ON THE DIFFERENTIAL GEOMETRY OF *G*-STRUCTURES

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INTRODUCTION

The goal of this work is to elaborate upon a general theory of first-order, real and complex G-structures that will be, at the same time, simple and as complete as possible. We will therefore not concentrate upon the case of a certain groups G whose properties are particularly rich (if not also known), but rather, we will try to isolate all that is common to those structures for some different groups G, or at least for large classes of them that are introduced naturally.

Let G be a subgroup of a real (or complex) linear group in m variables L_m (or CL_m). A G-structure on the manifold X of dimension m is determined by a *space of frames* that is a principal fiber subspace of the space E of real frames (or the space E^C of complex frames) on X. That is why the first chapter is dedicated to the study of principal fiber subspaces (PFSS). Certain definitions and classical constructions on (topological) fiber spaces are first recalled in a form that is adapted to our goal (§§ **1**, **2**). The G'-PFSS's of a fiber space H (X, G) are then defined and characterized (Prop. I, **3**.1). They correspond bijectively to the sections of H / G' (Prop. I, **3**.3). In § **5**, the notion of PFSS is analyzed in the differentiable case (Prop. I, **5**.1) and is characterized by the subsets H of H (X, G) that admit the structure of a differential PFSS. With an eye towards the study of subordinate structures that are common to two G-structures (Chap. III), we shall study the intersection of a G'-PFSS and a G''-PFSS in the topological case (§ **4**) and then in the differentiable case (§ **6**). One can say, on the whole, that the intersection is a Γ-PFSS ($\Gamma = G' \cap G''$) under the single condition (which is obviously necessary) that its projection should be X as long as G' / Γ (or G'' / Γ) is compact, and in the differentiable case, the property should be true, moreover, for "almost all pairs of subgroups G' and G''."

Chapter II first introduces (§§ **1**, **2**) the tool that will mainly be employed in Chapter III: Vector-valued differential forms and the calculations on those forms (in particular, the various "products" that constitute the extension to those forms of the laws of composition between their spaces of values). In § **3**, the notion of tensor that is associated to a tensorial form on a PFS H (X, G) is analyzed: It is a tensor on the fiber product $H \boxtimes E$ (E is the space of frames on X), and not on H itself. The main properties of connections are recalled in § **4**, at the end of which, we shall establish the fundamental formulas of differential geometry by the exclusive use of the algorithm that is introduced at the beginning of the chapter. The extension of that algorithm to the case of complex vectorial forms, and in particular, to the notion of associated tensor will define the subject of the last section.

Chapter III contains the main results of this treatise. The frame spaces and (real and complex) G-structures are first defined and several examples are analyzed (§ **1**). An important class of G-structures that contains almost-complex structures, almost-Hermitian, Riemannian, … is that of "structures defined by a tensor"; i.e., ones whose distinguished frame space is the subspace of the frames on E (E^C, resp.) for which a certain tensor on E (E^C, resp.) has a well-defined value (§ **2**). In § **3**, the equivalent and subordinate structures are studied: Theorem III.3, which is a simple application of the first chapter, indicates that the trivially necessary condition for two structures to have a common subordinate structure is generally sufficient. The spaces of frames H are characterized among the PFS's with base X by the existence of a "regular" tensorial 1-

form with values in R^m or C^m that is called the *fundamental 1-form* on H. *In the case of frame spaces,* the inclusion $H \subset E$ (E^C, resp.) permits one to define an associated tensor to any tensorial form on H with values in M that is a tensor on H itself (Prop. III.**4**.2). The correspondence between the tensorial form and associated form is bijective. The tensor and the form are coupled by a particularly-simple relation [(12), Chap. III, § **4**]. Nonetheless, that correspondence is defined in a canonical fashion only when H is a space of real frames (and arbitrary M) or when H is a space of complex frames and M is a complex vector space. That remark is essential to § **6**. § **5** is dedicated to the special properties of connections on frame spaces, the characterization of a manifold X that admits a G-structure by the existence of a linear connection whose holonomy group is a subgroup of G (Theorem III, § **5**.2), torsion, the generalized Ricci identity whose proof is carried out solely with the aid of the notion of associated tensor and the algorithm of Chapter II, the relation between the associated tensor to the absolute differential of a q-form and the "covariant derivative" of that form (Prop. III, § **5**). The *structure tensor* of a G-structure S (which generalizes the "torsion tensor of an almost-complex structure") characterizes the tensorial 2-forms on H of vectorial type with values in R^m or C^m that are the torsions of S-connections [Theorem (III, § **6**.1) and (III, § **6**.2)]. Nevertheless, that tensor is defined only for the *G-structures of the first kind*; i.e., real ones, or if they are complex, ones such that G is a complex Lie subgroup of CL_m . The peculiarity of complex structures of the second kind comes from the non-existence of a canonical associated tensor on a space of complex frames for a form with values in a real vector space. Sections **7** and **8** contain some calculations of the structure tensor and their identification with known invariants in the case of classical structures. We shall call a structure whose structure tensor is zero *almost-integrable*. For example, an almost-integrable almost-Hermitian structure is Kählerian.

In Chapter IV, we shall address some automorphisms and infinitesimal automorphisms of a real G-structure. In section **1**, the *transitive G-structures* are studied. The first-order Lie pseudogroups correspond bijectively to the classes of equivalent transitive G-structures. In § **2**, we shall show, in particular, that the *Cartan conditions* (Definition IV.**2**), which are necessary conditions for the existence of a transitive G-structure with given structure tensor (and which will be sufficient when G is involutive), translate simply into the tensorial character of the structure tensor, on the one hand, and the Bianchi identity, on the other (Prop. IV, § **2**.2). § **3** is dedicated to the involutive G-structures. If they are almost-integrable (almost-transitive, resp.) then they will be integrable (transitive, resp.) (Theorem IV, § **3**). We then give a necessary and sufficient condition for two Lie pseudogroups to be locally similar (Theorem IV, § **3**). In the last section, two problems that relate to infinitesimal automorphisms are posed that are studied with the aid of Hermann's lemma (Prop. IV, § **4**.2) in some particular cases. The main results are these: If S is a G-structure on X that is subordinate to an almost-integrable Riemannian structure then the infinitesimal isometries will also be infinitesimal automorphisms of S as long as X is compact or it does not admit a 2-form with vanishing covariant derivative (Theorem IV, § **4**.1). The theorem in (Chap. IV, § **4**.2) gives some conditions under which any infinitesimal affine transformation for an S-connection will be an infinitesimal automorphism of S.

This brief summary calls for some remarks: At the Colloque International de Géométrie différentielle du C.N.R.S. (Strasbourg 1953), three presentations attracted attention to the general notion of a G-structure (S.S. Chern [**9**]) or “regular infinitesimal structure” (C. Ehresmann [**14**] and P. Libermann [**19**]). A large number of structures of differential geometry can be defined by being given a G-structure, on the one hand, while on the other hand, the Lie pseudogroups (Élie Cartan’s “infinite continuous groups”) are conveniently defined to be pseudogroups of local automorphisms of some of them. It would then seem useful to elaborate upon a general theory of G-structures, and that is what we have attempted to do, while appealing to the results and examples of the cited authors, among other things. Hence, in [**9**], S. S. Chern introduced the “first invariants” of the structure, whereas in [**19**], P. Libermann could avoid using them because for the structures that she studied “one can impose the torsion canonically”. Our *structure tensor* (Chap. III, § **6**) specifies the nature of the first invariants, gives then a global definition, and permits one to characterize the torsion forms of S-connections. Similarly, numerous results will be generalizations of the known results of other authors that can often be simplified.

The G-structures to which this study is dedicated are the first-order, real and complex G-structures. It is necessary for us to clarify that choice. The example of almost-complex structures and subordinate structures, whose definitions and calculations are greatly simplified by the introduction of complex frames, leads us to define and study complex G-structures. At the same time, a first example of a complex G-structure that is not equivalent to a real one was studied by G. Legrand in his thesis [**18**]. He did not reveal the fundamental difference between real and complex G-structures (and that justifies their simultaneous study *a posteriori*), unless it appeared in the definition of the structure tensor. Although the complex G-structures of the first kind behave like real structures, the extension of the procedure to complex G-structures of the second kind leads to tensors that are not sufficient to characterize the torsion of S-connections. On the contrary, the natural setting for the study of higher-order G-structures (cf., Y. Matsushima [**24**]) is certainly C. Ehresmann’s *theory of jets*, while the first-order structures can be studied with the more classical methods of differential geometry (fiber spaces, connections, exterior differential calculus), which, when conveniently adapted by the use of the algorithm in Chapter II, in particular, will lead to some very simple calculations and formulations. That difference in methods already justifies an autonomous study of the first-order structures, which is required by their very importance, in our opinion.

We have been led to consider that all of the G-structures that are equivalent to a given structure S (viz., spaces of frames that are deduced from those of S by right-translation on E or E^C) will define the same infinitesimal structure, and we have called the class of structures that are equivalent to S a $\mathcal{C}$-structure, where $\mathcal{C}$ denotes the class of subgroups that are conjugate to G in L_m (or CL_m). Most of the properties studied here will be properties of $\mathcal{C}$-structures. We have been able to define a *sheaf of $\mathcal{C}$-structures*, and that is undoubtedly the direction in which this work will find the most immediate progress.

The bibliography that is placed at the end of this treatise is hardly exhaustive. We have based it upon the principle that it shall include only the works that were cited explicitly here.

CHAPTER I

FIBER SPACES
PRINCIPAL FIBER SUBSPACES

1. – Definitions and notations.

We call a locally-trivial principal fiber space with topological structure group a *principal fiber space* (PFS). More precisely:

DEFINITION I.1.1 ([1]) – *A PFS H* (X, G) *with base X and structure group G is defined as follows:*

a) *H and X are topological spaces* ([2]), *and G is a topological group.*

b) *H is endowed with a continuous map p from H onto X that admits a lift in a neighborhood of any* $x \in X$.

Such a lift, which is a continuous map σ of a neighborhood U of X into H such that $p \circ \sigma$ is the identity on U is called a *local section of H over U.* $H_x = p^{-1}(x)$ is the *fiber* over x.

c) *G acts on H on the right*; i.e., one has a continuous map $H \times G \to H$, $(z, g) \mapsto z \cdot g$, such that $(z, g) \cdot g' = z \cdot (gg')$ and $z \cdot e = z$ ($g, g' \in G$, e is the identity in G): The partial map Dg of H onto itself $z \mapsto z \cdot g$ – or *right-translation by g* – is then a homeomorphism of H such that $Dg'\, Dg = Dgg'$. *The right translations of G that respect the fibers are simply-transitive on the fibers.*

d) *The continuous bijective map* Φ_U *of* $U \times G$ *on* $H_U = p^{-1}(U)$, $(x, g) \mapsto \sigma(x) \cdot g$ *that is associated with a local section* σ *over U is a homeomorphism that one calls a* ***local chart*** *on H over U that is associated with* σ.

We say *fiber space* (FS) to mean a locally-trivial fiber space with topological structure group that can be defined as follows:

DEFINITION I.1.2. – *E* (X, G, F) *is a FS with base X, structure group G, fiber F if:*

a) *Let E, X, and F be topological spaces, and let G be a topological group that acts effectively on F on the left by way of* $(g, y) \mapsto g \cdot y$ ($g \in G$, $y \in F$). *If the partial map of F into F that takes* $y \mapsto g \cdot y$ *is also denoted by g then one will have* $g \circ g' = gg'$.

([1]) This is Steenrod's [**26**] classical definition, but put into a form that will be convenient for what follows.

([2]) A topological space will always be assumed to be separable.

b) *E is endowed with a continuous map p from E onto X* ([3]); $E_x = p^{-1}(x)$ is the *fiber* over x.

c) *Any* $x \in X$ *possesses an open neighborhood U that is endowed with a local chart* Φ_U ; i.e., a homeomorphism of $U \times F$ onto $E_U = p^{-1}(U)$ such that $p \circ \Phi_U(x, y) = x$, $x \in X$, $y \in F$.

d) *If U and V are open subsets of X that are endowed with local charts* Φ_U *and* Φ_V, *and if* $U \cap V \neq \varnothing$ *then there will exist a continuous map s of* $U \cap V$ *into G such that the change of chart* $\Phi_V^{-1} \circ \Phi_U$ *is the map of* $(U \cap V) \times F$ *onto itself* $(x, y) \mapsto (x, s(x) \cdot y)$.

One briefly says that $p : E \to X$ is a *fibration* if E is a FS, in the precise sense that was just defined, with base X and projection p. A PFS is obviously a FS with a group G that acts upon the fiber G by left translation. g will denote the element $g \in G$, as well as the action of left-translation by g, which will nonetheless be denoted by L_g when there is some risk of ambiguity: $L_g \cdot g' = g \cdot g'$ $(g, g' \in G)$.

2. – Various constructions on fiber spaces.

Let E be a FS, and let Φ_U^x be the restriction of the local chart Φ_U to $\{x\} \times G$. We say *frame at the point x of the fiber structure on E* to mean any homeomorphism of F onto E_x:

$$h = \Phi_U^x \circ g , \quad g \in G, x \in X.$$

It follows from axiom *d*) in the definition (I.1.2) that this definition is independent of the choice of the local chart Φ_U because if $x \in U \cap V$ then:

$$\Phi_U^x = \Phi_V^x \circ s(x) \qquad \text{and} \qquad h = (\Phi_V^x \circ s(x)) \circ g = \Phi_V^x \circ (s(x) g), \quad s(x) \cdot g \in G.$$

Let $\hat{E}_x$ be the set of frames in E at x, and let $\hat{E} = \bigcup_{x \in X} \hat{E}_x$. It is classical – and immediate, with the aid of Definition (I.1.1) – that the charts $\hat{\Phi}_U$ that are associated with the local charts Φ_U of E:

$$\hat{\Phi}_U : U \times G \to \hat{E}, \qquad (x, g) \mapsto \Phi_U^x \circ g , \qquad x \in X, \; g \in G$$

define the structure of a PFS $\hat{E}(X, G)$ on $\hat{E}$. We remark simply that the right-translation by g on $\hat{E}$ is $D_g \cdot h = h \circ g$. When $\hat{E}$ is endowed with that structure, it is called the *associated PFS to E.*

([3]) The projection of a FS E onto its base will be generally denoted by p, even if one is simultaneously dealing with several FS's. When there is some risk of ambiguity, one can be more specific and denote it by p_E .

In particular, that construction applies to a PFS H. If Φ_U is the chart on H that is associated with the local section σ then a *frame in H at x* is a map $\Phi_U^x \circ g = \Phi_U^x \circ L_g$ ($g \in G$) from G to H_x . Now:

$$\Phi_U^x \circ L_g(g') = \Phi_U^x(gg') = \Phi_U(x, gg') = \sigma(x) \cdot gg' = (\sigma(x) \cdot g)\, g' = \Phi_U(x, g) \cdot g' ;$$

i.e., since $\Phi_U^x \circ g = \hat{\Phi}_U(x, g)$:

$$\hat{\Phi}_U(x, g) \cdot g' = \Phi_U(x, g) \cdot g'.$$

The dot on the left-hand side of this denotes the action (on the left) of the fiber-type G on the frame, and the one on the right-hand side denotes the action (on the right) of G on H. That remark shows that there exists a bijective map of H onto $\hat{H}$ that can be defined on any pair of associated charts by:

$$\Phi_U(x, g) \mapsto \hat{\Phi}_U(x, g),$$

and which will be, as a result, a homeomorphism. It is, moreover, a G-isomorphism of the PFS (cf., I.3), and that will permit one to identify $\hat{H}$ with H, where $h \in H$ is identified with the frame $\hat{h} \in \hat{H} : g \in G \to h \cdot g \in H$. The notation $\hat{h}$ will sometimes be employed in what follows when it is necessary to distinguish $\hat{h}$ from h.

Let E (X, G, F) be an FS and let $H = \hat{E}$ be the associated PFS. Let $h \in H$ be a homeomorphism of F onto E_{ph} , $y \in F \mapsto h \cdot y \in E_{ph}$. For $h \in H$, $g \in G$, $y \in F$, one will then have:

$$(h \cdot g) \cdot y = (D_g h) \cdot y = (h \circ g)\, y = h \cdot (g \cdot y),$$

which will imply, in particular, that:

$$(h \cdot g) \cdot (g^{-1} \cdot y) = h \cdot [h \cdot (g^{-1} y)] = h \cdot y.$$

That remark permits one to identify E with the quotient $F \times H$ by the equivalence relation:

$$(y, h) \sim (g \cdot y, h \cdot g^{-1}), \qquad y \in F, h \in H, g \in G.$$

Consequently, one can construct E by starting from F and H.

More generally, let F be a topological space upon which G acts by way of $\mathcal{R}$ (G). If G is not assumed to act effectively on F then let N be a distinguished closed subgroup of elements of G that leave all of the points of F invariant; G / N will then act effectively on F.

DEFINITION I.2. – *Let* F (H) *be the quotient of* $F \times H$ *by the equivalence relation:*

$$(y, h) \sim (\mathcal{R}(g) \cdot y, h \cdot g^{-1}), \qquad y \in F, h \in H, g \in G.$$

There is a fiber structure $F(H)$ $[X, G / N, F]$. *One says that it is the FS that is obtained by modeling F on H* ([4]), or *the FS that is associated with H of type* $(F, \mathcal{R}(G))$.

Let α be the canonical projection of $F \times H$ onto $F(H)$; the projection of $F(H)$ onto X is defined by:

$$p_{F(H)}(\alpha(y, h)) = p_H(h),$$

and the fiber structure of $F(H)$ that is defined by the charts Ψ_U that are associated with the sections σ of H over U:

$$\Psi_U(x, y) = \alpha(y, \sigma(x)), \qquad x \in U \subset X, y \in F;$$

the PFS $\widehat{F(H)}$ is isomorphic to the PFS that is the quotient H / N (see below).

Let H be a PFS with group G, and let $G' \subset G$ be a closed subgroup; the relation:

$$h \sim h' \qquad \text{if} \qquad h' = h \cdot g', \quad h \in H, \quad h' \in H, \quad g' \in G'$$

is an equivalence relation in H. Let H / G be the topological space that is the quotient of H by that relation, and let π be the canonical map $H \to H / G'$. H / G' is naturally endowed with a projection onto X that is defined by:

$$p_{H/G'}(\pi(h)) = p_H(h).$$

Since G acts naturally on $L = G / G'$, one can model L on H. Let α be the canonical projection of $L \times H$ onto $L(H)$, and y_0 be the point of L that is defined by the class G'. Since:

$$\alpha(y_0, h \cdot g') = \alpha(g' y_0, hg' \cdot g'^{-1}) = \alpha(y_0, h), \qquad g' \in G', \quad h \in H,$$

the relation:

$$f(\pi(h)) = \alpha(y_0, h)$$

defines a map of H / G' into $L(H)$. One can show that f, which is bijective and respects the projections, is a homeomorphism, so:

PROPOSITION I.2.1. – *Let G' be a closed subgroup of the structure group G of a PFS H. The quotient space H / G', where G' acts on H by right-translations in G, is identified with the FS L* (H), *which is modeled on H by the homogeneous space* $L = G/G'$, *upon which G acts naturally.*

If N is a distinguished subgroup of G then the fiber structure on H / N that is specified by Proposition (I.2.1) will be a principal fiber structure with group G / N, and the canonical map π of H onto H / N *will be a homomorphism of PFS's that is compatible with the canonical homomorphism ρ of G onto G / N* (cf., I.3). Hence, if G' is not a

([4]) Cf., Aragnol [**1**], Chap. 1, **2**; with Aragnol's terminology, one says that $X \times F$ is *modeled on H.*

distinguished subgroup of G and G_0' is not the largest subgroup of G' that is invariant in G then H / G_0' will be a PFS with group G / G_0' that is identified with the PFS $\widehat{H / G'}$ that is associated with H / G'.

Proposition (I.2.1) can be completed if one makes the supplementary hypothesis that the canonical projection $G \to G / G'$ is a fibration. We say briefly that *G' is a subgroup (L. T.) of G*; it will necessarily be closed.

PROPOSITION I.2.2 (super-fibration theorem). – *Let G be a topological group, let G' and G'' be subgroups such that $G'' \subset G' \subset G$, and let H be a PFS with group G. If G' is a subgroup (L. T.) of G then each map in the commutative diagram below will be a fibration:*

$$\begin{tikzcd}
H \arrow[r, "\pi''"] \arrow[d, "p"'] \arrow[dr, "\pi'" near start] & H / G'' \arrow[d, "q"] \arrow[dl, "p''"' near start] \\
X & H / G' \arrow[l, "p'"']
\end{tikzcd}$$

That proposition results from the preceding and the proof of the property for the single map q.

Finally, we need the notion of a fiber product:

DEFINITION I.2.2. – *Let E (X, G, F) and E' (X, G', F') be two FS's. The **fiber product** $E \boxtimes E'$ is the subspace of $E \times E'$ that projects onto the diagonal of $X \times X$; it is endowed with a natural structure of a FS with base X and group $G \times G'$ that acts trivially on the fiber $F \times F'$.*

In particular, if E and E' are PFS's then the same thing will be true for $E \boxtimes E'$.

3. – Homomorphisms and subspaces of principal fiber spaces.

Let H (X, G) and $H'(X, G')$ be PFS's with the same base X. An *X-homomorphism f* of H into H' that is compatible with a homomorphism ρ of the topological group G into the topological group G' is a continuous map of H into H' such that $p_{H'} \circ f = p_H$, and:

(1) $$f(z \cdot g) = f(z) \cdot \rho(g), \qquad z \in H,\ g \in G.$$

Most often, we shall simply say "homomorphism." If $G' = G$ and ρ is the identity representation of G then f will be a *G-isomorphism.* If $H' = H$ then f will be an automorphism of H (X, G).

If f satisfies simply (1) then it will be a *representation* of H in H' that is compatible with ρ, and a *G-representation* if $G' = G$, while ρ is the identity representation of G.

Since $p_{H'} f(z \cdot g) = p_{H'} f(z)$, f will then be associated with a continuous map μ of X into itself that is defined by $p_{H'} \circ f = \mu \circ p_H$: One says that f *induces* μ on the base.

DEFINITION I.3.1. – *Let G' be a topological subgroup of G. A G'-**principal fiber subspace*** (G'-PFSS) *of* $H(X, G)$ *is a PFS* $H'(X, G')$ *such that:*

1. *The space H' is a subspace of H that is endowed with the induced topology.*

2. *The projection p' is the restriction of p to H'.*

3. *The right-translation by $g' \in G'$ is the restriction to H' of the right-translation $D_{g'}$ that acts on H.*

The following characterization will be useful in what follows:

PROPOSITION I.3.1. – *In order for a subset H' of H to be a G'-PFSS of H, it is necessary and sufficient that the restriction p' of p to H' should enjoy the following properties:*

1. $p'(H') = X$.

2. $p'^{-1}(x) = z \cdot G'$ if $z \in H'$ and $x = p' \cdot z$.

3. *p' admits a local lift in the neighborhood of any $x \in X$ (that is continuous for the induced topology).*

Those conditions, which are obviously necessary, are indeed sufficient: Let H' be a subset of H that satisfies those properties; give it the induced topology. Axiom *a*) of the definition (I.1.1) will then be verified at the same as *b*), since p', which is the restriction of a continuous map to a subspace, is continuous and enjoys property 3. As for Axiom *c*), it is verified because the map of $H' \times G'$ onto H':

$$(z, g') \mapsto z \cdot g', \qquad z \in H', \qquad g' \in G',$$

which is well-defined, from property 2, is once more continuous, since it is the restriction of a continuous map to a subspace. Finally, Axiom *d*) is verified because if σ is a local section of H' over U then it will also be a local section of H in such a way that the map of $U \times G$ onto H_U, $(x, g) \mapsto \sigma(x) \cdot g$ is a local chart of H and its restriction to $U \times G'$ (which is the restriction of a homeomorphism to a subspace) is again a homeomorphism.

One deduces from that proposition the:

COROLLARY. – *The image of a PFS $H'(X, G)$ by a homomorphism f of H into $H(X, G)$ that is compatible with the representation ρ of G' in G is a $\rho(G')$-PFSS of H.*

Indeed, if H' is closed then $H'_x = H_x \cap H'$ will be closed in H_x ($x \in X$). Now, G is homeomorphic to H_x under the homeomorphism $\hat{z}$:

$$g \mapsto z \cdot g \qquad (z \in H', g \in G)$$

that transforms G' into H'_x in such a way that G' will be closed in G. The converse is obtained immediately with the aid of local charts on H.

PROPOSITION I.3.3. – *If G' is a subgroup (L. T.) of G then the G'-PFSS's of a PFS H (X, G) will correspond bijectively to sections of H / G'.*

Keep the notations of Proposition (I.2.1) and let $H'(X, G') \subset H(X, G)$ be given. H' defines a map f of X into H / G' by $f(x) = \pi(H'_x)$, since if $z_1 \in H'_x$ and $z_2 \in H'_x$ then $z_2 = z_1 \cdot g'$, for $g' \in G$ and $\pi(z_1) = \pi(z_2)$. f is continuous because f will factorize into a product of continuous maps: $f = \pi \circ \sigma_U$ on an open set U that is endowed with a local section σ_U ; it will then be a section of H / G'. Conversely, let f be such a section, and let $H' = \bigcup_{x \in X} \pi^{-1}(f(x))$. The subset $H' \subset H$, when endowed with the induced topology, will satisfy properties 1 and 2 of Proposition (I.3.1) in an obvious way. In order to show that H' is a G'-PFSS of H, it remains to show that p' admits local lifts in the neighborhood of any $x \in X$. Now, if σ_U is a section of H and ρ is the canonical map of G onto $L = G / G'$ then one will have the local chart Ψ_U on H / G':

$$(x, y) \mapsto \Psi_U(x, y) = \pi(\sigma_U(x) \cdot g) \quad \text{if} \quad x \in X, \quad y \in L, \quad \rho(g) = y.$$

There will then exist a continuous function on U with values in L, $y \mapsto y(x)$ such that:

$$f(x) = \Psi_U(x, y(x)), \qquad x \in U.$$

Let $x_0 \in U$, where O is a neighborhood of $y(x_0)$ in L that is endowed with a lift $y \mapsto s(y) \in G$, which exists since G' is a subgroup (L.T.). $V = y^{-1}(O) \cap U$ is an open neighborhood of x_0, and for $x \in V$, one will have $y(x) = \rho[s(y(x))]$, so:

$$f(x) = \Psi_U(x, \rho[s(y(x))]) = \pi[\sigma_U(x) \cdot s(y(x))];$$

i.e., for $x \in V$, $\sigma_V(x) = \sigma_U(x) \cdot s(y(x)) \in H'_x$. Since σ_V is obviously continuous, it constitutes a lift of p' over V. Q.E.D.

That proposition is nothing but another form of a well-known theorem of C. Ehresmann [**12**] ([5]), because the notion of a G'-PFSS is, in fact, equivalent to that of the restriction to G' of the structure group G. Indeed, the existence of a G'-PFSS $H'(X, G')$ $\subset H(X, G)$ derives from the possibility of restricting the structure group of H to G'.

([5]) See also J. Frenkel [**15**], § **16**.

Conversely, if that operation were possible then that would signify that there exists a PFS $K'(X, G')$ that is "equivalent" to H by enlarging the structure group that is given in a PFS $K(X, G)$. However, in our language, that would mean that $K'(X, G')$ would be a G'-PFSS of $K(X, G)$, and that it would be G-isomorphic to $H(X, G)$. The image of $K'(X, G')$ under that isomorphism would then be a G'-PFSS of H.

4. – Intersection of principal fiber subspaces.

Let G' and G'' be subgroups of G, and let $H'(X, G')$ and $H''(X, G'')$ be PFSS's of the PFS $H(X, G)$. We shall now study their intersection.

PROPOSITION I.4.1. – *If* $z' \in H'_x$ *and* $z'' = z' \cdot g \in H''_x$ $(x\ X, g \in G)$ *then in order for* $H_x \cap H''_x \neq \varnothing$, *it is necessary and sufficient that* $g \in G' \cdot G''$. *Hence, if* $z \in H'_x \cap H''_x$ *then one will have* $H'_x \cap H''_x = z \cdot \Gamma$ *or* $\Gamma = G' \cap G''$.

Indeed, if $H'_x \cap H''_x \neq \varnothing$ then let $z \in H'_x \cap H''_x$. Since $z \in H'_x$, $z = z' \cdot g'$ $(g' \in G')$, and since $z \in H''_x$, $z = z'' \cdot g''$ $(g'' \in G'')$. One will then have $z'' \cdot g'' = z' \cdot g'$, or $z'' = z' \cdot (g' \cdot g''^{-1})$. Hence, $g = g' \cdot g''^{-1} \in G' \cdot G''$, and conversely. On the other hand, if z_1, $z_2 \in H'_x \cap H''_x$ then $z_2 = z_1 \cdot \gamma$. Since z_1, $z_2 \in H'_x$ (H''_x, resp.), $\gamma \in G'$ (G'', resp.) in such a way that $\gamma \in \Gamma$, and conversely. Hence, $H'_x \cap H''_x = z_1 \cdot \Gamma$.

Suppose, to simplify matters, that $p(H' \cap H'') = X$. $K = H' \cap H''$ will then satisfy conditions 1 and 2 of Proposition (I.3.1), and in order for K to be a Γ-PFSS of H, it would be necessary and sufficient that it should admit local sections in a neighborhood of any $x \in X$. Let U be an open subset of X that is endowed with local sections σ' (σ'', resp.) of H' (H'', resp.). One will have:

$$\sigma''(x) = \sigma'(x) \cdot g(x), \qquad x \in U,$$

in which $x \mapsto g(x)$ is a continuous map of U into G with values in $G' \cdot G''$, since $H'_x \cap H''_x \neq \varnothing$. If K is a PFSS then U will admit an open covering $\{U_\alpha\}$ that is endowed with local sections ρ_α of K over U_α. ρ_α will also be a local section of $H' \supset K$ (H'', resp.), in such a way that there will exist a continuous function g'_α (g''_α, resp.) $U_\alpha \to G'$ (G'', resp.) such that:

$$\rho_\alpha(x) = \sigma'(x) \cdot g'_\alpha(x) = \sigma''(x) \cdot g''^{-1}_\alpha(x), \qquad x \in U_\alpha;$$

hence:

$$g(x) = g'_\alpha(x) \cdot g''_\alpha(x). \tag{1}$$

Conversely, if U admits a covering $\{U_\alpha\}$ that is endowed with continuous g'_α (g''_α, resp.) with values in G' (G'', resp.) that satisfy (1) then one will have:

$$\sigma''(x) = \sigma'(x) \cdot g'_\alpha(x) \cdot g''_\alpha(x)$$

in U_α, in such a way that:

$$\rho_\alpha(x) = \sigma'(x) \cdot g'_\alpha(x) = \sigma''(x) \cdot g''^{-1}_\alpha(x)$$

is a common local section to H' and H'' over U_α; i.e., a local section of K. That leads us to pose the:

DEFINITION I.4.1. – *A continuous map f of a topological space Y into a topological group G with values in* $G' \cdot G''$ *(*G' *and* G'' *are subgroups of G) is called* ***locally factorizable*** *in* $G' \cdot G''$ *if there exists an open covering* $\{Y_\alpha\}$ *of Y that is endowed with continuous maps* g'_α *(*g''_α*, resp.) of* Y_α *into* G' *(*G''*, resp.) such that for* $y \in Y_\alpha$*, one will have* $f(y) = g'_\alpha(y) \cdot g''_\alpha(y)$.

We have established:

PROPOSITION I.4.2. – *Let* $H'(X, G')$ *and* $H''(X, G'')$ *be PFSS's of* $H(X, G)$*, and let* $\{U_\alpha\}$ *be an open covering of X that is endowed with local sections* σ'_A *(*σ''_A*, resp.) of* H' *(*H''*, resp.) that are coupled by* $\sigma''_A(x) = \sigma'_A(x) \cdot g_A(x)$*. In order for* $K = H' \cap H''$ *to be a PFSS of H, it is necessary and sufficient that the functions* g_A *should take their values in* $G' \cdot G''$ *and be locally factorizable.*

That will always be true, with the single reservation that:

$$p\,(H' \cap H'') = X$$

if G' and G'' satisfy the following property:

DEFINITION I.4.2. – *A pair* G', G'' *of subgroups of the topological group G is called* ***regular*** *if any continuous map of a topological space into G with values in* $G' \cdot G''$ *is locally factorizable.*

PROPOSITION I.4.3. – *If one is given a topological group G and two subgroups* G' *and* G'' *then in order for the intersection of a* G'*-PFSS* H' *and a* G''*-PFSS* H'' *of a PFS* $H(X, G)$ *to be a PFSS as long as* $p\,(H' \cap H'')$ ***(and for any*** X, H, H', H''*), it is necessary and sufficient that the pair* G', G'' *should be a regular pair of subgroups of G.*

That condition, which is sufficient from Proposition (I.4.2), is in fact necessary if one is given a topological space X and a continuous function $g : X \to G' \cdot G''$. Let $H = X \times G$, $\sigma'(x) = (x, e)$. σ' is a section of H, and $H'(X, G') = \bigcup_{x \in X} \sigma'(x) \cdot G'$ is a PFSS (Prop. I.3.1). σ'', as defined by $\sigma'' = \sigma'(x) \cdot g(x)$, is a section of H, and $H''(X, G'') = \bigcup_{x \in X} \sigma''(x) \cdot G''$ is a PFSS. Hence, $p\,(H' \cap H'') = X$, since $g(x) \in G' \cdot G''$ for any $x \in X$. If

$H' \cap H''$ is a PFSS then from Proposition (I.4.2), g will be locally factorizable. Since X and g are arbitrary, the proposition is established.

We shall now seek to find the conditions under which a pair of subgroups will be regular. Let G', G'' be a regular pair of subgroups of the topological group G. The identity map of $G' \cdot G''$ is, in particular, locally factorizable, and there exists an open covering $\{\mathcal{O}_\alpha\}$ of $G' \cdot G''$ and some continuous maps g'_α (g''_α, resp.) of $\mathcal{O}_\alpha$ into G' (G'', resp.) such that for $g \in \mathcal{O}_\alpha$, one will have:

$$g = g'_\alpha(g) \cdot g''_\alpha .$$

Therefore let f be a continuous map of the topological space Y into $G' \cdot G''$, so the $Y_\alpha = f^{-1}(\mathcal{O}_\alpha)$ will form an open covering of Y that is endowed with continuous maps $\bar{g}'_\alpha = g'_\alpha \circ f$ ($\bar{g}''_\alpha = g''_\alpha \circ f$, resp.) into G' (G'', resp.), and for $y \in Y_\alpha$, one will have $f(y) = g'_\alpha(g) \cdot g''_\alpha$; i.e., the pair G', G'' is regular when the identity map on $G' \cdot G''$ is locally factorizable.

$G' \cdot G''$ is a saturated subspace of G for the left equivalence relation modulo G, and if the canonical map π of G onto G / G'' is a fibration then upon restricting the fiber structure on G to $B = \pi(G' \cdot G'')$, $G' \cdot G''$ will admit a principal fiber structure with group G''. Let V be an open subset of B that is endowed with a local section s, $y \mapsto s(y) \in G' \cdot G''$. Since the pair G', G'' is regular, s will be locally factorizable, and one will have:

$$s(y) = g'(y) \cdot g''(y), \quad g'(y) \in G', \qquad g''(y) \in G''$$

locally in V, since the functions g' and g'' are continuous. Now:

$$y = \pi(s(y)) = \pi(g'(y) \cdot g''(y)) = \pi(g'(y)),$$

in such a way that g' is a local section with values in G'. Conversely, suppose that the fiber structure of $G' \cdot G''$ admits a local section s with values in G' in a neighborhood U of $y_0 = \pi(e)$, where one can suppose that $s(y_0) = e$. There exists such a section in the neighborhood of any $y_1 \in B$. Indeed, $y_1 = \pi(g'_1 \cdot g''_1) = \pi(g'_1)$ ($g'_1 \in G'$, $g''_1 \in G''$), and the left-translation $g \mapsto g'_1 \cdot g$ is a homeomorphism of G that preserves $G' \cdot G''$, and its restriction to $G' \cdot G''$, which is endowed with the induced topology, is once more a homeomorphism. Similarly, the map $y \mapsto g'_1 \cdot y$ ($y \in B$) is a homeomorphism of $B \cdot V = g'_1 \cdot U$ is a neighborhood of y_1, and the map of V into $G' \cdot G''$:

$$y \in V \mapsto g_1'^{-1} \cdot y \in U \mapsto s(g_1'^{-1} \cdot y) \mapsto g'_1 \cdot s(g_1'^{-1} \cdot y) = t(y)$$

is continuous and has values in G'. Since:

$$\pi(t(y)) = g_1' \cdot \pi(s(g_1'^{-1} \cdot y)) = g_1' \cdot (g_1'^{-1} \cdot y) = y,$$

t will be the aforementioned local section of V. Hence, the map Φ_V of $V \times G''$ onto $\pi^{-1}(V)$:

$$(y, g'') \mapsto t(y) \cdot g'', \quad y \in V,\ g'' \in G''$$

will be a local chart on the fiber structure of $G' \cdot G''$, in such a way that there will exist a continuous function g_V'' onto $\pi^{-1}(V)$ with values in G'' such that:

$$g = \Phi_V(\pi(g),\ g_V''(g)) = t(\pi(g) \cdot g_V''(g)), \quad g \in \pi^{-1}(V).$$

t takes its values in G', which signifies that the identity map of $G' \cdot G''$ is factorizable in $\pi^{-1}(V)$, and since the $\pi^{-1}(V)$ cover $G' \cdot G''$, that the couple G', G'' is regular. Therefore:

PROPOSITION I.4.4. – *In order for a pair G', G'' of subgroups of the topological group G to be regular, it is necessary and sufficient that the identity map of $G' \cdot G''$ should be locally factorizable. If one of the subgroups G'' (G', resp.) is (L.T.) then it is necessary and sufficient that the fibration of G', G'' by the left classes modulo G'' (right modulo G', resp.) should admit a local section with values in G' (G'', resp.).*

The latter condition expresses the idea that the restriction π of π' to G admits local lifts. Let $x \in B$, $g' \in \pi'^{-1}(x)$, $g_1' \in \pi'^{-1}(x)$. Hence, $g_1' = g' \cdot g''$, $g'' \in G''$, so $g'' \in G' \cap G'' = \Gamma$. Since conversely one has $\pi'(g' \cdot \Gamma) = \pi'(g') = y$, $\pi'^{-1}(x) = g' \cdot \Gamma$ ($y \in B$) will be a left class modulo Γ. Hence, one finds that one has defined a bijective map f' of G'/Γ onto B by:

$$f'(g' \cdot \Gamma) = \pi'(g') = g' \in G',$$

in such a way that if q is the canonical map of G' onto G'/Γ then one will have the commutative diagram:

$$\begin{tikzcd} & G' \arrow[dl, "\pi'"'] \arrow[dr, "q"] & \\ B & & G'/\Gamma \arrow[ll, "f'"'] \end{tikzcd}$$

Since q is open and π' is continuous, f' will be continuous. If s is a lift of p over $V \in B$ then $\sigma \circ f'$ will be a lift of q over $f'^{-1}(V)$ that is open in G'/Γ, in such a way that q is a fibration. Conversely, if q admits local lifts, and if f' is a homeomorphism, moreover (which one cannot generally state), then π' will admit local lifts, and the pair G', G'' will be regular. If f'' denotes the bijective map of G''/Γ onto $G' \cdot G''/G'$ that is defined analogously to f' then one will deduce that:

THEOREM I.4. – *In order for a pair G', G'' of subgroups (L.T.) of a topological group G to be regular, it is necessary and sufficient that $\Gamma = G' \cap G''$ should be a subgroup (L.T.) of G' and G''. If one of the maps f' or f'' defined above is a homeomorphism then that condition will be sufficient; in particular, it will be true in the following two cases:*

1. G' (*or* G'') *is open in* G.

2. G' / Γ (*or* G' / Γ) *is compact.*

In order to prove that, it remains to be shown that f' or f'' is bi-continuous in the two indicated cases; we shall show it for f'.

