point (x_0, y_0, z_0) that is situated on the symmetry axis.

In that case, as was stated at the beginning of my study, the form (14.*b*) will reveal itself to be the only admissible one (always excluding the linear case) in *any* space that is devoid of potentiating masses.

NOTE

One ordinarily proves the *necessity*, but not the *sufficiency* of the condition equations for the quantities α, β, γ, λ, μ, ν that were cited in § **1**. Given the importance of those equations in the context of the present work, I therefore deem it opportune to add a deduction of those equations that will clearly establish the property that they represent conditions that are not only necessary, but also sufficient, for the existence of the three components of the displacement u, v, w.

Recall from the general theory of the deformations of a continuous medium that along with the cited components α, β, γ, λ, μ, ν, the three quantities that are defined by the equations:

$$(a) \qquad \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2p, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 2q, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2r$$

will also intervene with no less authority and will represent the *components of rotation* of the particle that surrounds the point (x, y, z). Now, the system of nine equations that one obtains by combining the six equations (2) in § **1** with the preceding three (*a*) will give one the values of all of the first derivatives of the three components of the displacement u, v, w, and those values will be the following:

$$(b) \qquad \begin{cases} \dfrac{\partial u}{\partial x} = \alpha, & \dfrac{\partial u}{\partial y} = \dfrac{\nu}{2} - r, & \dfrac{\partial u}{\partial z} = \dfrac{\mu}{2} + q, \\ \dfrac{\partial v}{\partial x} = \dfrac{\nu}{2} + r, & \dfrac{\partial v}{\partial y} = \beta, & \dfrac{\partial v}{\partial z} = \dfrac{\mu}{2} - p, \\ \dfrac{\partial w}{\partial x} = \dfrac{\mu}{2} - q, & \dfrac{\partial w}{\partial y} = \dfrac{\lambda}{2} + p, & \dfrac{\partial w}{\partial z} = \gamma. \end{cases}$$

Consider the first three of those equations, which provide the values of the first derivatives of the function u. If one supposes that the quantities that enter into their right-hand sides to be *given* then in order for there to exist a function u that satisfies those three equations, it will be necessary and sufficient that three known relations should be satisfied, which can be written as follows:

$$-\frac{\partial q}{\partial y} - \frac{\partial r}{\partial z} = \frac{1}{2}\left(\frac{\partial \mu}{\partial y} - \frac{\partial \nu}{\partial z}\right), \qquad \frac{\partial q}{\partial y} = \frac{\partial \alpha}{\partial z} - \frac{1}{2}\frac{\partial \mu}{\partial x}, \qquad \frac{\partial r}{\partial x} = \frac{1}{2}\frac{\partial \nu}{\partial x} - \frac{\partial \alpha}{\partial y}.$$

One deduces the two analogous triples of necessary and sufficient conditions for the existence of the other two functions *v* and *w* from this by cyclic permutation. However, if one performs that permutation on only the first of the preceding three conditions and then sums corresponding sides of the three equations thus-obtained then one will find that (*):

$$\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z} = 0,$$

such that the first of the aforementioned three conditions can be written more simply as:

$$\frac{\partial p}{\partial x} = \frac{1}{2}\left(\frac{\partial \mu}{\partial y}-\frac{\partial \nu}{\partial z}\right).$$

In that way, one will obtain the following system of differential relations between the nine functions α, β, γ, λ, μ, ν, p, q, r :

