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On the resistance conditions for elastic bodies

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In the French version of the *Teoria dell'elasticità* by **Clebsch**, which was edited and annotated by the celebrated **de Saint-Venant**, who performed a new and noteworthy service for the study of that very important theory with that publication, one will find, in a final note to **§ 31** (pp. 252-282), a summary of the method that was proposed by the same **de Saint-Venant** a long time before for studying the limits of resistance of elastic bodies. That method differs from the one that will generally follow, and which was also emphasized by **Clebsch**, by the principle upon which it is based, which consists of assigning a maximum limit to the *dilatations*, as well as *tensions*.

In order to justify his new principle, **de Saint-Venant** cited, in particular, the very simple case of a rectangular parallelepiped that is stretched by a unit force along one, two, or all three directions of its figure axes. He then observed that, whereas the maximum tension is, by hypothesis, the same in all three cases, the maximum dilatation will be greater along the first one than along the second one, and similarly, it will be greater along the second than the third, so it would seem obvious to conclude that the danger of disintegration will be greater in the first case than in the second and third cases.

Now, that conclusion does not seem as legitimate to me as one might perhaps believe on a first glance. As is known, the stretching of a body in the direction that points longitudinally is accompanied by a contraction in any transverse direction, which is a contraction that is partially prevented, or also changed into a dilatation, when transverse stretchings are simultaneously superimposed upon the body. It will then follow that the molecular cohesion is weakened in the longitudinal direction by more in the first case than in the second, but it is also strengthened in the transverse case more than in the second case than in the first. Therefore, it is not easy, if not impossible, to decide *a priori* whether one effect prevails over the other.

However, if one cannot formulate any precise conclusion regarding that, I nonetheless think that one can assume that it is obvious, based upon precisely the example that **de Saint-Venant** adopted so opportunely, that the true measure of the resistance that the cohesion of an elastic body presents must not be inferred from either its maximum tension alone or its maximum dilatation alone, but it must be the result of *all* of the tensions or *all* of the dilatations that prevail in the neighborhood of any point in the body acting together.

Now, those tensions and dilatations, which are each represented by six distinct components, are both coupled to each other by linear relations that express the idea that the six components of tension are the derivatives with respect to the six components of the deformation of a single quadratic function that is composed of those second components, or that the six components of the deformation are the derivatives with respect to the six components of the tension of an analogous function that composed of the last components. That single function, which has the same value in the two different forms that it can take in either case, is the so-called *elastic potential*, and its distinguishing property is that it represents the *energy* per unit volume that the elastic body possesses in the neighborhood of the point in question, which is an energy that is equivalent to the work per unit volume that must be performed on the body during the restitution of the natural state from the current state, so it is the work that must be performed by external forces in order to take the given unit of volume in the natural state of elastic coaction to its current state.

Based upon that, it seems obvious to me that the true measure of the resistance that the molecular cohesion presents at any point of the body must be given by the value that the unit elastic potential assume at that point, and that the latter value, along with that of one tension or one dilatation, should prescribe a maximum limit beyond which the body is in danger of disintegrating, which is a limit that naturally differs according to whether one deals with local or distant disintegration.

That conclusion, which is already justified intrinsically by the dynamical significance of the elastic potential, is made even more obviously plausible by an analytic property of that potential, which must certainly depend upon the aforementioned significance, although a rigorous proof of that dependency is still not known (*).

I would like to allude to the property that the aforementioned potential has that it is an *essentially-positive* quadratic function, i.e., a function that cannot be annulled unless *all* of its six variables are zero, and that it will remain greater than zero for any other sextuple of real values of those variables. By virtue of that property, one cannot impose a limit on the value of the elastic potential without, at the same time, imposing a limit on that of *each* component of either the tension or the deformation, so the use of that potential as a measure of the elastic resistance will not intrinsically contradict the criterion that one infers by either considering only its tensions or only its deformations. In practice, the criterion that one infers from the potential will then have the great advantage that it does not require the preliminary solution to any equation. and one will be reduced to a discussion of a formula that cannot present any sign ambiguities.