1. In order for f'^{-1} to be continuous, it is necessary and sufficient that π' should be open, or even that the saturation of G'' by an open subset of G' should be open in $G' \cdot G''$. That will be the case if an open subset of G' is an open subset of G ; i.e., if G' is open in G.

2. Since G'' is a subgroup (L.T.) of G, it will be closed, so G / G'' and $B \subset G / G''$ will be separable. Hence, if G' / Γ is compact then f' will be continuous, since it is a continuous bijection of a compactum onto a separable space.

5. – Principal fiber spaces and subspaces in the differentiable case.

"Differentiable" will always mean "an arbitrary class of differentiability (including analytic) that is compatible with the givens," and the actual class will be specified only when necessary.

DEFINITION I.5.1. – *A differentiable principal fiber space (a differentiable FS, resp.) is defined by Definition (I.1.1) (I.1.2., resp.) when one supposes, moreover, that the base X, as well as the connected components of H and F, are differentiable manifolds and that G is a Lie group. All maps that enter into the definition are differentiable; homeomorphisms are differentiable and regular (viz., they have non-zero Jacobians).*

REMARK I.5. – Let H be a *set* that is endowed with a projection p onto the differentiable manifold X, let $\mathcal{R}$ be an open covering of X, and for any $U \in \mathcal{R}$, let there be a bijective map Φ_U of $U \times G$ onto $p^{-1}(U)$, such that:

a) $p \circ \Phi_U(x, g) = x$, $x \in U$, $g \in G$.

b) For any pair $U \in \mathcal{R}$, $V \in \mathcal{R}$, $U \cap V \neq \emptyset$, there exists a differentiable function $s_{U,V}$ on $U \cap V$ with values in G such that:

$$\Phi_V^{-1} \circ \Phi_U (x, g) = (x, s_{U,V}(x) \cdot g).$$

Hence, there exists a unique structure of a differentiable PFS $H(X, G)$ with projection p such that the Φ_U are local charts.

The constructions of paragraph I.2 (associated PFS, model, quotient by a closed subgroup of the structure group, fiber product) lead to differentiable FS's. The theorem of super-fibration (Prop. I.2.2) is valid for closed subgroups $G'' \subset G' \subset G$ with no supplementary hypotheses, since the spaces and maps in the diagram are all differentiable. Finally, the homomorphisms of a PFS are defined as in paragraph (I.3), when f is simply supposed to be differentiable and regular.

Nonetheless, the most natural notion of differentiable PFSS differs from that of PFSS in the topological case (Def. I.3.1), to the extent that a submanifold differs from a subspace: It is generally endowed with a topology that is different from the induced topology. More precisely, in what follows, a *submanifold* will refer to a regularly-embedded manifold with no double points ([6]), and one will say that a *submanifold is proper* if its topology coincides with the induced topology. Similarly, we say *Lie subgroup* of a Lie group G to mean an abstract subgroup G' of G that is itself a Lie group, and its connected component of the identity (for the proper topology of G') is an analytic subgroup of G ([7]). If G' is a proper submanifold of G then G' will be a *proper Lie subgroup.*

DEFINITION I.5.2. – *Let $H(X, G)$ be a differentiable PFS and let G' be a Lie subgroup of G. A differentiable G'-PFSS of $H(X, G)$ is a differentiable PFS $H'(X, G')$ for which:*

a) *The subordinate differentiable manifold H' is a submanifold of H* (or if H and H' are not connected then each connected component of H' is a submanifold of a connected component of H).

b) *The projection p' is the restriction of the projection p to H'.*

c) *The right-translation by $g' \in G'$ is the restriction to H' of the right-translation $D_{g'}$ that acts on H.*

([6]) For example, one can take the definition that was given by Chevalley ([**11**], pp. 85, def. 1) for analytic submanifolds by supposing that the givens are merely differentiable.

([7]) One can deduce the construction of all topologies on an abstract subgroup G' of G for which it will be a Lie subgroup of G from a theorem of Yamabe [**27**] on the arc-wise connected subgroups of a Lie group. Those topologies correspond bijectively to the distinguished subgroups K of G' that are arc-wise connected in G. If K is given then the corresponding topology $\mathcal{T}(G', K)$ of G' will admit the arc-wise connected components of e in the open neighborhoods of e for the induced topology on K as a fundamental system of neighborhoods of the identity e. One of those topologies is coarser than all of the other ones. It is obtained by taking K to be the arc-wise connected component G'_0 of e in G' for the induced topology. In particular, the former will coincide with the induced topology if G' is closed.

With that definition, G' is endowed with a well-defined structure of a Lie subgroup of G, which is associated with a topology $\mathcal{T}$. Let $\mathcal{T}_1$ be a topology that is coarser than $\mathcal{T}$ for which G' will again be a Lie subgroup of G, and let ρ be the identity homomorphism from G', endowed with $\mathcal{T}$, to G', endowed with $\mathcal{T}_1$ (which will be denoted by G_1'); it is a continuous homomorphism of Lie groups, and thus analytic. *There exists a unique structure of a differentiable PFS $H'(X, G_1')$ on H' such that the identity map of H is a homomorphism of differentiable PFS's that is compatible with ρ and takes $H'(X, G')$ onto $H'(X, G_1')$, and H' is once more a differentiable PFSS of H for that structure.* Indeed, let $\{\Phi_U\}$ be a family of charts that cover H' for the structure $H'(X, G')$. In order for the identity map f of H' to be a homomorphism, it is necessary that the Φ_U (more precisely, the $f \circ \Phi_U$) should once more be charts for $H'(X, G_1')$, because one must have:

$$f \circ \Phi_U(x, g) = f[\Phi_U(x, e) \cdot g] = f[\Phi_U(x, e)] \cdot g.$$

Since f is differentiable, $f[\Phi_U(x, e)]$ will be a differentiable local section of $H(X, G_1')$ over U, and:

$$\Phi_{1,U}: \qquad (x, g) \mapsto f[\Phi_U(x, e)] \cdot g$$

must then be a local chart, from axiom d) of Definition (I.1.1). Now, if $s_{U,V}$ is the function on $U \cap V$ with values in G' that is associated with a change of local coordinates $\Phi_V^{-1} \circ \Phi_U = \Phi_{1,V}^{-1} \circ \Phi_{1,U}$ (cf., Remark I.5.1) then it will again be a differentiable map into G_1', since ρ is analytic, in such a way that the charts $\{\Phi_{1,U}\}$ will effectively define a structure $H'(X, G_1')$ on H'. In order to show that $H'(X, G_1')$ is a differentiable PFSS of H, it remains to show that H' will be a submanifold when it is endowed with the differentiable structure that is subordinate to $H'(X, G_1')$. Let j be the identity map of H' into H. The map σ_U of U into H that is defined by $\sigma_U(x) = j \circ \Phi_U(x, e)$ is a differentiable local section of H, and:

$$\tilde{\Phi}_U: \qquad (x, g) \mapsto \sigma_U(x) \cdot g, \quad x \in U, g \in G$$

is a local chart of H whose restriction to $U \times G'$ is $j \circ \Phi_U$, from axiom c) of Definition (I.5.2). In the charts Φ_U (or $\Phi_{1,U}$) and $\tilde{\Phi}_U$, j is the map:

$$(x, g') \in U \times G' \mapsto (x, g') \in U \times G,$$

which is a map that is differentiable and regular for G_1', as well as for G', since G_1' is a Lie subgroup of G'. That concludes the proof.

Since $p'^{-1}(U)$ ($p^{-1}(U)$, resp.) is an open subset of $H'(X, G')$ [$H(X, G)$, resp.], the associated charts Φ_U and $\tilde{\Phi}_U$ will show, moreover, that H' is a proper submanifold of H if and only if G' is a proper Lie subgroup of G.

In particular, upon taking $\mathcal{T}_1$ to be the coarsest topology of G' for which it is a Lie subgroup of G ([8]), one will get a *minimal structure of a differentiable PFSS of H for H′.* The latter topology on G' is not an induced topology, in general, so the topological PFS that is subordinate to $H'(X, G'_1)$ will not generally be a topological PFSS of the topological PFS that is subordinate to H. Since the subspace H' of H obviously satisfies the hypotheses of Proposition (I.3.1), the differentiable local sections of $H'(X, G'_1)$ will provide local lifts of p' that are continuous for the induced topology, and H' will also admit the structure of a topological PFSS of H. The latter coincides with the structure that is subordinate to $H'(X, G'_1)$ if and only if H' is a proper submanifold of H or G' is a proper Lie subgroup of G. In particular, the same thing will be true if H' is closed in H, since from Proposition (I.3.2), G' will then be closed in G. Hence:

PROPOSITION I.5.1. – *Let $H'(X, G')$ be a differentiable PFSS of $H(X, G)$ that is endowed with its minimal structure. In order for the structure of a topological PFSS subordinate to $H(X, G')$ to coincide with the structure of a topological PFSS on $H(X, G)$ on H', it is necessary and sufficient that G' should be a proper Lie subgroup of G (or, what is equivalent, that H' should be a proper submanifold of H). One will then say that H' is a* ***proper PFSS*** *of H; in particular, that will be the case if H' is closed.*

We shall now establish the analogue of Proposition (I.3.1).

PROPOSITION I.5.2. – *Let G' be a Lie subgroup of G, and let $H(X, G)$ be a differentiable PFS. In order for a subset H' of H to admit the structure of a G'-PFSS of G, it is necessary and sufficient that the restriction p' of p to H' should enjoy the following properties:*

1. $p'(H') = X$.

2. $p'^{-1}(x) = z \cdot G'$ *if* $z \in H'$ *and* $x = p \cdot z$.

3. *p' admits local lifts that are differentiable sections of H.*

We shall show that these conditions are sufficient. Let $\mathcal{R}$ be an open covering of X for which each $U \in \mathcal{R}$ is endowed with a differentiable local section σ_U with values in H'. For $x \in U \cap V$, $U \in \mathcal{R}$, $V \in \mathcal{R}$, one will have $\sigma_U(x) = \sigma_V(x) \cdot s_{U,V}(x)$. $s_{U,V}$ is a differentiable map of $U \cap V$ into G that takes its values in G' from hypothesis 2: It is a differentiable map into G'([9]). Let Φ_U be the map of $U \times G'$ onto $p'^{-1}(x)$:

([8]) See note ([7]) on page 16.

([9]) This will follow from Lemma (I.6.2), when G' is considered to be an integral of the Pfaff system that is composed of the field of planes that is generated by left translation when one starts with the Lie algebra $\underline{G}'$ of G'.

$$\Phi_U (x, g') = \sigma_V (x) \cdot g', \qquad g' \in G', \qquad x \in U ;$$

The change of charts $\Phi_V^{-1} \circ \Phi_U$ is the map:

$$(x, g') \mapsto (x, s_{U,V} (x) \cdot g'), \qquad x \in U,\ g' \in G',$$

and consequently, the collection $\{\Phi_U\}$ will define a structure on H' of a differentiable PFS $H'(X, G')$ with projection p'. In order to establish the proposition completely, it remains to be established that H' is a submanifold of H with that structure, which derives from a consideration of the charts of H that are associated with the same sections σ_U of H.

Since the subgroups G' of a Lie group for which $G \to G / G'$ is an analytical fibration are identical to its closed subgroups, one can deduce the following proposition from Proposition (I.5.2) by the same proof that permitted one to establish Proposition (I.3.3) when starting from (I.3.1):

PROPOSITION I.5.3 ([10]). – *The* ***closed*** *differentiable PFSS's of a differentiable PFS H (X, G) correspond bijectively to the differentiable sections of the space H / G', where G' is an arbitrary closed subgroup of G.*

6. – Intersection of closed differentiable principal fiber subspaces.

The same analysis as in the beginning of section I.4 leads to the following definitions and propositions:

DEFINITION I.6.1. – *A differentiable map of a manifold Y into a Lie group G with values in $G' \cdot G''$ (G' and G'' are Lie subgroups of G) is called* ***differentiably locally factorizable in*** $G' \cdot G''$ *if it is locally factorizable in the sense of the definition* (I.4.1) *and the factors g'_α (g''_α, resp.) are differentiable maps into G' (G'', resp.).*

PROPOSITION I.6.1. – *Let $H'(X, G')$ and $H''(X, G'')$ be differentiable PFSS's of $H(X, G)$, and let $\{V_A\}$ be an open covering of X that is endowed with differentiable local sections σ'_A of H' (sections σ''_A of H'', resp.) that are coupled by $\sigma''_A(x) = \sigma'_A(x) \cdot g_A (x)$* ($x \in V_A$). *In order for $K = H' \cap H''$ to be a differentiable PFSS of H, it is necessary and sufficient that the functions g_A should take their values in $G' \cdot G''$ and be differentiably locally factorizable.*

Recall the proof of Proposition (I.4.2) with its notations and our new hypotheses. If we suppose that K is a differentiable PFSS then ρ_A will be a differentiable local section of H with values in H' (H'', resp.), and the functions g'_α (g''_α, resp.) will be differentiable maps into G with values in G' (G'', resp.). g'_α (g''_α, resp.) will then be differentiable

([10]) Cf., J. Frenkel [**15**], Proposition 19.2.

maps into G' (G'', resp.) ([11]), and g will then be differentiably locally factorizable. The converse will follow from Proposition (I.5.2).

DEFINITION I.6.2. – *A pair G', G'' of Lie subgroups of G is called* ***generic*** *if any differentiable map into G with values in $G' \cdot G''$ is differentiably locally factorizable.*

PROPOSITION I.6.2. – *Given a pair of Lie subgroups G', G'' of G, in order for the intersection of two differentiable PFSS's $H'(X, G')$ and $H''(X, G'')$ of the differentiable PFS $H(X, G)$ to be a PFSS of H such that $p(H' \cap H'') = X$ (**and that must be true for all** X, H, H', H''), it is necessary and sufficient that the pair G', G'' should be a generic pair of subgroups of G.*

We shall now establish some sufficient conditions for a pair of closed subgroups of a Lie group G to be generic. In order to do that, we shall need the:

LEMMA I.6.1. – *Let V_n be a proper submanifold of a differentiable manifold W_m, and let g be a differentiable map of a manifold U_p into W_m that takes its values in V_n. g will then be a differentiable map of U_p into V_n.*

Indeed, if one supposes only that V_n is a submanifold that is not necessarily proper then for any $q \in V_n$ there will exist a neighborhood $\mathcal{O}$ of V_n (for the proper topology on V_n) that is endowed with local coordinates X^i ($i = 1, 2, \ldots, n$) and cubic for those coordinates (i.e., $|X^i - X_0^i|) < b$), and a neighborhood $\mathcal{O}'$ on W_m that is endowed with local coordinates z^α ($\alpha = 1, 2, \ldots, m$) and cubic for the z^α (i.e., $|z^\alpha - z_0^\alpha| < b$), such that $\mathcal{O} \subset \mathcal{O}'$ and the restriction to $\mathcal{O}$ of the identity map f of V_n into W_m is:

$$z^i = X^i, \qquad z^{n+k} = 0 \qquad (k = 1, \ldots, m - n).$$

If V_n is a proper submanifold, moreover, then since any open subset of V_n is the trace on V_n of an open subset of W_m, one can restrict $\mathcal{O}$ and $\mathcal{O}'$ in such a way that $\mathcal{O} = \mathcal{O}' \cap V_n$. Hence, let $r \in U_p$, $g(r) = q$, and let g_1 be the map into V_n that is defined by g. Since $g^{-1}(\mathcal{O}')$ is an open subset of U_p, there will exist an open neighborhood ω of r in U_p that is endowed with local coordinates x^a ($a = 1, \ldots, p$) such that $g(\omega) \subset \mathcal{O}'$. Since $g(\omega) \subset V_n$, $g(\omega) \subset V_n \cap \mathcal{O}' = \mathcal{O}$, and the restriction of g to ω will be expressed by:

$$z^i = g^i(x^1, \ldots, x^p) \qquad (i = 1, \ldots, n), \qquad z^{n+k} = 0 \qquad (k = 1, \ldots, m - n),$$

in which the functions g^i are differentiable for $|x^a - x_0^a| < b$. One will then have $g_1(\omega) = g(\omega) \subset \mathcal{O}$, and the restriction of g_1 to ω will be expressed by:

([11]) Cf., footnote ([9]), pp. 18.

$$X^i = g^i(x^1, x^2, \dots, x^p) \qquad (i = 1, \dots, n)\,.$$

Q.E.D.

Recall the notations of I.4, but with the present hypotheses. The natural projection q of G' onto G'/Γ defines a structure on G' of an analytic PFS $G'(G'/\Gamma, \Gamma)$. The associated fiber space with fiber G'', when Γ acts on G'' by left translation, is the analytic FS $G''(G')$ that is obtained by taking the quotient of $G'' \times G'$ by the equivalence relation ρ (Def. I.2.1):

$$(g'', g') \sim (\gamma \cdot g'', g' \cdot \gamma^{-1}), \qquad g' \in G', \quad g'' \in G'', \quad \gamma \in \Gamma.$$

The map of $G'' \times G'$ into G:

$$(g'', g') \mapsto g' \cdot g''$$

passes to the quotient, since $(g'\gamma^{-1}) \cdot (\gamma g'') = g' \cdot g''$. If α is the natural map of $G'' \times G'$ onto $G''(G')$ then the map f of $G''(G')$ into G :

$$\alpha(g'', g') \mapsto g' \cdot g''$$

that one obtains will be a bijection onto $G' \cdot G''$, which will define the structure of an analytic manifold on that set, in particular. Let p be the projection of $G''(G')$ onto the base G'/Γ :

$$p \circ \alpha(g'', g') = q(g') = g' \cdot \Gamma,$$

so

$$f' \circ p \circ \alpha(g'', g') = f'(g' \cdot \Gamma) = g'G''.$$

On the other hand, $\pi \circ f \circ \alpha(g'', g') = \alpha(g' g'') = g' \cdot G''$; i.e.:

$$\pi \circ f = f' \circ p.$$

If i denotes the injection of G' in G then one will have the commutative diagram:

$$(D) \qquad \begin{tikzcd} & G''(G') \arrow[r, "f"] \arrow[dd, "p"'] & G \arrow[dd, "\pi"] \\ G' \arrow[urr, "i"] \arrow[dr, "q"'] & & \\ & G'/\Gamma \arrow[r, "f'"] & G/G'' \end{tikzcd}$$

Since f' is injective, it will also define an analytic manifold structure on its image $B = \pi(G' \cdot G'')$. We shall show that f and f' *are everywhere-regular analytic maps* and that, as a result, *$G' \cdot G''$ (B, resp.) will be an analytic submanifold of G (G/G'', resp.).*

Let $x_0 = q(e)$, and let U be an open neighborhood of x_0 that is endowed with an analytic section s of the fibration q, with $s(x_0) = e$. $f = \pi \circ i \circ s$ in U : The latter is then a composition of analytic maps, so f will be analytic in U. We let $\underline{\varphi}$ be the linear map that is tangent to a map φ, denote the vector space that is tangent to G'/Γ at x_0 by T_{x_0}, and let $\underline{G}$ ($\underline{G}', \underline{G}''$, $\underline{\Gamma}$, resp.) denote the tangent spaces at e to G (G', G'', Γ, resp.). Let n be the dimension of T_{x_0}. The fact that $q \circ s$ = identity on U leads to the fact that $\underline{s}(T_{x_0})$ is n-dimension and is supplementary to $\underline{\Gamma}$. Since $\underline{i}$ is injective, one can identify $\underline{G}'$ and $\underline{i}(\underline{G}')$. $\underline{s}(T_{x_0}) \subset \underline{G}'$ implies that $\underline{s}(T_{x_0}) \cap \underline{G}'' \subset \underline{G}' \cap \underline{G}'' = \underline{\Gamma}$, so $\underline{s}(T_{x_0}) \cap \underline{G}'' \subset \underline{s}(T_{x_0})$ $\cap\ \underline{\Gamma}$, which is zero, since $\underline{s}(T_{x_0})$ is supplementary to $\underline{\Gamma}$. Hence, $\underline{s}(T_{x_0}) \cap \underline{G}'' = 0$, and since $\underline{G}''$ is the kernel of $\underline{\pi}$, dim $\underline{\pi}\left(\underline{s}(T_{x_0})\right)$ = dim $\underline{s}(T_{x_0}) = n$; i.e., upon returning to the complete notations, $\underline{f}'(T_{x_0}) = \underline{\pi} \circ \underline{i} \circ \underline{s}(T_{x_0})$ is n-dimension. f' is then regular at the point x_0 . By homogeneity, one then deduces that f' is everywhere-analytic and regular. Indeed, G' acts on both G'/Γ and G/G'', and transitively on G'/Γ, since its actions are analytic isomorphisms of the two spaces that commute with f'.

The section s of G' on U is associated with the analytic chart Φ on $G''(G')$ [cf., Def. I.2.1]:

$$U \times G'' \to G''(G'), \qquad (x, g'') \mapsto \Phi(x, g'') = \alpha(g'', s(x)),$$

so

(1) $$f \circ \Phi(x, g'') = s(x) \cdot g'',$$

which shows that f is analytic in the open subset $\Phi(U \times G'')$. If T_e is the tangent to $G''(G')$ at the point $\Phi(x_0, e) = f^{-1}(e)$ then it will follow from (1) that:

(2) $$\underline{f}(T_e) = \underline{s}(T_{x_0}) + \underline{G}''.$$

Now, we have shown that $\underline{s}(T_{x_0})$ is transversal to $\underline{G}''$ and that the right-hand side of (2) is a direct sum, so:

$$\dim \underline{f}(T_e) = \dim T_{x_0} + \dim \underline{G}'' = \dim G''(G'),$$

which shows that f is regular at the point $f^{-1}(e)$. On the other hand, since $\underline{G}'' = \underline{s}(T_{x_0}) + \underline{\Gamma}$ and $\underline{\Gamma} \subset G''$, one will see that:

(3) $$\underline{f}(T_e) = \underline{G}' + G''.$$

Finally, in order to show that f is everywhere analytic and regular, we shall show that the analytic isomorphism φ of G :

$$g \mapsto \varphi(g) = g_1' \cdot g \cdot g_1'', \qquad g_1' \in G', \qquad g_1'' \in G'' \text{fixed}, \qquad g \in G,$$

which will leave $G' \cdot G''$ invariant, induces an analytic isomorphism. From (1), the restriction of the induced transformation $f^{-1} \circ \varphi \circ f$ to $\Phi(U \times G'')$ is defined by:

(4) $$f^{-1} \circ \varphi \circ f \circ \Phi(x, g'') = f^{-1} \circ \varphi(s(x), g'') = f^{-1}(g_1' \cdot s(x) \cdot g \cdot g_1'').$$

Let σ be the section of $G' \to G' / \Gamma$ over $V = g_1' \cdot U$:

$$y \mapsto \sigma(y) = g_1' \cdot s(g_1' \cdot y),$$

and let Ψ be the associated chart of $G''(G')$:

$$(y, g'') \mapsto \Psi(y, g'') = \alpha(g'', \sigma(y)), \quad y \in V, \; g'' \in G''.$$

One will then have $f \circ \Psi(y, g'') = \sigma(y) \cdot g''$, and if $x \in U$ then:

(5) $$f \circ \Psi(g_1' \cdot x, g'' \cdot g_1'') = \sigma(g_1' \cdot x) \cdot g'' \cdot g_1'' = g_1' \cdot s(x) \cdot g'' \cdot g_1''.$$

Upon comparing (4) and (5), one will get:

$$f^{-1} \circ \varphi \circ f \circ \Phi(x, g'') = \Psi(g_1' \cdot x, g'' \cdot g_1''),$$

which means that in the analytic charts Φ and Ψ of $G''(G')$, $f^{-1} \circ \varphi \circ f$ is expressed by:

$$(x, g'') \mapsto (g_1' \cdot x, g'' \cdot g_1''),$$

which is obviously an analytic isomorphism. Q.E.D.

The following proposition will serve to establish the two theorems that we have in mind:

PROPOSITION I.6.3. – *Any differentiable map into $G' \cdot G''$, when it is endowed with the analytic structure was just defined, is differentiably locally factorizable.*

Such a map h is a differentiable map of a manifold W into $G''(G')$. Let $\{\mathcal{O}_\alpha\}$ be an open covering of G' / Γ, with each $\mathcal{O}_\alpha$ being endowed with an analytic local section σ_α of $G \to G' / \Gamma$ and $G''(G')$ being endowed with the associated chart Ψ_α :

$$(x, g'') \mapsto \Psi_\alpha(x, g'') = \alpha(g'', \sigma_\alpha(x)), \qquad x \in \mathcal{O}_\alpha, \qquad g'' \in G''.$$

The $W_\alpha = h^{-1} \circ p^{-1}(\mathcal{O}_\alpha)$ constitute an open covering of W. The restriction h_α of h to W_α is a differentiable map into $p^{-1}(\mathcal{O}_\alpha)$; i.e., there exist differentiable maps x_α of W_α into $\mathcal{O}_\alpha$ (g_α'' of W_α into G) such that:

$$h_\alpha(z) = \Psi_\alpha(x_\alpha(z), g_\alpha''(z)) \quad \text{for} \quad z \in W_\alpha.$$

One then has $f(h_\alpha(z)) = f[\alpha(g''_\alpha(z)),\ \sigma_\alpha(x_\alpha(z))] = \sigma_\alpha(x_\alpha(z))\cdot g''_\alpha(z))$, so since σ_α is an analytic map in G', $g'_\alpha = \sigma_\alpha \circ x_\alpha$ will be a differentiable map into G', which establishes the proposition.

Consider the diagram (*D*) once more. Since p and π are open maps and the image under f of a set that is saturated for p will be saturated for π, it is obvious that if f is a homeomorphism (onto a subspace) then the same thing will be true for f'. Conversely, suppose that f' is a homeomorphism. Let s be a section of $G' \to G'/\Gamma$ over an open subset U. $V = f'(U)$ is an open subset relative to $B = \pi(G'\cdot G'')$, and $\sigma = i\circ s\circ f'^{-1}$ is a lift of π over V that is continuous for the induced topology. It will then be a local section of the restriction to B of the topological PFS that is subordinate to $G \to G/G''$. In the local charts that are associated with s and σ, f will translate into:

$$(x, g'') \mapsto (f'(x), g''), \qquad x \in U,\ g'' \in G'',$$

and as a result, f will be a homeomorphism. When f and f' are homeomorphisms, $G'\cdot G''$ will be a proper submanifold of G, and [Lemma (I.6.1)] a differentiable map g of a manifold W into G with values in $G'\cdot G''$ will be a differentiable map into $G'\cdot G''$ [more precisely, $h = f^{-1}\circ g$ will be a differentiable map into $G''(G')$]. As a result of Proposition (I.6.3), g will be differentiably locally factorizable. We have proved:

THEOREM I.6.1. – *Let G' and G'' be closed subgroups of the Lie group G. If $G'\cdot G''$ is a proper subgroup of G then the pair G', G'' will be generic. In particular, that will be the case in the following two cases:*

1. *G' (or G'') is an open subset in G.*

2. *G'/Γ (G''/Γ, resp.) is compact.*

The double classes $V_g = G'\cdot g\cdot G''$, ($g \in G$) are also analytic submanifolds of G, since $V_g = g\cdot(g^{-1}\cdot G'\cdot g)\cdot G''$. From formula (3), the tangent space at e to $(g^{-1}\cdot G'\cdot g)\cdot G''$ is $\underline{(\mathrm{ad}\, g^{-1})}\,\underline{G}' + \underline{G}''$, and the tangent space at g to V_g will then be:

$$T_g = \underline{L_g}\,\underline{(\mathrm{ad}\, g^{-1}}\cdot\underline{G}' + \underline{G}'') = \underline{D}_g\cdot\underline{G}' + \underline{L}_g\cdot\underline{G}'' = \underline{G}'\cdot g + g\cdot\underline{G}'', \tag{6}$$

with notations that should be clear. The V_g constitute an analytic foliation $\mathcal{F}$ on V. A point $g_0 \in G$ will be called *regular for the foliation* $\mathcal{F}$ if there exists an open neighborhood $\mathcal{O}$ of g_0 such that $\dim V_g = \dim V_{g_0}$, or $\dim T_g = \dim T_{g_0}$, for $g \in \mathcal{O}$. The set Ω of regular points will then be an open set. It is saturated for the foliation $\mathcal{F}$ because if g_0 is regular and $g_1 \in V_{g_0}$ then there will exist $g'_1 \in G'$ and $g''_1 \in G''$ such that $g_1 =$

$g_1' \cdot g_0 \cdot g_1''$. $\mathcal{O}_1 = g_1' \cdot \mathcal{O} \cdot g_1''$ is an open neighborhood of g_1, and for $g_2 \in \mathcal{O}_1$ there will exist a $g \in \mathcal{O}$ such that $g_1 = g_1' \cdot g \cdot g_1''$, in such a way that $V_{g_2} = V_g$ and:

$$\dim V_{g_2} = \dim V_{g_0} = \dim V_{g_1}.$$

From (6), the regularity condition is equivalent to:

$$\dim\left(\underline{\mathrm{ad}\, g^{-1}}\right)\underline{G}' \cap \underline{G}'' = \dim\left(\underline{\mathrm{ad}\, g_0^{-1}}\right)\underline{G}' \cap \underline{G}'' \quad \text{for} \qquad g \in \mathcal{O}.$$

$\left(\underline{\mathrm{ad}\, g^{-1}}\right)\underline{G}'$ depends upon g analytically. Consequently, once a basis for $\underline{G}$ has been chosen, the preceding condition will signify that the rank $r(g)$ of a certain homogeneous linear system S whose coefficients are analytic functions on G is constant in $\mathcal{O}$. The minors of order $r(g_0) + 1$ of S are therefore analytic functions on G that are zero on an open set $\mathcal{O}$. They are identically zero on the connected component C_{g_0} of G, in such a way that for any $g \in C_{g_0}$, one will have $r(g) \le r(g_0)$. If $g_1 \in C_{g_0}$ is regular then the same argument will show that $r(g) \le r(g_1)$ for $g \in C_{g_0}$. Those two inequalities will imply that $r(g_0) = r(g_1)$; i.e., that if g_0 is regular then:

(7) $$r(g_0) = \sup_{g \in C_{g_0}} r(g) = r,$$

namely:

(8) $$\dim\left(\underline{\mathrm{ad}\, g_0^{-1}}\right)\underline{G}' \cap \underline{G}'' = \inf_{g \in C_{g_0}} \dim\left(\underline{\mathrm{ad}\, g^{-1}}\right)\underline{G}' \cap \underline{G}''.$$

Conversely, if g_0 satisfies (7) then there will be a minor of S of order ρ that is not zero at g_0. That minor will be non-zero in an open neighborhood $\mathcal{O} \subset C_{g_0}$ of g_0, in such a way that for $g \in \mathcal{O}$, $r(g) \ge \rho$, and as a result of (7), one will have $r(g) = \rho$. There will then be an identity between the points of $C_{g_0} \cap \Omega$ and the ones that satisfy (7). That will show that Ω is not empty, and that the complement of $C_{g_0} \cap \Omega$ in C_{g_0} is the set of zeroes of the minors of S of order ρ, which are analytic functions on C_{g_0} that are not all identically zero. $C_{g_0} \cap \Omega$ will then be dense in C_{g_0}, and Ω will be everywhere dense in G.

Suppose that $e \in \Omega$. One will then have that $G' \cdot G'' \subset \Omega$ and that Ω is an open neighborhood of $G' \cdot G''$. Furthermore, to simplify the presentation, suppose that Ω and $G' \cdot G''$ are connected (otherwise the same argument would apply to the connected components), that Ω is a connected open subset of G that is an analytic manifold with $G' \cdot G''$ as a submanifold, and that the V_g $(g \cdot \Omega)$ and their tangent planes T_g all have the same dimension. Since, from formula (6), T_g depends analytically upon g, the field of planes T, $g \in \Omega \to T_g$ will define a completely-integrable Pfaff system on Ω for which the V_g, and in particular, $V_e = G' \cdot G''$, are the maximal integrals. Now:

LEMMA I.6.2. – *Let T be an analytic Pfaff system on an analytic manifold* Ω, *and let V be an integral manifold of T that is denumerable. If h is a differentiable map of class* C^s (s = 1, 2, …, ∞, ω) *of a manifold W in* Ω *with values in V then h will be a differentiable map* C^s *in V.*

That proposition was proved in Chevalley [**11**] (Chap. III, § **9**, pp. 94, Prop. 1) for $s = \omega$. It can be extended with no changes to an arbitrary s. Since G is connected, it will be denumerable, and its submanifold $G' \cdot G''$ will also be so (*ibid.*, Prop. 2). One can then apply Lemma (I.6.2) to $V = G' \cdot G''$. A C^s-differentiable map h of W into G with values $G' \cdot G''$ is a differentiable map into Ω, and as a result, into $G' \cdot G''$. Proposition (I.6.3) will then show that h is differentiably locally factorizable.

Finally, let $g_1 \in G$, and let $g_1^{-1} \cdot G' \cdot g_1$, G'' be a pair of closed subgroups. Let V^1 be the double classes, let $\mathcal{F}^1$ be the foliation, and let T^1 be the field of planes that are defined by starting with that pair. $\mathcal{F}^1$ and T^1 are deduced from $\mathcal{F}$ and T by right-translation by g_1^{-1}:

$$V_g^1 = (g_1^{-1} \cdot G' \cdot g_1) \cdot g \cdot G'' = g_1^{-1} \cdot V_{g_1 g}, \qquad \text{so} \qquad T_g^1 = g_1^{-1} \cdot T_{g_1 g}.$$

As a result, in order for e to be regular for $\mathcal{F}^1$, it is necessary and sufficient that g_1 should be that way for $\mathcal{F}$. One has thus established the:

THEOREM I.6.2. – *The pairs of closed subgroups of the Lie group G are "almost always" generic in the following sense: Let* G', G'' *be a pair of closed subgroups. The set of* $g \in G$ *such that the pair* ad $g^{-1} \cdot G'$, G'' *is generic contains an open subset that is everywhere-dense in G. In order for that pair to be generic, it is sufficient that g should be regular for the foliation of G by the double classes* $G' \cdot g \cdot G''$.

EXAMPLES. – Let r, r', r'' be the dimensions of G, G', G'', resp. The regularity condition is certainly realized at the point e if:

a) Upon utilizing the condition (8), dim $\underline{G}' \cap \underline{G}''$ = 0; i.e., if $\Gamma = \underline{G}' \cap \underline{G}''$ is discrete.

b) Upon using the condition (7), $r(e) = (r - r') + (r - r'')$, namely, $r' + r'' - (r - r(e)) = r$; i.e.:

$$\dim \underline{G}' + \dim \underline{G}'' - \dim \underline{G}' \cap \underline{G}'' = \dim \underline{G}$$

or

$$\dim (\underline{G}' + \underline{G}'') = \dim \underline{G} \qquad \text{or} \qquad \underline{G}' + \underline{G}'' = \underline{G}.$$

c) One of the groups G' (or G'') is distinguished, in which case, $(\underline{\mathrm{ad}\,g^{-1}} \cdot \underline{G}') \cap \underline{G}'' = \underline{G}' \cap \underline{G}''$ has a constant dimension and $\Omega = G$.

d) $\Gamma = G' \cap G''$ is distinguished, because $G' \supset \Gamma$ will then imply that $(\underline{\mathrm{ad}\,g^{-1}})\underline{G}' \supset \underline{\Gamma}$, so $(\underline{\mathrm{ad}\,g^{-1}})\underline{G}' \cap \underline{G}'' \supset \underline{\Gamma}$, and:

$$\dim\,(\underline{\mathrm{ad}\,g^{-1}})\underline{G}' \cap \underline{G}'' \geq \dim\,(\underline{\Gamma} = \underline{G}' \cap \underline{G}''),$$

which implies the regularity of *e*, from (8).

REMARK. – The criterion of genericity that is given by Theorem (I.6.3) is hardly necessary. For example, let ([12]) $G = CL_m$, $G'' = L_m$, $G' = CL\,(n_1, n_2)$ be matrix groups, and the last of them is defined by:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad A \in CL_{n_1}, \qquad B \in CL_{n_2} \qquad (n_1 + n_2 = m).$$

One easily sees that the point *e* is not regular for the foliation $\mathcal{F}$ that is associated with the pair G', G''; however:

PROPOSITION I.6.3. – *The pair* L_m, $CL\,(n_1, n_2)$ $(m = n_1 + n_2)$ *is a generic pair of subgroups of* CL_m.

We first remark that in order for a pair $G', G'' \subset G$ to be generic, it is sufficient that there should exist a neighborhood $\mathcal{O}$ of *e* in $G' \cdot G''$ for the induced topology such that any differentiable map into *G* with values in $\mathcal{O}$ is locally differentiably factorizable. Indeed, if *f* is a differentiable map of *Y* into *G* with values in $G' \cdot G''$, and if $g_1 \cdot g_1 \in f(Y)$ then $V = g_1' \cdot \mathcal{O} \cdot g_1''$ will be a neighborhood of $g_1' \cdot g_1''$, and let $W = f^{-1}(V)$ be an open subset of *Y*. Let φ be the restriction of *f* to *W*, and let $\psi = L_{g_1'}^{-1} \circ D_{g_1''}^{-1} \circ \varphi$. There will then exist two differentiable functions g' (g'', resp.) onto *W* with values in G' (G'', resp.) such that for $y \in W$, one will have $\psi(y) = g'(y) \cdot g''(y)$. Hence, $\varphi(y) = g_1' \cdot g'(y) \cdot g_1'' \cdot (y)\; g''$, which shows that φ is differentiably factorizable. Since the neighborhoods *W* cover *Y*, *f* will be differentiably locally factorizable, and the pair G', G'' will be generic.

