“Osservazioni sulla Nota precedente,” Rend. Real. Accad. Lincei, Classe di sci. fis., mat. e nat. (5) **1** (1892), 141-142.

Observation on the preceding note (†)

By member EUGENIO BELTRAMI

Translated by D. H. Delphenich

It is known that if the six components of the deformation that corresponds to a system of displacements (*u*, *v*, *w*) are represented by:

$$\alpha = \frac{\partial u}{\partial x}, \qquad \lambda = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},$$

$$\beta = \frac{\partial v}{\partial y}, \qquad \mu = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$

$$\gamma = \frac{\partial w}{\partial z}, \qquad \nu = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

then if those six functions $\alpha, \beta, \gamma, \lambda, \mu, \nu$ are supposed to be given *a priori*, they must satisfy six necessary and sufficient conditions in order for it to be possible to determine them as consequences of the three functions *u*, *v*, *w* upon which the preceding formulas depended. Those six conditions are:

$$A = 0, \quad B = 0, \quad C = 0, \quad L = 0, \quad M = 0, \quad N = 0,$$

in which:

$$A = \frac{\partial^2 \lambda}{\partial y \, \partial z} - \frac{\partial^2 \beta}{\partial z^2} - \frac{\partial^2 \gamma}{\partial y^2}, \qquad \text{etc.,}$$

$$L = \frac{\partial^2 \alpha}{\partial y \, \partial z} + \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial y} - \frac{\partial \nu}{\partial z} \right), \quad \text{etc.}$$

(†) Translator: The preceding note in this Journal volume (pp. 137-141) was by G. Morera: “Soluzione generale delle equazioni indefinite dell’equilibrio di un corpo continuo.”

Having assumed that, it is easy to verify that the following three identity relations exist between the latter six expressions *A*, *B*, *C*, *L*, *M*, *N* ([1]):

$$\frac{\partial A}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial M}{\partial z}=0,$$

$$\frac{\partial N}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial L}{\partial z}=0,$$

$$\frac{\partial M}{\partial x}+\frac{\partial L}{\partial y}+\frac{\partial C}{\partial z}=0,$$

because those relations will then have the same form as the indefinite equations of equilibrium of a continuous body that is not acted upon by external forces. It will then be legitimate to satisfy those equations by setting:

$$X_x = A, \qquad Y_y = B, \qquad Z_z = C,$$

$$Y_z = L, \qquad Z_x = M, \qquad X_y = N,$$

and taking $\alpha, \beta, \gamma, \lambda, \mu, \nu$ to be *six entirely-arbitrary* functions of x, y, z.

The solution that was obtained in a different way by prof. Morera corresponds to the particular hypotheses:

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \lambda = U, \quad \mu = V, \quad \nu = W.$$

The six components of stress X_x , Y_y , … are necessarily subject to certain conditions when they correspond to the internal forces that are generated by deformation alone, since in that case they can be expressed in a completely-determined way (that depends upon the nature of the body) by means of three components of the displacement u, v, w. In the case of isotropy, those conditions, which are immediate consequences of the ones that were described above, are extremely simple; indeed, they have the following form:

$$\frac{\partial^2 P}{\partial x^2} + C\,\Delta_2\,X_x = 0, \qquad \frac{\partial^2 P}{\partial y^2} + C\,\Delta_2\,Y_y = 0, \qquad \frac{\partial^2 P}{\partial z^2} + C\,\Delta_2\,Z_z = 0,$$

$$\frac{\partial^2 P}{\partial y\,\partial z} + C\,\Delta_2\,Y_z = 0, \qquad \frac{\partial^2 P}{\partial z\,\partial x} + C\,\Delta_2\,Z_x = 0, \qquad \frac{\partial^2 P}{\partial x\,\partial y} + C\,\Delta_2\,X_y = 0,$$

in which:

$$P = X_x + Y_y + Z_z ,$$

([1]) Those relations trace their origin to the ones that are known to couple the three components of rotation, since (as I established in an additional note to my paper "Sull'interpretazione meccanica delle formole di Maxwell") the six conditions $A = 0$, etc., that were set down are nothing but the integrability conditions for the differentials of those components.

and C is a constant. (Properly speaking, $C = 1 + \eta$, in which η is the known *contraction ratio*, as it was called in the French translation of Clebsch's treatise.)

The last condition supposes the absence of external forces. For the sake of brevity, I have omitted a discussion of the analogous conditions for the case in which that force exists and has the components X, Y, Z.

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