$$(c)\qquad \left\{\begin{array}{lll}
\dfrac{\partial p}{\partial x}=\dfrac{1}{2}\left(\dfrac{\partial \mu}{\partial y}-\dfrac{\partial \nu}{\partial z}\right), & \dfrac{\partial p}{\partial y}=\dfrac{1}{2}\dfrac{\partial \lambda}{\partial y}-\dfrac{\partial \beta}{\partial z}, & \dfrac{\partial p}{\partial z}=\dfrac{\partial \gamma}{\partial y}-\dfrac{1}{2}\dfrac{\partial \lambda}{\partial z}, \\[2ex]
\dfrac{\partial q}{\partial x}=\dfrac{\partial \alpha}{\partial z}-\dfrac{1}{2}\dfrac{\partial \mu}{\partial x}, & \dfrac{\partial q}{\partial y}=\dfrac{1}{2}\left(\dfrac{\partial \nu}{\partial z}-\dfrac{\partial \lambda}{\partial x}\right), & \dfrac{\partial q}{\partial z}=\dfrac{1}{2}\dfrac{\partial \mu}{\partial z}-\dfrac{\partial \gamma}{\partial x}, \\[2ex]
\dfrac{\partial r}{\partial x}=\dfrac{1}{2}\dfrac{\partial \nu}{\partial x}-\dfrac{\partial \alpha}{\partial y}, & \dfrac{\partial r}{\partial y}=\dfrac{\partial \beta}{\partial x}-\dfrac{1}{2}\dfrac{\partial \nu}{\partial y}, & \dfrac{\partial r}{\partial z}=\dfrac{1}{2}\left(\dfrac{\partial \lambda}{\partial x}-\dfrac{\partial \mu}{\partial y}\right).
\end{array}\right.$$

That system of equations contains the necessary and sufficient conditions for the existence of the three functions *u*, *v*, *w* to satisfy the nine conditions (*b*) or the six equations (2) in § **1** and the three equations (*a*) in this Note.

Having said that, consider just the *six components of deformation* α, β, γ, λ, μ, ν to be *given*. If three functions *u*, *v*, *w* exist that satisfy the equations (2) in § **1** then there will certainly also exist the three functions *p*, *q*, *r* that are defined by equations (*a*) of this Note. Since the derivatives of the last three functions are coupled to the α, β, γ, λ, μ, ν by the nine equations (*c*), one needs for them to satisfy the integrability conditions that result from the latter nine equations and that reduce to the following six:

(*) That well-known relation already results from the defining formulas (*a*). However, in the present context, it will be necessary to make it known that it is included in the nine integrability conditions that we spoke of.

$$
(d) \quad \begin{cases}
\dfrac{\partial^2 \beta}{\partial z^2} + \dfrac{\partial^2 \gamma}{\partial y^2} = \dfrac{\partial^2 \lambda}{\partial y \, \partial z}, \quad 2\dfrac{\partial^2 \alpha}{\partial y \, \partial z} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mu}{\partial y} + \dfrac{\partial \nu}{\partial z} - \dfrac{\partial \lambda}{\partial x}\right), \\[2ex]
\dfrac{\partial^2 \gamma}{\partial x^2} + \dfrac{\partial^2 \alpha}{\partial z^2} = \dfrac{\partial^2 \mu}{\partial z \, \partial x}, \quad 2\dfrac{\partial^2 \beta}{\partial z \, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \nu}{\partial z} + \dfrac{\partial \lambda}{\partial x} - \dfrac{\partial \mu}{\partial y}\right), \\[2ex]
\dfrac{\partial^2 \alpha}{\partial y^2} + \dfrac{\partial^2 \beta}{\partial x^2} = \dfrac{\partial^2 \nu}{\partial x \, \partial y}, \quad 2\dfrac{\partial^2 \gamma}{\partial x \, \partial y} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial \lambda}{\partial x} + \dfrac{\partial \mu}{\partial y} - \dfrac{\partial \nu}{\partial z}\right),
\end{cases}
$$

which are precisely the ones that were cited in § **1**. When those conditions are satisfied, there will indubitably exist three functions p, q, r that satisfy the nine equations (*c*), but one has already seen that if those nine equations are satisfied by the nine functions α, β, γ, λ, μ, ν, p, q, r then there will exist three functions u, v, w that satisfy the conditions (2) in § **1** and (*a*) in the present Note. Therefore, the six conditions (*d*), which are obviously *necessary* for the existence of three functions u, v, w that satisfy only the equations (2) in § **1**, are also *sufficient.*

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