In the case of perfectly isotropic bodies, the elastic potential Π is expressed as a function of the six components of tension by the following formula:

$$2\,E\,\Pi = (t_{xx} + t_{yy} + t_{zz})^2 + 2(1+\eta)(t_{yz}^2 + t_{zx}^2 + t_{xy}^2 - t_{yy}\,t_{zz} - t_{zz}\,t_{xx} - t_{xx}\,t_{yy}),$$

in which the symbols for the tensions and the ones for the two isotropy constants E, η are the same ones that **de Saint-Venant** used. We observe, in passing, that in this case, the *essentially-positive* character of Π is exhibited by the equivalent expression:

$$2\,E\,\Pi = (1+\eta)\,[(s\,\Pi - t_{xx})^2 + (s\,\Pi - t_{yy})^2 + (s\,\Pi - t_{zz})^2 + 2(t_{yz}^2 + t_{zx}^2 + t_{xy}^2)]\ ,$$

in which:

(*) See a noteworthy article by **Lipschitz** in Bd., 78 of **Borchardt**'s Journal, where the property in question is deduced from the postulate of the *stability* of any free vibratory motion.

$$s = \frac{\sqrt{1+\eta} \pm \sqrt{1-2\eta}}{3\sqrt{1+\eta}} .$$

Whenever one has:

$$-1 < \eta < \tfrac{1}{2} ,$$

the value of s will be real, and the three binomials:

$$s\,\Pi - t_{xx} , \qquad s\,\Pi - t_{yy} , \qquad s\,\Pi - t_{zz}$$

can be annulled simultaneously only when:

$$t_{xx} = t_{yy} = t_{zz} = 0 .$$

The preceding limitation on η coincides exactly with an analogous condition that **Green** stated (pp. 246 of his *Mathematical Papers*), and in my opinion was not proved rigorously by **Ferrers** (*ibid*., pp. 330).

Assume, with **de Saint-Venant**, that the following equations exist for the cylindrical or prismatic bodies that are ordinarily considered:

$$t_{xx} = 0 , \qquad t_{yy} = 0 , \qquad t_{zz} = 0 .$$

The condition for cohesion is then given by:

$$\Pi_0 \leq \frac{t_{zz}^2 + 2(1+\eta)(t_{yz}^2 + t_{zx}^2)}{2E} ,$$

in which Π_0 is the maximum value of the potential Π. Let R_0, T_0 denote the maximum values of the unit tensions according to whether the body is subject to only longitudinal tension or only torsion, resp. One will then have the following relations between Π_0, R_0, T_0 :

$$R_0^2 = 2\,E\,\Pi_0 , \qquad\qquad (1+\eta)T_0^2 = E\,\Pi_0 ,$$

by virtue of which the preceding condition can be written:

$$\frac{t_{zz}^2}{R_0^2} + \frac{t_{yz}^2 + t_{zx}^2}{T_0^2} \leq 1 , \tag{a}$$

while that will give rise to the necessary relation between the maximum values R_0, T_0 of the two types of tensions:

$$T_0 = \frac{R_0}{\sqrt{2(1+\eta)}} . \tag{a′}$$

That relation is different from the one that **de Saint-Venant** obtained, who found that:

$$T_0 = \frac{R_0}{1+\eta} .$$

The value that this latter formula assigns to the ratio $T_0 : R_0$ is greater than the one that results from (a′), since η is always < 1 / 2.

We now pass on to the case of bodies endowed with only transverse isotropy, which is a case that was considered more especially by **de Saint-Venant**. The expression for the potential by means of tension is given by the formula:

$$2\,\Pi = \frac{t_{zz}^2 - 2\eta\, t_{zz}(t_{xx}+t_{yy})}{E} + \frac{t_{xx}^2 + t_{yy}^2 - 2\eta'\, t_{xx}\, t_{yy} + 2(1+\eta')t_{xy}^2}{E'} + \frac{t_{yz}^2 + t_{zx}^2}{G} ,$$

in which E, E' are the two elastic moduli (viz., longitudinal and transverse, resp.), G is the tangential elastic coefficient, and η, η' are the coefficients that determine the transverse contraction that is due to a longitudinal and a transverse dilatation, respectively. There is also a third coefficient η'' that determines the longitudinal contraction that is due to a transverse dilatation, but that coefficient depends upon the other constants by means of the relation:

$$\eta'' = \frac{E'\eta}{E} .$$

If one also confines oneself to considering the cylindrical or prismatic bodies that admit the **de Saint-Venant** relations then the cohesion condition will reduce to:

$$2\,\Pi_0 \le \frac{t_{zz}^2}{E} + \frac{t_{yz}^2 + t_{zx}^2}{G} .$$

One therefore has (a) once more, but with the single difference that now the maximum values R_0, T_0 of the two types of tensions are longer related by (a′), but by:

$$T_0 = R_0 \sqrt{\frac{G}{E}} , \tag{a″}$$

in which the ratio $G : E$ can take arbitrary values, so the ratio $T_0 : R_0$ can also assume an arbitrary value, at least *a priori*.

Finally, in the more general case of a cylindrical body that is endowed with three elastic axes, one of which points in the longitudinal direction, the condition for cohesion will again have the simple form:

$$\frac{t_{zz}^2}{R_0^2} + \frac{t_{yz}^2}{T_x^2} + \frac{t_{zx}^2}{T_y^2} \leq 1 , \tag{b}$$

in which R_0, T_x, T_y are the maximum values of the tensions t_{zz}, t_{yz}, t_{zx} . Those values are coupled with each other by the relations:

$$T_x = R_0 \sqrt{\frac{G_x}{E}} , \qquad y_x = R_0 \sqrt{\frac{G_y}{E}} , \tag{b′}$$

in which E is the longitudinal elastic modulus, and G_x, G_y are the tangential elastic coefficients. When $t_{zz} = 0$, the preceding condition will agree with the one that **de Saint-Venant** gave on pp. 272.

Let us seize this opportunity to show how one can very easily obtain the complete determination of the tensions in cylindrical bodies with *elliptic* sections, even when those bodies have three elastic axes that are parallel to the axes of the figure (*). The elementary process that I shall use consists of proving that one can satisfy all of the conditions of the problem by taking the six components of the tension to be just as many *functions of degree two* of the coordinates x, y, z.

Let:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

be the equation of the lateral cylindrical surface. The conditions that the components of the tension must satisfy at any point of that surface are:

$$\frac{x\,t_{xx}}{a^2} + \frac{y\,t_{xy}}{b^2} = 0 , \qquad \frac{x\,t_{xy}}{a^2} + \frac{y\,t_{yy}}{b^2} = 0 , \qquad \frac{x\,t_{xz}}{a^2} + \frac{y\,t_{yz}}{b^2} = 0 ,$$

and they all have the form:

$$\frac{x\varphi}{a^2} + \frac{y\psi}{b^2} = 0 .$$

Now, one easily sees that the most general form of two functions of degree two φ and ψ that are constrained to satisfy that equation at any point of the cylindrical surface is the following:

(*) The method is also applicable to the case in which *only the longitudinal axis* is an elastic axis, but for the sake of brevity, I shall confine myself to the aforementioned simpler case.

$$\varphi = \frac{H\Gamma}{2} - \frac{yA}{b^2}, \qquad \psi = \frac{H\Gamma}{2} + \frac{xA}{a^2},$$

in which H and K are two constants, A is a *linear* function of x, y, z, and Γ is the function of degree two:

$$\Gamma = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Given that, if one observes that in the first two equations for the surface, the component t_{xy} will alternately play the role of ψ and φ, then one will immediately find that the first five components of the tension can have only the following forms:

$$t_{xx} = \frac{A\Gamma}{2} - a^2 D\,\frac{y^2}{b^2}, \quad t_{xy} = \frac{B\Gamma}{2} + D\,x\,y, \qquad t_{yy} = \frac{C\Gamma}{2} - b^2 D\,\frac{x^2}{a^2}, \quad t_{xz} = \frac{H\Gamma}{2} - \frac{y\Lambda}{b^2},$$

$$t_{y^2} = \frac{H\Gamma}{2} + \frac{x\Lambda}{a^2},$$

in which A, B, C, D, H, K are constants, and Λ, Γ have the meanings they were given above.