Return to the case in which the groups are the ones that were indicated at the beginning of this remark. The set $\mathcal{O}$ of matrices:

([12]) We shall generally use the notations of C. Chevalley [**11**] for the classical groups. Nonetheless, the group $Gl\,(n, \mathrm{R})$ [$Gl\,(n, \mathrm{C})$, resp.] will be denoted by L_n (CL_n, resp.).

$$g = \begin{pmatrix} C & D \\ E & F \end{pmatrix} \in CL_m \,, \qquad \text{in which} \qquad C \in CL_{n_1} \qquad \text{and} \qquad F \in CL_{n_2}$$

is a neighborhood of the identity. Let f be a differentiable map into G with values in $\mathcal{O} \cap G' \cdot G''$: $f(y) = \begin{pmatrix} C(y) & D(y) \\ E(y) & F(y) \end{pmatrix}$, in which the four partial matrices are differentiable functions. For any $y \in Y$, there exist $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in CL\,(n_1, n_2)$ and $\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in L_m$ such that:

$$\begin{pmatrix} C & D \\ E & F \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} AP & BQ \\ AR & BS \end{pmatrix},$$

so $C = AP$ (P is then a regular matrix), and $E = AR$, in such a way that $C^{-1} E = P^{-1} R$; i.e., that $C^{-1} E$ is real. Similarly, $F^{-1} D = S^{-1} Q$ is real. Now, one has:

$$\begin{pmatrix} C & D \\ E & F \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & F \end{pmatrix} \cdot \begin{pmatrix} E_{n_1} & F^{-1}D \\ C^{-1}E & E_{n_2} \end{pmatrix}.$$

The two matrices in the right-hand side are differentiable and belong to $CL\,(n_1, n_2)$ and L_m , which establishes the proposition.

CHAPTER II

DIFFERENTIAL FORMS
WITH VALUES IN A VECTOR SPACE
CONNECTIONS

As far as this chapter is concerned, the reader can refer to A. Lichnerowicz [**22**], and we shall adopt most of his definitions and notations, as well as A. Aragnol [**1**]. We think that the systematic use of vector-valued forms that are defined globally on a principal fiber space and the operations that one can define on those forms permit a particularly simple presentation of the questions of differential geometry that are coupled with the theory of connections. Without giving a formal presentation of those methods here, we wish to develop certain rules in sufficient detail and with sufficient rigor (although from an intentionally "naïve" viewpoint) for us to be able to apply them as often as possible in the rest of this study.

To simplify the presentation, we shall employ the same notation in this chapter for a differentiable map and its tangent linear map. On the other hand, in all of this work, the summation over repeated indices will not be indicated, in general.

1. – Forms with values in a vector space.

Let V be a differentiable manifold, let T_x be the tangent vector space to V at the point x, and let T_x^* be its dual. On the other hand, let M be a real vector space of finite dimension m. An *exterior form with values in M at the point x* is a linear map φ_x of $\wedge T_x$ into M ; i.e., an element of the vector space $M \otimes \wedge T_x^*$. If a basis $\{e_A\}$ $(A = 1, 2, \ldots, m)$ is chosen in M then φ_x can be written:

$$\varphi_x = \sum_{A=1,2,\ldots,m} e_A \otimes \varphi_x^A = e_A \otimes \varphi_x^A, \tag{1}$$

in which the φ_x^A belong to $\wedge T_x^*$; i.e., they are scalar exterior forms on T_x that we call the *components of φ_x in the basis* $\{e_A\}$. Conversely, any finite sum such as (1) will determine an exterior form with values in M, even if the vectors e_A do not constitute a basis for M. Let:

$$e_A = e_{A'} M_A^{A'}, \qquad\qquad e_{A'} = e_A M_{A'}^A$$

be the formulas for passing from the basis $\{e_A\}$ to the basis $\{e_{A'}\}$, in which the matrix $\left(M_{A'}^A\right)$ is consequently the inverse of $\left(M_A^{A'}\right)$. From the bilinearity of the tensor product:

$$\varphi_x = e_A \otimes \varphi_x^A = e_{A'} M_A^{A'} \otimes \varphi_x^A = e_{A'} \otimes M_A^{A'} \varphi_x^A,$$

so the relation between the components of φ_x in the two bases will be:

$$\varphi_x^{A'} = M_A^{A'} \varphi_x^A . \tag{2}$$

If an exterior form φ_x with values in M is defined at any point $x \in V$ then if its components φ_x^A depend differentiably upon x and have class C^s, moreover [from (2), that will be independent of the chosen basis for M], we say that the collection $\{\varphi_x\}$ defines an *exterior differential form* φ *on* V *with values in* M *of class* C^s, and we further write:

$$\varphi = e_A \otimes \varphi^A, \tag{3}$$

in which φ^A is the (scalar) exterior differential form whose restriction to T_x is φ_x^A. We will further have the relation:

$$\varphi^{A'} = M_A^{A'} \varphi^A \tag{4}$$

between the differential forms $\varphi^{A'}$ and φ^A, which are the components of φ in the bases $\{e_{A'}\}$ and $\{e_A\}$, resp.

We shall now utilize the same notation φ, φ^A for φ_x , φ_x^A, resp. If the φ^A are homogeneous and have the same degree q [which is an intrinsic property, from (4)] then φ will be a *q-form with values M*. If φ is not homogeneous then it will once more have an intrinsic character, since the decomposition $\varphi = \sum \varphi_q$ is a sum of homogeneous forms that are obtained by performing that decomposition on the components; one lets $\bar{\varphi}$ denote the form $\bar{\varphi} = \sum (-1)^q \varphi_q$.

The value of the form φ for $\mathcal{T} \in \wedge T_x$ is:

$$\varphi(\mathcal{T}) = e_A < \varphi^A, \mathcal{T} >, \tag{5}$$

in which $< .,\ . >$ denotes the canonical bilinear form on $(\wedge T_x) \times (\wedge T_x^*)$. By abuse of language, it will sometimes be convenient to use the following notation:

$$\varphi(\mathcal{T}) = < \varphi, \mathcal{T} >. \tag{6}$$

For a decomposable q-vector, $\mathcal{T} = \mathcal{T}_1 \wedge \mathcal{T}_2 \wedge \ldots \wedge \mathcal{T}_q$, one will also use the notation:

$$\varphi(\mathcal{T}) = \varphi(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_q) . \tag{7}$$

Let μ be a differentiable map of W into V; one can define the *inverse image* $\mu^* \varphi$ to be the form on W with values in M such that:

(8) $$\mu^*\varphi = e_A \otimes \mu^*\varphi^A.$$

It will follow from (4) that this definition is intrinsic. If $\mathcal{T}_y \in \wedge T_x$, $y \in W$ then, from (5) and (6), one will have:

$$<\mu^*\varphi, \mathcal{T}_y> = e_A <\mu^*\varphi^A, \mathcal{T}_y> = e_A <\varphi^A, \mu\,\mathcal{T}_y>,$$

namely:

(9) $$<\mu^*\varphi, \mathcal{T}_y> = <\varphi, \mu\,\mathcal{T}_y>.$$

Finally, let d be the symbol of exterior differentiation. It once more follows from (4) that the form with values in M :

(10) $$d\varphi = e_A \otimes d\varphi^A$$

does not depend upon the basis $\{e_A\}$: It is the *exterior differential* of φ.

The operators μ^* and d are linear over R, and from their definition in a base on M, they will satisfy the usual relations:

(11) $$(\mu_2 \circ \mu_1)^* = \mu_1^*\mu_2^*,$$

(12) $$d\mu^* = \mu^* d,$$

(13) $$d \cdot d = 0.$$

2. – Composition of vector-valued forms.

A) Let L, M, P be three finite-dimensional vector spaces, and let a bilinear map of L M into P be denoted by:

$$l, m \mapsto (l, m), \qquad l \in L, \quad m \in M.$$

One can associate a form (Φ, φ) with values in P to any pair that consists of a form Φ with values in L and a form φ with values in M, in such a fashion that if:

$$\varphi = e_i \otimes \varphi^i, \qquad \text{where } e_i \in M \text{ are finite in number,}$$

and

$$\Phi = h_\alpha \otimes \Phi^\alpha, \qquad \text{where } h_\alpha \in M \text{ are finite in number,}$$

one will have:

(1) $$(\Phi, \varphi) = (h_\alpha, e_i) \otimes \Phi^\alpha \wedge \varphi^i.$$

Indeed, it suffices to take $\{e_i\}$ ($\{h_\alpha\}$, resp.) to be a basis for M (L, resp.), as well as a basis $\{f_\alpha\}$ for P. If $(h_\alpha, e_i) = C^a_{\alpha_i}\, f_a$ then, from (1), (Φ, φ) will necessarily be:

(2) $$(\Phi, \varphi) = f_a \otimes C^a_{\alpha_i}\, \Phi^\alpha \wedge \varphi^i,$$

and by changing bases, one can verify the fact that (Φ, φ) thus-defined depends upon only Φ and φ. The operation (Φ, φ) enjoys the following properties, which are obvious from (2):

a) It satisfies formula (1) for any decomposition of φ and Φ into sums of tensor products.

b) If Φ is a q-form and φ is a q-form then (Φ, φ) will be a $(q + q')$-form.

c) It is bilinear:

$$(3) \qquad (\lambda_1 \Phi_1 + \lambda_2 \Phi_2, \varphi) = \lambda_1 (\Phi_1, \varphi) + \lambda_2 (\Phi_2, \varphi),$$

$$(4) \qquad (\Phi, \lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 (\Phi, \varphi_1) + \lambda_2 (\Phi, \varphi_2), \qquad \lambda_1, \lambda_2 \in \mathrm{R}.$$

d) For a differentiable map μ of W into V:

$$(5) \qquad \mu^* (\Phi, \varphi) = (\mu^*\Phi, \mu^*\varphi),$$

e) If Φ is a q-form then:

$$(6) \qquad d (\Phi, \varphi) = (d\Phi, \varphi) + (\bar{\Phi}, d\varphi) = (d\Phi, \varphi) + (-1)^q (\Phi, d\varphi) .$$

We shall now study the operations (Φ, φ) for differentiable bilinear maps $l, m \mapsto (l, m)$: Those operations will then enjoy properties *a*), *b*), *c*), *d*), *e*) above, which we shall not recall.

B) *Product of a vectorial form with a homomorphism form.* – Let M and P be as above, and let $L = \mathcal{L}(M, P) = P \otimes M^*$ be the vector space of linear maps of M into P. We let $h \cdot X$ denote the transform of $X \in M$ by $h \in L$: The map $h, X \mapsto h \cdot X$ is a bilinear map of $L \times M$ into P, and if φ (Φ, resp.) is a form with values in M [in $\mathcal{L}(M, P)$, resp.] then the form $\Phi \cdot \varphi$ will be a form with values in P that is well-defined by paragraph A).

If M (P, resp.) is referred to the basis $\{e_A\}$ ($\{f_a\}$, resp.) then refer $\mathcal{L}(M, P)$ to its basis $\{\varepsilon_a^A\}$ that is associated with the preceding two and is defined by:

$$(7) \qquad \varepsilon_a^A \cdot e_B = \delta_B^A f_a \qquad (\delta_B^A \text{ is the Kronecker symbol}).$$

One will then have:

$$\varphi = e_B \otimes \varphi^B, \qquad \Phi = \varepsilon_a^A \otimes \Phi_A^a$$

for any form φ (Φ, resp.), so:

$$\Phi \cdot \varphi = \varepsilon_a^A \cdot e_B \otimes \Phi_A^a \wedge \varphi^B = \delta_a^A f_B \otimes \Phi_A^a \wedge \varphi^B ,$$

namely:

(8) $$\Phi \cdot \varphi = f_A \otimes \Phi_A^a \wedge \varphi^B .$$

Case in which Φ *is a* 0-*form.* Let:

$$\Phi = h_\alpha \otimes \Phi^\alpha \quad \text{and} \qquad \varphi = e_i \otimes \varphi^i$$

be arbitrary decompositions into sums of tensor products.

From formulas (1) and [(5), § **1**], for $\mathcal{T}_x \in T_x$, one must have:

$$<\Phi \cdot \varphi, \mathcal{T}_x> = h_\alpha \cdot e_i < \Phi_x^\alpha \wedge \varphi_x^i, \mathcal{T}_x >,$$

and since the Φ_x^α are scalars:

$$<\Phi \cdot \varphi, \mathcal{T}_x> = (h_\alpha \cdot \Phi_x^\alpha) \cdot e_i < \varphi_x^i, \mathcal{T}_x > = \Phi_x \cdot < \varphi, \mathcal{T}_x > ;$$

i.e., with the simplest notations:

(9) $$<\Phi \cdot \varphi, \mathcal{T}_x> = \Phi_x \cdot \varphi(\mathcal{T}_x) .$$

If Φ is 0-form, moreover (i.e., a well-defined homomorphism of M into P), then it will be convenient for what follows to use the special notation:

$$h \cdot \varphi = h(\varphi) ,$$

and formulas (1), (5), (6) will become:

(10) $$h(e_i \otimes \varphi^i) = h(e_i) \otimes \varphi^i ;$$

hence, from [(5), § **1**]:

$$< h(\varphi), \mathcal{T} > = h(e_i) < \varphi^i, \mathcal{T} > = h(\varphi(\mathcal{T})) ,$$

(11) $$\mu^* h(\varphi) = h(\mu^* \varphi),$$
(12) $$dh(\varphi) = h(d\varphi) .$$

C) *Case in which* $P = M$ *and* $L = \mathcal{L}(M)$ is a vector space of endomorphisms on M. If g and $h \in \mathcal{L}(M)$ then the product $g \cdot h$ of endomorphisms will be a bilinear function with values in $\mathcal{L}(M)$. Paragraph A) will then permit one to define the product $\Psi \cdot \Phi$ of two forms Ψ and Φ with values in $\mathcal{L}(M)$, which is a product that again has its values in $\mathcal{L}(M)$. If one refers M to a basis $\{e_A\}$, and $\mathcal{L}(M)$ to the corresponding basis $\{\varepsilon_B^A\}$, such that:

(13) $$\varepsilon_B^A \cdot e_C = \delta_C^A \cdot e_B ,$$

then one will have $\Phi = \varepsilon_B^A \otimes \Phi_A^B$ [we say that $\left(\Phi_A^B\right)$ is the matrix of Φ in the basis $\{e_A\}$)] and $\Psi = \varepsilon_D^C \otimes \Psi_C^D$, and we can deduce the rule for calculating $\Psi \cdot \Phi$ with the aid of the components of Ψ and Φ from the multiplication table in $\mathcal{L}(M)$:

$$\varepsilon_D^C \cdot \varepsilon_B^A = \delta_B^C \varepsilon_D^A, \tag{14}$$

namely:

$$\Psi \cdot \Phi = \varepsilon_D^A \otimes \Psi_B^D \wedge \Phi_A^B; \tag{15}$$

i.e., the matrix of the product is the product of the matrices. In addition to having the usual properties in paragraph A), that product is doubly-associative:

$$\Psi \cdot (\Phi \cdot \varphi) = (\Psi \cdot \Phi) \cdot \varphi, \tag{16}$$
$$\Theta \cdot (\Psi \cdot \Phi) = (\Theta \cdot \Psi) \cdot \Phi, \tag{17}$$

in which φ has values in M, while Θ, Ψ, Φ have values in $\mathcal{L}(M)$. Of course, one has a product $\Phi_2 \cdot \Phi_1$ of a form Φ_1 with values in $\mathcal{L}(M, P)$ with a form Φ_2 with values in $\mathcal{L}(P, Q)$, and it will have analogous properties.

D) *Case in which M is a Lie algebra L.* – Since the bracket of the algebra L is a bilinear function with values in L, paragraph A) will permit one to extend the bracket operation to forms with values in L. If:

$$\Phi = e_A \otimes \Phi^A \quad \text{and} \quad \Psi = e_B \otimes \Psi^B$$

then formula (1) will become:

$$[\Phi, \Psi] = [\varepsilon_A, \varepsilon_B] \otimes (\Phi^A \wedge \Psi^B). \tag{18}$$

Besides the properties in paragraph A), that bracket will enjoy a commutation property:

$$[\Phi, \Psi] = (-1)^{qq'+1} [\Psi, \Phi] \tag{19}$$

and satisfy a generalized Jacobi identity:

$$(-1)^{qq''}[\Phi, [\Psi, \Theta]] + (-1)^{q'q} [\Phi, [\Psi, \Theta]] + (-1)^{q'q''}[\Phi, [\Psi, \Theta]] = 0, \tag{20}$$

in which q, q', q'' are the degrees of Φ, Ψ, Θ, respectively.

In particular, consider $L = \mathcal{L}(M)$. While keeping the notations of the preceding paragraph, one will have:

$$\Phi = \varepsilon_B^A \otimes \Phi_A^B, \qquad \Psi = \varepsilon_D^C \otimes \Psi_C^D,$$
$$[\Phi, \Psi] = [\varepsilon_B^A, \varepsilon_D^C] \otimes (\Phi_A^B \wedge \Psi_C^D).$$

Now, from (14):

$$[\varepsilon_B^A, \varepsilon_D^C] = \delta_B^A\, \varepsilon_D^C - \delta_B^C\, \varepsilon_D^A,$$

so:

$$[\Phi, \Psi] = \varepsilon_B^C \otimes (\Phi_A^B \wedge \Psi_C^A) - \varepsilon_B^A \otimes (\Phi_A^B \wedge \Psi_C^D),$$

or, from (15):

(21) $$[\Phi, \Psi] = \Phi \cdot \Psi - (-1)^{qq'}\, \Psi \cdot \Phi.$$

In particular, if Φ is a form of odd degree then one will have:

(22) $$[\Phi, \Phi] = 2\, \Phi \cdot \Phi.$$

Finally, let h be a fixed representation of the Lie algebra L in the Lie algebra L_1 . In particular, it is a homomorphism of vector spaces and it satisfies the properties in paragraph B). Furthermore, from (10) and (18):

$$h([\Phi, \Psi]) = h([\varepsilon_A, \varepsilon_B]) \otimes \Phi^A \wedge \Psi^B = [h(\varepsilon_A), h(\varepsilon_B)] \otimes \Phi^A \wedge \Psi^B$$
$$= [h(\varepsilon_A) \otimes \Phi^A, h(\varepsilon_B) \otimes \Psi^B];$$

i.e.:

(23) $$h([\Phi, \Psi]) = [h(\Phi), h(\Psi)].$$

E) Finally, let $\omega = e_i \otimes \omega^i$ be a 1-form with values in M. We let $\overset{q}{\wedge}\omega$ denote the q-form with values in $\overset{q}{\wedge} M$ whose restriction to T_x is $\overset{q}{\wedge}\omega_x$. In order to calculate its components, it will suffice to calculate its value for a decomposable q-vector. Now, by the definition of the exterior power of a linear map:

$$\begin{aligned}
\overset{q}{\wedge}\omega\,(\mathcal{T}_1 \wedge \ldots \wedge \mathcal{T}_q) &= \omega(\mathcal{T}_1) \wedge \ldots \wedge \omega(\mathcal{T}_q) \\
&= \omega^{i_1}(\mathcal{T}_1)\, \omega^{i_2}(\mathcal{T}_2) \cdots \omega^{i_q}(\mathcal{T}_q)\, e_{i_1} \wedge \cdots \wedge e_{i_q} \\
&= \frac{1}{q!}\varepsilon_{l_1 \cdots l_q}^{i_1 \cdots i_q}\, \omega^{l_1}(\mathcal{T}_1) \cdots \omega^{l_q}(\mathcal{T}_q)\, e_{i_1} \wedge \cdots \wedge e_{i_q} \\
&= \frac{1}{q!} < \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}, \mathcal{T}_1 \wedge \cdots \wedge \mathcal{T}_q > e_{i_1} \wedge \cdots \wedge e_{i_q} \\
&= \sum_{i_1 < \cdots < i_q} < \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}, \mathcal{T}_1 \wedge \cdots \wedge \mathcal{T}_q > e_{i_1} \wedge \cdots \wedge e_{i_q},
\end{aligned}$$

and from [(5), § **1**], that will show that the components of $\overset{q}{\wedge}\omega$ in the basis $e_{i_1} \wedge \cdots \wedge e_{i_q}$ ($i_1 < \ldots < i_q$) for $\overset{q}{\wedge} M$ are $\omega^{l_1} \wedge \cdots \wedge \omega^{l_q}$; one will then have:

$$\overset{q}{\wedge}\omega = \sum_{i_1 < \cdots < i_q} e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes \omega^{i_1} \wedge \cdots \wedge \omega^{i_q} \tag{24}$$
$$= \frac{1}{q!} e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}.$$

3. – Tensors and tensor-valued forms on a principal fiber space.

We shall use the terminology of A. Lichnérowicz that we shall first recall briefly. Let $H(X, G)$ be a PFS, let M be a vector space, and let $\mathcal{R}$ be a linear representation of G in M. *A tensor on H of type* $\mathcal{R}(G)$ *with values in M* is a continuous function t on H with values in M such that:

$$t(z \cdot g) = \mathcal{R}(g^{-1}) \cdot t(z), \qquad g \in G,\ z \in H.$$

There is a canonical isomorphism between the vector space of tensors on H of type $\mathcal{R}(G)$ with values in M, and the vector space of sections of the FS $M(H)$ that is obtained by modeling M on H, while G operates on M by way of $\mathcal{R}(G)$ (cf., Def. I.2). The section that corresponds to t under that isomorphism is:

$$x \mapsto \alpha(t(z), z), \qquad (pz = z),$$

which is well-defined, since:

$$(t(z \cdot g), z \cdot g) = (\mathcal{R}(g^{-1})\, t(z), z \cdot g) \sim (t(z), z).$$

If $H(X, G)$ is a differentiable PFS, moreover, then we let Θ_h (V_h, resp.) denote the tangent space to H at $h \in H$ (to the fiber H_{ph} at h, resp.). A q-form Λ on H with values in M said to have type $\mathcal{R}(G)$ if:

$$D_g^* \Lambda = \mathcal{R}(G) \cdot \Lambda. \tag{1}$$

It is a tensorial q-form of type $\mathcal{R}(G)$ if one also has:

$$\Lambda(\mathcal{T}) = 0 \qquad \text{whenever} \qquad p\mathcal{T} = 0, \qquad \mathcal{T} \in \overset{q}{\wedge}\Theta_h, \tag{2}$$

in which p represents the q^{th} exterior power of the tangent linear map to p at the point h, which is a simplified notation that will be used systematically in this paragraph for all linear maps and their exterior powers. Consider a tensor over H to be a 0-form, and let $\{U_\alpha\}$ be an open covering of H that is endowed with local sections s_α such that:

(3) $$s_\beta(x) = s_\alpha(x) \cdot g_{\alpha\beta}(x) \qquad \text{for any} \qquad x \in U_\alpha \cap U_\beta,$$

in which $g_{\alpha\beta}$ is a differentiable map into G. One will establish that the local forms (functions, resp.) on X with values in M:

(4) $$\Lambda_\alpha = s_\alpha^* \Lambda$$

are coupled in $U_\alpha \cap U_\beta$ by:

(5) $$\Lambda_\beta = \mathcal{R}(g_{\alpha\beta}^{-1}) \cdot \Lambda_\alpha,$$

and conversely, a family of local forms Λ_α that satisfy (5) will determine a unique form Λ on H in M of type $\mathcal{R}(G)$ by the single condition (4). That property consists of the remark that a tensorial form is well-defined by the form that it induces on the submanifolds that are transversal to the fibers that constitute the images of the sections. One sees, more generally, that it is well-defined by the tensorial form that it induces on a PFS $H' \cap H$.

Let f be a X-homomorphism of $H'(X, G')$ in $H(X, G)$ that is compatible with a homomorphism $\rho : G' \to G$. $f^*\Lambda$ is a tensorial q-form on H' with values in M and type $\mathcal{R}'(G')$, where $\mathcal{R}' = \mathcal{R} \circ \rho$. Conversely, if one is given a tensorial form Ψ on H of type $\mathcal{R}'(G')$ with values in M then if there exists a linear representation $\mathcal{R}$ of G in M such that $\mathcal{R}' = \mathcal{R} \circ \rho$ (and for that to be true, if ρ is surjective then it will suffice that the kernel of $\mathcal{R}'$ should be contained in that of ρ), there will exist a well-defined tensorial form Λ on H of type $\mathcal{R}(G)$ such that $\Psi = f^*\Lambda$. We say that Ψ is *projectable onto H along* Λ (although H' only projects *into H*).

We shall now specify the notation of the *tensor associated with a tensorial form* Λ. Let E be the PFS of linear frames on the manifold X (cf., Chap. III, § **1**) with structure group $L_n = Gl(n, \mathrm{R})$. $z \in E$ is an isomorphism of R_n onto T_x. Let $(h, z) \in H \boxtimes E$ ($p_H h = p_E z = x$) (Def. I.2.2). Λ defines a linear map $t(h, z)$ of $\overset{q}{\wedge} R_n$ into M by way of:

(6) $$\begin{cases} t(h,z)\cdot u = \Lambda(\mathcal{T}_h), \\ \text{if} \quad \mathcal{T}_h \in \overset{q}{\wedge}\Theta_h, \quad u \in \overset{q}{\wedge} R_n, \quad z\cdot u = p\cdot\mathcal{T}_h. \end{cases}$$

That map is well-defined, since from (2), the value of $\Lambda(\mathcal{T}_h)$ depends upon only that of $p\mathcal{T}_h$, and that p is surjective. Since linearity is obvious, $t(h, z) \in M \otimes \overset{q}{\wedge} R_n^*$. I say that the function t, $(h, z) \mapsto t(h, z)$ is a tensor on $H \boxtimes E$. Indeed, calculate $t(h \cdot g, z \cdot l)$, $l \in L_n$, $g \in G$:

$$\begin{cases} t(h \cdot g, z \cdot l) \cdot u = \Lambda(\mathcal{T}_{hg}), \\ \text{if} \quad u \in \overset{q}{\wedge} R_n, \quad \mathcal{T}_{hg} \in \overset{q}{\wedge} \Theta_{hg}, \quad (z \cdot l) \cdot u = p\mathcal{T}_{hg}. \end{cases}$$

Let $v = l \cdot u \in \overset{q}{\wedge} R_n$, so $z \cdot v = p \cdot \mathcal{T}_{hg} = p \cdot D_g^{-1} \mathcal{T}_{hg}$, in which $D_g^{-1} \mathcal{T}_{hg} \in \Theta_h$; consequently, from (6):

$$t(h, z) \cdot v = \Lambda\left(D_g^{-1} \cdot \mathcal{T}_{hg}\right) = \mathcal{R}(g) \cdot \Lambda(\mathcal{T}_{hg})$$

and

$$\Lambda(\mathcal{T}_{hg}) = \mathcal{R}(g^{-1}) \cdot [t(h, z) \cdot lu] = t(h \cdot g, z \cdot l) \cdot u,$$

or

$$[\mathcal{R}(g^{-1}) \circ t(h, z) \circ l]\, u = t(h \cdot g, z \cdot l) \cdot u$$

for any $u \in \overset{q}{\wedge} R_n$; i.e., upon returning to the complete notations:

$$t(h \cdot g, z \cdot l) = [R(g^{-1}) \otimes \overset{q}{\wedge} l\,] \cdot t(h, z), \tag{7}$$

in which t is a tensor of type $\rho(G \times L_n)$ with $\rho(g, l) = \mathcal{R}(g) \otimes \overset{q}{\wedge} l^{-1}$. One sees immediately with the aid of sections that it is differentiable of the same class as Λ.

One can give a very simple form to the relation between Λ and t. We first remark that the relations (6) are equivalent to:

$$\Lambda(\mathcal{T}_h) = t(h, z) \cdot z^{-1} \cdot p \cdot \mathcal{T}_h, \quad \text{in which} \qquad p = p_H .$$

Let f (g, resp.) be the natural projection of $H \boxtimes E$ onto H (E, resp.). One obviously has $p_H \circ f = p_E \circ g$. $\Psi = f^{*}\Lambda$ is a tensorial form on $H \boxtimes E$ that is projectable onto H whose givens are equivalent to those of Λ.

$$\Psi(\mathcal{T}_{(h,z)}) = \Lambda(f(\mathcal{T}_{(h,z)}) = t(h, z) \cdot z^{-1} p_H (f(\mathcal{T}_{(h,z)}) = t(h, z) \cdot z^{-1} p_E\, g(f(\mathcal{T}_{(h,z)})) .$$

Now, $z^{-1}p_H$ is the fundamental 1-form θ on E (cf., Chap. III.**2**) and:

$$z^{-1} p_E\, g(f(\mathcal{T}_{(h,z)}) = \langle g^{*}\theta, \mathcal{T}_{(h,z)} \rangle,$$

in such a way that:

$$\Psi(\mathcal{T}_{(h,z)}) = t(h, z) \cdot \langle g^{*}\theta, \mathcal{T}_{(h,z)} \rangle = \langle t \cdot g^{*}\theta, \mathcal{T}_{(h,z)} \rangle,$$

and from the relation (9), § **2**, which expresses that $\Psi = t \cdot g^*\theta$. Upon returning to the complete notations, we state the:

DEFINITION II.3. – *If* Λ *is a* q*-form on the PFS* $H(X, G)$ *of type* $\mathcal{R}(G)$ *with values in* M *then the associated tensor to* Λ *– namely,* $t\,\Lambda$ *– will be the tensor on* $H \boxtimes E$ *with values in* $M \otimes \overset{q}{\wedge} \mathrm{R}^{n*}$ *and of type* $\rho(G \times L_n)$*, in which* $\rho(g, l) = \mathcal{R}(g) \otimes \overset{q}{\wedge} l^{-1}$*, which is defined uniquely by the relation:*

$$f^*\Lambda = (t\,\Lambda) \cdot \left(\overset{q}{\wedge} g^*\theta\right). \tag{8}$$

From formula (24), § **2**, the components of the three forms that enter into (8) in a basis $\{e^A\}$ of M, the canonical basis for R^n, and the associated bases in the other space, are coupled by the explicit formulas:

$$\begin{aligned}(f^*\Lambda)^A &= \sum_{i_1<\cdots<i_q} (t\Lambda)^A_{i_1\cdots i_q}\,((g^*\theta)^{i_1} \wedge \cdots \wedge (g^*\theta)^{i_1}) \\ &= \frac{1}{q!}(t\Lambda)^A_{i_1\cdots i_q}\, g^*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_1}),\end{aligned}$$

in the second line of which $(t\Lambda)^A_{i_1\cdots i_q}$ is antisymmetric in $i_1, i_2, \ldots, i_q$.

In the right-hand side of (8), one finds the product of a tensorial q-form by a tensorial 0-form, and that product will be a tensorial q-form. More generally:

PROPOSITION II.3.1. – *If* φ *is a tensorial form of type* $\rho(G)$ *with values in the vector space* M *and* Φ *is a tensorial form with values in* $\mathcal{L}(M, P)$ *of type* $\mathcal{I}(G)$*, where* $\mathcal{I}(g) = \mathcal{R}(g) \otimes \rho(g^{-1})$*, then the form* $\Phi \cdot \varphi$ *will be a tensorial form of type* $\mathcal{R}(G)$ *with values in* P*.*

Indeed, from formula (5), § **2**:

$$D_g^*(\Phi \cdot \varphi) = (D_g^*\Phi) \cdot (D_g^*\varphi) = (\mathcal{I}(g^{-1}) \cdot \Phi) \cdot (\rho(g^{-1}) \cdot \varphi).$$

Now, $\mathcal{I}(g^{-1}) \cdot \Phi = \mathcal{R}(g^{-1}) \cdot \Phi \cdot \rho(g)$ [which is a product of the constant 0-form $\rho(g)$ with values in $\mathcal{L}(M)$ by the form Φ with values in $\mathcal{L}(M, P)$, times the constant 0-form $\mathcal{R}(g^{-1})$ with values in $\mathcal{L}(P)$, and from [C], § **2**], that product is associative], and by associativity:

$$D_g^*(\Phi \cdot \varphi) = (\mathcal{R}(g^{-1}) \cdot \Phi \cdot \rho(g)) \cdot \rho(g^{-1}) \cdot \varphi = \mathcal{R}(g^{-1}) \cdot (\Phi \cdot \varphi),$$

so $\Phi \cdot \varphi$ has type $\mathcal{R}(G)$. In order to show that:

$$< \Phi \cdot \varphi, \mathcal{T}_h > = 0,$$

moreover, for any $\mathcal{T}_h \in \Theta_h$ such that $p\mathcal{T}_h = 0$, it will suffice to show that this is true for decomposable $\mathcal{T}_h$, and that will be obvious for the components $\Phi^a_A \wedge \varphi^A$ of $\Phi \cdot \varphi$ when one expresses them in a basis for Θ_h whose first vectors generate V_h.

Proposition (II.3.1) contains a converse to the definition of the associated tensor:

PROPOSITION II.3.2. – While preserving the notations of Definition (II, § **3**), *if λ is a tensor on $H \boxtimes E$ with values in $M \otimes \overset{q}{\wedge} \mathrm{R}^{n*}$ and type $\rho\,(G \times L_n)$ then there will exist a unique tensorial q-form of type $\mathcal{R}\,(G)$ with values in M whose associated tensor is λ.*

Indeed, it is immediate that $\overset{q}{\wedge} g^*\theta$ is a tensorial q-form on $H \boxtimes E$ with values in $\overset{q}{\wedge} \mathrm{R}^{n*}$ and type $\rho_1\,(G \times L_n)$, in which $\rho_1\,(g, l) = \overset{q}{\wedge} l$. If one lets $\mathcal{R}'\,(G \times L_n)$ denote the representation in M such that $\mathcal{R}'\,(g, l) = \mathcal{R}(g)$ then $\rho\,(g, l) = \mathcal{R}'\,(g, l) \otimes \rho_1\,(g^{-1}, l^{-1})$. One can then apply Proposition (II.3.1) to the form $\psi = \lambda \cdot \left(\overset{q}{\wedge} g^*\theta \right)$, which is a tensorial q-form on $H \boxtimes E$ with values in M of type $\mathcal{R}'\,(G \times L_n)$. One the other hand, since f is a homomorphism of $H \boxtimes E$ into H that is compatible with the trivial homomorphism $\mathcal{I} : G \times L_n \to G$, and $\mathcal{R}' = \mathcal{R} \circ \mathcal{I}$, ψ will be projectable onto H along a tensorial form Λ of type $\mathcal{R}\,(G)$ with values in M, and $\lambda = t\,\Lambda$.

4. – Connections.

A) Let $H\,(X, G)$ be a differentiable PFS. For the time being, H will not be identified with its associated PFS $\hat{H}$ (Chap. I), and let $\hat{z} \in \hat{H}$ be the differentiable homomorphism of G onto H_z :

$$\hat{z} : \quad g \mapsto z \cdot g, \qquad g \in G, z \in H.$$

Its tangent linear map at the identity e of G – namely, $\underline{\hat{z}}$ – is an isomorphism of the Lie algebra $\underline{G}$ of G onto V_z. For $\lambda \in \underline{G}$, we further write $\underline{\hat{z}}(\lambda) = z \cdot \lambda$. Let β be the 1-form *on the fibers* of H (and not on H itself) whose restriction to V_z is the inverse isomorphism $\underline{\hat{z}}^{-1}$. On each fiber, it is a form of type "the adjoint representation of G" (we shall say, moreover, briefly, "adjoint type") :

$$D^*_g\beta = (\mathrm{ad}\, g^{-1}) \cdot \beta.$$

An infinitesimal connection g on H (X, G) is defined by the given of a differential 1-form π on H with values in $\underline{G}$ that has adjoint type and its restriction to the fibers will coincide with β.

The latter condition and a simple dimensional consideration will show that the vector subspace $\mathcal{H}_z = \pi_z^{-1}(0)$ of Θ_z is supplementary to V_z, which then decomposes into a direct sum:

$$\Theta_z = V_z \oplus \mathcal{H}_z .$$

That decomposition defines two projectors in Θ_z :

V : $\Theta_z \to V_z$, which is called the "vertical part,"

$\mathcal{H}$: $\Theta_z \to \mathcal{H}_z$, which is called the "horizontal part."

Following the conventions that were employed before, we shall again let the same letter V ($\mathcal{H}$, resp.) denote the extensions of those operations to $\wedge \Theta_z$.

Finally, the field of planes $z \mapsto \mathcal{H}_z$, or the *connection field*, depends differentiably on z and is invariant under the right-translations by G. Conversely, one shows that a field of planes $\mathcal{H}_z$ that is supplementary to V_z and enjoys the latter properties will define an infinitesimal connection on H.

The connection field defines a Pfaff system on H: If it is completely integrable then the connection γ will be called *integrable*. A path in H that is the image of a segment [0, 1] by a differentiable map is called a *horizontal path* if it is an integral manifold of that system. The holonomy group ψ_z (restricted holonomy group σ_z) at the point $z \in H$ of the connection is the set of $g \in G$ such that $z \cdot g$ is connected with z by a horizontal path (a horizontal path whose projection on X is a loop that is homotopic to 0). One knows that σ_z , which is an arcwise-connected subgroup, is an analytic subgroup of G and ([13]) that it is the arcwise-connected component of the identity of ψ_z . ψ_z is then a Lie subgroup of G (cf., Chap. I.5).

One calls the set of all $z' \in H$ that can be joined to a point $z \in H$ by a horizontal path the *holonomy sheet* H'_z *at z*. Let p' be the restriction of p to H'_z.

1. $p'(H'_z) = X$, since X is arcwise-connected and there exists a horizontal path over any path in X.

2. One knows ([13]) that if $z \in H'_z$ then $\psi_{z'} = \psi_z$, so $p'^{-1}(p\, z') = z' \cdot \psi_z$.

3. p' admits local lifts that are differentiable local sections of H. They will appear in the construction of a special local section of H for the connection ([14]), and the

([13]) A. Lichnerowicz [**22**], pp. 65.
([14]) A. Lichnerowicz [**22**], pp. 117.

differentiability of the section is deduced from the differentiability of the solutions of a system of ordinary differentiable equations in regard to initial givens. From Proposition (I.5.2), that will show that H'_z is a differentiable ψ_z-PFSS of H. That will give one an example of a PFSS, but one does not know if it is closed, or even proper, in general.

B) *Comparing connections.* – Let f be an X-homomorphism of $H'(X, G')$ into $H(X, G)$ that is compatible with the homomorphism ρ of G' into G. Let γ' be a connection on H' will the form π', and let $\mathcal{H}'$ be its connection field. Let $\mathcal{H}_z = f\,\mathcal{H}_{z'} \subset \Theta_z$ be a vector subspace at the point $z = f(z') \in H$. It follows from the single relation $p \circ f = p'$ that $\mathcal{H}_z$ will be supplementary to V_z. $\mathcal{H}_z$ is defined for any $z \in f(H')$ and uniquely, since if $z = f(z'_1)$ then one will have:

$$z'_1 = z' \cdot g' \qquad (g' \in G'), \qquad \text{with} \quad f(z' \cdot g') = z = f(z') \cdot \rho(g') = z \cdot \rho(g'),$$

in such a way that $\rho(g') = e$ and $D_{\rho(g')}$ = identity on H. Hence, $\mathcal{H}'_{z'_1} = D_{g'}\mathcal{H}'_{z'}$, since $\mathcal{H}'$ (viz., the connection field) is invariant under right-translation on H, and:

$$f\,\mathcal{H}'_{z'_1} = (f \circ D_{g'})\mathcal{H}'_{z'} = (D_{\rho(g')} \circ f)\,\mathcal{H}'_{z'} = f\,\mathcal{H}'_{z'}.$$

One likewise shows that the field $\mathcal{H}$ on $f(H') \subset H$ is invariant under right-translation by $\rho(G') \subset G$. It then extends by right-translation to a field that is defined on any H that we will again denote by $\mathcal{H}$. That field, which is invariant under right-translation, by construction, depends differentiably of z (one sees this with the aid of a local chart on H'): It then defines a connection $\gamma = f(\gamma')$ on H that we call the *image of γ' under the homomorphism f*. Let $\tilde{\rho}$ be the representation of the Lie algebra $\underline{G}'$ in $\underline{G}$ that is defined by ρ (tangent linear map at the point e'), and let π be the form of connection γ. One easily sees that:

$$f^{*}\pi = \tilde{\rho}(\pi'), \tag{1}$$

and that relation characterizes π.