If one substitutes those expressions in the first two equations of equilibrium:

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} = 0, \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} = 0$$

then one will find the conditions:

$$A = a^2 D, \qquad C = b^2 D, \qquad B = 0, \qquad \frac{\partial \Lambda}{\partial z} = 0.$$

When one takes them into account and adds the value of the last component t_{zz}, which results from the third equation of equilibrium:

$$\frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} = 0,$$

one will get the following expressions:

$$t_{xx} = \frac{a^2 D}{2}\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2}\right), \qquad t_{xy} = D\,x\,y, \qquad t_{yy} = \frac{b^2 D}{2}\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2}\right),$$

$$t_{xz} = \frac{H+Q}{2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\frac{y}{b^2}(P\,x+Q\,y+R)\,,$$
$$t_{yz} = \frac{K-P}{2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\frac{x}{a^2}(P\,x+Q\,y+R)\,,$$
$$t_{zz} = \left(\frac{H\,x}{a^2}+\frac{K\,y}{b^2}\right)z+\varphi(x,y)\,.$$

Those expressions satisfy all of the equilibrium equations and the one that relates to the lateral surface, no matter what the constants D, H, K, P, Q, R, and the function $\varphi(x, y)$ of degree two in just x, y.

Having said that, suppose that the body has three elastic axes in the directions x, y, z, and let:

$$\Pi = \tfrac{1}{2}\{At_{xx}^2+Bt_{yy}^2\,Ct_{zz}^2+2A'\,t_{yy}\,t_{zz}+2B'\,t_{zz}\,t_{xx}+2C'\,t_{xx}\,t_{yy}+A''\,t_{yz}^2+B''\,t_{zx}^2+C''\,t_{xy}^2\}$$

represent the unit potential, when expressed in terms of the components of tension. It results from that the six components of the deformation, with the symbols the **de Saint-Venant** used, will have the expressions:

$$d_x = At_{xx}+C'\,t_{yy}+B'\,t_{zz}\,, \qquad g_{yz} = A''\,t_{yz}\,,$$
$$d_y = C'\,t_{xx}+Bt_{yy}+A'\,t_{zz}\,, \qquad g_{xz} = B''\,t_{zx}\,,$$
$$d_z = B'\,t_{xx}+A'\,t_{yy}+Ct_{zz}\,, \qquad g_{xy} = C''\,t_{xy}\,.$$

In order to be able to determine the three components of a displacement u, v, w that can generate the given components of deformation d_x, d_y, d_z, g_{yz}, g_{xz}, g_{xy}, as is known, the six second-order differential equations that result from the following two must be satisfied:

$$\frac{\partial^2 d_y}{\partial z^2}+\frac{\partial^2 d_z}{\partial y^2} = \frac{\partial^2 g_{yz}}{\partial y\,\partial z}\,, \qquad 2\,\frac{\partial^2 d_x}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial g_{yz}}{\partial x}+\frac{\partial g_{zx}}{\partial y}+\frac{\partial g_{xy}}{\partial z}\right),$$

with circular permutations of x, y, z. If one substitutes the preceding values of d_x, d_y, …, and t_{xx}, t_{yy}, … in those six equations then one will find that:

$$C\,\frac{\partial^2\varphi}{\partial x^2} = D\left(\frac{3b^2A}{a^2}+B'\right), \qquad C\,\frac{\partial^2\varphi}{\partial y^2} = D\left(A'+\frac{3a^2B'}{b^2}\right),$$
$$\frac{D}{C}\left\{\frac{3a^2(AC-B'^2)}{b^2}+\frac{3b^2(BC-A'^2)}{a^2}-2(A'\,B'-C\,C')+C\,C'\right\}=0\,,$$
$$2A'\,H = A''\,Q+\frac{a^2B''}{b^2}(H+3Q)\,,$$

$$2B'K = -B''P + \frac{b\,A''}{a^2}(K - 3P) \ ,$$
$$C\frac{\partial^2\varphi}{\partial x\,\partial y} = 0 \ .$$

Now, the coefficient C, which is the inverse of the modulus of longitudinal elasticity, cannot be zero. The expression:

$$\frac{3a^2(AC - B'^2)}{b^2} + \frac{3b^2(BC - A'^2)}{a^2} - (A'B' - CC') + CC'$$

can be annulled only for some particular values of the ratio $a : b$ (*). Therefore, one needs to set:

$$D = 0 \ , \qquad \frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial x\,\partial y} = \frac{\partial^2\varphi}{\partial y^2} = 0 \ .$$