In particular, if $H'(X, G') \subset H(X, G)$ is a G'-PFSS of H then the preceding study will apply to f (ρ, resp.), which is the identity map of H' into H (G' into G, resp.). In either case, we say that $\gamma = f(\gamma')$ is *the extension of the H'-connection γ' to H*, or if no ambiguity is possible, that γ is an *H'-connection*. Formula (1) then expresses the idea that π' is the form that is induced on H' by π.

Now, let f be a G-representation (Chap. I, § **3**) of $H'(X', G)$ in $H(X, G)$ that induces the map $\mu: X' \to X$. One likewise sees that if π is a connection form on H then $\pi' = f^{*}\pi$ will be a connection form on H', since the connection field $\mathcal{H}'$ will then be projectable by f along $\mathcal{H}$.

Finally, if γ_1 and γ_2 are two connections on H (X, G), with forms π_1 and π_2, then $\pi_2 - \pi_1 = u$ will be a tensorial 1-form on H with values in $\underline{G}$ and adjoint type. Conversely, if u is a tensorial 1-form of that type then $\pi_1 + u$ will be a connection form on H.

C) *Absolute differential. Fundamental formulas.* – If Λ is a tensorial q-form on $H(X, G)$ of type $\mathcal{R}$ (G) with values in M then, from formulas [(1), § **3**] and [(12), § **2**], one will get $D_g^* \, d\Lambda = \mathcal{R}$ $(g^{-1}) \cdot d\Lambda$ by exterior differentiation. If γ is a connection on H with field $\mathcal{H}$, and the associated projectors in Θ_z are $\mathcal{H}$ and V then it will be obvious that the $(q + 1)$-form that is defined at the point z :

$$(\nabla\Lambda)_z = (d\Lambda)_z \circ \mathcal{H}$$

is a tensorial $(q + 1)$-form of type $\mathcal{R}$ (G) with values in M. It is the *absolute differential of* Λ.

One establishes (for example, by using a local chart on H) the global expression for $\nabla\Lambda$:

$$\nabla\Lambda = d\Lambda + \tilde{\mathcal{R}}(\pi) \cdot \Lambda, \tag{2}$$

in which $\tilde{\mathcal{R}}$ denotes the representation of $\underline{G}$ in $\mathcal{L}$ (M) that is defined by $\mathcal{R}$. The term $\tilde{\mathcal{R}}(\pi) \cdot \Lambda$ then denotes the product (§ **2**) of a form with values in M times a form with values in the space $\mathcal{L}$ (M) of endomorphisms of M.

The *curvature form* Ω of the connection γ on H is the tensorial 2-form with values in $\underline{G}$ and adjoint type:

$$\Omega = d\pi + \tfrac{1}{2}[\pi, \pi] \, . \tag{3}$$

Its absolute differential $\nabla\Omega$ is the tensorial 3-form:

$$\nabla\Omega = d\,\Omega + \tilde{\mathcal{R}}(\pi) \cdot \Omega,$$

in which $\tilde{\mathcal{R}}$ is the adjoint representation of $\underline{G}$. If $\lambda, \mu \in \underline{G}$ then:

$$\tilde{\mathcal{R}}(\lambda) \cdot \mu = [\lambda, \mu],$$

so

$$\tilde{\mathcal{R}}(\lambda) \cdot \mu = [\lambda, \mu], \qquad (\S\ \mathbf{2})$$

and:

$$\begin{aligned} \nabla\Omega &= \tfrac{1}{2} d\,[\pi, \pi] + [\pi, d\pi] + \tfrac{1}{2}[\pi, [\pi, \pi]] \\ &= \tfrac{1}{2}[d\pi, \pi] + \tfrac{1}{2}[\pi, d\pi] + \tfrac{1}{2}[\pi, [\pi, \pi]] \, . \end{aligned}$$

Now, $d\pi$ has even degree, so $[\pi, d\pi] = -[d\pi, \pi]$ [(19), § **2**], and when the Jacobi identity [(20), § **2**] is applied to three equal forms, it will give $[\pi, [\pi, \pi]] = 0$. One will then have:

$$\nabla\Omega = d\,\Omega + [\pi, \Omega] = 0, \tag{4}$$

which is the *Bianchi identity* for curvature.

If Λ is a tensorial form of type $\mathcal{R}\,(G)$ on H then we shall calculate its second absolute differential $\nabla^2\Lambda = \nabla\,(\nabla\Lambda)$. Since $\nabla\Lambda$ has the same type as Λ, the formula (2) will give $\nabla^2\Lambda = d\,(\nabla\Lambda) + \tilde{\mathcal{R}}(\pi)\cdot\nabla\Lambda$, in which:

$$\begin{aligned} d\,\nabla\Lambda &= d\,(\tilde{\mathcal{R}}(\pi)\cdot\Lambda) = ((d\tilde{\mathcal{R}}(\pi))\cdot\Lambda - \tilde{\mathcal{R}}(\pi)\cdot d\Lambda, && [(6), \S\ \mathbf{2}] \\ &= \tilde{\mathcal{R}}(d\pi)\cdot\Lambda - \tilde{\mathcal{R}}(\pi)\cdot d\Lambda, && [(12), \S\ \mathbf{2}] \end{aligned}$$

and

$$\tilde{\mathcal{R}}(\pi)\cdot\nabla\Lambda = \tilde{\mathcal{R}}(\pi)\cdot d\Lambda + \tilde{\mathcal{R}}(\pi)\cdot(\tilde{\mathcal{R}}(\pi)\cdot\Lambda)\,.$$

Now:

$$\begin{aligned} \tilde{\mathcal{R}}(\pi)\cdot(\tilde{\mathcal{R}}(\pi)\cdot\Lambda) &= (\tilde{\mathcal{R}}(\pi)\cdot\tilde{\mathcal{R}}(\pi))\cdot\Lambda && [(16), \S\ \mathbf{2}] \\ &= \tfrac{1}{2}[\tilde{\mathcal{R}}(\pi)\cdot\tilde{\mathcal{R}}(\pi)]\cdot\Lambda && [(22), \S\ \mathbf{2}] \\ &= \tfrac{1}{2}[\tilde{\mathcal{R}}([\pi,\pi])\cdot\Lambda\,. && [(23), \S\ \mathbf{2}] \end{aligned}$$

One will then have:

$$\begin{aligned} \nabla^2\Lambda &= \left(\tilde{\mathcal{R}}(d\pi)\cdot\Lambda - \tilde{\mathcal{R}}(\pi)\cdot\Lambda\right) + \left(\tilde{\mathcal{R}}(\pi)\cdot d\Lambda + \tfrac{1}{2}\tilde{\mathcal{R}}([\pi,\pi])\cdot\Lambda\right) \\ &= \tilde{\mathcal{R}}(d\pi)\cdot\Lambda + \tfrac{1}{2}\tilde{\mathcal{R}}([\pi,\pi])\cdot\Lambda \\ &= \left(\tilde{\mathcal{R}}(d\pi)\cdot\Lambda + \tfrac{1}{2}\tilde{\mathcal{R}}([\pi,\pi])\right)\cdot\Lambda && [(3), \S\ \mathbf{2}] \\ &= \tilde{\mathcal{R}}\left(d\pi + \tfrac{1}{2}[\pi,\pi]\right)\cdot\Lambda\ , && [(4), \S\ \mathbf{2}] \end{aligned}$$

so finally:

$$\nabla^2\Lambda = \tilde{\mathcal{R}}(\Omega)\cdot\Lambda\,. \tag{5}$$

We shall now calculate the absolute differential of the tensorial form $\Phi\cdot\varphi$ that was defined in Proposition (II.3.1). Φ is a form with values in $\mathcal{L}\,(M, P)$ of type $\mathcal{I}\,(G)$, in which $\mathcal{I}\,(g) = \mathcal{R}\,(g)\otimes\rho\,(g^{-1})$, and:

$$\nabla\Phi = d\,\Phi + \tilde{\mathcal{I}}(\pi)\cdot\Phi. \tag{6}$$

If $\{\varepsilon_\rho\}$ ($\{h_\alpha\}$, resp.) is a basis for $\underline{G}$ [$\mathcal{L}\,(M, P)$, resp.] then one will have:

$$p = \varepsilon_\rho\otimes\pi^\rho, \quad \tilde{\mathcal{I}}(\pi) = \tilde{\mathcal{I}}(\varepsilon_\rho)\otimes\pi^\rho, \quad \Phi = h_\alpha\otimes\Phi^\alpha,$$

so:

(7) $$\tilde{\mathcal{I}}(\pi)\cdot\Phi = \tilde{\mathcal{I}}(\varepsilon_\rho)\cdot h_\alpha \otimes \pi^\rho \wedge \Phi^\alpha,$$

in which the forms π^ρ and Φ^α are scalar forms. One will then be reduced to calculating $\tilde{\mathcal{I}}(\lambda)\cdot h$ [$\lambda \in \underline{G}$, $h \in \mathcal{L}(M, P)$]. By definition, for $u \in$ R:

$$\tilde{\mathcal{I}}(\lambda)\cdot h = \lim_{u\to 0}\frac{1}{u}\,(\mathcal{I}(\exp \lambda u)\cdot h - h).$$

Now:

$$\tilde{\mathcal{I}}(g)\cdot h = \mathcal{R}(g)\cdot h\cdot \rho(g^{-1}),$$

so

$$\begin{aligned}\mathcal{I}(\exp \lambda u)\cdot h &= \mathcal{R}(\exp \lambda u)\cdot h\cdot \rho(\exp(-\lambda u))\\ &= \exp \mathcal{R}(\lambda u)\cdot h\cdot \exp(\tilde{\rho}(-\lambda u))\\ &= [[\tilde{\mathcal{R}}(0)+u\,\tilde{\mathcal{R}}(\lambda)+\cdots]\cdot h\cdot[\tilde{\rho}(0)-u\,\tilde{\rho}(\lambda)+\cdots]\\ &= h+u[\tilde{\mathcal{R}}(\lambda)\cdot h - h\cdot\tilde{\rho}(\lambda)]+\cdots,\end{aligned}$$

and finally, $\tilde{\mathcal{I}}(\lambda)\cdot h = \tilde{\mathcal{R}}(\lambda)\cdot h - h\cdot\tilde{\rho}(\lambda)$. Upon referring this to (7), and then (6), if Φ has degree p then one will get:

$$\begin{aligned}\tilde{\mathcal{I}}(\pi)\cdot\Phi &= [\tilde{\mathcal{R}}(\varepsilon_\rho)\cdot h_\alpha - h_\alpha\cdot\tilde{\rho}(\varepsilon_\rho)]\otimes \pi^\rho\wedge\Phi^\alpha\\ &= \tilde{\mathcal{R}}(\pi)\cdot\Phi - [h_\alpha\cdot\tilde{\rho}(\varepsilon_\rho)]\otimes(-1)^p\,\Phi^\alpha\wedge\pi^\rho;\end{aligned}$$

i.e.:

$$\tilde{\mathcal{I}}(\pi)\cdot\Phi = \tilde{\mathcal{R}}(\pi)\cdot\Phi - (-1)^p\,\Phi\cdot\tilde{\rho}(\pi),$$

so

$$\nabla\Phi = d\,\Phi + \tilde{\mathcal{R}}(\pi)\cdot\Phi - (-1)^p\,\Phi\cdot\tilde{\rho}(\pi).$$

On the other hand, since:

$$\nabla\varphi = d\varphi + \tilde{\rho}(\pi)\cdot\varphi,$$

one will have:

$$\begin{aligned}\nabla\Phi\cdot\varphi + (-1)^p\,\Phi\cdot\nabla\varphi &= d\,\Phi\cdot\varphi + (-1)^p\,\Phi\cdot d\varphi + \tilde{\mathcal{R}}(\pi)\cdot\Phi\cdot\varphi\\ &= d\,(\Phi\cdot\varphi) + \tilde{\mathcal{R}}(\pi)\cdot(\Phi\cdot\varphi),\end{aligned}$$

namely:

(8) $$\nabla(\Phi\cdot\varphi) = \nabla\Phi\cdot\varphi + \bar{\Phi}\cdot\nabla\varphi,$$

if Φ is not supposed to be homogeneous.

5. – Complex vectorial forms.

Let N be a real vector space, let N^{C} be its complexification, let M be a complex vector space, and let all of them be finite-dimensional. If f is a linear map (over C) of N^{C} into M, so $f \in M \otimes_{\mathrm{C}} (N^{\mathrm{C}})^*$, then its restriction $\underline{f}$ to $N \subset N^{\mathrm{C}}$ will be a linear map (over R) into M : $\underline{f} \in M \otimes_{\mathrm{R}} N^*$: Conversely, if $g \in M \otimes_{\mathrm{R}} N^*$ then it will extend by linearity to complex numbers into a C-linear map $\bar{g}$ of N^{C} into M. Any $u \in N^{\mathrm{C}}$ can be written $u = x + iy$ ($x, y \in N$; $i = \sqrt{-1}$), so it will suffice to set $\bar{g}(u) = g(x) + i\, g(y)$ and $\bar{g} \in M \otimes_{\mathrm{C}} (N^{\mathrm{C}})^*$. Hence, the map $g \mapsto \bar{g}$ will be a canonical isomorphism of vector spaces over C that takes $M \otimes_{\mathrm{R}} N^*$ to $M \otimes_{\mathrm{C}} (N^{\mathrm{C}})^*$. If one takes M to be the complex field then one will find that the space $\mathrm{C} \otimes_{\mathrm{R}} N^*$ of forms on N with complex values is canonically isomorphic to the dual $(N^{\mathrm{C}})^*$ of N^{C}.

If $N = \wedge\, T_x$ then $N^{\mathrm{C}} = \wedge\, T_x^{\mathrm{C}}$ (T_x is the tangent vector space to the manifold V at x). Let φ_x be a (real) exterior form at x with values in M. It is written $\varphi_x = e_j \otimes \varphi_x^j$ ($j = 1, 2, \ldots, 2p$), where $\{e_j\}$ is a basis on M over R, and the φ_x^j are exterior forms with real values. Now, let $\bar{\varphi}_x$ be the extension of φ_x to T_x^{C}, and let $\{e_A\}$ be a basis on M over C ($A = 1, 2, \ldots, p$). Since $\bar{\varphi}_x \in M \otimes_{\mathrm{C}} (\wedge T_x^{\mathrm{C}})^*$, one will have $\bar{\varphi}_x = e_A \otimes \varphi_x^A$, where the φ_x^A are (complex) exterior forms on T_x^{C}. When considered to be the restriction of $\bar{\varphi}_x$ to T_x, φ_x can then be written as $\varphi_x = e_A \otimes \underline{\varphi_x^A}$, where the $\underline{\varphi_x^A}$, which are restrictions of the φ_x^A to T_x, are exterior forms in T_x with complex values. From now on, we shall identify φ_x and $\bar{\varphi}_x$ (φ_x^A and $\underline{\varphi_x^A}$, resp.), in such a way that *an exterior form with complex values (in M) at the point x will be written:*

$$\varphi_x = e_A \otimes \varphi_x^A, \tag{1}$$

in which the φ_x^A are exterior forms on T_x with complex values, and φ_x can be interpreted as either a (real) linear map of $\wedge\, T_x$ into M or a (complex) linear map of $\wedge T_x^{\mathrm{C}}$ into M.

Finally, if φ is an exterior differential form on V with values in M then one can again write:

$$\varphi_x = e_A \otimes \varphi^A, \tag{2}$$

in which the φ^A are exterior differential forms with complex values. As in section 1, that notation signifies simply that the restriction of φ to T_x is given by (1).

With these clarifications, all of the operations that were studied in sections 1 and 2 can be expressed formally with the aid of complex components that were defined by (2) just as they were expressed in the real case.

One likewise has the equivalent of Definition (II.3) and Proposition (II.3.2):

PROPOSITION II.5. – *Let* Λ *be a tensorial q-form on H* (*X*, *G*) *with values in a complex vector space M and type* $\mathcal{R}$ (*G*). The complex tensor that is associated with Λ (viz., $t\,{}^{\mathrm{C}}\Lambda$) *is the tensor on* $H \boxtimes E$ *with values in* $M \otimes_{\mathrm{C}} \overset{q}{\wedge} \mathrm{C}^{m*}$ *and type* $\rho\,(G \times CL_m)$, *in which* $\rho\,(g, l) = \mathcal{R}\,(g) \otimes \overset{q}{\wedge} \mathrm{C}^{m*}$ *is defined uniquely by:*

$$F^*\Lambda = (t\,{}^{\mathrm{C}}\Lambda)\left(\overset{q}{\wedge} G^*\theta^{\mathrm{C}}\right). \tag{2}$$

Conversely, if $t\,{}^{\mathrm{C}}\Lambda$ *is a given tensor of that type then it will be the tensor that is associated with a well-defined form* Λ *on H by* (2).

In that statement, E^{C} is the space of complex frames on X (Chap. III, § **1**), and θ^{C} is its fundamental form. F (G, resp.) is the canonical map of $H \boxtimes E^{\mathrm{C}}$ onto H (E^{C}, resp.). We remark that if Λ defines a real tensorial form with values in the real vector space that is subordinate to M then $t\,\Lambda$ will be a tensor on $H \boxtimes E$ with values in $M \otimes_{\mathrm{R}} \overset{q}{\wedge} \mathrm{R}^{m*}$. One sees that $t\Lambda$ is the restriction of $t\,{}^{\mathrm{C}}\Lambda$ to $H \boxtimes E \subset H \boxtimes E^{\mathrm{C}}$, which is meaningful, since there is a canonical isomorphism of $M \otimes_{\mathrm{C}} \overset{q}{\wedge} \mathrm{C}^{m*}$ onto $M \otimes_{\mathrm{R}} \overset{q}{\wedge} \mathrm{R}^{m*}$.

CHAPTER III

SPACES OF FRAMES. *G*-STRUCTURES.

1. – Spaces of real or complex frames.

Let $T = T(X, \mathrm{R}^m)$ be the FS of tangent vectors to the differentiable manifold X of class C^s. The associated PFS $E = E(X) = \hat{T}(X, L_m)$ is a PFS that is differentiable of class C^{s-1}. Let $z \in E$ be a frame (cf., I.3) at the point $x \in X$ of the fiber structure on T; i.e., an isomorphism of vector spaces of R^m on T_x ($x = p\,z$). z can be identified with the image $\{e_i\}$ of the canonical basis $\{f_i\}$ on R^m under z, in such a way that E will be identified with the space of bases of the vector space T_x ($x \in X$). We shall utilize both interpretations. E will be called the *space of real linear frames in X*, or more simply, the *space of frames in X*.

The inverse homomorphism $\varphi = z^{-1}$, $T_x \in \mathrm{R}^m$ is a *coframe* at x. It is a 1-form at the point x with values in R^m (cf., II.1). Its components φ^i in the canonical basis for R^m are the scalar 1-forms on T_x that are defined by:

$$\varphi = f_j \otimes \varphi^j. \tag{1}$$

Hence, $f_i = z^{-1}(z \cdot f_i) = \langle \varphi, e_i \rangle = f_j \langle \varphi^j, e_i \rangle$, so $\langle \varphi^j, e_i \rangle = \delta_i^j$; i.e., the m forms φ^j are linearly independent and constitute the basis for T_x^* that is dual to the basis $\{e_i\}$, which is a basis that one can then identify with φ, and for that reason, it will be called the coframe that is *dual to the frame z*. Conversely, a basis $\{\varphi^j\}$ for T_x^* will determine a coframe by (1), and the inverse frame $z = \varphi^{-1}$ will be the *dual frame to* φ.

Let T_x^{C} be the complexification of T_x and $T^{\mathrm{C}} = \bigcup_{x\in X} T_x^{\mathrm{C}}$; let E_x^{C} be the set of bases (over C) of T_x^{C} and $E^{\mathrm{C}} = \bigcup_{x\in X} E_x^{\mathrm{C}}$. Any basis for T_x is a basis for T_x^{C} over C, in such a way that $E_x^{\mathrm{C}} \supset E_x$ and $E^{\mathrm{C}} \supset E$. The group CL_m acts on the right on each E_x^{C} by way of:

$$z = \{e_A\} \in E_x^{\mathrm{C}}, \qquad l = (l_B^A) \in CL_m \mapsto \{e_A\, l_B^A\} = z \cdot l \in E_x^{\mathrm{C}},$$

so one sees immediately (Remark I.5) that E^{C} is naturally endowed with the structure of a PFS $E^{\mathrm{C}}(X, CL_m)$ for which E is an L_m-PFSS of E^{C}.

Let α be the canonical projection of $\mathrm{C}^m \times E^{\mathrm{C}}$ onto the model space $\mathrm{C}^m(E^{\mathrm{C}})$ (cf., Definition I.2), while CL_m acts naturally on C^m. It follows from the inclusion $\mathrm{R}^m \times E \subset \mathrm{C}^m \times E^{\mathrm{C}}$, where E is a PFSS of E^{C} and C^m is the complexification of R^m, that on the one hand, $\alpha(\mathrm{R}^m \times E) = T$ and on the other hand, the fiber of $\mathrm{C}^m(E^{\mathrm{C}})$ at x is the complexification of the fiber T_x of T; i.e., that $\mathrm{C}^m(E^{\mathrm{C}}) = T^{\mathrm{C}}$, which then has a fiber structure $T^{\mathrm{C}}(X, CL_m, \mathrm{C}^m)$. Since CL_m is effective on C^m, the associated PFS $\widehat{T^{\mathrm{C}}}$ is nothing but E^{C}, and any $z \in E^{\mathrm{C}}$ will be identified with an isomorphism of C^m onto T_x^{C}. That is why $E^{\mathrm{C}} = E^{\mathrm{C}}(X)$ will be called the *space of complex frames on X*.

The inverse isomorphism $\varphi = z^{-1}$, $T_x^{\mathrm{C}} \to \mathrm{C}^m$, will be further called the *complex coframe at x that is dual to z*. It is a 1-form on T_x^{C} with values in the complex vector space C^m. It is then identified (cf., II.5) with a linear map of T_x into C^m, and if $\{f_j\}$ is the canonical basis C^m then it can be further written:

$$\varphi = f_j \otimes \varphi^j, \tag{1}$$

in which this time the φ^j are forms on T_x with complex values. The same calculation as in the real case will show that those forms are linearly independent over the complex numbers. Conversely, m linear forms at x with complex values that are linearly-independent over C will determine a coframe φ by way of (1) whose inverse frame is the *frame that is dual to* φ.

If h is a differentiable local section of E (E^{C}, resp.) over an open subset U, and $\theta_x = h(x)^{-1}$ is the coframe that is dual to $h(x)$ then, by abuse of language, the 1-form on U with values in R^m (C^m, resp.) whose restriction to the point x is θ_x will be called the *coframe on U that is dual to h*. Its components θ^j ($j = 1, 2, \ldots, m$) are m real Pfaff forms (with complex values, resp.) that are linearly-independent over R (C, resp.) on any U, and conversely, m such forms will be the components of a coframe on U.

In particular, if $h(x)$ is the natural frame at x of a system of local coordinates x^i ($i = 1, 2, \ldots, m$) on U then the dual coframe θ on U will have components dx^i, in such a way that $d\theta = 0$. Conversely, a coframe θ on U such that $d\theta = 0$ is locally a natural coframe for the local coordinates. Those remarks extend to frames and complex coframes upon calling a a system of m differentiable functions with complex values on $U \subset X$ that are independent over the complex numbers a *local system of complex coordinates* on X.

DEFINITION III.1. – *We call any differentiable principal fiber subspace for the space E of linear frames (the space* E^{C} *of complex frames, resp.) on X the* ***space of frames (space of complex frames****, resp.)* ***on the differentiable manifold*** *X. One calls the structure S = S (G, H) that is determined by the given of a space of frames H (space of complex frames, resp.) on X with structure group G a G-****structure*** *(****complex*** *G-****structure****, resp.).*

G is then a Lie subgroup of L_m (CL_m, resp.). S is said to have class C^r if H is a PFSS of E (E^{C}, resp.) of class C^r. $z \in H_x$ ($x \in X$) can be called a *distinguished frame on X at x for the structure S*, or more briefly, a *distinguished frame of S*. The dual coframe to a distinguished frame is a *distinguished coframe*. A G-structure ($H \subset E$) will often be called a *real G-structure*, as opposed to a *complex G-structure* ($H \subset E^{\mathrm{C}}$).

It follows from Proposition (I.5.2.) that H (and S) can be determined by a family $\{U_\alpha, h_\alpha\}$, where $\{U_\alpha\}$ is an open covering of X, and h_α is a local section of E (E^{C}, resp.) over U_α, with:

$$h_\beta(x) = h_\alpha(x) \cdot g_{\alpha\beta}(x) \qquad \text{for} \qquad x \in U_\alpha \cap U_\beta, \tag{2}$$

in which $g_{\alpha\beta}$ is a differentiable function on $U_\alpha \cap U_\beta$ with values in G. Hence, if θ_α is the coframe on U_α that is dual to h_α then one will have:

$$\theta_\alpha = g_{\alpha\beta} \cdot \theta_\beta \tag{3}$$

in $U_\alpha \cap U_\beta$, with the notations of Chapter II. Conversely, let $\{U_\alpha, \theta_\alpha\}$ be a family, where $\{U_\alpha\}$ is a covering of X, and let θ_α be a coframe on U_α, and let those coframes be coupled in $U_\alpha \cap U_\beta$ by (3). That will determine a G-structure on X.

The latter way of determining a G-structure is the most conventional way of determining one locally; cf., S. S. Chern [**9**]. A large number of more-or-less classical G-structures in differential geometry can be determined by the given of a G-structure; one will find some examples below. The monograph by P. Libermann [**20**] includes an abundant list.

First examples. –

a) The PFS of orthonormal frames on a Riemannian manifold X defines an $O(m)$-structure, and conversely.

b) *The complex (real, resp.) almost-product structures* [or π-structures (π_{R}-structures, resp.)]. – These were envisioned by D. C. Spencer [**25**] and studied in detail by G. Legrand [**18**] and independently by the author. If dim $X = m = n_1 + n_2$ then a π-structure (π_{R}-structure, resp.) on X is defined when one is given two fields of complex (real, resp.) vectors subspaces T_i of T_x^{C} (T_x, resp.) that have dimension n_i ($i = 1, 2$) and are supplementary. The bases for T_x^{C} (T_x, resp.) whose first n_1 vectors belong to T_1 and whose following n_2 vectors belong to T_2 constitute a space of complex (real, resp.) frames with structure group $CL(n_1, n_2)$ [$L(n_1, n_2)$, resp.] (cf., Chap. I, § **6**). Conversely, a $CL(n_1, n_2)$-structure on X will determine a π-structure; a real $L(n_1, n_2)$-structure will determine a π_{R}-structure.

c) A π-structure for which T_2 is the complex conjugate of T_1 (so $n_1 = n_2 = n$, and $m = 2n$) defines an almost-complex structure (cf., [**22**], § **101**). An adapted basis in T_x^{C}, which is composed of a basis $\{\varepsilon_\alpha\}$ ($\alpha, \beta = 1, 2, \ldots, n$) in T_1 and the complex-conjugate basis $\{\varepsilon_{\alpha^*} = \mathcal{T} \cdot \varepsilon_\alpha\}$ ($\alpha^* = \alpha + n$) in T_2, defines a space of frames $E^b(X) \subset E^{\mathrm{C}}$ that has the group CL_m^b of matrices:

$$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}, \qquad \text{where } A \in CL_n, \qquad \bar{A} = \text{complex-conjugate of } A,$$

which is a group that is isomorphic to CL_n, for its structure group. Conversely, a CL_m^b–structure S determines an almost-complex structure if and only if for any $x \in X$ there exists a distinguished frame $\{\varepsilon_i\}$ ($i = 1, 2, \ldots, 2n$) such that $\varepsilon_{\alpha^*} = \mathcal{T} \cdot \varepsilon_\alpha$ for $\alpha = 1, 2, \ldots,$

n. In particular, an almost-complex structure on X is perfectly determined by its space $E^b(X)$.

The *real bases that are adapted* to the almost-complex structure are the bases for T_x that are deduced from the preceding ones by:

$$e_\alpha = \frac{1}{\sqrt{2}}(\varepsilon_\alpha + \varepsilon_{\alpha^*}), \qquad\qquad e_{\alpha^*} = \frac{i}{\sqrt{2}}(\varepsilon_\alpha - \varepsilon_{\alpha^*}).$$

They constitute a *space of real frames* $E^a(X) \subset E$ whose structure group CL_n^a is the real representation of CL_n in L_{2n} ; i.e., the group of matrices:

$$\begin{pmatrix} B & C \\ -C & B \end{pmatrix}, \qquad B, C \text{ are real matrices with } \quad B + iC = A \in CL_n\,.$$

Conversely, a real CL_n^a–structure on X will determine an almost-complex structure.

d) Let $X = G / H$ be a homogeneous space of the Lie group G, let p be its canonical projection, and let E be the PFS of frames on X. If K_g denotes the action of $g \in G$ on X then that action will prolong to E: If $z \in E_x$, then $\underline{K_g} \circ z = gz \in E_{gx}$ ($\underline{K_g}$ is the tangent linear map to K_g). Let z_0 be fixed such that $p_E \cdot z_0 = pe = x_0$. Let $\tilde{H}$ be the linear isotropy group of X at x_0 , and let $\tilde{H}_{z_0} \subset L_m$ be the group $z_0^{-1} \cdot \tilde{H} \cdot z_0$, which is isomorphic to $\tilde{H}$. The map f of G into E, $g \mapsto g \cdot z_0 = f(g)$, is an X-homomorphism of the PFS that is compatible with the homomorphism:

$$\rho : H \to L_m\,, \quad h \in H \mapsto z_0^{-1} \cdot \underline{K_h} \cdot z_0 \in \tilde{H}_{z_0} \subset L_m\,;$$

indeed:

$$f(g \cdot h) = (g \cdot h)\, z_0 = \underline{K_{gh}} \circ z_0 = \underline{K_g} \circ \underline{K_h} \circ z_0\,,$$

namely:

$$f(g \cdot h) = \underline{K_g} \circ z_0 \circ (z_0^{-1} \circ \underline{K_h} \circ z_0) = f(g) \cdot \rho(h)\,.$$

The image $P_{z_0}(X) = f(G)$ is then (Proposition I.5.2.) a $\rho(H)$-PFSS of E ; i.e., a space of frames with group $\tilde{H}_{z_0}$. In particular, if G / H is a reductive homogeneous space then $\tilde{H}$ will be isomorphic to H, ρ and f will be isomorphisms, and G will be isomorphic to $P_{z_0}(X)$:

PROPOSITION III.1. – *If $X = G / H$ is a homogeneous space with Lie group G then it will be naturally endowed with an $\tilde{H}_{z_0}$–structure, where $\tilde{H}_{z_0}$ is the representation of the linear isotropy group $\tilde{H}$ in a frame z_0 on X at the point $x_0 = pe$. The corresponding space* *of frames will be homomorphic to the PFS $G \to G / H$: It will be isomorphic to it if G / H is reductive.*

2. – *G*-structures defined by a tensor.

The first three examples above belong to the same schema. Let $\mathcal{R}$ be a linear representation of L_m in a vector space M and let $G \subset L_m$ be a subgroup that leaves $u \in M$ invariant. On the other hand, let $S(G, H)$ be a G-structure on X. The constant map $H \to u$ is a tensor on H with values in M and type $\mathcal{R}(G)$, which then extends (Chap. II, § **3**) to a tensor t on E with values in M and type $\mathcal{R}(L_m)$: That tensor takes its values in the intransitivity class M_u of u for $\mathcal{R}(L_m)$, since if $z \in E$ then there will exist $z' \in H$ such that $z = z' \cdot l$ ($l \in L_m$), and one will then have:

$$t(z) = t(z' \cdot l) = \mathcal{R}(l^{-1}) \cdot t(z) = \mathcal{R}(l^{-1}) \cdot u \in M_u .$$

Conversely, suppose that G is *the largest subgroup of* L_m that *leaves* u *invariant* (G will then be closed in L_m), and let t be a tensor on E of type $\mathcal{R}(L_m)$ with values in M_u. Let $H \subset E$ be the set of frames z such that $t(z) = u$.

1. $p(H) = X$, because if $z_1 \in E_x$ then $t(z_1) \in M_u$. There will then exist $l_1 \in L_m$ such that $t(z_1) = \mathcal{R}(l_1) \cdot u$ and $t(z_1 \cdot l_1) = \mathcal{R}(l_1^{-1}) \cdot t(z_1) = u$, in such a way that $z_1 \cdot l_1 \in H$.

2. Let $z, z' \in H_x$, $z' = z \cdot l$ ($l \in L_m$) and $t(z') = \mathcal{R}(l^{-1})\, t(z)$; i.e., $\mathcal{R}(l^{-1})\, u = u$ and $l \in G$. One will then have $H_x = z \cdot G$.

From Proposition (I.5.2), in order for H to be a G-PFSS of E, it is necessary and sufficient, moreover, that E should admit local sections with values on H. Let us analyze that last condition. Let π be the canonical map $L_m \to L_m / G$ and let f be the injection $L_m / G \to M$, $l \cdot G \mapsto \mathcal{R}(l)\, u \in M$ (which is a bijective differentiable map on M_u). f is analytic and everywhere regular, in such a way that M_u will be an analytic submanifold of M. Let $\tilde{M}_u$ denote that submanifold and identify it with L_m / G by way of f in such a way that:

$$\pi(l) = l \cdot G = \mathcal{R}(l) \cdot u . \tag{1}$$

t, which is a differentiable map into M that takes its values in M_u, is not necessarily a differentiable map into $\tilde{M}_u$. Suppose that H is a PFSS of E, and let V be an open subset of X that is endowed with a section z with values in H. For $x \in V$, $l \in L_m$, $t(z(x) \cdot l) = \mathcal{R}(l^{-1}) \cdot u$; i.e., in the chart on E that is associated with the section z, the map t, $E_V \to \tilde{M}_u$ is expressed by:

$$(x, l) \mapsto \pi(l^{-1}),$$

which is a map that is therefore differentiable. In order for H to be a PFSS, it is necessary then that t should be a differentiable map, not only in M, but also in $\tilde{M}_u$. That condition is sufficient. Indeed, let V be an open subset of X that is endowed with a section s of E. $t \circ s = g$ is a differentiable map of V into $\tilde{M}_u$, and if one restricts V in such a fashion that g (V) is included in an open subset of $L_m / G = \tilde{M}_u$ then $l = \sigma \circ g$ will be a differentiable map of V into L_m. $x \mapsto z(x) = s(x) \cdot l(x)$ is a local differentiable section of E over V and:

$$t(z(x)) = t(s(x) \cdot l(x)) = \mathcal{R}(l(x)^{-1}) \cdot t(s(x)) = \mathcal{R}(l(x)^{-1}) \cdot g(x),$$

and since $\pi \circ \sigma$ = identity on L_m / G, $g(x) = \pi(l(x)) = R(l(x))\, u$, from (1), so $t(z(x)) = u$, and z will take its values on H. If one takes Lemma (I.6.1) into account then one will have established:

PROPOSITION III.2. – *Let $\mathcal{R}$ be a linear representation of L_m (CL_m, resp) in a vector space M, and let G be the subgroup of L_m (CL_m, resp.) that leaves $u \in M$ invariant, while M_u is the intransitivity class of u by L_m (CL_m, resp.), which is endowed with its analytic structure of a homogeneous space L_m / G. Being given a G-structure on V_m is equivalent to being given a tensor on E (V_m) [E^C (V_m), resp.] of type $\mathcal{R}$ (L_m) [$\mathcal{R}(CL_m)$, resp.] with values in M_u, provided that t is a differentiable map into M_u (**and not just in** M). The latter condition is always realized if M_u is a proper submanifold of M, and in particular, if L_m / G is compact.*

Except for the compact case, the problem of the existence of distinguished local sections is always well-posed then. Recall the examples of the preceding paragraph.

a) Let M be the space of bilinear forms on R^m, and let u be the bilinear form $x, y \in \mathrm{R}^m \mapsto \sum_{i=1,\dots,m} x^i y^i$ whose matrix in the canonical basis for R^m is the identity matrix: One will then have $G = O(m)$, while M_u will be the set of symmetric, positive-definite, bilinear forms. t is the "metric tensor" that defines the Riemannian structure on V_m that is associated with the $O(m)$-structure. If the differentiable tensor t with values in M is given then the existence of orthonormal local sections is proved directly by constructing such a section by starting from an arbitrary local section of E and applying a procedure that does not affect its continuity, such as the "Schmidt orthogonalization procedure."

b) Let $M = \mathcal{L}(\mathrm{C}^m)$, and let $\mathcal{R}(l)$ ($l \in CL_m$) be the canonical transformation $h \in M \mapsto \mathcal{R}(l) \cdot h = l^{-1} \cdot h \cdot l$, while $CL(n_1, n_2)$ is the subgroup of CL_m that leaves the matrix:

$$u = \begin{pmatrix} E_{n_1} & 0 \\ 0 & -E_{n_2} \end{pmatrix}$$

invariant. M_u is the set of automorphisms of C^m whose square is the identity, and whose space of proper vectors that correspond to the proper value + 1 is n_1-dimensional; t is the tensor on $E^{\mathrm{C}}(V_m)$ that defines an automorphism of T_x^{C} at each point $x \in V_m$ whose square is the identity. Being given t is equivalent to being given a π-structure ([15]).

c) An almost-complex structure that is determined by a real CL_n^a–structure can be defined in an analogous fashion: $M = \mathcal{L}(\mathrm{R}^{2n})$; $\mathcal{R}(l)$ ($l \in CL_m$) is once more the canonical transformation. CL_n^a is then the subgroup of L_{2n} that leaves the matrix:

$$u = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

invariant. M_u is the set of automorphisms of R^{2n} with the identity for their squares; t is the almost-complex tensor.

3. – Equivalent and subordinate *G*-structures.

A)

DEFINITION III.3.1. – *Let $S = S(G, H)$ be a (real or complex) structure. A structure $S' = S'(G', H')$ is said to be* ***equivalent to*** *S if there exists $l \in L_m$ (CL_m, resp.) such that $H' = H \cdot l$. A complex S is said to be* ***equivalent to a real one*** *if it admits an equivalent real structure. If S is real (complex, resp.) then an S'-structure that is equivalent to S ($H' = H \cdot l$) will once more be a real (complex, resp.) G-structure if and only if l belongs to the normalizer of $N(G)$ ($N^{\mathrm{C}}(G)$, resp.) of G in L_m (CL_m, resp.): One will then say that S is a* ***real (complex, resp.)*** *G-structure that is* ***associated with*** *S.*

The structure S of the frame space $H' = H \cdot l$ has a structure group that is conjugate to G in CL_m, namely, $G = l^{-1} \cdot G \cdot l$, because if $z \in H_x$, $H_x = z \cdot G$ then $= z \cdot G \cdot l = z \cdot l\ (l^{-1} \cdot G \cdot l)$ ($x \in X$). In order to have $G' = G$, it will then be necessary and sufficient that $l \in N(G)$ [$N^{\mathrm{C}}(G)$, resp.].

We see in the examples above and in Chapter IV that equivalent structures must be considered to be things that define the same infinitesimal structure on X. If a class $\mathcal{C}$ of conjugate subgroups of L_m is given then one can call the set of all G'-structures that are equivalent to a given G-structure ($G, G' \in \mathcal{C}$) a *$\mathcal{C}$-structure*. Each of the G'-structures can

([15]) Cf., G. Legrand [**18**].

then be envisioned to be a "representative" of the $\mathcal{C}$-structure. The problem of determining all possible G-structures on X is then solved by Proposition (I.5.3) when the given G is closed in L_m. They correspond bijectively to the differentiable sections of the space E / G. The problem of determining all $\mathcal{C}$-structures can then be posed as follows: Once a representative $G \in \mathcal{C}$ is chosen, as long as $N(G) \neq G$, a given $\mathcal{C}$-structure will admit several representatives that are G-structures, and those structures will be associated. The group $N = N(G) / G$ acts on E / G, as well as on the sheaf $\mathcal{F}$ of germs of differentiable sections of E / G, which one can call the *sheaf of germs of G-structures.* If N is endowed with the discrete topology then the quotient space $\mathcal{F} / N$ will again be a sheaf, and if q is the canonical map $\mathcal{F} \to \mathcal{F} / N$ then in order for two sections of $\mathcal{F}$ to define two associated G-structures, it is necessary and sufficient that they should have the same image under q. One can then call $\mathcal{F} / N$ the *sheaf of $\mathcal{C}$-structures on X*: There is a bijective correspondence between the $\mathcal{C}$-structures on X and the sections of the sheaf that have a lift to $\mathcal{F}$. The same analysis will obviously be valid in the complex case.

Examples. – Recall some of the examples in § **1**, with the same notations.

a) The various $\tilde{H}_{z_0}$-structures that are defined on a homogeneous space G / H (example *d*) are equivalent: If z_0 is replaced by $z_1 = z_0 \cdot l$ ($l \in L_m$) and f is replaced with f_1 then one will have:

$$f_1(g) = \underline{K_g} \circ z_1 = \underline{K_g} \circ z_1 \circ l = f(g) \cdot l,$$

so

$$P_{z_1}(X) = P_{z_0}(X) \cdot l .$$

b) The CL_n^b-structure S^b and the CL_n^a-structure S^a that are defined by an almost-complex structure (example *c*) are equivalent, because if $z = \{\varepsilon_\alpha , \varepsilon_{\alpha^*}\} \in E^b(X)$ is a complex adapted basis and $z' = \{e_\alpha , e_{\alpha^*}\} \in E^a(X)$ is the corresponding real basis then one will have:

$$z' = z \cdot l, \qquad \text{in which} \qquad l = \frac{1}{\sqrt{2}}\begin{pmatrix} E_n & E_n \\ i\,E_n & -i\,E_n \end{pmatrix}.$$

Hence, $E^a(X) = E^b(X) \cdot l$ and $CL_m^a(X) = l^{-1} \cdot CL_m^b(X) \cdot l$, which one can verify immediately. If S^a is real then S^b will be equivalent to real.

c) The existence of G-structures that are not equivalent to real ones is obvious: It will suffice that $\dim G > m^2$ in order for a G-structure to not be equivalent to a real one. Hence, *a π-structure* (example *b*) *will never be equivalent to a real one.*

B)

DEFINITION III.3.2. – *Let S (G, H) and S' (G', H') be two structures: If $H \subset H'$ (hence, $G \subset G'$) then one will say that S is **subordinate to** S' or that S' is an **extension** of S.*

If G' is given then the problem of the existence of a G-structure that is subordinate to S' is solved by Proposition (I.5.3). If one is given another structure S'' (G'', H'') then one can pose the problem of the existence of a structure S that is subordinate to both S' and S'': If $H' \cap H''$ is again a space of frames then it will define such a structure with group $\Gamma = G' \cap G''$ (the largest one). Conversely, if $H \subset H' \subset H''$ defines a common subordinate structure then $H' \cap H'' = H \cdot \Gamma$ will also determine one. Our problem is then reduced to this one: Is $H' \cap H''$ a space of frames? It was in order to solve that problem that we carried out our study in section (I.6). Proposition (I.6.2) and Theorem (I.6) then permit us to state:

THEOREM III.3. – *In order for there to exist a common subordinate structure to a G'-structure S' and a G''-structure S'' over X, it is necessary that they should admit a common distinguished frame at each point $x \in X$. That condition is sufficient if the pair G', G'' is a generic pair of subgroups of L_m (CL_m, resp.); for example, if G' / Γ (G'' / Γ, resp.) is compact ($G = G' \cap G''$). In particular, in order for a complex G-structure S to be the extension of a real structure, it is necessary that S should admit a distinguished real frame at each point. That condition will be sufficient if the pair G, L_m is a general pair of subgroups of CL_m, and in particular, if G / Γ (L_m / Γ is compact).*

The existence condition for a distinguished frame at x that is common to S' and S'', $H'_x \cap H''_x \neq \emptyset$, can be put into the form $H'_x \subset H''_x \cdot G'$; it is realized for any $x \in X$ if:

$$H' \subset H'' \cdot G'. \tag{1}$$

In particular, in order for the complex G-structure S to admit a real distinguished frame at any point, it is necessary and sufficient that:

$$H \subset E \cdot G. \tag{2}$$

If $z_x \in E_x$ then the fiber of $E \cdot G$ at x will be $z_x \cdot L_m \cdot G$, and as long as $L_m \cdot G \neq CL_m$ (for example, $\dim G < m^2$), one can state that *a complex G-structure is not generally the extension of a real structure.* On the contrary, the condition (2) will always be realized if and only if:

$$L_m \cdot G = CL_m .$$

Since that condition implies that the pair L_m, G is a generic pair of subgroups [Theorem (I.6.2), example b], such a G-structure will always be the extension of a real structure.

Examples. –

a) Since $O(m)$ is compact, if an arbitrary structure $S(G, H)$ over X admits a distinguished orthonormal frame at any point then, from the preceding theorem, S will admit a structure that is subordinate to the group $G \cap O(m)$. In the most common cases, that fact can result directly from the manner by which the distinguished orthonormal frame is determined (although the proof is omitted by most authors), and here it follows from a general theorem.

b) An almost-Hermitian structure that is subordinate to an almost-complex structure on X of dimension $2n$ can be determined by a space of frames $\varepsilon^b(X) \subset E^b(X)$ that has a structure group in the form of the group $U^b(n) \subset CL_n^b$ of matrices:

$$\begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}, \qquad A \in U(n),$$

which is isomorphic to $U(n)$, and $U^b(n) = CL_n^b \cap U(2n)$. It can just as well be determined by the space of real frames $\varepsilon^a(X) = \varepsilon^b(X) \cdot l$ [l is the matrix defined in example *b*) of paragraph A)], whose group is $U^a(n) = O(2n) \cap CL_n^a$. It is therefore the largest subordinate structure that is common to the Riemannian structure that is defined by the orthonormal frames $\varepsilon^a(X) \cdot O(2n)$ and the almost-complex structure that is defined by the adapted real frames $\varepsilon^a(X) \cdot CL_n^a = E^a(X)$. Conversely, when one is given a Riemannian metric on X and an almost-complex operator $\mathcal{J}$ such that the operator $\mathcal{J}_x$ at any point x is Hermitian with respect to the metric Φ_x [$\Phi_x(\mathcal{J}_x v, \mathcal{J}_x w) = \Phi_x(v, w)$ for any $v, w \in T_x$], the fact that the space $\varepsilon^a(X)$ of real adapted frames at each point of those two structures is indeed a "space of frames" supposes that one has a proof (which is generally omitted) that there exist local sections of $E(X)$ that are both orthonormal and adapted to the almost-complex structure. Our theorem (which is applicable, since one of the groups is compact) reduces that proof to a proof of the existence of local sections that are orthonormal for the metric Φ, on the one hand, and local sections that are adapted to the almost-complex structure, on the other.

c) In order for a π-structure S to be the extension of a π_R-structure, it is necessary and sufficient that S should admit a distinguished real frame at each point. That is obvious here, because since the field of planes T_i ($i = 1, 2$) is differentiable, as well as the field T_x, the field of planes $T_i \cap T_x$ will also be differentiable. That can also result from Theorem III.3, since the pair $CL(n_1, n_2)$, L_m of subgroups of CL_m is generic (Proposition I.6.4).

C)

DEFINITION III.3.3. – *A G'-structure S' is said to be* ***subordinate in the larger sense*** *to a structure S if it is subordinate to a structure that is equivalent to S (or equivalent to a structure that is subordinate to S). S will then be an* ***extension in the larger sense*** *of S'.*

In particular, let us study the conditions under which a complex structure S (G, H) is an extension in the larger sense of a real structure. In order for that to be true, it is necessary and sufficient that there should exist an $l \in CL_m$ such that $S'(l \cdot G \cdot l^{-1}, H \cdot l^{-1})$ should admit a real subordinate structure. From (1), it is then necessary that there should exist an l such that $H \cdot l^{-1} \subset E \cdot (l \cdot G \cdot l^{-1})$, namely:

$$H \subset E \cdot l \cdot G. \tag{3}$$

That condition is sufficient if the pair L_m , $l \cdot G \cdot l^{-1}$ is a generic pair of subgroups of CL_m .

The sets $E \cdot l \cdot G$, $l \in CL_m$ (double classes modulo $E : G$) define equivalence classes over E^{C} that correspond bijectively to the double classes of CL_m modulo $L_m : G$. If there exist more than one such class (i.e., if $L_m \cdot G \neq CL_m$) then *there will surely exist complex G-structures that do not admit real structures that are subordinate in the larger sense.* In that sense, the complex G-structures constitute a true generalization of the real G-structures.

4. – Characterization of a space of frames by the fundamental 1-form.

DEFINITION III.4.1. – *Let H (X, G) be a space if real (complex, resp.) frames on the m-dimensional manifold X. One says the* ***fundamental 1-form on*** *H to mean the 1-form ω with values in R^m (C^m, resp.) that makes the vector (complex vector, resp.) $\mathcal{T}_z$ that is tangent to H at the point z correspond to the vector:*

$$\omega(\mathcal{T}_z) = z^{-1} \cdot p\, \mathcal{T}_z \in \mathrm{R}^m \qquad (\mathrm{C}^m, \text{resp.}) . \tag{1}$$

The fundamental 1-form on E (E^{C}, resp.) will be denoted by θ (θ^{C}, resp.). The restriction of the form on E that is induced by θ^{C} to the real tangent vectors coincides with θ. If $H \subset E$ (E^{C}, resp.) then the fundamental 1-form ω on H will be the form that is induced by θ (θ^{C}, resp.) on H. When no ambiguity is possible, θ^{C} will be again denoted by θ.

The 1-form ω that was defined in Definition (III.4.1) satisfies the two properties:

$$D_g^* \omega = g^{-1} \cdot \omega, g \in G, \tag{2}$$

$$\omega(\mathcal{T}) = 0 \qquad \Leftrightarrow \qquad p\, \mathcal{T} = 0 \tag{3}$$

($\mathcal{T}$ is a real or complex vector that is tangent to H). Indeed, if $\mathcal{T}_z$ is tangent to H at the point z then $D_g\mathcal{T}_z$ will be tangent to the point $z \cdot g$, and:

$$\omega(D_g\mathcal{T}_z) = (z \cdot g)^{-1}\, p\,(D_g\mathcal{T}_z) = g^{-1}\, z^{-1}\, p\, \mathcal{T}_z = g^{-1}\, \omega(\mathcal{T}_z),$$

from (2). On the other hand, $\omega(\mathcal{T}_z) = 0 \Leftrightarrow z^{-1}(p\,\mathcal{T}_z) = 0$, which is equivalent to $p\,\mathcal{T}_z = 0$, since z is an isomorphism of R^m (C^m, resp.) with T_{px} (T_{pz}^{C}, resp.); ω will then be a tensorial 1-form: We interpret the last property by saying that the 1-form is *regular* and its type by saying that it is a *vectorial 1-form*.

Let s be a section of H over the open subset $U \subset X$; $s^*\omega$ is a 1-form over U with values in R^m (C^m, resp.). If $\mathcal{T}_x \in T_x$ (T_x^{C}, resp.) then $s^*\omega(\mathcal{T}_x) = (s^{-1}(x) \cdot p)\,(s\,\mathcal{T}_x) = s^{-1}(x)\,(\mathcal{T}_x)$; i.e., $(s^*\omega)_x = s^{-1}(x)$ and $s^*\omega$ is the coframe that is dual to the section s. That remark can serve as the definition of ω (cf., Chap. II.3).

The fundamental form characterizes the spaces of frames:

PROPOSITION III.4.1. – *Let G be a Lie subgroup of L_m (CL_m, resp.). In order for a PFS H (X, G) to be G-isomorphic to a space of frames on X, it is necessary and sufficient that it can be endowed with a tensorial 1-form ω with values in* R^m (C^m, *resp.*) *that satisfies* (2) *and* (3). *There will then exist a unique homomorphism f of H into E* (X) (E^{C} (X), *resp.*) *that is compatible with the identity map of G into L_m (CL_m, resp.) and is such that:*

$$f^*\theta = \omega, \tag{4}$$

in which θ is the fundamental 1-form on E (E^{C}, resp.*).*

First, let f be a G-isomorphism of H (X, G) onto a space of frames $H'(X, G) \subset E$. It is a homomorphism into E, and one knows (Chap. II.3) that $\omega = f^*\theta$ is a tensorial 1-form on H of the same type; ω is regular since:

$$\omega(\mathcal{T}_z) = 0 \quad\Leftrightarrow\quad \theta(f\,\mathcal{T}_z) = 0 \quad\Leftrightarrow\quad p_E\, f\,\mathcal{T}_z = 0 \quad\Leftrightarrow\quad p_H\,\mathcal{T}_z = 0\ (p_H = p_E \circ f)\,.$$

Conversely, if H (X, G) satisfies the hypotheses of Proposition (III.4.1) then suppose that there exists a G-isomorphism f of H into E such that $f^*\theta = \omega$. If $\mathcal{T}_h \in T_h$ ($h \in H$) then $\omega(\mathcal{T}_h) = \theta(f\,\mathcal{T}_h)$, and since $f\,\mathcal{T}_h \in T_{f(h)}$, one will then have $\omega(\mathcal{T}_h) = [f(h)]^{-1}\, p_E\, f\,\mathcal{T}_h$; i.e.:

$$\omega(\mathcal{T}_h) = [f(h)]^{-1}\, p_E\, \mathcal{T}_h\,. \tag{5}$$

Let t_ω be the tensor associated with ω (Def. II.3), which is a tensor on $H \boxtimes E$ with values in $\mathrm{R}^{m*} \otimes \mathrm{R}^m$ of type $\rho\,(G \times L_m)$ such that $\rho\,(g, l) = g \otimes l^{-1}$; i.e., such that $t_\omega\,(h \cdot g, z \cdot l) = g^{-1} \cdot t_\omega\,(h, z) \cdot l$. It can be defined by the formulas [(6), Chap. II.3]:

(6)
$$\begin{cases} t\,(h,z)\cdot u = \omega(\mathcal{T}_h), \\ \text{if } \mathcal{T}_h \in T_h, \quad (h,z) \in H \otimes E, u \in R^m \ \text{ then } \ z \cdot u = p_H \mathcal{T}_h. \end{cases}$$

As a result of (3), $t_\omega\,(h, z) \cdot u = 0 \Leftrightarrow u = 0$, and $t_\omega\,(h, z) \in L_m$. (6) implies that $t_\omega(h,z)\cdot z^{-1} p_H \mathcal{T}_h = \omega(\mathcal{T}_h)$, in such a way that (5) is equivalent to:

(7)
$$t_\omega(h,z)\cdot z^{-1} p_H \mathcal{T}_h = [f\,(h)]^{-1}\, p_H \cdot \mathcal{T}_h\,,$$

and since $p_H \cdot \mathcal{T}_h = \mathcal{T}_{ph}$, (7) will be equivalent to the equality between operators:

(8)
$$f\,(h) = z \cdot [t_\omega\,(h, z)]^{-1},$$

which is meaningful, since $t_\omega\,(h, z) \in L_m$. Up to now, we have established that in order for a map f of H into E to satisfy (4), it is necessary and sufficient that it should satisfy (8). Now, the right-hand side of (8) does not depend upon z, but only upon h, because if:

$$(h, z') \in H \boxtimes E, \qquad p_E\, z' = p_H\, h = p_E\, z \qquad \text{and} \qquad z' = z \cdot l \qquad (l \in L_m)$$

then

$$z' \cdot [t_\omega\,(h, z')]^{-1} = z \cdot l \cdot [t_\omega\,(h, z \cdot l\,)]^{-1} = (z \cdot l)\,[t_\omega\,(h, z)]^{-1} = z \cdot [t_\omega\,(h, z)]^{-1}.$$

(8) then defines a unique map f of H into E. That map is differentiable, since (8) can be written:

$$f\,(h) = s\,(ph)\,[t_\omega \cdot (h, s\,(ph))]$$

over an open subset $U \subset X$ that is endowed with a differentiable section s of E, and t_ω is itself a differentiable function on $H \boxtimes E$. Finally, f is a homomorphism because:

$$f\,(h \cdot g) = z\,[t_\omega\,(h \cdot g, z)]^{-1} = z\,[g^{-1} \cdot t_\omega\,(h, z)]^{-1} = z\,[t_\omega\,(h, z)]^{-1} \cdot g = f\,(h) \cdot g.$$

The proof is completed immediately by applying Proposition (I.5.3). It extends with no modifications to spaces of complex frames provided that one utilizes the complex tensor $t^{\mathrm{C}}\omega$ that is associated with ω and has its values in CL_m.

We have established (Chap. II, §§ **3** and **5**) a bijective correspondence between tensorial forms on a PFS and associated tensors. We have seen that the property (3) of ω is equivalent to t_ω taking its values in L_m. Since the tensors of a certain type on a PFS correspond bijectively to the sections of a certain associated FS, Proposition (III.4.1) will have the:

COROLLARY. – *Let H (X, G) be a PFS, in which G is a Lie subgroup of L_m (CL_m, resp.) (m =* dim *X). The structures of the space of frames on H correspond bijectively to the sections of the fiber space with fiber L_m (CL_m, resp.) that is associated with $H \boxtimes E$ ($H \boxtimes E^{\mathbb{C}}$, resp.), while $G \times L_m$ ($G \times CL_m$, resp.) acts on the fiber by way of:*

$$(g, l), t \mapsto g^{-1} \cdot t \cdot l, \quad g \in G; l, t \in L_m \qquad (CL_m \text{, resp.}).$$

For a tensorial form Λ on a space of frames H (X, G), one can define a simpler notion of an associated tensor than the one on an arbitrary PFS. First, suppose that H is a space of real frames and that Λ has values in a real vectorial space M ; $t\,\Lambda$ will then be a tensor on $H \boxtimes E$. Consider the following maps:

$$i : H \to E, \qquad \text{inclusion,}$$

$$j : H \to H \boxtimes E, \qquad h \in H \mapsto (h, h) \in H \boxtimes E,$$

which is a map that identifies H with the diagonal of the PFSS $H \boxtimes H \subset H \boxtimes E$;

$$f : H \boxtimes E \to H, \qquad (h, z) \mapsto h, \quad h \in H, z \in E, \; p_H\, h = p_E\, z\,,$$

$$g : H \boxtimes E \to E, \qquad (h, z) \mapsto z, \quad h \in H, z \in E, \; p_H\, h = p_E\, z\,.$$

One then has:

(9) $$f \circ i = \text{identity on } H \quad \text{and} \quad g \circ j = i.$$

$t'\Lambda = j^* t\,\Lambda$ is a tensor on H, since j is a homomorphism of a PFS. It follows from (9) that $\Lambda = j^* t\,\Lambda$, and formula [(8), Chap. II, § **3**] will become:

$$\Lambda = j^*\left[(t\Lambda) \cdot \overset{q}{\wedge}\, g^*\theta\right] = (j^* t\,\Lambda) \cdot \left(\overset{q}{\wedge}\, j^*(g^*\theta)\right)$$

or

$$\Lambda = (t'\Lambda) \cdot \left(\overset{q}{\wedge}\, j^* g^*\theta\right),$$

namely, from (9):

$$\Lambda = (t'\Lambda) \cdot \left(\overset{q}{\wedge}\, i^*\theta\right),$$

or, since $i^*\theta$ is nothing but the fundamental form ω on H:

(10) $$\Lambda = (t'\Lambda) \cdot \left(\overset{q}{\wedge}\, \omega\right).$$

Conversely, an application of Proposition (II.3.1) shows that if λ is a tensor on H with values in $M \otimes \overset{q}{\wedge} \mathrm{R}^{m*}$ and type $\rho_1(G)$ [$\rho_1(g) = \mathrm{R}(g) \otimes \overset{q}{\wedge} g^{-1}$] then $\Lambda = \lambda \cdot \left(\overset{q}{\wedge} \omega\right)$ will be a tensorial q-form such that $\lambda = t'\Lambda$.

Similarly, if H is a space of complex frames and Λ has values in the *complex vector space M* then $t\,^{\mathrm{C}}\Lambda$ will be a tensor on $H \boxtimes E^{\mathrm{C}}$. The inclusion $H \subset E^{\mathrm{C}}$ permits one to define maps $I, J, \ldots$ that are analogous to $i, j, \ldots$, and $t'^{\mathrm{C}}\Lambda = J^*\, t\,^{\mathrm{C}}\Lambda$ will be a tensor on H that is coupled to Λ by a formula that is analogous to (10), since the correspondence between Λ and $t'^{\mathrm{C}}\Lambda$ is once more bijective.

If H is a *space of real frames* and M is a *complex vector space* then one will find that two associated tensors on H can be defined according to whether one uses the inclusion $H \subset E$ (which defines $t'\Lambda$ to have values in $M \otimes_{\mathrm{R}} \overset{q}{\wedge} \mathrm{R}^{m*}$) or $H \subset E^{\mathrm{C}}$ (which defines $t'^{\mathrm{C}}\Lambda$ to have values in $M \otimes_{\mathrm{C}} \overset{q}{\wedge} \mathrm{C}^{m*}$): The remark that was made in Chap. II, § **5** shows that these two tensors will coincide modulo the canonical identification of $M \otimes_{\mathrm{C}} \overset{q}{\wedge} \mathrm{C}^{m*}$ with $M \otimes_{\mathrm{R}} \overset{q}{\wedge} \mathrm{R}^{m*}$.

On the contrary, if H is a *space of complex frames* and M is a *real vector space* [which does not admit a complex structure for which the $\mathcal{R}(g)$ are linear transformations *over* C] *then one cannot define an associated tensor that corresponds bijectively to* Λ *on H itself* by a formula that is analogous to (10). Indeed, suppose that such a tensor λ is defined such that:

$$\Lambda = \lambda \cdot \left(\overset{q}{\wedge} \omega\right), \tag{11}$$

so one will then have that for $h \in H$, it is necessary that $l(h) \in M \otimes_{\mathrm{R}} \overset{q}{\wedge} \mathrm{C}^{m*}$. Now, Λ is defined only on the space Θ_h that is tangent to H at h and not on its complexification Θ_h^{C}: $\lambda(h)$ is then restricted by (II) only by way of the condition:

$$\begin{aligned}\Lambda(\mathcal{T}_h) &= \lambda(h) \cdot \left\langle \overset{q}{\wedge} \omega, \mathcal{T}_h \right\rangle, \quad \mathcal{T}_h \in \overset{q}{\wedge} \Theta_h \\ &= \lambda(h) \cdot h^{-1} \cdot p\, \mathcal{T}_h\end{aligned}$$

(with the simplified notations of Chap. II). When $\mathcal{T}_h$ describes $\overset{q}{\wedge} \Theta_h$, $p\mathcal{T}_h$ will describe $\overset{q}{\wedge} h^{-1} T_x$ ($x = ph$) and $h^{-1} p\mathcal{T}_h$ will describe a (real) m-dimensional vector subspace of C^m: $\lambda(h)$ will not be determined completely by (11) then.

In order to define Λ by a tensor on H, one can meanwhile proceed as follows: Let M be a complex vector space such that $M' \supset M$ and $M' = M + iM$, and $\mathcal{R}(g)$ extends to a complex automorphism of M', moreover. For example, one can take M' to be the complexification of M . (That is not necessarily the most convenient choice.) Λ will then be a complex tensorial form with values in M' and type $\mathcal{R}(G)$ that takes its values in $M \subset M'$ for *real tangent vectors.* Λ will now be associated with the tensors $t'^{\mathbb{C}}\Lambda$ ($t^{\mathbb{C}}\Lambda$, resp.) on H ($H \boxtimes E^{\mathbb{C}}$, resp.) with values in $M' \otimes_{\mathbb{C}} \overset{q}{\wedge} \mathbb{C}^{m*}$, as well as with $t\,\Lambda$ on $H \boxtimes E$, which has values in $M' \otimes_{\mathbb{R}} \overset{q}{\wedge} \mathbb{R}^{m*}$. Since the tensors $t'^{\mathbb{C}}\Lambda$ correspond bijectively to q-forms Λ on H with values in M', we seek to characterize the ones for which Λ has values in M. Now, $t'^{\mathbb{C}}\Lambda$ is determined bijectively by $t^{\mathbb{C}}\Lambda$, whose restriction to $H \boxtimes E$ is $t\,\Lambda$ (Chap. II.5), in such a way that $t'^{\mathbb{C}}\Lambda$ is determined by $t\,\Lambda$. In order for Λ to have values in M, it is necessary and sufficient that $t\,\Lambda$ should have values in $V = M \otimes_{\mathbb{R}} \overset{q}{\wedge} \mathbb{R}^{m*} \subset M' \otimes_{\mathbb{R}} \overset{q}{\wedge} \mathbb{R}^{m*}$; $t^{\mathbb{C}}\Lambda$ and $t'^{\mathbb{C}}\Lambda$ will then take their values in the orbit of V under CL_m , which is an orbit that is not generally a (real) vector subspace of $M' \otimes_{\mathbb{C}} \overset{q}{\wedge} \mathbb{C}^{m*}$, in such a way that the condition on $t'^{\mathbb{C}}\Lambda$ generally translates into a *nonlinear condition.*

That is why in what follows we will speak of the tensor on H that is associated with a form Λ on H with values in M only if *H is a space of real frames and M is arbitrary* or when *H is a space of complex frames and M is complex.* There will be no ambiguity then, and that tensor will always be denoted by $t\,\Lambda$. We state:

PROPOSITION III.4.2. – *Let H* (X, G) *be a space of real (complex, resp.) frames with a fundamental form* ω, *and let M be a vector space (complex vector space, resp.). The q-forms* Λ *on H with values in M correspond bijectively to the tensors on H with values in* $M \otimes_{\mathbb{R}} \overset{q}{\wedge} \mathbb{R}^{m*}$ ($M \otimes_{\mathbb{C}} \overset{q}{\wedge} \mathbb{C}^{m*}$, resp.) *and type* $\rho_1(G)$, *in which* $\rho_1(g) = \mathcal{R}(g) \otimes \overset{q}{\wedge} g^{-1}$. *The tensor that corresponds to* Λ *is the associated tensor* $t\,\Lambda$ *on H, which is defined by:*

(12)
$$\Lambda = (t\,\Lambda) \cdot \left(\overset{q}{\wedge}\omega\right)$$

In the basis $\{e_A\}$ for M and the canonical basis on $\mathbb{R}^{m*}$ ($\mathbb{C}^{m*}$, resp.), (12) can be written:

(13)
$$\Lambda^A = \frac{1}{q!}(t\Lambda)^A_{i_1\cdots i_q}\,\omega^{i_1} \wedge \cdots \wedge \omega^{i_q} ,$$

in which the ω^i components of ω are linearly-independent global forms on H, and the components $(t\Lambda)^A_{i_1\cdots i_q}$ of the associated tensor are functions with real (complex, resp.) values that are supposed to be antisymmetric with respect to the indices i.

If $H = E$ ($E^{\mathbb{C}}$, resp.) then one will recover the usual notion of the canonically-associated tensor by taking the inverse image of (13) by some local section. In particular, if one applies (12) to ω itself then one will get $\omega = (t\,\omega) \cdot \omega$, which shows that $t\omega$ is the constant tensor on H which equals the identity on L_m (Cl_m, resp.).

5. – Connections on spaces of frames.

A) We shall call a connection on E (X) (E^{C} (X), resp.) a *linear (complex linear, resp.) connection* on X. If S (G, H) is a given G-structure then we shall call a connection on H an *H-connection* or *S-connection*, indifferently. When only the group $G \subset L_m$ is given, we will call an arbitrary H-connection a *G-connection*.

Let γ be an H-connection, and let $\hat{\gamma}$ be its extension to E (E^{C}, resp.); their forms are π and $\hat{\pi}$, respectively. A path in H that is horizontal for γ will also be horizontal for $\hat{\gamma}$, since π is the form that is induced on H by $\hat{\pi}$. The uniqueness of the horizontal path for a connection over a given path in X with a given origin z will then lead to the fact that the holonomy sheet with origin $z \in H$ is the same for the two connections, and consequently the holonomy groups for the two connections, as well; in particular, $\hat{\psi}_z \subset G$. Conversely, if Γ is a linear connection and the holonomy sheet H'_z of the holonomy group $\psi_z \in G$ at a point $z \in E$ (E^{C}, resp.) is a differentiable ψ_z-PFSS (Chap. II, § **4**) then $H = H'_z \cdot G$ will define G-structure S. On the other hand, since the holonomy field of Γ at any point $z' \in H'_z$ is tangent to H'_z, and therefore to H, one will see immediately that the holonomy field of Γ is tangent to H at any point of H, so Γ will be the extension of an H-connection. We have established:

THEOREM III.5.1. ([16]). – *In order for there to exist a G-structure S in X, it is necessary and sufficient that there should exist a linear connection* Γ *on X whose holonomy group for a frame* $z \in E$ (X) (E^{C}(X), *resp.) is a subgroup of G;* Γ *will then be the extension of an S-connection.*

Now, let G be a group such that G-structures can be defined by a tensor t with values in M in the sense of Proposition (III.2), and recall its notations. If G is the subgroup of $g \in L_m$ such that $\mathcal{R}(g) \cdot u = u$ then $\underline{G}$ will be the subalgebra of $\lambda \in \underline{L_m}$ such that $\tilde{\mathcal{R}}(\lambda) \cdot u = 0$.

Let S (G, H) be a G-structure, let γ be an S-connection, and let $\hat{\gamma}$ be the linear connection that is an extension of γ, while π and $\hat{\pi}$ are the respective forms and ∇ and $\hat{\nabla}$ are the corresponding absolute differentials, and let t be the tensor on E that defines the structure. From [(2), Chap. II.4], one has:

$$\hat{\nabla} t = dt + \tilde{\mathcal{R}}(\hat{\pi}) \cdot t,$$

so if i is the injection $H \to E$ then:

([16]) This result contains those of [**22**], § **118** and [**18**], Chap. III, § **5**.

$$
\begin{aligned}
i^*\hat{\nabla} t &= di^* t + \tilde{\mathcal{R}}(i^*\hat{\pi}) \cdot i^* t \\
&= di^* t + \tilde{\mathcal{R}}(\pi) \cdot i^* t = \nabla i^* t,
\end{aligned}
$$

in which $i^* t$ is constant on H and equal to u :

$$
(1) \qquad i^*\hat{\nabla} t = \tilde{\mathcal{R}}(\pi) \cdot u \; ;
$$

i.e., if $\pi = \varepsilon_\rho \otimes \pi^\rho$ ($\{\varepsilon_\rho\}$ is a basis for $\underline{G}$) then:

$$
i^*\hat{\nabla} t = \tilde{\mathcal{R}}(\varepsilon_\rho) \cdot u \otimes \pi^\rho = 0,
$$

which implies that $\hat{\nabla} t = 0$.

Conversely, let $\hat{\gamma}$ be a linear connection whose form is $\hat{\pi}$ and is such that $\hat{\nabla} t = 0$. $\pi = i^*\hat{\pi}$ is a 1-form on H of adjoint type with values in $\underline{L_m}$ whose restriction to the fibers of H coincides with the form β that relates to H (Chap. II, § **4**.A) because the right translations by G on H are the restrictions of the right translations of G that act on E (Def. I.5.2). In order for π to be a connection form on H, it will then suffice, moreover, that it should take its values in $\underline{G}$. Let $\{\varepsilon_\rho , \varepsilon_a\}$ be a basis for $\underline{L_m}$ that is obtained by completing the basis $\{\varepsilon_\rho\}$ for $\underline{G}$: $\pi = \varepsilon_\rho \otimes \pi^\rho + \varepsilon_a \otimes \pi^a$. $i^*\hat{\nabla} t$ is once more given by (1) on H, and is zero, by hypothesis; now:

$$
\tilde{\mathcal{R}}(\pi) \cdot u = \tilde{\mathcal{R}}(\varepsilon_\rho) \cdot u \otimes \pi^\rho + \tilde{\mathcal{R}}(\varepsilon_a) \cdot u \otimes \pi^a = \tilde{\mathcal{R}}(\varepsilon_a) \cdot u \otimes \pi^a,
$$

since $\varepsilon_\rho \in \underline{G}$. Our hypothesis then will imply that:

$$
(2) \qquad \tilde{\mathcal{R}}(\varepsilon_a) \cdot u \otimes \pi^a = 0.
$$

The vectors $\tilde{\mathcal{R}}(\varepsilon_a) \cdot u$ are linearly independent, because $\sum_a \mu_a \tilde{\mathcal{R}}(\varepsilon_a) \cdot u = 0$ implies that $\sum_a \tilde{\mathcal{R}}(\mu^a \varepsilon_a) \cdot u = 0$; i.e., $\mu^a \varepsilon_a \in \underline{G}$, which is absurd. Consequently, (2) will imply the vanishing of the form π^a, and π will have values in $\underline{G}$. We have then established:

THEOREM III.5.2. – *If the G-structure S on X can be defined by the tensor t on E* (*X*) *(*E^{C} (*X*)*, resp.) then the necessary and sufficient condition for a linear connection (complex connection, resp.) on X to be the extension of an S-connection is that the absolute differential of t under that connection should be zero.*

That theorem contains the characterization of the linear connections that are Euclidian for a given metric ([**22**], § **51**), as well as almost-complex ([22], § **109**), and almost-Hermitian. (An almost-Hermitian structure is defined by two tensors, namely, the metric tensor and the almost complex tensor, so it will belong to our class of structures: Two tensors can be considered to be just one tensor with values in the direct sum of their spaces of value spaces.) It likewise contains the characterization of the complex linear connections that are π-connections for a given π-structure ([**18**], Chap. II, § **6**) and complex linear connections that are almost-Hermitian connections in the larger sense for an almost-Hermitian structure in the larger sense ([17]) (*ibid.*, Chap. III, § **4**).

B) *Torsion.* – In these two paragraphs, $H = H(X, G)$ will be a space of frames that is endowed with a well-defined connection γ; K will denote either of the fields R or C, according to whether H is real or complex, resp. The existence of the fundamental 1-form ω with values in K^m on H implies the existence of a supplementary invariant for the connection γ, namely, the *torsion form:*

$$\Sigma = \nabla\omega = d\omega + \pi \cdot \omega \tag{2}$$

(since the representation of G in K^m that defines the type of ω is its representation as a linear group in K^m). The tensor $t\,\Sigma$ that is associated with Σ on H is the *torsion tensor* of γ. Naturally, Σ and $t\,\Sigma$ are, respectively, the form and the tensor that are induced on H by the torsion:

$$\hat{\Sigma} = \hat{\nabla}\theta = d\theta + \hat{\pi} \cdot \theta \tag{3}$$

of the linear connection $\hat{\gamma}$, which is an extension of γ, and its canonically-associated tensor $t\hat{\Sigma}$. The identity [(5), Chap. II, § **4**] provides the absolute differential of Σ:

$$\nabla\Sigma = \nabla^2\omega = \Omega \cdot \omega. \tag{4}$$

That is the *Bianchi identity for torsion*.

C) *Covariant derivative. Generalized Ricci identity.* – If Λ is a tensorial q-form (q = 0, 1, …, m) on the space of frames $H(X, G)$ with values in the vectorial space M (which is real if H is real and complex if H is complex) then we shall call the tensor on H :

$$D\Lambda = t\,\nabla t\,\Lambda \tag{5}$$

the *covariant derivative of* Λ.

The action of the covariant derivative is intrinsic to the spaces of frames, and it differs from the absolute differentiation that acts on any differentiable PFS.

If $t\Lambda$ is a tensor of type $\mathcal{R}(g) \otimes \overset{q}{\wedge} g^{-1}$ with values in $M \otimes_{\mathrm{K}} \overset{q}{\wedge}(\mathrm{K}^{m*})$ then $\nabla t\,\Lambda$ will be a 1-form of the same type, and $D\Lambda$ will be a tensor $\mathcal{R}(g) \otimes \overset{q}{\wedge} g^{-1} \otimes g^{-1}$ with values in

([17]) An almost-Hermitian structure in the larger sense is defined by G. Legrand [**18**] by the given of a complex metric Φ and a field of operators $\mathcal{J}$ on T_x^{C} with identity squares, such that $\Phi(\mathcal{J}u, \mathcal{J}v) = -\Phi(u, v)$, $u, v \in T_x^{\mathrm{C}}$.

$M \otimes_K \overset{q}{\wedge}(K^{m*}) \otimes_K K^{m^*}$. Since $\nabla t\,\Lambda$ is a 1-form, formula [(12), § **4**], which defines the associated tensor, can be written:

$$\nabla t\,\Lambda = (D\Lambda)\cdot\omega. \tag{6}$$

When Λ is a tensor W, since $tW = W$, (5) will become $DW = t\,\nabla W$, and (6) will become:

$$\nabla W = (DW)\cdot\omega. \tag{7}$$

One will recover the usual notion of covariant derivative in that case, and if $H = E$, or even if Λ is a form of identity type (viz., the inverse image of a form on X). On the contrary, when $q > 0$, $t\,\nabla\Lambda$ will be a tensor with values in $M \otimes \overset{q+1}{\wedge} K^{m^*}$ and type $R(g) \otimes \overset{q+1}{\wedge} g^{-1}$, so there will then be no reason for it to coincide with $D\Lambda$. Upon applying the differentiation formula for a product of the type [(8), Chap. II, § **4**], one will deduce from the relation [(12), § **4**]:

$$\Lambda = (t\,\Lambda)\cdot\left(\overset{q}{\wedge}\omega\right),$$

that:

$$\nabla\Lambda = (\nabla t\,\Lambda)\cdot\left(\overset{q}{\wedge}\omega\right) + (t\,\Lambda)\cdot\left(\nabla\overset{q}{\wedge}\omega\right). \tag{8}$$

From (6), that formula, which can also be written:

$$\nabla\Lambda = (\nabla t\,\Lambda)\cdot\left(\overset{q}{\wedge}\omega\right) = (D\Lambda\cdot\omega)\cdot\left(\overset{q}{\wedge}\omega\right) + (t\,\Lambda)\cdot\left(\nabla\overset{q}{\wedge}\omega\right), \tag{9}$$

will permit one to calculate the absolute differential as a function of the covariant derivative and $\nabla\overset{q}{\wedge}\omega$.