The first of those conditions carries with it the equations:

$$t_{xx} = t_{xy} = t_{yy} = 0 \ ,$$

which constitute the point of departure for the **de Saint-Venant** process. The remaining three components of tension take the following expressions:

$$\left.\begin{aligned} t_{xz} &= \frac{H+Q}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \frac{y}{b^2}(P\,x + Q\,y + L'), \\ t_{yz} &= \frac{K-P}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) + \frac{x}{b^2}(P\,x + Q\,y + L'), \\ t_{zz} &= \left(\frac{H\,x}{a^2} + \frac{K\,y}{b^2}\right) z + H'\,x + K'\,y + L, \end{aligned}\right\} \qquad \text{(c)}$$

in which the constants H, K, L, H', K', L' are arbitrary, while the P, Q will still be determined by two of the conditions that were just obtained, which can be written:

$$\left.\begin{aligned} a^2B''(H + 3Q) - b^2(2A'H - A''Q) = 0, \\ b^2A''(K - 3P) - a^2(2B'H + B''P) = 0. \end{aligned}\right\} \qquad \text{(c}'\text{)}$$

(*) Moreover, one can prove that this expression will remain positive for any *real* value of the ratio $a : b$.

The six arbitrary constants H, K, L, H', K', L' are coupled directly with the forces that act upon the cylinder. Indeed, let X, Y, Z denote the components of the force, and let M_x, M_y, M_z be those of the moment that results from the transport of the external force to the origin of the coordinates. If one sets $\pi\, a\, b = \sigma$:

$$\left.\begin{aligned} X &= \frac{H\sigma}{4}, & Y &= \frac{K\sigma}{4}, & Z &= L\sigma, \\ M_x &= \frac{K' b^2 \sigma}{4}, & M_y &= -\frac{H' a^2 \sigma}{4}, & M_z &= \frac{L'\sigma}{4}. \end{aligned}\right\} \tag{c″}$$

One has the following six equations for determining the components of the displacement u, v, w :

$$\frac{\partial u}{\partial x} = B' t_{zz}, \qquad \frac{\partial v}{\partial y} = A' t_{zz}, \qquad \frac{\partial w}{\partial z} = C t_{zz},$$

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = A'' t_{yz}, \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = B'' t_{xz}, \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

whose integration is always possible and presents no difficulty. For example, in the case of simple torsion, one finds that:

$$u = -\frac{L' y z}{2}\left(\frac{A''}{a^2} + \frac{B''}{b^2}\right), \qquad v = \frac{L' x z}{2}\left(\frac{A''}{a^2} + \frac{B''}{b^2}\right), \qquad w = \frac{L' x y}{2}\left(\frac{A''}{a^2} - \frac{B''}{b^2}\right).$$

For a body that is endowed with only transverse isotropy, one has:

$$A = B = \frac{1}{E'}, \quad C = \frac{1}{E}, \qquad A' = B' = -\frac{\eta}{E'}, \qquad C' = -\frac{\eta'}{E'},$$

$$A'' = B'' = \frac{1}{G}, \qquad C'' = \frac{2(1+\eta')}{E'},$$

and the preceding constants that were denoted by P, Q will have the following values:

$$P = \frac{E b^2 + 2G\eta a^2}{E(a^2 + 3b^2)} K, \qquad Q = -\frac{E a^2 + 2G\eta b^2}{E(b^2 + 3a^2)} H,$$

from which one deduces that:

$$H + Q = \frac{E(2a^2 + b^2) - 2G\eta b^2}{E(b^2 + 3a^2)} H, \quad K - P = \frac{E(a^2 + 2b^2) - 2G\eta a^2}{E(a^2 + 3b^2)} K,$$

which allow one to express the tensions (c) by means of only six essential components.

Addendum. – After writing down the preceding discussion, I recognized, with great pleasure, that the objection that I had raised against the ultimate ways of establishing the conditions of cohesion had been formulated, in almost the same language, by the late engineer **Castigliano** on page 128, *et seq.*, of his *Théorie de l'équilibre des systèmes élastiques*. I am happy to think that the learned engineer, who had recognized all of the importance of the concept of elastic potential, would have probably approved of my proposal that it should serve as a foundation, as well as my deduction of the aforementioned conditions.

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