Let us first apply this to the case in which $\Lambda = \nabla\Phi$ [Φ is a tensorial $(q-1)$-form]:

$$\nabla^2\Phi = ((D\nabla\Phi)\cdot\omega)\cdot\left(\overset{q}{\wedge}\omega\right) + (t\,\nabla\Lambda)\cdot\left(\nabla\overset{q}{\wedge}\omega\right).$$

Since, on the other hand, $\nabla^2\Phi = \tilde{\mathcal{R}}(\Omega)\cdot\Phi$ [(5), Chap. II, § **4**], one will obtain the identity:

$$((D\nabla\Phi)\cdot\omega)\cdot\overset{q}{\wedge}\omega = \tilde{\mathcal{R}}(\Omega)\cdot\Phi - (t\,\nabla\Lambda)\cdot\left(\nabla\overset{q}{\wedge}\omega\right) \qquad (\text{degree } \Phi = q-1), \tag{10}$$

to which one can give the name of the *generalized Ricci identity*. Indeed, take the case in which Φ is a tensor W ($q = 1$). Since $W = tW$ and $DW = t\,\nabla W$, one will have:

$$D\nabla W = t\,\nabla t\,\nabla t\,W = t\,\nabla t\,DW = D^2W,$$

and (10) can then be written:

$$(D^2W \cdot \omega) \cdot \omega = \tilde{\mathcal{R}}(\Omega) \cdot W - (DW) \cdot \Sigma\,. \tag{11}$$

That formula is the *Ricci identity*, properly speaking (which is more general than the usual identity, moreover, since the representation $\mathcal{R}$ is arbitrary). In order to see that, we write it out explicitly when $W = V$ is a vector field by taking the usual notations for the components of the covariant derivative. (11) is written:

$$(D^2V \cdot \omega) \cdot \omega = \Omega \cdot V - (DV) \cdot \Sigma, \tag{12}$$

namely:

$$(\nabla_\lambda \nabla_\mu V^i\, \omega^\lambda) \wedge \omega^\mu = \Omega^i_j \cdot V^j - \nabla_k V^i\, \Sigma^k. \tag{13}$$

Since:

$$\Omega^i_j = \tfrac{1}{2} R^i_{j,\lambda\mu}\, \omega^\lambda \wedge \omega^\mu \qquad \text{and} \qquad \Sigma^k = -\, S^{\;k}_{\lambda\mu}\, \omega^\lambda \wedge \omega^\mu,$$

with the usual normalizations, and $R^i_{j,\lambda\mu}$ and $S^{\;k}_{\lambda\mu}$ are antisymmetric in λ and μ, one will get:

$$\tfrac{1}{2}(\nabla_\lambda \nabla_\mu V^i - \nabla_\mu \nabla_\lambda V^i)\, \omega^\lambda \wedge \omega^\mu = \tfrac{1}{2}(R^i_{j,\lambda\mu} V^j + \nabla_k V^i \cdot S^{\;k}_{\lambda\mu})\, \omega^\lambda \wedge \omega^\mu, \tag{14}$$

so finally, one will have:

$$\nabla_\lambda \nabla_\mu V^i - \nabla_\mu \nabla_\lambda V^i = R^i_{j,\lambda\mu} V^j + \nabla_k V^i \cdot S^{\;k}_{\lambda\mu}\,.$$

In order to now obtain the explicit form for (9), we calculate $\nabla \overset{q}{\wedge} \omega$. It follows from the definition of $\overset{q}{\wedge}\omega$ (Chap. II.2., E) and the tensorial character of ω that $\overset{q}{\wedge}\omega$ is a tensorial form of type $\overset{q}{\wedge}(G)$ and:

$$\nabla\left(\overset{q}{\wedge}\omega\right) = d\left(\overset{q}{\wedge}\omega\right) + \widetilde{\overset{q}{\wedge}}(\pi) \cdot \left(\overset{q}{\wedge}\omega\right),$$

in which $\{e_i\}$ is the canonical basis for K^m. Formula [(24), Chap. II, § **2**] gives the expression for $\overset{q}{\wedge}\omega$ with the aid of the components ω^i of ω in that basis:

$$\overset{q}{\wedge}\omega = \frac{1}{q!} e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes \omega^{i_1} \wedge \cdots \wedge \omega^{i_q},$$

in which:

$$d\left(\overset{q}{\wedge}\omega\right) = \frac{1}{q!}\sum_{k=1}^{q}(-1)^{k-1}\, e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes \omega^{i_1} \wedge \cdots \wedge d\omega^{i_1} \wedge \cdots \wedge \omega^{i_q}. \tag{15}$$

On the other hand, an easy calculation will show that in the basis $B = \{ e_{i_1} \wedge \cdots \wedge e_{i_q}, i_1 < i_2 \ldots < i_q\}$ for $\overset{q}{\wedge} K^m$, the operation $\widetilde{\overset{q}{\wedge}}(\lambda)$ ($\lambda \in \underline{L_m}$, $\underline{CL_m}$, resp.) will be given by its components ([18]):

$$(\widetilde{\overset{q}{\wedge}}(\lambda))^{l_1\cdots l_q}_{i_1\cdots i_q} = \sum_{h=1}^{q} \varepsilon^{l_1\cdots l_q}_{i_1\cdots \underset{\hat{i}_h}{p} \cdots i_q} \lambda^p_{i_h},$$

in which $\varepsilon^{l_1\cdots l_q}_{i_1\cdots i_q}$ is the indicator of the permutation and λ^p_r are the components of λ in the canonical basis for $\underline{L_m}$ ($\underline{CL_m}$, resp.). The components of $\varphi = \left(\widetilde{\overset{q}{\wedge}}(\pi)\cdot \overset{q}{\wedge}\omega\right)$ in the basis B will be then:

$$\begin{aligned}
\varphi^{l_1\cdots i_q} &= \sum_{i_1<\cdots<i_q} \widetilde{\overset{q}{\wedge}}(\pi)^{l_1\cdots l_q}_{i_1\cdots i_q} \cdot \left(\overset{q}{\wedge}\omega\right)^{i_1\cdots i_q} \\
&= \sum_{k=1}^{q} \sum_{i_1<\cdots<i_q} \varepsilon^{l_1\cdots l_q}_{i_1\cdots \underset{\hat{i}_k}{p}\cdots i_q} \pi^p_{i_k} \wedge \omega^{l_1} \wedge \cdots \wedge \omega^{l_q} \\
&= \frac{1}{q!}\sum_{k=1}^{q} \varepsilon^{l_1\cdots l_q}_{i_1\cdots \underset{\hat{i}_k}{p}\cdots i_q}\ \pi^p_{i_k} \wedge \omega^{l_1} \wedge \cdots \wedge \omega^{l_q},
\end{aligned}$$

or, upon moving ω^{l_k} and p to the first position:

$$\varphi^{l_1\cdots i_q} = \frac{1}{q!}\sum_{k=1}^{q} \varepsilon^{l_1\cdots l_q}_{i_1\cdots \underset{\hat{i}_k}{p}\cdots i_q}\ (\pi^p_{i_k} \wedge \omega^{l_k}) \wedge \omega^{l_1} \wedge \cdots \wedge \omega^{l_{k-1}} \wedge \omega^{l_{k+1}} \wedge \cdots \wedge \omega^{l_q},$$

and then, upon suppressing the summation over k:

([18]) A sequence of indices $i_1 \ldots \underset{\hat{i}_h}{p} \ldots i_q$ represents the sequence that is obtained by replacing i_h with p in the sequence $i_1 \ldots i_h \ldots i_q$. Similarly, the sequence $i_1 \ldots \hat{i}_h \ldots i_q$ represents the sequence $i_1 \ldots i_q$ when the term i_h is suppressed.

$$\varphi^{l_1\cdots i_q} = \frac{1}{(q-1)!}\varepsilon^{l_1\cdots l_q}_{pi_2\cdots i_q}\,(\pi^p_r\wedge\omega^r)\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_q}\,.$$

The sum of terms in that sum for which $p = l_1$ is:

$$\frac{1}{(q-1)!}\varepsilon^{l_1 l_2\cdots l_q}_{l_1 i_2\cdots i_q}\,(\pi^{l_1}_r\wedge\omega^r)\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_q} = (\pi^{l_1}_r\wedge\omega^r)\wedge\omega^{l_2}\wedge\cdots\wedge\omega^{l_q}\,,$$

and upon proceeding similarly for each index l, one will get:

$$\varphi^{l_1\cdots i_q} = \sum_{k=1}^{p}(-1)^{k-1}\omega^{l_1}\wedge\cdots\wedge\omega^{l_{k-1}}\wedge(\pi^{l_k}_r\wedge\omega^r)\wedge\omega^{l_{k+1}}\wedge\cdots\wedge\omega^{l_q}\,.$$

Since that expression is antisymmetric with respect to l :

$$\varphi = \sum_{i_1<\cdots<i_q} e_{i_1}\wedge\cdots\wedge e_{i_1}\otimes\varphi^{i_1\cdots i_q} = \frac{1}{q!}e_{i_1}\wedge\cdots\wedge e_{i_1}\otimes\varphi^{i_1\cdots i_q}\,,$$

and upon comparing (15) and (16), one will obtain the components of $\left(\nabla\overset{q}{\wedge}\omega\right)$ in B):

(17) $$\left(\nabla\overset{q}{\wedge}\omega\right)^{i_1\cdots i_q} = \sum_{k=1}^{p}(-1)^{k-1}\omega^{l_1}\wedge\cdots\wedge(d\omega^{l_k}+\pi^{l_k}_r\wedge\omega^r)\wedge\cdots\wedge\omega^{l_q}$$
$$= \sum_{k=1}^{p}(-1)^{k-1}\omega^{l_1}\wedge\cdots\wedge\Sigma^{l_k}\wedge\omega^{l_{k+1}}\wedge\cdots\wedge\omega^{l_q}\,.$$

We now calculate each of the terms in (9) in the basis $\{e_A\}$ for M:

(18) $$(\nabla\Lambda)^A = \frac{1}{(q+1)!}(t\nabla\Lambda)^A_{i_0 i_1\cdots i_q}\,\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}\,,$$

(19) $$\left[(D\Lambda\cdot\omega)\cdot\left(\overset{q}{\wedge}\omega\right)\right]^A = \frac{1}{q!}\Big((D\Lambda)^A_{i_0 i_1\cdots i_q}\,\omega^{i_0}\Big)\wedge\omega^{i_1}\wedge\cdots\wedge\omega^{i_q}$$
$$= \frac{1}{(q+1)!}\left(\sum_{l=0}^{q}(-1)^l(D\Lambda)^A_{i_0 i_1\cdots\hat{i}_l\cdots i_q}\right)\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}\,,$$

in order to render the coefficient of $\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}$ antisymmetric with respect to the i's.

On the other hand:

(20) $$\left[(t\Lambda\cdot\omega)\cdot\left(\nabla\overset{q}{\wedge}\omega\right)\right]^A$$
$$= \frac{1}{q!}(t\Lambda)^A_{l_1\cdots l_q}\cdot\left(\nabla\overset{q}{\wedge}\omega\right)^{l_1\cdots l_q}$$
$$= \frac{1}{q!}(t\Lambda)^A_{l_1\cdots l_q}\sum_{k=1}^{1}(-1)^{k-1}\omega^{l_1}\wedge\cdots\wedge\omega^{l_{k-1}}\wedge\Sigma^{l_k}\wedge\omega^{l_{k+1}}\wedge\cdots\wedge\omega^{l_q}$$
$$= \frac{1}{(q-1)!}(t\Lambda)^A_{p i_1\cdots i_q}\Sigma^p\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_q}$$
$$= -\frac{1}{(q-1)!}(t\Lambda)^A_{p i_2\cdots i_q}S^p_{i_0 i_1}\omega^{i_0}\wedge\omega^{i_1}\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_q}$$
$$= \frac{1}{q!}\sum_{l=1}^{q}(-1)^l S^p_{i_0 i_l}(t\Lambda)^A_{p i_1\cdots\hat{i}_l\cdots i_q}\,\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}$$
$$= \frac{1}{(q+1)!}\sum_{\substack{k<l\\ k,l\,=\,0,1,\ldots,q}} 2\cdot(-1)^{k+l}S^p_{i_k i_l}(t\Lambda)^A_{p i_0\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_q}\,\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}$$

by an antisymmetrization calculation of classical type. Hence, since the coefficients of $\omega^{i_0}\wedge\cdots\wedge\omega^{i_q}$ are all antisymmetric in (18), (19), and (20), (9) will become:

(21) $$(t\Lambda)^A_{i_0 i_1\cdots i_q} - \frac{1}{(q+1)!}\sum_{l=0}^{q}(-1)^l(D\Lambda)^A_{p i_0\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_q} = \sum_{\substack{k<l\\ k,l\,=\,0,1,\ldots,q}} 2(-1)^{(k+l)}S^p_{i_k i_l}(t\Lambda)^A_{p i_0\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_q}.$$

PROPOSITION III.5. – *The tensor on a space of frames that is associated with the absolute differential of a q-form* Λ *is equal to the antisymmetrization of the covariant derivative of* Λ *in a connection with zero torsion. When the torsion is not zero, it will differ from it by a term that is bilinear in the torsion and the associated tensor to* Λ *according to formula* (21).

Application. – If α is a scalar q-form on X then $\Lambda = p^*\alpha$ will be a q-form of identity type and $\nabla p^*\alpha = dp^*\alpha = p^*d\alpha$, in such a way that upon taking inverse images of the two sides of (21) under an arbitrary local section and utilizing the classical notations, the relation that couples the exterior differential of a form to its covariant derivative will be:

(22) $$(d\alpha)_{i_0 i_1\cdots i_q} - \sum_{l=0}^{q}\nabla\alpha_{i_l i_0\cdots\hat{i}_l\cdots i_q} = \sum_{\substack{k<l\\ k,l\,=\,0,1,\ldots,q}} 2(-1)^{k+l}S^p_{i_k i_l}\,\alpha_{p i_0\cdots\hat{i}_k\cdots\hat{i}_l\cdots i_q}.$$

6. – Structure tensor of a G-structure.

A) Let S be a G-structure on the frame space H and let ω be its fundamental form. One lets K (KL_m, resp.) denote the real field (the group L_m, resp.) when S is real or the complex field (CL_m, resp.) when S is complex.

Let γ and γ' be H-connections with forms π, π', resp., and let Σ, Σ', resp., be their torsions. The torsion tensors are tensors on H with values in $P = \mathrm{K}^m \otimes_{\mathrm{K}} \overset{2}{\wedge} \mathrm{K}^{m*}$ and type $\mathcal{R}(G)$, in which $\mathcal{R}$ is the representation of KL_m in P such that:

$$\mathcal{R}(l) = l \otimes \overset{2}{\wedge} l^{-1}, \qquad l \in KL_m .$$

$\pi' - \pi = u$ is a tensorial 1-form on H of adjoint type with values in $\underline{G}$. *If S is complex then suppose that $\underline{G}$ is a complex vector subspace of $\underline{CL_m}$.* u will then (§ **4**) have an associated tensor $tu = \xi$ on H (and being given the latter is equivalent to being given u) with values in $N = N_G = \underline{G} \otimes_{\mathrm{K}} \mathrm{K}^{m*}$, and is such that:

$$u = \xi \cdot \omega. \tag{1}$$

ξ has type $\mathcal{Q}(G)$, where $\mathcal{Q}$ is the representation of KL_m in $\mathcal{N} = \underline{KL_m} \otimes_{\mathrm{K}} \mathrm{K}^{m*}$ such that $\mathcal{Q}(l) = \mathrm{ad}\, l \otimes l^{-1}$ [$N_G \subset \mathcal{N}$ is obviously an invariant subspace for $\mathcal{Q}(G)$].

Since $\mathcal{N} = K^m \otimes_{\mathrm{K}} \mathrm{K}^{m*} \otimes_{\mathrm{K}} \mathrm{K}^{m*}$, P can be considered to be the quotient of $\mathcal{N}$ by the subspace $\mathcal{I}$ that is generated by the elements $x \otimes f \otimes f$ ($x \in K^m$, $f \in K^{m*}$). Let $-\mathcal{A}$ be the natural projection $\mathcal{N} \to \mathcal{N} / \mathcal{I} = P$, $x \otimes f \otimes f' \mapsto x \otimes (f \wedge f')$. In the canonical basis $\{e_i\}$ for K^m (and the associated bases on the other spaces), $-\mathcal{A}$ translates into:

$$t^i_{jk} \mapsto (t^i_{jk} - t^i_{kj}),$$

and it is obvious from the definitions that for any $l \in KL_m$:

$$\mathcal{A} \circ \mathcal{Q}(l) = \mathcal{R}(l) \circ \mathcal{A} . \tag{2}$$

From (1), the components of u in a basis $\{\varepsilon_\rho = \left(a^i_{j\rho}\right)\}$ for $\underline{G}$ (a basis over K) will be given by:

$$u^\rho = \xi^\rho_k \omega^k \tag{3}$$

(ξ^ρ_k are the components of ξ in the basis $\varepsilon_\rho \otimes x^i$ for N_G). From the definition of torsion [(2), § **5**], one gets:

$$\Sigma' - \Sigma = (\pi' - \pi) \cdot \omega = u \cdot \omega$$

$$
\begin{aligned}
&= (\varepsilon_\rho \otimes u^\rho) \cdot (e_k \otimes \omega^k) \\
&= (\varepsilon_\rho \cdot e_k) \otimes u^\rho \wedge \omega^k) \\
&= a^i_{k\rho} e_i \otimes \xi^\rho_j \omega^j \wedge \omega^k \\
&= e_i \otimes \tfrac{1}{2}(a^i_{k\rho}\, \xi^\rho_j - a^i_{j\rho}\, \xi^\rho_k)\, \omega^j \wedge \omega^k.
\end{aligned}
$$

Since the expression in parentheses is antisymmetric in j, k, from [(13), § **4**], that will give the components of $t\,(\Sigma' - \Sigma)$ in the canonical basis for P :

(4) $$(t\Sigma' - t\Sigma)^i_{jk} = a^i_{k\rho}\, \xi^\rho_j - a^i_{j\rho}\, \xi^\rho_k\,.$$

Since the components of ξ in the canonical basis for $\mathcal{N}$ are $t^i_{jk} = a^i_{j\rho}\, \xi^\rho_k$, (4) translates into:

$$t\,\Sigma' - t\,\Sigma = \mathcal{A} \circ \xi\,,$$

or upon letting A denote the restriction of $\mathcal{A}$ to $N_G \subset \mathcal{N}$:

(5) $$t\,\Sigma' - t\,\Sigma = A \circ \xi\,.$$

Since N_G is invariant under $\mathcal{Q}\,(G)$, (2) will become:

(6) $$A \circ \mathcal{Q}\,(g) = \mathcal{R}\,(g) \circ A, \qquad g \in G.$$

Let $V_G = A\,(N_G)$, $M = M_G = P \,/\, V_G$, and let α be the natural projection $P \to M_G$. From (6), V_G is invariant under $\mathcal{R}\,(G)$ in such a way that the representation $\mathcal{R}\,(G)$ passes to the quotient. Let ρ be the representation of G thus-obtained in M ; it is defined by:

(8) $$\rho\,(g) \circ \alpha = \alpha \circ \mathcal{R}\,(g)\,.$$

From (6), $\alpha \circ t\,\Sigma' = \alpha \circ t\,\Sigma$ is then a function t_S on H with values in M (which is defined globally and independently of the connection) and is such that:

$$t_S\,(z \cdot g) = \alpha\,(t\,\Sigma\,(z \cdot g)) \qquad (z \in H, g \in G),$$

namely:

$$t_S\,(z \cdot g) = \alpha\,(\mathcal{R}\,(g^{-1}) \cdot t\,\Sigma\,(z)) = \rho\,(g^{-1}) \cdot t_S\,(z)\,.$$

t_S is then a tensor on H with values in M and type $\rho\,(G)$ that depends upon only the structure: We call it the *structure tensor* on S.

B) Conversely, let Σ_1 be a vectorial 2-form on H with values in K^m. Is it the torsion Σ' of an H-connection γ'? Suppose that the necessary condition $\alpha \circ t\,\Sigma_1 = t_S$ is satisfied, and let γ be an arbitrary H-connection. With the preceding notations, the determination of γ' is equivalent to that of the tensorial 1-form $u = \pi' - \pi$, which is itself equivalent (Prop. III.4.2) to the determination of its associated tensor ξ with values on N. The condition imposed on γ' that $\Sigma' = \Sigma_1$ is equivalent to $\Sigma' - \Sigma = \Sigma_1 - \Sigma$, in which the right-hand side is a given tensorial 2-form of the same type Σ'', and from (5), the condition $\Sigma' - \Sigma = \Sigma''$ is itself equivalent to:

(9) $$A \circ \xi = t\,\Sigma''.$$

On the other hand, the hypothesis that $\alpha \circ t\,\Sigma_1 = t_S = \alpha \circ t\,\Sigma$ is equivalent to:

(10) $$\alpha \circ t\,\Sigma'' = 0.$$

Let $N(H)$ [$P(H)$, $M(H)$, resp.] be the fiber that is obtained by modeling (Def. I.2) N (P, M, resp.) on H, while G acts on the fiber by way of $\mathcal{Q}(G)$ [$\mathcal{R}(G)$, $\rho(G)$, resp.], and let $\widetilde{N(H)}$ [$\widetilde{P(H)}, \widetilde{M(H)}$, resp.] be the sheaf of germs of sections of that space. The exact sequence of homomorphisms of vector spaces:

$$N \xrightarrow{A} P \xrightarrow{\alpha} M \to 0$$

corresponds to an exact sequence of sheaf homomorphisms:

$$\widetilde{N(H)} \xrightarrow{\tilde{A}} \widetilde{P(H)} \xrightarrow{\tilde{\alpha}} \widetilde{M(H)} \to 0,$$

and since all of those sheaves are sheaves of germs of sections of FS's with vector fibers, that sequence will itself correspond to the exact sequence of cohomology groups:

(11) $$\mathcal{H}_0(X, \widetilde{N(H)}) \xrightarrow{\tilde{A}_0} \mathcal{H}_0(X, \widetilde{P(H)}) \xrightarrow{\tilde{\alpha}_0} \mathcal{H}_0(X, \widetilde{M(H)}) \to 0\,.$$

The tensors on H with values in N (P, M, resp.) and type $\mathcal{Q}(G)$ [$\mathcal{R}(G)$, $\rho(G)$, resp.] correspond bijectively to the sections of $N(H)$ [$P(H)$, $M(H)$, resp.]; i.e., to the elements of the cohomology groups $\mathcal{H}_0(X, \widetilde{N(H)})$ [$\mathcal{H}_0(X, \widetilde{P(H)})$, $\mathcal{H}_0(X, \widetilde{M(H)})$, resp.]. Therefore, let $s \in \mathcal{H}_0(X, \widetilde{N(H)})$ be the element that corresponds to ξ, and let $\sigma \in \mathcal{H}_0(X, \widetilde{P(H)})$ be the one that corresponds to $t\,\Sigma''$; (9) and (10) are equivalent to:

(12) $$\tilde{A}_0(s) = \sigma,$$

(13) $$\tilde{\alpha}_0(\sigma) = 0.$$

Since the sequence (11) is exact, the image of $\tilde{A}_0$ will coincide with the kernel of $\tilde{\alpha}_0$, and s that satisfy (12) will exist as long as σ satisfies (13). We have then shown the existence of the connection γ'.

C) Those calculations and proofs do not extend to the case in which S is complex, so $\underline{G}$ is not a complex vector subspace of $\underline{CL_m}$. First of all, if $\underline{G}$ does not admit the structure of a complex Lie algebra then the ad g ($g \in G$) will not be automorphisms of the complex vector space of $\underline{G}$, and the forms u on H with values in $\underline{G}$ of adjoint type will not admit the associated tensor on H (cf., § **4**). Similarly, if $\underline{G}$ admits such a structure that is not induced by the one on $\underline{CL_m}$ then the value space $\underline{G} \otimes_{\mathrm{C}} \mathrm{C}^{m*}$ for associated tensors to u will not be identified with a complex subspace of $\underline{CL_m} \otimes_{\mathrm{C}} \mathrm{C}^{m*}$, and the proof will once more break down.

One can then consider proceeding as in § **4**: Let $\underline{G}' = \underline{G} + i\,\underline{G}$. The ad g ($g \in G$) are automorphisms of the complex vector space of $\underline{G}'$, so let u be the associated tensor to ξ with values on $\underline{G}' \otimes_{\mathrm{C}} \mathrm{C}^{m*} \subset \mathcal{N}$. The calculations of paragraph A) will then remain valid and will further permit one to define a "structure tensor" t'_S. However, the ones in B) will not be valid, since ξ is generally restricted by some supplementary nonlinear conditions (§ **4**). Indeed, there is no reason for that tensor t'_S to characterize the S-connections, either. Let G' be the connected subgroup of CL_m that is generated by $\underline{G}'$ and suppose that G is itself connected. One will then have $G' \supset G$, and S will be subordinate to a G'-structure S', so t'_S will be nothing but the structure tensor of S, in such a way that it will characterize the torsions of the S'-connections, which are not S-connections, in general (cf., § **8**, D).

D) We shall now state the results that have been obtained and infer some first consequences.

DEFINITION III.6. – *A G-structure is said to be of* ***the first kind*** *if it is real or if it is complex and G is a complex Lie subgroup of CL_m. It will be of* ***the second kind*** *in all of the other cases.*

We remark that the condition that G is a complex Lie subgroup of CL_m (i.e., the injection $G \to CL_m$ is a complex analytic map) is equivalent to the condition that $\underline{G}$ is a complex vector subspace of $\underline{CL_m}$.

THEOREM III.6.1. – *Let G be a Lie subgroup of L_m or a complex Lie subgroup of CL_m. In the former case,* K *denotes the real field, while it denotes the complex field in the latter. That group is canonically associated with a* K*-linear representation ρ of G in a vector space M_G, which is the image of* $P = \mathrm{K}^m \otimes_{\mathrm{K}}$ *under a homomorphism α in such a fashion that:*

1. *For any G-structure S of the first kind, one can define a tensor t_S of type* ρ (G) *with values in M_G on the space H of distinguished frames of S* : *t_S is the* ***structure tensor*** *on S.*

2. *The necessary and sufficient for a vectorial 2-form Σ on H to be the torsion of an S-connection is that:*

$$\alpha \circ t\,\Sigma = t_S\,.$$

In certain cases, that theorem will take on a much simpler form, which is contained in the known theorems, as we will see:

THEOREM III.6.2. – *If G satisfies the same hypotheses as in Theorem* (III.6.1) *then suppose, moreover, that the subspace $V = \alpha^{-1}(0)$ of P admits a supplement W that is likewise invariant under $\mathcal{R}(G)$. For any G-structure S of the first kind and any S-connection, the torsion tensor (on H) will be the sum of two tensors, $t\,\Sigma = (t\,\Sigma)_V + (t\,\Sigma)_W$ with values in V and W, respectively:*

1. $(t\,\Sigma)_W$ *does not depend upon the connection and is identified with t_S [$\rho(G)$ is identified with the induced representation of $\mathcal{R}(G)$ on W].*

2. $(t\,\Sigma)_V$ *depends upon only the connection and can be chosen arbitrarily by a convenient choice of connection. In particular, there always exists an S-connection whose torsion tensor reduces to $t_S = (t\,\Sigma)_W$ exactly.*

Under the hypotheses of that theorem, since $(t\,\Sigma)_V$ and $(t\,\Sigma)_W$ are tensors with values in P and type $\mathcal{R}(G)$, one can make an analogous statement that gives a decomposition of the torsion form as a sum of two forms. On the other hand, let π be the connection form of one of the connections for which $(t\,\Sigma)_V = 0$. Hence, from the definition of torsion, one will have:

$$(13)\qquad \left\{\begin{aligned} & d\omega = -\pi\cdot\omega + (t_S)\cdot\left(\overset{2}{\wedge}\,\omega\right), \\ \text{or}\quad & d\omega^i = -\pi^i_j \wedge \omega^j + \tfrac{1}{2}(t_S)^i_{jk}\,\omega^j \wedge \omega^k, \\ \text{or}\quad & d\omega^i = a^i_{j\rho}\,\omega^j \wedge \pi^\rho + \tfrac{1}{2}(t_S)^i_{jk}\,\omega^j \wedge \omega^k \end{aligned}\right.$$

globally on H, with the notations of A), and conversely, if π is an H-connection, and if one can write $d\omega$ in the form (13), while t_S is a tensor with values in W, then t_S will be the structure tensor of S.

Under only the hypothesis of Theorem (III.6.1), let W_1 be a supplement to V that is not invariant, in general. For a given H-connection of the form π, one will have a decomposition $t\,\Sigma = (t\,\Sigma)_V + (t\Sigma)_{W_1}$. If $(t\Sigma)_{W_1}(z) = C(z)$ is the projection of $t_S(z)$ onto W_1 under the natural projection $q : P / V \to W_1$ then C will be a function on H that is independent of the connection, and one can write:

$$(14)\qquad d\omega^i = -\,\pi^i_j \wedge \omega^j + \tfrac{1}{2}(t\Sigma)^{\ \ i}_{V,\,jk}\,\omega^j \wedge \omega^k + \tfrac{1}{2}(C)^i_{jk}\,\omega^j \wedge \omega^k\,.$$

Since $(t\,\Sigma)_V(z) \in V$ for any $z \in H$, the equations for the unknowns η_k^ρ ($\rho = 1, \dots, r$; $k = 1, \dots, m$):

$$(t\Sigma)_{V,\,jk}^{\;\;i} = a_{k\rho}^i \eta_j^\rho - a_{j\rho}^i \eta_k^\rho \qquad (i, j, k = 1, \dots, m \,;\, j < k)$$

are compatible, and since they have constant coefficients, one can find their differentiable solutions $\eta_k^\rho(z)$ (which are unique only if the map A is injective). Since $\pi_j^i = a_{j\rho}^i \pi^\rho$, (14) will then become:

$$d\omega^i = \omega^j \wedge a_{j\rho}^i (\pi^\rho - \eta_k^\rho \omega^k) + \tfrac{1}{2} C_{jk}^i \,\omega^j \wedge \omega^k$$

or

(15)
$$\begin{cases} d\omega^i = \omega^j \wedge a_{j\rho}^i \pi'^\rho + \tfrac{1}{2} C_{jk}^i \,\omega^j \wedge \omega^k, \\ \text{namely,} \quad d\omega = -\pi' \cdot \omega + C \cdot \overset{2}{\wedge}\omega. \end{cases}$$

In (15), $\pi = \varepsilon_\rho \otimes \pi'^\rho$ is a global form on H, but it is not defined canonically. Furthermore, it is not a connection form on H, since otherwise C would be the torsion tensor of that connection, which is contrary to the hypothesis that W_1 is not invariant under $\mathcal{R}(G)$. Finally, although one has written some relations on H such as (15), in which the function C takes its values in W_1, C is the projection onto W_1 of the structure tensor, which is then defined perfectly.

In particular, if one takes W_1 to be a supplement to V that is generated by a subset of the canonical basis for P, and if one lets $t_{j''k''}^{i''}$ denote the coordinates of P, which are zero on W_1, and the other one by $t_{j'k'}^{i'}$ then (15) can be written:

(16)
$$d\omega^i = \omega^j \wedge a_{j\rho}^i \pi'^\rho + \tfrac{1}{2} C_{j'k'}^{i'} \,\omega^{j'} \wedge \omega^{k'}.$$

If one sets $F_{jk}^i = a_{k\rho}^i \eta_j^\rho - a_{j\rho}^i \eta_k^\rho$ ($\eta_k^\rho \in \mathrm{K}$) ($i, j, k = 1, \dots, m$; $j < k$) then the indices $\left(\begin{smallmatrix} i'' \\ j''k'' \end{smallmatrix}\right)$ will be characterized by the property that the linear forms $F_{j''k''}^{i''}$ are linearly-independent and maximal in number. The $C_{j'k'}^{i'}$ are the *primary invariants of the structure*, as they were defined by S S. Chern in [**9**] for the real case. Equations (13), (15), or (16) can be called the *structure equations* of S.

Let r be the dimension of G, and let s be the rank of A. One has $\dim N = mr$ and $\dim P = m^2 (m-1)/2$. s, which is less than or equal to those two numbers, will be equal to the rank of the system of $m^2 (m-1)/2$ linear forms F_{jk}^i. One can specify that:

COROLLARY III.6.1. – *If A is surjective then its torsion can be chosen arbitrarily.*

In particular, for any G-structure S of the first kind there will always exist an S-connection with zero torsion. In that case, $s = m^2 (m-1)/2$, which demands that $r \geq m(m-1)/2$.

COROLLARY III.6.1. – *If A is injective* ([19]) *then being given the torsion* Σ *will determine the connection, provided that* $\alpha \circ t\Sigma = t_S$.

That will come about if $s = mr$ and one then demands that $r \leq m\,(m - 1) / 2$. If one satisfies the conditions for applying Theorem (III.6.2), moreover, then any G-structure will admit a *canonical connection* whose torsion tensor coincides with the structure tensor (cf., § **8**, ex. B, E).

COROLLARY III.6.3. – *If A is bijective*, which supposes that $r = m\,(m - 1) / 2$, so if it is closed, then from some results of Weyl-Cartan (cf., W. Klingenberg [**28**]), G will be the orthogonal or special orthogonal group of a definite or indefinite non-degenerate quadratic form, and the torsion will be arbitrary and determined uniquely by the connection. In particular, there is a *canonical connection with zero torsion.*

COROLLARY III.6.4. – *In order for a G-structure S of the first kind to admit an S-connection with zero torsion, it is necessary and sufficient that the structure tensor should be zero.*

7. – Calculating the structure tensor.

A) *Equivalent structures.* – Let S and S' be equivalent structures for the frame spaces H and $H' = H \cdot l$, respectively, with groups G and $G' = l^{-1} \cdot G \cdot l$, resp. One either considers S and S' to be real ($l \in L_m$, K = R) or S to be complex of the first kind. Hence, for any $l \cdot CL_m$, S' will be complex of the first kind (K = C).

The various spaces and maps will be denoted as they were in § **6**, with an index that depends upon S or G (e.g., $N' = N'_G$, …). Hence:

$$N' = \underline{G}' \otimes_{\mathrm{K}} \mathrm{K}^{m*} = l^{-1} \cdot \underline{G} \cdot l \otimes_{\mathrm{K}} \mathrm{K}^{m*} = \mathcal{Q}\,(l^{-1})\,N$$

and

$$V' = \mathcal{A}\,(N') = \mathcal{A} \circ \mathcal{Q}\,(l^{-1})\,(N) = \mathcal{R}\,(l^{-1})\,\mathcal{A}\,(N) = \mathcal{R}\,(l^{-1})\,V,$$

in such a way that:

a) Since the rank of $A = \dim V$, one will have that the rank of A = rank of A', and the two structures will be in the same situation for the corollaries in § **6** to be applicable.

b) The automorphism $\mathcal{R}\,(l^{-1})$ of P passes to the quotient to give an isomorphism $\tilde{\rho}(l^{-1}) : M = P / V \to M' = P / V'$ such that $\alpha' \circ \mathcal{R}\,(l^{-1}) = \tilde{\rho}(l^{-1}) \circ \alpha$.

([19]) In the real case, that hypothesis will be equivalent to this one: The first group is deduced from G = identity (S. S. Chern [**9**]) or G has finite type of degree 2 (P. Libermann). That remark then contains a theorem of P. Libermann from [**21**]. It is also Cherm's property (C) in [**10**].

Let γ be a *S*-connection, and let $\hat{\gamma}$ be the linear (complex linear, resp.) connection that is an extension of γ. $\hat{\gamma}$ is also the extension of an *S*-connection γ' (from the invariance of the connection under right-translations), and if Σ and Σ' are the torsions (on H and H', resp.) of γ and γ', resp., then:

$$t\,\Sigma'\,(z\cdot l) = \mathcal{R}\,(l^{-1})\;t\,\Sigma\,(z) \qquad (z \in H),$$

so

$$t_S\,(z\cdot l) = \alpha'\circ t\,\Sigma'(z\cdot l) = \alpha'\circ \mathcal{R}(l^{-1})\,t\,\Sigma'(z) = \bar{\rho}\,(l^{-1})\circ\alpha\,t\,\Sigma(z) = \bar{\rho}\,(l^{-1})\,t_S\,(z)\,.$$

One then has the following relation between the two structure tensors:

$$D_l^*\,t_{S'} = \bar{\rho}\,(l^{-1})\,t_S\,. \tag{1}$$

In particular, if S' is a *structure that is associated with* S then $l = n \in N\,(G)$ [$N^{\mathrm{C}}(G)$, resp.]. $G' = G$ implies that $M' = M$, $\alpha' = \alpha$, and $\bar{\rho}\,(l^{-1})$ is the automorphism of M that is defined by:

$$\alpha\circ\mathcal{R}\,(n^{-1}) = \bar{\rho}\,(n^{-1})\circ\alpha\,. \tag{2}$$

$\bar{\rho}$ is a representation of $N\,(G)$ in M whose restriction to $G \subset N\,(G)$ is ρ.

B) *Subordinate structures.* – With the same notations, let S be subordinate to S'. Hence, $G \subset G'$ will imply that $N \subset N'$ and $V = \mathcal{A}\,(V) \subset V' = \mathcal{A}\,(N')$ in such a way that:

a) If A is surjective then A' will also be so.

b) If A' is injective then A will be so: i.e., *if the torsion determines the connection for a G-structure then the same thing will be true for the subordinate structures.*

The inclusions $V \subset V' \subset P$ imply the existence of a projection $\alpha_1 : P / V \to P / V'$, such that $\alpha_1 \circ \alpha = \alpha'$ and $\alpha_1\circ\rho\,(g) = \rho'(g)\circ\alpha'$ ($g \in G$). Let i be the injection $H \to H'$ ($H \subset H'$), let γ be an H-connection with form π, and let γ' be its extension to H' with form π'. One will then have that $\omega = i^*\omega'$ and $\pi = i^*\pi'$ will imply that $\Sigma = i^*\Sigma'$ and $t\,\Sigma = i^*t\,\Sigma'$, resp., so since $t_S = \alpha\circ t\,\Sigma$:

$$\alpha_1\circ\, t_S = \alpha_1\circ\alpha\circ t\,\Sigma = \alpha'\circ\, i^*t\,\Sigma' = i^*\alpha'\circ\, t\,\Sigma,$$

namely:

$$\alpha_1\circ\, t_S = i^*t_{S'}, \tag{2}$$

in which $t_{S'}$ is determined by its restriction $i^*t_{S'}$ to H', so (2) defines $t_{S'}$.

Meanwhile, if S is the largest subordinate structure that is common to S' and S'' ($H = H' \cap H''$), $N = N' \cap N''$, then since $\mathcal{A}(N' \cap N'') \neq \mathcal{A}(N) \cap \mathcal{A}(N')$, the structure tensors to S' and S'' will not generally suffice to determine the one on S (cf., § **8**, E).

C) *Local calculations.* – Let $U \subset X$, let s be a distinguished local section, and let θ_U be the dual coframe on U. The field of tangent planes to the image of the section s and their right-translates by G will determine a connection on H_U ; it can be extended to a connection on H with a form π. For that connection:

$$\Sigma = d\omega + \pi \cdot \omega,$$
$$s^*\Sigma = s^* d\omega + s^*\pi \cdot s^*\omega = d\theta_U ,$$

since, by definition, $s^*\pi = 0$; one will then have the components:

$$(s^*\Sigma)^i = d\theta_U^i = \tfrac{1}{2}(C_U)^i_{jk}\, \theta_U^j \wedge \theta_U^k , \tag{3}$$

in which the $(C_U)^i_{jk}$, which are antisymmetric in j, k, determine a map $C_U : U \to P$. On the other hand, one deduces from $\Sigma = (t\,\Sigma) \cdot \overset{2}{\wedge}\, \omega$ that:

$$s^*\Sigma = (s^*\, t\,\Sigma) \cdot \overset{2}{\wedge}\, \theta_U \qquad \text{or} \qquad (s^*\Sigma)^i = \tfrac{1}{2}(s^* t\,\Sigma)^i_{jk}\, \theta_U^j \wedge \theta_U^k \ . \tag{4}$$

A comparison of (3) and (4) finally gives:

$$s^* t_S = s^*(\alpha \circ t\,\Sigma) = \alpha \cdot s^*\, t\,\Sigma = \alpha \cdot C_U ,$$

so

PROPOSITION III.7. – *If θ_U is a distinguished coframe on U then the expression for t_S in the dual coframe thus-obtained will be:*

$$(t_S)_U = \alpha \cdot C_U ,$$

in which $C_U : U \to P$ is the map that is defined by $d\theta_U^i = \tfrac{1}{2}(C_U)^i_{jk}\, \theta_U^j \wedge \theta_U^k$.

COROLLARY III.7. – *An integrable G-structure has a zero structure tensor.*

The converse of this, which is false, in general (cf., § **8**, B), is true in numerous cases (cf., § **8**, C, D, F, and Chap. IV, § **3**). We also pose the:

DEFINITION III.7. – *A G-structure is called* ***almost-integrable*** *if its structure tensor is zero.*

8. – Applications and examples.

A) Let $X = G / H$ be a reductive homogeneous space. Its Cartan connection ([20]) determines a connection with zero torsion for each of the $\tilde{H}_{z_0}$–structures S that were defined in Proposition (III.1): *Those structures will then have zero structure tensors.*

B) *Real* $O(m)$*-structures.* – (K = R) Let $\underline{G}$ be the subalgebra of matrices $(A^i_j) \in \underline{L_m}$ such that such that $A^i_j + \bar{A}^i_j = 0$; N is then the subspace of $t \in \mathcal{N}$ with coordinates t^i_{jk} such that $t^i_{jk} + t^j_{ik} = 0$. Since, on the other hand:

$$\dim G = \frac{m(m-1)}{2}, \qquad \dim N = \frac{m^2(m-1)}{2} = \dim P,$$

and in order to show that A is bijective, it will suffice to show that $A^{-1}(0) = 0$. Now, the system of equations that defines $A^{-1}(0)$:

$$t^i_{jk} + t^j_{ik} = 0, \qquad t^i_{kj} - t^i_{jk} = 0$$

is Cramerian (cf., calculating the Christoffel symbols) and $A^{-1}(0) = 0$. Some consequences are:

a) $M = 0$. Any $O(m)$-structure S admits a canonical S-connection with zero torsion, namely, the Riemannian connection. Furthermore (§ **6**, B), for any group $G' \subset O(m)$, one will again have $M' = 0$, and so on for $G' = L_m$, …

b) Since any closed subgroup $G' \subset O(m)$ is compact, the subspace $V_{G'} \subset P$ that is invariant under $\mathcal{R}(G)$ will admit a supplement W that is likewise invariant, and Theorem (III.6.2) will apply to real G'-structures S'. In particular, there will be an S'-connection whose torsion coincides with the structure tensor on S', and since A', like A, is injective, that connection will be unique.

PROPOSITION III.8.1. – *For any G'-structure S' that is subordinate to a Riemannian structure, there exists a* ***canonical*** *S'-connection whose torsion coincides with the structure tensor. In order for the structure tensor to be zero, it is necessary and sufficient that this canonical connection should coincide with the Riemannian connection.*

C) Let S be a real (K = R) or complex (K = C) *almost-product structure* $G = KL(n_1, n_2)$. Employ the indices $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n_1$ and $\alpha', \beta', \gamma', \ldots = n_1 + 1, \ldots, n_1 + n_2$. Upon denoting a basis for K^m (the dual basis, resp.) by $\{e_i\}$ ($\{f^j\}$, resp.), $\underline{G}$ is the algebra of matrices over K:

([20]) A. Lichnerowicz [**23**], § **37**.

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and it will have $\{\varepsilon_\alpha^\beta = e_\alpha \otimes f^\beta;\ \varepsilon_{\alpha'}^{\beta'} = e_{\alpha'} \otimes f^{\beta'}\}$ for a basis. Hence, there will be a basis for N:

$$\{e_\alpha \otimes f^\beta \otimes f^\gamma,\ e_\alpha \otimes f^\beta \otimes f^{\gamma'},\ e_{\alpha'} \otimes f^{\beta'} \otimes f^\gamma,\ e_{\alpha'} \otimes f^{\beta'} \otimes f^{\gamma'}\},$$

and then for $V = A\,(N)$:

$$\{e_\alpha \otimes f^\beta \otimes f^\gamma\,(\beta < \gamma),\ e_\alpha \otimes f^\beta \otimes f^{\gamma'},\ e_{\alpha'} \otimes f^{\beta'} \otimes f^\gamma,\ e_{\alpha'} \otimes f^{\beta'} \otimes f^{\gamma'}\,(\beta' < \gamma')\}.$$

V then has the supplement W, which is likewise invariant under G and has a basis:

$$\{e_\alpha \otimes f^{\beta'} \otimes f^{\gamma'}\,(\beta' < \gamma'),\ e_{\alpha'} \otimes f^\beta \otimes f^\gamma\,(\beta < \gamma)\}.$$

t_S is then the tensor that is determined by only the components $t^\alpha_{\beta'\gamma'}$, $t^{\alpha'}_{\beta\gamma}$ of the torsion tensor of no particular S-connection. From [(13), § **6**], for a connection whose torsion reduces to t_S, one will have:

$$\begin{cases} d\omega^\alpha = -\pi^\alpha_\beta \wedge \omega^\beta + \frac{1}{2} t^\alpha_{\beta'\gamma'}\, \omega^{\beta'} \wedge \omega^{\gamma'}, \\ d\omega^{\alpha'} = -\pi^{\alpha'}_{\beta'} \wedge \omega^{\beta'} + \frac{1}{2} t^{\alpha'}_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma, \end{cases}$$

so

$$\text{(1)} \qquad \begin{cases} d\omega^\alpha \equiv \frac{1}{2} t^\alpha_{\beta'\gamma'}\, \omega^{\beta'} \wedge \omega^{\gamma'} & (\mathrm{mod}\ \omega^\beta), \\ d\omega^{\alpha'} \equiv \frac{1}{2} t^{\alpha'}_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma & (\mathrm{mod}\ \omega^{\beta'}), \end{cases}$$

which permits us to identify our structure tensor with the "torsion tensor of the almost-product structure" that was defined by G. Legrand [**18**], and the results of § **6** contain some of those results.

D) Now $\mathcal{J}$ be an *almost-complex structure* (with the notations of § **1**) ($m = 2n$). CL_n^b is not a complex Lie subgroup of CL_{2n}, and our theory will not apply to the complex CL_n^b-structure S that determines $\mathcal{J}$. One easily sees that $\underline{CL_n^b} + i\underline{CL_n^b} = \underline{CL(n,n)}$, and since $CL(n, n)$ is connected, the smallest structure of the first kind to which S is subordinate will be the complex almost-product structure that it determines. Now, in $E^b(X)$, (1) will become:

$$\text{(2)} \qquad \begin{cases} d\omega^\alpha \equiv \frac{1}{2} t^\alpha_{\beta^*\gamma^*}\, \omega^{\beta^*} \wedge \omega^{\gamma^*} & (\mathrm{mod}\ \omega^\beta), \\ d\omega^{\alpha^*} \equiv \frac{1}{2} t^\alpha_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma & (\mathrm{mod}\ \omega^{\beta^*}), \end{cases}$$

which is a relation that characterizes the "almost-complex torsion." It is then identified with the structure tensor of the π-structure that is defined by $\mathcal{J}$.

It is meanwhile obvious that the single condition $T^{\alpha}_{\beta*\gamma*} = t^{\alpha}_{\beta*\gamma*}$, $T^{\alpha*}_{\beta\,\gamma} = t^{\alpha*}_{\beta\,\gamma}$ does not suffice to characterize the tensors T with values in P that are the torsions of an almost-complex connection. However, we shall see that *in this very special case*, the supplementary conditions that were mentioned in § **6**, C) are expressed simply.

Let $i, j, k = 1, 2, \ldots, 2n$ be indices, and let π^i_j be the components of an S^b-connection π (in the canonical basis for $\underline{CL_{2n}}$). From the definition of CL^b_n, $\pi^{\alpha}_{\beta*} = \pi^{\alpha*}_{\beta} = 0$ and $\pi^{\alpha*}_{\beta*} = \overline{\pi^{\alpha}_{\beta}}$. Since, on the other hand, $\omega^{\alpha*} = \overline{\omega^{\alpha}}$, one will have:

$$\Sigma^{\alpha} = d\omega^{\alpha} + \pi^{\alpha}_{\beta} \wedge \omega^{\beta} \quad \text{and} \quad \Sigma^{\alpha*} = d\omega^{\alpha*} + \pi^{\alpha*}_{\beta*} \wedge \omega^{\beta*} = \overline{\Sigma^{\alpha}}$$

for the torsion. A first condition for a vectorial 2-form on $E^b(X)$ with components Σ^i to be the torsion of an S-connection is then: $\Sigma^{\alpha*} = \overline{\Sigma^{\alpha}}$. The forms that satisfy this are determined bijectively by the 2-form $\hat{\Sigma}$ on $E^b(X)$ with values in C^n of vectorial type (CL^b_n acts on C^n by way of the group that is isomorphic to CL_n) whose components are $\hat{\Sigma}^{\alpha} = \Sigma^{\alpha}$. $t\hat{\Sigma}$ has values in $P_1 = \mathrm{C}^n \otimes \overset{2}{\wedge}(\mathrm{C}^{2n})^*$ (coordinates t^{α}_{jk}) and type $\mathcal{R}_1\left(CL^b_n\right)$.

On the other hand, if one compares the torsions of the two connections then one will get $\Sigma'^{\alpha} - \Sigma^{\alpha} = u^{\alpha}_{\beta} \wedge \omega^{\beta}$, in which the 1-forms u^i_j are the components of $u = \pi' - \pi$ in the basis for $\underline{CL_{2n}}$. The forms u^{α}_{β} alone determine u, since $u^{\alpha}_{\beta*} = u^{\alpha*}_{\beta} = 0$, $u^{\alpha*}_{\beta*} = \overline{u^{\alpha}_{\beta}}$. They also constitute the components of a 1-form $\hat{u}$ with values in $\underline{CL_n}$ and adjoint type that admits an associated tensor λ on H with values in $N_1 = \underline{CL_n} \otimes \mathrm{C}^{2n*} = \mathrm{C}^n \otimes \mathrm{C}^{n*} \otimes \mathrm{C}^{2n*}$ and type $\mathcal{Q}_1(CL^b_n)$, and one will have $\hat{u} = \lambda \cdot \omega$, namely, $u^{\alpha}_{\beta} = \lambda^{\alpha}_{\beta\gamma}\,\omega^{\gamma} + \lambda^{\alpha}_{\beta\gamma*}\,\omega^{\gamma*}$. If $\mathcal{N}_1 = \mathrm{C}^n \otimes \mathrm{C}^{2n*} \otimes \mathrm{C}^{2n*}$ then N_1 will be identified with the complex vector subspace of $\mathcal{N}_1$ for which $\lambda^{\alpha}_{\beta*\gamma*} = 0$. One will again have a projection – $\mathcal{A}_1 : \mathcal{N}_1 \to P$ ($t^{\alpha}_{jk} \mapsto t^{\alpha}_{jk} - t^{\alpha}_{kj}$) and the commutation relation $\mathcal{A}_1 \circ \mathcal{N}_1(g) = \mathcal{N}_1(g) \circ \mathcal{A}_1$. Hence:

$$\begin{aligned} \Sigma'^{\alpha} - \Sigma^{\alpha} &= (\lambda^{\alpha}_{\beta\gamma}\,\omega^{\gamma} + \lambda^{\alpha}_{\beta\gamma*}\,\omega^{\gamma*}) \wedge \omega^{\beta} \\ &= \tfrac{1}{2}(\lambda^{\alpha}_{\beta\gamma} - \lambda^{\alpha}_{\beta\gamma})\,\omega^{\beta} \wedge \omega^{\gamma} - \lambda^{\alpha}_{\beta\,\gamma*}\,\omega^{\beta} \wedge \omega^{\gamma*}, \end{aligned} \tag{3}$$

which is further written $\widehat{t\Sigma'} - \widehat{t\Sigma} = A_1 \circ \lambda$ (A_1 is the restriction of $\mathcal{A}_1$ to N_1). One then sees that the theory can be developed as in § **6**, but in the converse sense. Moreover, $V_1 = A(N_1)$ is the subspace of P_1 whose equation is $t^{\alpha}_{\beta*\gamma*} = 0$, which admits the invariant supplement W whose equations are $t^{\alpha}_{\beta\gamma} = t^{\alpha}_{\beta\gamma*} = 0$. One then finds that one has established the:

PROPOSITION III.8.2. – *The "almost-complex torsion" tensor t of an almost-complex structure is nothing but the structure tensor for the π-structure that it* *determines. In order for a vectorial 2-form Σ on the space E^b (X) of adapted complex frames to be the torsion of an almost-complex connection, it is necessary and sufficient that it should satisfy the two conditions:*

1. $$(t\,\Sigma)^{\alpha}_{\beta*\gamma*} = t^{\alpha}_{\beta*\gamma*},$$

2. $$\Sigma^{\alpha*} = \overline{\Sigma^{\alpha}}.$$

One can get that proposition by a simple application of Theorem (III.6.2) upon utilizing the space of *real* frames E^a (X). It seems more interesting to us to exhibit, on the one hand, the peculiarities of an almost-complex structure among the complex *G*-structures, and on the other hand, the kind of difficulties that one will encounter for the complex *G*-structures of the second kind.

E) For an *almost-Hermitian structure* that is defined by its space of adapted complex frames ε^b (X) $\subset E^b$ (X) (with the notations of § **3**), one will be in an entirely analogous situation. While preserving the notations of the preceding paragraph, the components of $\hat{u}$ will be restricted by the supplementary condition $u^{\alpha}_{\beta} + \overline{u^{\beta}_{\alpha}} = 0$, and those of λ will be restricted by $\lambda^{\alpha}_{\beta\gamma} + \overline{\lambda^{\beta}_{\alpha\gamma*}} = 0$ ($\lambda^{\alpha}_{\beta\gamma*} + \overline{\lambda^{\beta}_{\alpha\gamma}} = 0$). From (3), one will then have:

$$\Sigma'^{\alpha} - \Sigma^{\alpha} = \tfrac{1}{2}(\overline{\lambda^{\beta}_{\alpha\gamma*}} - \overline{\lambda^{\gamma}_{\alpha\beta*}})\,\omega^{\beta}\wedge\omega^{\gamma} - \lambda^{\alpha}_{\beta\gamma*}\,\omega^{\beta}\wedge\omega^{\gamma}.$$

Without entering into the details, one will see that the invariant subspace V_2 of P_1 in which $\widehat{t\,\Sigma'} - \widehat{t\,\Sigma}$ takes its values will have the equations:

$$t^{\alpha}_{\beta*\gamma*} = 0, \qquad t^{\alpha}_{\beta\gamma} = \overline{t^{\gamma}_{\alpha\beta*}} - \overline{t^{\beta}_{\alpha\gamma*}},$$

and it will admit the supplement *W*, which will likewise be invariant and have the equation $t^{\alpha}_{\beta\gamma*} = 0$. Hence, for any connection on ε^b (X), the torsion:

$$\Sigma^{\alpha} = \tfrac{1}{2}a^{\alpha}_{\beta\gamma}\,\omega^{\beta}\wedge\omega^{\gamma} + a^{\alpha}_{\beta\gamma*}\,\omega^{\beta}\wedge\omega^{\gamma*} + \tfrac{1}{2}a^{\alpha}_{\beta*\gamma*}\,\omega^{\beta*}\wedge\omega^{\gamma*} \tag{4}$$

will admit the decomposition:

$$\Sigma^{\alpha} = (\Sigma^{\alpha})_W + (\Sigma^{\alpha})_{V_2}, \tag{5}$$

in which:

$$(\Sigma^{\alpha})_W = \tfrac{1}{2}(a^{\alpha}_{\beta\gamma} - \overline{a^{\gamma}_{\alpha\beta*}} + \overline{a^{\beta}_{\alpha\gamma*}})\,\omega^{\beta}\wedge\omega^{\gamma} + \tfrac{1}{2}a^{\alpha}_{\beta*\gamma*}\,\omega^{\beta*}\wedge\omega^{\gamma*} \tag{6}$$

$$(\Sigma^{\alpha})_{V_2} = \tfrac{1}{2}(\overline{a^{\gamma}_{\alpha\beta*}} - \overline{a^{\beta}_{\alpha\gamma*}})\,\omega^{\beta}\wedge\omega^{\gamma} + a^{\alpha}_{\beta\gamma*}\,\omega^{\beta}\wedge\omega^{\gamma*}\,.$$

Furthermore, since upon passing to real frames on $\varepsilon^a(X)$, one can apply Proposition (III.8.1), one will see that one can recover the existence of the second canonical connection for an almost-Hermitian structure [which is the unique almost-Hermitian connection whose torsion tensor consists of only terms of type (2, 0) and (0, 2)], and one can state, more precisely:

PROPOSITION III.8.3. ([21]). – *The structure tensor of an almost-Hermitian structure can be identified with the torsion tensor of the second canonical connection. In order for a vectorial 2-form* (4) *to be the torsion of an almost-Hermitian connection, it is necessary and sufficient that the term* (6) *in the decomposition* (5) *should coincide with the torsion of the second canonical connection of the structure. An integrable almost-Hermitian manifold is Kählerian.*

Indeed, one knows that the latter property is equivalent to the coincidence of the second canonical connection with the Riemannian connection.

F) Let G be the group that S. S. Chern pointed out in [**9**] of real matrices of the form:

$$g = \begin{pmatrix} u & v & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (u > 0). \tag{7}$$

dim $X = 3$. Let $i, j, k = 1, 2, 3$ be indices. Let $\underline{G}$ be the subalgebra of $\underline{L_3}$ with equations $A^i_2 = A^i_3 = 0$, and let $N = \underline{G} \otimes \mathrm{R}^3$ be the subspace of $\mathcal{N}$ with equations $t^i_{2k} = t^i_{3k} = 0$. $V = A(N)$ is the subspace of P with equations $s^i_{23} = 0$. The supplementary subspace W_1 with equations $s^i_{12} = s^i_{13} = 0$ is not invariant under $\mathcal{R}(G)$. For $g \in G$ and $s \in W_1$, one will have:

$$\left(\mathcal{R}(g)s\right)^p_{23} = g^p_i\, s^i_{jk}(g^{-1})^j_2\,(g^{-1})^k_3 = g^p_i\, s^i_{23}(g^{-1})^2_2\,(g^{-1})^3_3 - g^p_i\, s^i_{23}(g^{-1})^3_2\,(g^{-1})^2_3 = g^p_i\, s^i_{23}\,;$$

i.e., if one identifies W_1 with R^3 upon setting $\lambda^i = s^i_{23}$ then the quotient representation ρ in W_1 will be nothing but the representation of G as a linear group on R^3 : *The structure tensor will then be a vector field.*

([21]) Cf., A. Lichnerowicz [**22**], § **112**, **114**, and S. S. Chern [**10**] and W. Klingenberg [**28**].

CHAPTER IV

AUTOMORPHISMS OF A *G*-STRUCTURE

1. – Local automorphisms.

A) *Image and inverse image of a G-structure.* – Let X and X' be differentiable manifolds of the same dimension m, and let E and E' (E^{C} and E'^{C}, resp.) be their real (complex, resp.) frame spaces. If μ is a regular map (viz., of rank m everywhere) of X into X' then its tangent linear map $\underline{\mu}_x$ at the point x will be an isomorphism of T_x with $T_{\mu x}$ (T_x^{C} with $T_{\mu x}^{\mathrm{C}}$, resp.). For $z \in E_x$ (E_x^{C}, resp.):

$$\tilde{\mu}(z) = \underline{\mu}_x \circ z \tag{1}$$

will then be an isomorphism of R^m onto $T_{\mu x}$ (C^m with $T_{\mu x}^{\mathrm{C}}$, resp.) and $\tilde{\mu}(z) \in E'_{\mu x}$ ($E'^{\mathrm{C}}_{\mu x}$, resp.). Hence, $\tilde{\mu}$ will be a map of E into E' (E^{C} into E'^{C}, resp.), and one will see immediately from (1) that:

$$\tilde{\mu}(z \cdot g) = \tilde{\mu}(z) \cdot g \qquad g \in L_m\ (CL_m, \text{ resp.}), \tag{2}$$

$$p_{E'} \circ \tilde{\mu} = \mu \circ p_E ; \tag{3}$$

i.e., $\tilde{\mu}$ is an L_m-representation of E in E' (a CL_m-representation of E^{C} in E'^{C}, resp.) (Chap. I, § **3**): $\tilde{\mu}$ *is the prolongation of* μ *to* E (E^{C}, resp.).

Let θ' (θ, resp.) be the fundamental form on E' (E, resp.): $< \tilde{\mu}^*\theta', \mathcal{T}_z > \, = \, <\theta', \tilde{\mu}\mathcal{T}_z>$. Since $\tilde{\mu}\mathcal{T}_z$ is tangent to E' at the point $\tilde{\mu}(z)$, it will follow from (1) and (3) that:

$$< \tilde{\mu}^*\theta', \mathcal{T}_z > \, = [\tilde{\mu}(z)]^{-1} \circ p_{E'}(\tilde{\mu}\mathcal{T}_z) = [\underline{\mu} \circ z]^{-1} \circ \mu \circ p_E(\mathcal{T}_z) = z^{-1}\, p_E(\mathcal{T}_z) = \, <\theta, \mathcal{T}_z>,$$

namely:

$$\tilde{\mu}^*\theta' = \theta. \tag{4}$$

Conversely, (4) characterizes the representations $\tilde{\mu}$ of E in E' that are prolongations of a map μ of X into X'.

Let S (G, H) be a G-structure on X. The image $\tilde{\mu}(H)$ is not generally a PFSS, because if $\mu(x_1) = \mu(x_2)$ then $\tilde{\mu}H_{x_1} \cap \tilde{\mu}H_{x_2} = \varnothing$, in general. From Proposition (I.5.2), it will nonetheless be true when μ is a homeomorphism of X onto X'. Indeed:

1. $\tilde{\mu}(H_x) = \tilde{\mu}(z \cdot G) = \tilde{\mu}(z) \cdot G$ if $z \in H_x$, $x = pz \in X$.

2. If σ is a local section of H on V then $\sigma' = \tilde{\mu} \circ \sigma \circ \mu^{-1}$ will be a local section of $\tilde{\mu}(H)$ on μ (V), which is differentiable because $\tilde{\mu}$ and μ^{-1} are, as well, under our hypotheses.

$\tilde{\mu}(H)$ then determines a *G'-structure S' over X'* that is *the image of S under the homeomorphism μ.* One will denote $S' = \mu \cdot S$.

μ is regular, as before, but not necessarily a homeomorphism, so now let a G-structure $S'(G, H')$ be given on X'. The set $H = \bigcup_{x \in X} H_x$, in which:

$$(5) \qquad H_x = (\tilde{\mu}_x)^{-1} H'_{\mu x} \qquad (x \in X)$$

($\tilde{\mu}_x$ denotes the restriction of $\tilde{\mu}$ to H_x) is a G-PFSS that is differentiable, from Proposition (I.5.2), because:

1) $H_x = z_x \cdot G$ if $z_x \in H_x$, from (2).

2) If σ' is a differentiable section of H' over U then $\sigma = (\tilde{\mu}_x)^{-1} \circ \sigma' \circ \mu$ will be a differentiable local section of E with values in H (it suffices to remark that μ is locally a homeomorphism). H is the inverse image of H' by μ, and it will determine a *G-structure S that is the inverse image of S' under μ.* One will note that $S = \mu^* S'$ and $H = \mu^* H'$, so H is, moreover, the inverse image of H' in the sense of the theory of fiber spaces. Hence, in particular, any covering of a space X that is endowed with a G-structure will be canonically endowed with a G-structure that is the inverse image. If $\tilde{\mu}_H$ is the restriction of $\tilde{\mu}$ to H then $\tilde{\mu}_H$ will be a representation of $H = \mu^* H'$ on H' and one can deduce from (4) that:

$$(5)[sic] \qquad \tilde{\mu}_H^* \omega' = \omega.$$

Finally, it follows from (5) that if S' is determined by $\{V_\alpha, \theta_\alpha\}$, in which $\{V_\alpha\}$ is an open covering of X that is endowed with a distinguished coframe θ_α, then $S = \mu^* S'$ will be the G-structure on X that is determined by $\{ \mu^{-1}(V_\alpha), \mu^* \theta_\alpha \}$.

When μ is a homeomorphism, the relations $S' = \mu \cdot S$ and $S = \mu^* S'$ will be equivalent, and the preceding remark will permit one to determine S'.

B) Let X and X' be endowed with G-structures S and S', resp. An *isomorphism of S onto S'* is a regular differentiable homeomorphism μ of X onto X' such that $\mu S = S'$.

An *automorphism of S* is an isomorphism of S onto itself.

If U is an open subset of X then H_U will be a G-PFSS of E_U and will define the *G-structure S_U that is induced by S on U* (if i is the identity map $U \to X$ then $S_U = i^* S$). *A* *local isomorphism of S onto S′* with source U and target V (U is open in X, V is open in X') is an isomorphism of S_U onto S'_V. Two G-structures *S and S′ are locally isomorphic* if for any pair $x \in X$, $x' \in X'$, there exists a local isomorphism of S onto S' whose source contains x and whose target contains x'.

One defines a *local automorphism of S* similarly. The set $\Gamma(S)$ of local automorphisms S constitutes a *pseudogroup of transformations of X* ([22]).

Finally, recall ([23]) that S is called *locally homogeneous* if $\Gamma(S)$ acts transitively on X and *isotropic* if the prolongation $\tilde{\Gamma}$ of $\Gamma(S)$ acts transitively on each fiber H_x of H. It will then be locally homogeneous and isotropic if $\tilde{\Gamma}$ acts transitively on any H. By abuse of language, we say that S is *transitive.*

A *first-order transitive Lie pseudogroup* can be defined ([24]) to be the pseudogroup $\Gamma(S)$ of local automorphisms of a transitive real G-structure, or more restrictively ([25]), as the pseudogroup $\Gamma \subset \Gamma(S)$ of *analytic* local automorphisms of a real G-structure S that is itself analytic. We shall adopt the former definition, with the understanding that certain converses will not be true when the givens are analytic.

C) *Transitive G-structures.*

PROPOSITION IV.1.1: *A transitive complex G-structure is equivalent to a real one.*

Let $\tilde{f}$ be the prolongation of a regular, locally-differentiable homeomorphism f of X. If $z = z_1 \cdot l$, $z_1 \in E$, $l \in CL_m$ then $\tilde{f}(z) = \tilde{f}(z_1) \cdot l$, where $\tilde{f}(z_1) \in E$; i.e., if $z \in E \cdot l$ then $\tilde{f}(z) \in E \cdot l$. Let S be transitive, while $z_0 \in H$ is fixed and $z \in H$ is arbitrary. Hence, if $z_0 \in E \cdot l$, $z \in E \cdot l$, and $H \subset E \cdot l$ then there will exist $f \in \Gamma(S)$ such that $\tilde{f}(z_0) = z$. That is equivalent to $H \cdot l^{-1} \subset E$, and establishes the proposition.

Now, let S' be equivalent to S ($H' = H \cdot l$) and $\mu \in \Gamma(S)$. $\tilde{\mu}(H_U) = H_V$ and $H'_U = H_U \cdot l$ implies that:

$$\tilde{\mu}(H'_U) = \tilde{\mu}(H_U) \cdot l = H_V \cdot l = H'_V \;;$$

i.e., $\mu \in \Gamma(S)$. Hence:

PROPOSITION IV.1.2. – *Two equivalent structures admit the same automorphisms, so in particular, they will both be transitive.*

One can say that a $\mathcal{C}$-structure $\mathcal{J}$ is transitive if a G-structure $S \in \mathcal{J}$ ($G \in \mathcal{C}$) is transitive, and that property will be independent of the chosen representative S.

([22]) For the definition of a pseudogroup of transformations, see C. Ehresmann or S. S. Chern [**9**].
([23]) P. Libermann [**19**].
([24]) Y. Matushima [**24**].
([25]) C. Ehresmann [**14**] and P. Libermann [**19**].

It follows from Proposition (IV.1.1) that the automorphisms of a transitive complex *G*-structure are those of a real *G*-structure. From that standpoint, the introduction of complex *G*-structures will not bring anything new with it, and in what follows, a transitive *G*-structure can always be supposed to be real.

Proposition (IV.1.2) admits the following partial converses:

THEOREM IV.1 ([26]). – *Let S be a G-structure and let S′ be a G′-structure that admit the same local automorphisms:*

a) *If S is transitive then it will be subordinate to S′ in the large sense, and consequently, G will be conjugate to a subgroup of G′.*

b) *If S′ is also transitive then S′ and S will be equivalent.*

c) *If G′ = G then S′ will be associated with S.*

Indeed:

a) Let $z, z_1 \in H$ (viz., the space of distinguished frames on *S*). There exist $\varphi \in \Gamma(S)$ such that $\tilde{\varphi}(z) = z_1$ and $l \in CL_m$ such that $z' = z \cdot l \in H'$. Since $\varphi \in \Gamma(S)$, one will have $\tilde{\varphi}(z') = \tilde{\varphi}(z) \cdot l = z_1 \cdot l \in H'$. Hence, for any $z_1 \in H$, one will have $z_1 \cdot l \in H'$; i.e.:

$$H' \supset H \cdot l \,. \tag{6}$$

S is subordinate to *S′* in the large sense and that will imply that $G' \supset l^{-1} \cdot G \cdot l$.

b) If *S′* is also transitive then one will deduce from *a*) that $H \supset H' \cdot l^{-1}$, and when that is compared with (6), that will yield $H' = H \cdot l$ and $G' = l^{-1} \cdot G \cdot l$.

c) If $G' = G$ without supposing that *S′* is transitive then (6) will imply that $G' \supset l^{-1} \cdot G \cdot l$, so $l \in N(G)$ and *S′* will be associated with *H*.

That theorem shows, in particular, that if Γ is a first-order Lie pseudogroup then all of the *G*-structures with the aid of which it can be defined will be equivalent, or further, *the first-order Lie pseudogroups correspond bijectively to the* ***transitive*** *$\mathcal{C}$-structures.*

2. – Properties relating to the structure tensor.

A) Let *S* (*G*, *H*) be a *G*-structure on *X*, let μ be a regular map of *X* into *X′*, and let *S* = *S′*(*G*, *H′*) be the inverse image of *S* by μ. Let $\tilde{\mu}$ also denote the restriction of $\tilde{\mu}$ to *H*. It is a *G*-representation of *H′* in *H*, and if π is a connection form on *H* then $\tilde{\mu}^* \pi = \pi'$ will

([26]) D. Bernard [**4**].

be (Chap. II. § **4**) a connection form on H'. Let ω and ω' be the fundamental forms on H and H', resp. With some self-evident notations:

$$\Sigma = d\omega + \pi \cdot \omega,$$

$$\Sigma' = d\omega' + \pi' \cdot \omega' = d(\tilde{\mu}^*\omega) + (\tilde{\mu}^*\pi)\cdot(\tilde{\mu}^*\omega),$$

from [(4), § **1**], so:

(1) $$\Sigma' = \tilde{\mu}^*\Sigma .$$

Upon passing to the associated tensors, (1) will become:

$$(t\,\Sigma') \cdot \overset{2}{\wedge}\omega' = \tilde{\mu}^*\left((t\,\Sigma)\cdot \overset{2}{\wedge}\omega\right),$$

namely:

$$(t\,\Sigma') \cdot \overset{2}{\wedge}\omega' = (\tilde{\mu}^* t\,\Sigma)\cdot \overset{2}{\wedge}\tilde{\mu}^*\omega = (\tilde{\mu}^* t\,\Sigma)\cdot \overset{2}{\wedge}\omega',$$

which is equivalent to:

(2) $$t\,\Sigma' = \tilde{\mu}^* t\,\Sigma .$$

Finally, if α (Chap. III, § **6**) is the projection of P onto M then:

$$t_{S'} = \alpha \cdot t\,\Sigma' = \alpha \cdot \tilde{\mu}^* t\,\Sigma = \tilde{\mu}^*(\alpha \cdot t\,\Sigma) = \tilde{\mu}^* t_S \, .$$

PROPOSITION IV.2.1. – *If S' is the inverse image of S under μ then its structure tensor $t_{S'}$ will be the inverse image of t_S by $\tilde{\mu}$:*

(3) $$t_{\mu^* S} = \tilde{\mu}^* t_S \, .$$

If one applies that proposition to the local automorphisms of a *G*-structure *S* then one will see that if *S* is isotropic at x_0 then t_S will be constant on H_{x_0}. *If S is transitive then t_S will be constant on H.*

DEFINITION IV.2.1. – *A G-structure S is called* ***almost-transitive*** *if it has a constant structure tensor* ([27]).

B) *Almost-transitive G-structures. Cartan conditions.* – If *S* is almost-transitive then the constant value *t* of t_S will not be arbitrary in *M*. In particular, if t_S is a tensor of type $\rho(G)$ then *t* must be invariant under $\rho(G)$.

([27]) Those structures are called "integrable" by S. S. Chern [**9**] or Y. Matsushima [**24**], as well as by the author in [**2**] and [**5**].

Example. – Let S be a π-structure with the notations of (Chap. III, § **7**, C). One has:

$$(\rho(g)t)^{\alpha_1}_{\beta_1\gamma_1'} = g^{\alpha_1}_{\alpha}\, t^{\alpha}_{\beta'\gamma'}(g^{-1})^{\beta'}_{\beta_1'}(g^{-1})^{\gamma'}_{\gamma_1'}, \qquad g \in G.$$

If one takes g such that $g^{\alpha_1}_{\alpha} = \lambda\,\delta^{\alpha_1}_{\alpha}$, $g^{\alpha_1'}_{\alpha'} = \delta^{\alpha_1'}_{\alpha'}$ then one must have $(\rho(g)t)^{\alpha_1}_{\beta_1'\gamma_1'} = \lambda\, t^{\alpha_1}_{\beta_1'\gamma_1'}$. The condition $\rho(g)\, t = t$ then demands that $\lambda\, t^{\alpha_1}_{\beta_1'\gamma_1'} = t^{\alpha_1}_{\beta_1'\gamma_1'}$ for any λ, so $t = 0$. *An almost-transitive π-structure is necessarily almost-integrable (hence, integrable).* That result applies to almost-complex structures, in particular (cf., [**9**]).

The latter condition is not also sufficient. Some necessary – and in certain cases, sufficient – conditions for $t \in M$ to have the same value as the structure tensor of an almost-transitive G-structure have been determined by E. Cartan ([**6**] and [**7**]). We shall briefly recall them and then interpret them.

Let W_1 be a supplement to V: The c^i_{jk}, which are the components of the natural projection $c = q\,(t_S)$ of t_S onto W (Chap. III, § **6**, D) in the basis on P, are constants here. We have seen that there exist forms π'^{ρ} on H that satisfy the structure equations [(15), Chap. III, § **6**], namely:

$$d\omega^i = a^i_{j\rho}\,\omega^j \wedge \pi'^{\rho} + \tfrac{1}{2} c^i_{jk}\,\omega^j \wedge \omega^k. \tag{4}$$

The forms π'^{ρ}, along with the ω^i, constitute a basis for θ^*_z at any point $z \in H$; one will then have:

$$d\pi'^{\rho} = \tfrac{1}{2}\gamma^{\rho}_{\sigma\tau}\,\pi'^{\rho} \wedge \pi'^{\tau} + u^{\rho}_{\sigma i}\,\pi^{\sigma} \wedge \omega^i + \tfrac{1}{2} v^{\rho}_{ij}\,\omega^i \wedge \omega^j, \tag{5}$$

in which $\gamma^{\rho}_{\sigma\tau} = -\gamma^{\rho}_{\tau\sigma}$, $u^{\rho}_{\sigma i}$, $v^{\rho}_{ij} = -\,v^{\rho}_{ji}$ are functions on H. By exterior differentiation of (4), $d\,(d\omega^i) = 0$ will give:

$$a^i_{j\tau}\,a^j_{l\sigma} - a^i_{j\sigma}\,a^j_{l\tau} = \gamma^{\rho}_{\sigma\tau}\,a^i_{l\rho}, \tag{6}$$

$$a^i_{j\rho}\,c^j_{lm} + a^j_{i\rho}\,c^i_{mj} + a^j_{m\rho}\,c^i_{jl} = a^i_{m\sigma}u^{\sigma}_{\rho l} - a^i_{l\sigma}u^{\sigma}_{\rho m}, \tag{C_1}$$

$$c^i_{pk}\,c^p_{lm} + c^i_{pl}\,c^p_{mk} + c^i_{pm}\,c^p_{kl} = a^i_{k\rho}v^{\rho}_{lm} + a^i_{l\rho}v^{\rho}_{mk} + a^i_{m\rho}v^{\rho}_{kl}. \tag{C_2}$$

Since the relation (6) is nothing but $[\varepsilon_\tau, \varepsilon_\sigma] = \gamma^{\rho}_{\sigma\tau}\,\varepsilon_\rho$, the first equations are always compatible, because the $\gamma^{\rho}_{\sigma\tau}$ are the structure constants of the Lie group G in the basis $\{\varepsilon_\rho\}$.

DEFINITION IV.2.2. – *We say that $t \in M$ satisfies the* ***Cartan conditions relative to the group*** *G if the relations* (C_1) *and* (C_2) *are compatible for a basis* $\varepsilon_\rho = (a^i_{j\rho})$ *on* $\underline{G}$, *while the* c^i_{jk} *are the components of* $c = q\,(t)$.

As we have seen, a necessary condition for t to be the structure tensor of an almost-transitive structure is that $\rho\,(G)\,t = t$, so if $\lambda \in \underline{G}$ then $\tilde{\rho}(\lambda)t = 0$, namely, since $t = \alpha \cdot c$:

$$\tilde{\rho}(\lambda)\alpha \cdot c = 0, \qquad \lambda \in \underline{G}. \tag{7}$$

Since $\alpha \circ \mathcal{R}\,(g) = \rho\,(g) \circ \alpha\,(g \in G)$, one will also have $\alpha \circ \tilde{\mathcal{R}}(\lambda) = \tilde{\rho}\,(g) \circ \alpha$, and (7) will be equivalent to $\alpha\tilde{\mathcal{R}}(\lambda)c = 0$; i.e., $\tilde{\mathcal{R}}(\lambda)c \in V$. Since $V = A(N)$, that condition can be expressed as follows: There exists an $\xi \in N$ such that:

$$\tilde{\mathcal{R}}(\lambda)c = A\,(\xi). \tag{8}$$

Now:

$$(\tilde{\mathcal{R}}(\lambda)c)^i_{lm} = \lambda^i_j \cdot c^j_{lm} - c^i_{jm} \cdot \lambda^j_l - c^i_{lj} \cdot \lambda^j_m = \lambda^i_j\, c^j_{lm} + \lambda^j_l c^i_{mj} + \lambda^j_m\, c^i_{jl},$$

and (8) can be written:

$$\lambda^i_j\, c^j_{lm} + \lambda^j_l c^i_{mj} + \lambda^j_m\, c^i_{jl} = a^i_{m\rho}\,\xi^\rho_l - a^i_{l\rho}\,\xi^\rho_m, \tag{9}$$

in such a way that equations (C_1) are nothing but that condition (9) when it is applied to all of the elements ε_ρ of the basis for $\underline{G}$. (C_1) then expresses the idea that $\tilde{\rho}(\lambda)t = 0$ for any $\lambda \in \underline{G}$. That interpretation is not essentially different from the one that was given by E. Cartan in [**6**], § **36**, but it can be expressed more simply thanks to the notion of structure tensor.

Now, let Ω be the curvature of an *S*-connection. It is a tensorial 2-form with values in $\underline{G}$, and if $R = t\,\Omega$ is its associated tensor with values in $\underline{G} \otimes \overset{2}{\wedge} \mathrm{R}^*_m$ then one will have:

$$\Omega = R \cdot \overset{2}{\wedge}\,\omega.$$

The components Ω^ρ of Ω in $\{\varepsilon^\rho\}$ are then:

$$\Omega^\rho = \tfrac{1}{2} R^\rho_{lm}\,\omega^l \wedge \omega^m$$

(R^ρ_{lm} are the components of R), and since $\Omega = \varepsilon_\rho \otimes \Omega^\rho$, one can deduce the components of Ω in the canonical basis for $\underline{L_m}$:

(10) $$\Omega^i_j = a^i_{j\rho}\,\Omega^\rho = \tfrac{1}{2} a^i_{j\rho}\, R^\rho_{lm}\, \omega^l \wedge \omega^m .$$

As always, we let S be a transitive G-structure and address the case in which W_1 is invariant under $\mathcal{R}$ (G), in such a way that there will exist (Theorem III.6.2) an S-connection γ such that $t_\Sigma = t_S$ (= constant). We seek the explicit form for the Bianchi identity for that connection:

(11) $$\nabla\Sigma = \Omega \cdot \omega.$$

The two sides of (11) are tensorial 3-forms whose associated tensors must be calculated.

For the right-hand side, one will have:

$$\begin{aligned}(\Omega \cdot \omega)^i &= \Omega^i_k \wedge \omega^k = \tfrac{1}{2} a^i_{k\rho}\, R^\rho_{lm}\, \omega^l \wedge \omega^m \wedge \omega^k \\ &= \frac{1}{3!}(a^i_{k\rho}\, R^\rho_{lm} + a^i_{l\rho}\, R^\rho_{mk} + a^i_{m\rho}\, R^\rho_{kl})\, \omega^k \wedge \omega^l \wedge \omega^m ,\end{aligned}$$

namely:

(12) $$(t(\Omega\cdot\omega))^i_{klm} = a^i_{k\rho}\, R^\rho_{lm} + a^i_{l\rho}\, R^\rho_{mk} + a^i_{m\rho}\, R^\rho_{kl} .$$

For the left-hand side, $t\ \nabla\Sigma$ is given by Proposition (III.5). Nevertheless, formula (21) can be simplified, since $t\Sigma$ is constant, so $D\Sigma = t\nabla t\ \Sigma = 0$. The components of $t\ \Sigma = t_S$ are denoted by c^i_{jk}, as always, so one will also $c^i_{jk} = -\ 2S^i_{jk}$, and formula (21) will give:

(13) $$(t\nabla\Sigma)^i_{klm} = c^i_{jk}\, c^j_{lm} + c^i_{jl}\, c^j_{mk} + c^i_{jm}\, c^j_{kl} ,$$

in such a way that the Bianchi identity (11) can be written:

$$c^i_{jk}\, c^j_{lm} + c^i_{jl}\, c^j_{mk} + c^i_{jm}\, c^j_{kl} = a^i_{k\rho}\, R^\rho_{lm} + a^i_{l\rho}\, R^\rho_{mk} + a^i_{m\rho}\, R^\rho_{kl} ,$$

which shows that equations (C_2) are necessarily compatible in V^ρ_{lm} and admit the solution $V^\rho_{lm} = R^\rho_{lm}$. We state the:

PROPOSITION IV.2.2. – *The Cartan condition* (C_1) *expresses the invariance of* t *under* $\tilde{\rho}(\underline{G})$ *and can be written* $\tilde{\rho}(\underline{G})\cdot t = 0$. *Under the hypotheses of Theorem (III.6.2), the condition* (C_2) *is simply the translation of the Bianchi identity in terms of the associated tensor.*

3. – Involutive analytic *G*-structures ([28]).

A) Let S (G, H) be an almost-transitive analytic G-structure, and let μ be a local automorphism of S with source U and target V, while U is restricted in such a way that it can be endowed with local sections. If $\tilde{\mu}$ denotes the prolongation of μ to H, while ω_U and ω_V denote the restrictions of ω to H_U and H_V, resp., then will have $\tilde{\mu}^*\omega_V = \omega_U$ from [(5), § **1**]. Let s be a section $U \to H_U$, and let f be the map $U \to H_U \times H_V$, $x \mapsto (\sigma(x), \tilde{\mu}(\sigma(x)))$. Hence:

$$f^*(\omega_U - \omega_V) = \sigma^*\omega_U - \sigma^*\tilde{\mu}^*\omega_V = \sigma^*(\omega_U - \tilde{\mu}^*\omega_V) = 0,$$

so f will then define an integral manifold of the Pfaff system:

$$(1) \qquad \omega_U = \omega_V \qquad \text{or} \qquad \omega_U^i = \omega_V^i \qquad (i = 1, 2, \ldots, m),$$

which is an m-dimensional integral manifold "that does not introduce any relation between the ω_U^i " (or "with independent variables x^i " that are the local coordinates of x).

Conversely, such an integral manifold is identified by a map $f : U \to H_U \times H_V$, $x \mapsto (\sigma(x), g(x))$ such that $f^*(\omega_U - \omega_V) = 0$. That, in its own right, defines a map $\mu : U \to V$, $\mu = p \circ g$, which one easily verifies to be a local automorphism of S.

The system (1) is closed by adding the equations:

$$d\omega_U^i = d\omega_V^i ,$$

namely, from (15) (Chap. III, § **6**) and the hypothesis of almost-transitivity:

$$a_{j\rho}^i\, \omega_U^j \wedge \pi_U'^\rho + \tfrac{1}{2} c_{jk}^i\, \omega_U^j \wedge \omega_U^k = a_{j\rho}^i\, \omega_V^j \wedge \pi_V'^\rho + \tfrac{1}{2} c_{jk}^i\, \omega_V^j \wedge \omega_V^k ,$$

which are equations that can be written:

$$(2) \qquad a_{j\rho}^i\, \omega_V^j \wedge (\pi_V'^\rho - \pi_U'^\rho) = 0$$

when one takes (1) into account.

The involution criteria show that the involution of the closed system (1), (2) with respect to the independent variables $x^1, \ldots, x^m$ depends upon only the coefficient $a_{j\rho}^i$, and even that it depends upon only $\underline{G}$, and not upon the particular choice of the basis $\varepsilon_\rho = (a_{j\rho}^i)$. When those conditions are realized, *G will be called* ***involutive****.* A *G-structure S is involutive* if G is involutive.

One can then give the following statement to E. Cartan's third fundamental theorem:

([28]) We shall not recall the theory of differential systems in involution, but only refer the reader to [**6**], [**8**], and [**9**].

PROPOSITION IV.3.1. (E. Cartan). – *If G is involutive then the Cartan conditions will be sufficient for there to exist an almost-transitive analytic G-structure with structure tensor t.*

One see from the criteria, moreover, that if G is involutive then its conjugates in L_m will also be so, in such a way that one can speak of *involutive $\mathcal{C}$-structures,* or *involutive Lie pseudogroups.*

B) LEMMA IV.3. – *If G is involutive then if S and S' are two almost-integrable G-structures such that $t_S = t_{S'}$ then there will exist a local isomorphism of S onto S' that maps an arbitrary distinguished frame z on S to an arbitrary distinguished frame z' on S'. In particular, those structures are locally isomorphic.*

Indeed, while preserving the notations of A), the determination of such a local isomorphism μ amounts to the determination of an integral manifold "of dimension m with independent variables x^i " of the closed system (1), (2) that passes through the pair (z, z'). Since that system is in involution with respect to the x^i, because it does not have finite equations, any pair $(z, z') \in H_U \times H'_V$ is an integral point, and since the system of forms $\omega_U^i - \omega_V^i$ is everywhere of rank m, it is a regular integral point, and that will suffice to confirm the existence of our integral manifold. We will then deduce that:

THEOREM IV.3. ([29]). – *If G is involutive then an analytic G-structure S that is almost-transitive will be transitive, and an almost-integrable one will be integrable.*

The first assertion is an immediate consequence of the Lemma. On the other hand, let S be almost-integrable. The G-structure S' that is defined on R^m by the PFSS $H' = \{R_y \cdot G\}$, $y \in \mathrm{R}^m$, in which R_y is the natural frame at y with the canonical coordinates y^i is integrable and analytic. Since $t_S = t_{S'} = 0$, the lemma shows that for any $x \in X$, there exists a local isomorphism $f : U \subset X \to V \subset \mathrm{R}^m$, in which $x \in U$. Hence, if the coframe $dy = \{dy^i\}$ is distinguished for S then the coframe $\theta = f^*dy$ will be distinguished for S. Its components will be $\theta^i = f^*dy = d(f^*y^i) = dz^i$. If f is a regular differentiable homeomorphism then the functions $z^i = f^*y^i$ will be local coordinates on U. θ will then be a natural coframe for the distinguished local coordinates on S, which will then be integrable.

All of the cases of the integrability of analytic almost-integrable G-structures that were encountered up to now are included in the applications of the latter theorem, since their groups are involutive: e.g., almost-product structures, almost-complex ones, example F), Chap. III, § **8**, almost-symplectic structures ([30]).

C) *Locally-similar Lie pseudogroups* ([31]). – Let L be the subspace of $t \in M$ that satisfy the Cartan conditions for a certain group $G \subset L_m$. If S is almost-transitive then

([29]) D. Bernard [**2**] and [**5**].
([30]) $G = Sp(m, \mathrm{R})$. Cf., P. Libermann [**19**].
([31]) Results of [**2**], up to presentation.

from formula [(1), Chap. III, § **7**], the same thing will be true for any equivalent structure; viz., it is a property of the $\mathcal{C}$-structure $\mathcal{I}$. In particular, all of the *G*-structures S' that are associated with *S* are almost-transitive, and their structure tensors $t_{S'} \in L$ belong to the same intransitivity class of *M* under the group $G^* = \bar{\rho}(N(G))$ (Chap. III, § **7**, A).

Conversely, let S (G, H) and $S'(G, H')$ be almost-transitive structures on *X* and X' such that $t_{S'} = t_S$ modulo G^*. Let $n \in N(G)$ such that $t_{S'} = \bar{\rho}(n^{-1})\, t_S$. If $S''(G, H'')$ is defined by $H'' = H \cdot n$ then one will have $t_{S''} = \bar{\rho}(n^{-1})\, t_S = t_{S'}$, and if one supposes that *G* is involutive then, from Lemma (IV.3), S'' will be locally isomorphic to S'. One can then say that the condition $t_{S'} = t_S$ modulo G^* is necessary and sufficient for *S* to be equivalent to a structure that is locally-isomorphic to S'. Nonetheless, the result will become clearer when one expresses it in terms of the pseudogroups $\Gamma(S')$ and $\Gamma(S) = \mathrm{G}(S'')$.

Indeed, let $f : U \to V$ be a local isomorphism of S'' with S'. Let Γ_U (Γ_V, resp.) be the restriction of $\Gamma(S)$ to *U* (of $\Gamma(S')$ to *V*, resp.). If $g \in \Gamma_U$ then its transmutation $\varphi = f \circ g \circ f^{-1}$ will be a product of local isomorphisms of *G*-structures ($S' \to S'' \to S'' \to S'$), so it will be a local automorphism of S': $\varphi \in \Gamma'_V$. One deduces that $\Gamma'_V = f \circ \Gamma_U \circ f^{-1}$ from this. By analogy with transformation groups, one says that Γ_U and Γ'_V *are similar* and that:

DEFINITION IV.3. – *Two pseudogroups of transformations Γ on X, Γ' on X' are* ***locally similar*** *if for any pair $x \in X$, $x' \in X'$ there exists a neighborhood U of x and a neighborhood V of x' such that the restrictions Γ_U and Γ'_V are similar.*

If *S* and S' are locally isomorphic then one will see that $\Gamma(S)$ and $\Gamma(S')$ are locally similar. $t_S = t_{S'}$ modulo G^* then implies: "$\Gamma(S)$ is locally similar to $\Gamma(S')$."

Conversely, let *S* and S' be two almost-transitive *G*-structures such that $\Gamma(S)$ and $\Gamma(S')$ are locally similar. If $f : U \to V$ realizes the similarity of Γ_U *and* Γ'_V then the *G*-structure on *U* – namely, $f^*S'_V$ – will admit the same local automorphisms as S_U; it will then be associated with it (Theorem IV.1). One then deduces that $t_{S'} = t_S$ modulo G^*. If one calls t_S (or the components c^i_{jk} of $c = q(t_S)$) a "system of structure constants for $\Gamma(S)$" (E. Cartan) and the intransitivity class of t_S modulo G^* the "family of systems of structure constants for $\Gamma(S)$" (Matsushima [**24**]), or more simply, the *characteristic family* of $\Gamma(S)$ then one can state:

THEOREM IV.3. – *In order for two first-order, transitive, involutive Lie pseudogroups to be locally similar, it is necessary and sufficient that they should have the same characteristic family.*

One deduces from that theorem, along with Theorem (IV.1) and Proposition (IV.3.1), that:

COROLLARY. – *The first-order, transitive, involutive Lie pseudogroups correspond bijectively to the pairs that consist of a class of conjugate involutive linear groups and an* *intransitivity class of L under G^*, when a representative G has been chosen from each class.*

Example. – Let $m = 3$, and let G be the group that was indicated in paragraph F), Chap. III, § **8**. We have seen that one can choose the supplement W of V in such a fashion that $c = q\,(t_S)$ has components $\lambda^i = c^i_{23}$ ($i = 1, 2, 3$) and $c^i_{12} = c^i_{13} = 0$; i.e., that the structure equations are:

$$d\omega^i = \pi^i \wedge \omega^1 + \lambda^i\, \omega^2 \wedge \omega^3.$$

The Cartan condition is $\lambda^1 = 0$. L will then be a two-dimensional subspace of W with coordinates λ^2, λ^3. $N\,(G)$ is the group of matrices:

$$n = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \beta' & \gamma' \\ 0 & \beta'' & \gamma'' \end{pmatrix} \qquad (\det n \neq 0),$$

and $\rho\,(n)$ acts on L by:

$$(3) \qquad (\lambda^2, \lambda^3) \mapsto \left(\lambda'^2 = \frac{\beta'\lambda^2 + \beta''\lambda^3}{\beta'\gamma'' - \gamma'\beta''}, \lambda'^3 = \frac{\gamma'\lambda^2 + \gamma''\lambda^3}{\beta'\gamma'' - \gamma'\beta''}\right).$$

Formulas (3) define the group G^*. There are only two intransitivity classes with representatives $c_0 = (0, 0)$ and $c_1 = (1, 0)$.

c_0 corresponds to the Lie pseudogroups that are locally similar to the pseudogroup that acts on R^3 and has the finite equations:

$$X = f\,(x), \qquad Y = g\,(x) + y, \qquad Z = h\,(x) + z$$

(f, g, h are arbitrary analytic functions).

c_1 corresponds to the Lie pseudogroups that are locally similar to the pseudogroup that is defined in the half-space $y > 0$ of R^3 by the finite equations:

$$X = f\,(x), \qquad Y = g\,(x) \cdot y, \qquad Z = g\,(x) \cdot z + h\,(x)\,.$$

4. Infinitesimal automorphisms.

A) *Lie derivatives.* – Let η be a vector field or infinitesimal transformation (*i.t.*) on X. The local one-parameter group of transformations that it determines will be denoted by $\exp\,(t\eta)$ ([32]), and the Lie derivative with respect to η, by $\mathcal{L}\,(\eta)$.

If Φ (φ, resp.) is a form on X with values in $\mathcal{L}\,(M, P)$ (M, resp.) (Chap. II, § **2**) then since $\mathcal{L}\,(\eta)$ is a derivation of degree zero on scalar forms, one will see immediately that:

([32]) In all of this paragraph, the definitions of the notations that are not specified completely here can be found in A. Lichnerowicz [**23**].

(1) $$\mathcal{L}(\eta)(\Phi \cdot \varphi) = (\mathcal{L}(\eta)\Phi) \cdot \varphi + \Phi \cdot (\mathcal{L}(\eta)\varphi).$$

Just as a transformation μ of X has a prolongation $\tilde{\mu}$ to $E(X)$ (cf., § **1**), similarly, an *i.t.* η on X will have a prolongation $\tilde{\eta}$ to $E(X)$. It can be defined by:

$$\exp(t\tilde{\eta}) = \widetilde{(\exp(t\eta))}$$

and will consequently satisfy:

(2) $$p \circ \exp(t\tilde{\eta}) = \exp(t\eta) \circ p,$$

so

(3) $$\mathcal{L}(\tilde{\eta}) \circ p^* = p^* \circ \mathcal{L}(\eta).$$

If Φ is a tensorial form on $E(X)$ then the form $\mathcal{L}(\tilde{\eta})\Phi$ will be called the "Lie derivative of Φ with respect to η," and is often denoted by $\mathcal{L}(\eta)\Phi$. (We shall avoid using that notation.) Let θ_U be a coframe on $U \subset X$. It is a 1-form on U with values on R^m. $\mathcal{L}(\eta)\theta_U$ is also a 1-form with values on R^m then, in such a way that if z_U is the frame dual to θ_U then $(\mathcal{L}(\eta)\theta_U)_x \circ z_U(x)$ will be an endomorphism $a_U(x)$ of R^m and:

$$(\mathcal{L}(\eta)\theta_U)_x = a_U(x)\, z_U^{-1}(x);$$

i.e., there will exist a function $a_U : U \to \underline{L_m}$ such that:

(4) $$\mathcal{L}(\eta)\theta_U = a_U \cdot \theta_U.$$

Conversely, the local function a_U determines the transform of the coframe θ_U by a finite transformation of the one-parameter group that is generated by η:

(5) $$(\exp(t\eta)^*\theta_U)_x = \left(\exp\int_0^t a_U[x(\tau)]\,d\tau\right) \cdot (\theta_U)_x,$$

in which $x(\tau) = \exp(\tau\eta) \cdot x$ and in which the exp in the right-hand side denotes the exponential representation $\underline{L_m} \to L_m$.

Let g_U be the function on E_U with values in $L_m \subset \underline{L_m}$ that is defined by the local chart that is associated with z_U : $z = z_U(pz) \cdot g_U(z)$ for $z \in E_U$. If g_U^{-1} is the function $z \mapsto \left(g_U(z)^{-1}\right)$ then one will have the following representation of the fundamental form θ for $E(X)$ in E_U:

(6) $$\theta = g_U^{-1} \cdot p^*\theta_U .$$

Now, it follows from $\tilde{\mu}^*\theta = \theta$ [(4), § **1**] that $\mathcal{L}(\tilde{\mu})\theta = 0$, and upon applying (1), (3), and (4), the Lie derivative of (6) will give:

$$\mathcal{L}(\tilde{\eta})\theta = \left[\mathcal{L}(\tilde{\eta})g_U^{-1} + g_U^{-1}(p^*a_U)\right] p^*\theta_U ,$$

so

(7) $$\mathcal{L}(\tilde{\eta})g_U^{-1} = -g_U^{-1}(p^*a_U)$$

and

(8) $$\mathcal{L}(\tilde{\eta})g_U = (p^*a_U)\cdot g_U .$$

Let Φ be a tensorial form on *E* (*X*) with values in a vector space *M* and type $\mathcal{R}(L_m)$. One will then have a local representation of Φ in E_U that is analogous to (6):

(9) $$\Phi = \mathcal{R}(g_U^{-1})\cdot p^*\Phi_U , \quad \text{in which} \qquad \Phi_U = z_U^* \, \Phi .$$

One deduces from (7) by a simple calculation that:

(10) $$\mathcal{L}(\tilde{\eta})(g_U^{-1}) = -\mathcal{R}(g_U^{-1})\cdot\tilde{\mathcal{R}}(p^*a_U),$$

and then, from (9), that:

(11) $$\mathcal{L}(\tilde{\eta})\Phi = \mathcal{R}(g_U^{-1})\cdot p^*[\mathcal{L}(\eta)\Phi_U - \tilde{\mathcal{R}}(a_U)\cdot\Phi_U],$$

namely:

(12) $$(\mathcal{L}(\tilde{\eta})\Phi)_U = \mathcal{L}(\eta)\Phi_U - \tilde{\mathcal{R}}(a_U)\cdot\Phi_U .$$

Finally, if π is a connection form on *E* then one will have:

(13) $$\pi = (\text{ad } g_U^{-1})\cdot p^*\pi_U + g_U^{-1}\cdot dg_U$$

locally in *E*, so one will deduce that:

(14) $$\mathcal{L}(\tilde{\eta})\pi = (\text{ad } g_U^{-1})\cdot p^*[\mathcal{L}(\eta)\,\pi_U + [\pi_U, a_U] + da_U]$$

by a calculation that involves only formulas that were established already, in which $\pi_U = z_U^* \, \pi$. That exhibits the tensorial character of $\mathcal{L}(\tilde{\eta})\pi$, and can be written as:

(15) $$(\mathcal{L}(\tilde{\eta})\pi)_U = \mathcal{L}(\eta)\,\pi_U + [\pi_U, a_U] + da_U\,.$$

B) *Infinitesimal automorphisms (i.a.) of a G-structure S.* – An *i.t.* of X is an *i.a.* of S if $\exp(t\eta)$ is an automorphism of S for any t for which it is defined. If ω_U is a distinguished coframe of S on U then the function a_U with values in $\underline{L_m}$ that is defined by (4) such that $\mathcal{L}(\eta)\,\omega_U = a_U\,\omega_U$ will take its values in $\underline{G}$. Conversely, from (5), if a_U has values in $\underline{G}$ then $\exp(t\,\eta)^*\omega_U$ will be a distinguished coframe of S and η will be an *i.a.*

Among the G-structures that we have considered are (Chap. III, § **2**) the "G-structures that are defined by a tensor." Upon recalling those notations, we shall establish the:

PROPOSITION IV.4.1. – *If S is a G-structure that is defined by a tensor t on E* (X) *then in order for* η *to be an i.a. of S, it will be necessary and sufficient that* $\mathcal{L}(\tilde{\eta})\,t = 0$.

Indeed, from (12), one will have $(\mathcal{L}(\tilde{\eta})\,t)_V = \mathcal{L}(\eta)\,t_V - \tilde{\mathcal{R}}(a_V)\,t_V$ in an open subset $V \subset X$ that is endowed with a distinguished coframe ω_V of S. Since ω_V is a distinguished coframe, $t_V = u$ will be constant, and $\mathcal{L}(\eta)\,t_V = 0$. In order to have $\mathcal{L}(\tilde{\eta})\,t = 0$, it is necessary and sufficient then that $\tilde{\mathcal{R}}(a_V)\,u = 0$. Now, in order for that to be true, it is necessary and sufficient that $a_V(x) \in \underline{G}$; i.e., that η should be an *i.a.*. Q.E.D.

That shows, in particular, that there is an identity between the infinitesimal isometries of a Riemannian structure and the *i.a.* of the $O(m)$-structure that it determines.

Let $\hat{G}$ be the group of matrices $\lambda \cdot g$ (λ real > 0, $g \in G$) and let $\hat{S}$ be the $\hat{G}$-structure that is an extension of S. An automorphism (*i.a.*, resp.) of $\hat{S}$ can be called a *conformal transformation* (*infinitesimal conformal transformation, or i.c.t.*, resp.) of S. In particular, a conformal transformation μ is a "homothety" if there exists an $\lambda \in \mathrm{R}$ such that for any distinguished coframe ω_U of S, $(1/\lambda)\,\mu^*\omega_U$ will again be a distinguished coframe of S. In order for η to be an *infinitesimal homothety* – i.e., in order for $\exp(t\eta)$ to be a homothety for any t – it is necessary that $a_U = k\,I + \alpha_U$, $\alpha_U(x) \in \underline{G}$. From (5), that will suffice, because one will then have:

$$\int_0^t a_U[x(\tau)]\,d\tau = kt\,I + \int_0^t \alpha_U[x(\tau)]\,d\tau = kt\,I + \beta_U(x, t), \qquad \beta_U(x, t) \in \underline{G}\,,$$

and $\exp(kt\,I + \beta_U(x, t)) = e^{kt} \cdot \exp \beta_U(x, t)$, since $kt\,I$ and $\beta_U(x, t)$ commute. One will then have:

$$(\exp(t\,\eta)^*\,\omega_U)_x = e^{kt} \cdot \exp \beta_U(x, t)(\omega_U)_x\,, \qquad \text{in which } \exp \beta_U(x, t) \in G\,;$$

i.e., $e^{kt} \cdot (\exp(t\,\eta)^*\omega_U)$ will be a distinguished coframe and η will be an infinitesimal homothety.

The definitions that were given here again coincide with the usual notations in the Riemannian case.

Let $S' = S \cdot l$ be a G-structure that is equivalent to S. If ω_U is a distinguished coframe of S then $\omega'_U = l^{-1} \cdot \omega_U$ will be a distinguished coframe of S'. For an *i.t.* η, one has:

$$\mathcal{L}(\eta)\,\omega'_U = l^{-1} \cdot \mathcal{L}(\eta)\,\omega_U = l^{-1} \cdot a_U \cdot \omega_U = l^{-1} \cdot a_U \cdot l \cdot \omega'_U .$$

If $a_U \in \underline{G}$ ($\hat{\underline{G}}$, resp.) then $l^{-1} \cdot a_U \cdot l \in \underline{G}'$ ($\hat{\underline{G}}'$, resp.), in such a way that there will be an identity between the *i.a.*'s (*i.c.t.*'s, resp.) of the structures S and S'; they are the *i.a.*'s (*i.c.t.*'s, resp.) of the $\mathcal{C}$-structure that is determined by S and S'. The same thing will be true for infinitesimal homotheties.

If S' is an extension of S then any *i.a.* (*i.c.t*, resp.) of S will also be an *i.a.* (*i.c.t.*, resp.) of S'. The converse is obviously not true, in general, and leads one to pose the problem:

P_1 : *If S is subordinate to S′ then under what conditions can one assert that an i.a. of S′ will also be an i.a. of A* ?

If the G-structures S for a group G admit a canonical S-connection of the form π_S (i.e., such that $\tilde{\mu}^* \pi_{S'} = \pi_S$ for any isomorphism μ of a G-structure S on the G-structure S') then an *i.a.* of S will be an affine *i.t.* of π_S, since $\exp\,(t\tilde{\eta})^* \pi_S = \pi_S$ implies that $\mathcal{L}(\tilde{\eta})\,\pi_S = 0$. The converse, which is generally false, even when there exists a canonical S-connection, poses the problem:

P_2 : *If γ is an S-connection then when will an affine i.t. for γ be an i.a. for S* ?

These two problems, which are well-known in the case of the orthogonal group and some of its subgroups (cf., [**23**]) have been studied under very general hypotheses by R. Hermann ([**16**] and [**17**]). We shall conclude by recalling Hermann's method and deducing some results from it that are generally broader in scope than his are.

C) *Hermann's lemma.* – Let $G \subset G' \subset L_m$ be subgroups. Suppose that G is reductive in G, and let $\underline{G}' = \underline{G} \oplus M$ be a direct sum decomposition, where ad $(G)\, M \subset M$. Let S be G-subordinate to a G'-structure S', and let η be an *i.a.* of S'. If ω_U is a distinguished coframe of S then its Lie derivative will be $\mathcal{L}(\eta)\,\omega_U = a_U \cdot \omega_U$. Decompose a_U according to:

$$a_U = b_U + c_U , \qquad b_U(x) \in \underline{G} , \qquad c_U(x) \in M.$$

PROPOSITION IV.4.2. –

1. *The c_U define a tensor C of adjoint type – i.e., a field of endomorphisms of the tangent space to X – whose vanishing is the necessary and sufficient condition for η to be an i.a. of S.*

2. *If η is an affine i.t. of an S-connection γ, moreover, then C will have a zero covariant derivative under γ.*

If the change of distinguished coframe of S is defined in $U \cap V$ by $\omega_V = M_{UV}\, w_U$ (where M_{UV} is a function on $H \cap V$ with values on G) then the function a_U will transform into ([33]):

$$a_V = (\mathrm{ad}\, M_{UV})\, a_U + i\,(\eta)\,(dM_{UV}\,(M_{UV})^{-1}),$$

so, upon taking the parts on both sides that have their values in M :

$$c_V = (\mathrm{ad}\, M_{UV})\, c_U\ ,$$

which proves the first result.

If η is an affine *i.t.* of the S-connection γ with the form π then upon taking the parts of the two sides of the relation (15) that have their values in M, one will get:

$$[\pi_U, c_U] + dc_U = 0, \qquad \text{i.e.,} \qquad (\nabla C)_U = 0,$$

which establishes the second result.

D) *In regard to Problem* P_1, we shall establish the:

THEOREM IV.4.1. ([34]). – *If G a subgroup of O* (m) *and S is an* ***almost-integrable*** *G-structure on X then there will be an identity between the infinitesimal isometries of the Riemannian structure that is defined by S and the i.a.'s of S in the following cases:*

1. *X is compact.*

2. *X does not admit a 2-form with zero covariant derivative; for example, X is irreducible and does not admit a Kählerian structure.*

3. *X is irreducible and Kählerian with non-zero Ricci curvature.*

4. *X is complete and there is at least one point where it admits a non-degenerate Ricci curvature.*

G is reductive in O (m) for the decomposition $\underline{O(m)} = \underline{G} + M$, where M is the orthogonal complement to $\underline{G}$ in $\underline{O(m)}$ for the metric that is defined by $\left(\alpha_j^i\right)\cdot\left(\alpha_l'^k\right) = \sum_{i,j} \alpha_j^i\, \alpha_j'^i$. If S is almost-integrable then it will admit an S-connection γ with zero torsion, which will then induce a Euclidian connection with zero torsion; i.e., the Riemannian

([33]) $i\,(\eta)\,\Phi$ denotes the interior product of the form Φ with the vector field η.

([34]) Theorem 5 of [**17**] gives only the result that if X is compact then the second Betti number will be zero.

connection. If an infinitesimal isometry η is an affine *i.t.* for the Riemannian connection then it will be an affine *i.t.* for g, and one can apply the lemma to it.

Let ω_U be a distinguished coframe of S on $U \subset X$, and keep the notations of the lemma. We must show the vanishing of C under the various hypotheses. C is a tensor of adjoint type with values in $M \subset \underline{O(m)}$ with antisymmetric components C^i_j, and $\alpha_{ij} = g_{ik}C^k_j$ are the components of a 2-form α with zero covariant derivative (since $\nabla g = \nabla C = 0$). That proves the theorem under hypothesis 2). (For the example, see A. Lichnerowicz [**22**], pp. 266). Under the hypotheses 3) and 4), any 2-form with zero covariant derivative will determine an element of the Lie algebra of the homogeneous holonomy group (cf., A. Lichnerowicz [**22**], pp. 250 and [**23**], pp. 104), so $C_U(x) \subset \underline{\sigma_{z_U(x)}} \subset \underline{G}$, since $z_U(x)$, which is the frame dual to $(\omega_U)_x$, is a distinguished frame of S, and $C_U(x) \in \underline{G} \cap M$ implies that $C = 0$.

We shall now address the compact case and utilize the following abbreviated notations: ω^j are the components of ω_U, π^i_j are the components of $\pi_U = z^*_U\,\pi$ (π is the connection form for γ), a^i_j (C^i_j, resp.) are the components of a_U (C_U, resp.). If π is the Riemannian connection then one will have:

$$\pi^i_j + \pi^j_i = 0, \tag{16}$$

$$d\omega^i + \pi^i_j \wedge \omega^k = 0. \tag{17}$$

Now, $\mathcal{L}(\eta)\,\omega_U = a_U \cdot \omega_U$; i.e.:

$$di(\eta)\,\omega^i + i(\eta)\,d\omega^i = a^i_j\,\omega^j, \tag{18}$$

but $i(\eta)\,\omega^i = \eta^i$ are the components of η in the basis $z_U(x)$, and (17) will then give:

$$i(\eta)\,d\omega^i + (i(\eta)\pi^i_j)\cdot\omega^i - \pi^i_k\,\eta^k = 0.$$

(18) then becomes:

$$a^i_j\,\omega^j = d\eta^i + \pi^i_k\,\eta^k - (i(\eta)\pi^i_j)\,\omega^j = (\nabla_j\,\eta^i - i(\eta)\pi^i_j)\,\omega^j \tag{19}$$

so one will deduce that:

$$a^i_j = \nabla_j\,\eta^i - i(\eta)\pi^i_j. \tag{20}$$

First of all, the fact that η is an infinitesimal isometry is equivalent to saying that $a_U \in \underline{O(m)}$, and thus, to $a^i_j + a^j_i = 0$. Since $(i(\eta)\pi^i_j) \in \underline{O(m)}$, one will recover the necessary and sufficient condition for η to be an infinitesimal isometry:

(21) $$\nabla_j \eta^i + \nabla_i \eta^j = 0.$$

On the other hand, if γ is an S-connection and ω_U is a distinguished coframe for S then $(i(\eta)\pi^i_j) \in \underline{G}$. One will then deduce from (20), with some notations that should be obvious, that:

(22) $$C^i_j = (a^i_j)_M = (\nabla_j \eta^i)_M .$$

Let the vector field $\xi = C \cdot \eta$ have components $\xi^i = c^i_j \eta^j$ in $z_U(x)$; then:

$$\nabla_i \xi^i = c^i_j (\nabla_i \eta^j), \qquad \text{since} \quad \nabla_k c^i_j = 0,$$

and from (21):

$$\nabla_i \xi^i = -\sum_{i,j} c^i_j (\nabla_j \eta^i) = -\sum_{i,j} c^i_j (c^i_j + (\nabla_j \eta^i)_{\underline{G}}),$$

namely, from the orthogonality of M and G

$$\nabla_i \xi^i = -\sum_{i,j} (c^i_j)^2 = -c^2.$$

If X is orientable and v is the volume element then one will have:

$$0 = \int_X (\nabla_i \xi^i) v = -\int_X c^2 v \leq 0,$$

which demands that $C^2 = 0$ and $C = 0$. The proposition is then proved. One can drop the hypothesis that X must be orientable by possibly passing to the orientable covering of X, when it is endowed with the inverse images of the structures.

One can also deduce our theorem from a study of the Kostant group that is generated by η (cf., A. Lichnerowicz [**23**]).

E) *In regard to the problem P_2 .*

THEOREM IV.4. – *If G is reductive in L_m then let X be endowed with a G-structure S and an S-connection γ whose holonomy group is irreducible under the complex field. Any affine i.t. for γ will then be an infinitesimal homothety of S. Moreover, η will be an i.a. of S in the following cases:*

1. *G is invariant under homothety ($\hat{G} = G$).*

2. *X is compact, G is unimodular, and γ has zero torsion* (35).

(35) The last case of our theorem was the object of study in Theorem IV of [**17**] under some slightly-different hypotheses.

If the tensor field C has a zero covariant derivative then the operator C_x that it defines on T_x will belong to the centralizer of ψ_x (viz., the homogeneous holonomy group) in the algebra of endomorphisms of T_x (cf., [**22**], § **54**). Let $h \in \psi_x$ (i.e., $h \cdot C_x = C_x \cdot h$), and let $v \in T_x$ be a proper vector of C_x for the (real or complex) proper value k, and E_k is the space of proper vectors for the proper value k. One has:

$$C_x\, v = kv \qquad \text{and} \qquad C_x\,(hv) = h\, C_x\, v = hkv = khv\ ;$$

i.e., $v \in E_k$ implies that $hv \in E_k$. E_k is invariant under ψ_x , and as a result of the irreducibility, $E_k = T_x^{\mathrm{C}}$. $C_x\, v = k\, v$ for any $v \in T_x^{\mathrm{C}}$ and $C_x = k\,(v) \cdot I\,(x)$. If C has zero covariant derivative then one will get $k\,(x) = k$, which is constant on X, and $C = k \cdot I$. One will then have $a_U = b_U + k\, I$, $b_U \in \underline{G}$; i.e., η is an infinitesimal homothety.

If G is invariant under homothety (i.e., $\hat{G} = G$), moreover, then $C = k\, I$ will imply that $C \in \underline{G}$; i.e., $C = 0$, and η is an *i.a.* Finally, if G is unimodular then $b_U \in \underline{G}$ will imply that $b_U = 0$, and consequently, that tr a_U = tr $C_U = mk$. By a calculation that was made before [formula (20)], the vanishing of the torsion of γ would imply that:

$$a_j^i = \nabla_j\, \eta^i - i\,(\eta)\pi_j^i\,,$$

in which π is a form with values in $\underline{G}$, tr $\pi = \pi_i^i = 0$, and one has tr $a_U = \nabla_i\, \eta^i$; i.e.:

$$mk = \nabla_i\, \eta^i. \tag{23}$$

X is orientable, since G is unimodular. Hence, if it is compact and v is the volume elements then the integration of (23) will yield:

$$mk \int_X v = \int_X (\nabla_i\, \eta^i)\, v = 0,$$

so $k = 0$. One then infers that $C = 0$, in such a way that η will be an *i.a.* Q.E.D.

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