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On the theory of capillary phenomena

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The theory that Laplace proposed in order to explain capillary phenomena was vigorously criticized by Poisson. The objections of the celebrated geometer led him to propose a hypothesis that was not generally adopted, and which possibly still leaves many difficulties, moreover. Meanwhile, one must recognize that those objections have some basis to them and that there is an obvious contradiction in the manner by which Laplace envisioned the question. The hypothesis of an incompressible fluid whose disjoint molecules act upon each other according to a function of the distance is indeed impossible, mathematically speaking. However, it will suffice that this hypothesis, although not rigorously exact, is very close to reality in that its consequences agree with experiments. I shall add (and it seems that this simple observation had escaped Poisson) that Laplace, despite his inexact hypothesis on the incompressibility of fluids, had proposed a question that is not absurd from a mathematical viewpoint. The introduction of force, which he called "pressure" and whose value is determined precisely in such a manner that it does not change the volume, will indeed permit one to regard the fluid as incompressible. It is quite true that in a compressible, physical fluid, that pressure cannot be distinguished from the resultant of the molecular actions and must be calculated, as Poisson had often remarked, by means of the function that represents it. However, from the abstract viewpoint that the geometers assume, that function will define a force that is different in nature from the ones that one often introduces into mechanics under the name of *constraint forces,* and which in each case can give rise to analogous objections if one contests the right to introduce a question into the statement that involves the notion of rigid bodies, fixed axes, etc. In a word, if one refuses to replace a question of physics with an abstract question that is analogous to it, but not identical.

The principle of virtual velocities, when applied to a system that is defined in an analogous manner, will permit one to avoid that apparent difficulty that comes from the constraint forces, which will acquire a new force in the theory of capillary phenomena by the introduction of molecular forces that are produced in reality and which one meanwhile regards as being completely independent of them. There is no doubt that this is the reason why Gauss, in his beautiful paper on capillary phenomena, was inclined to take the principle of virtual velocities as the sole basis for his argument. However, as he himself said, he had, at the same time, the objective of giving an example of the application of the calculus of variations to a question that relates to multiple integrals. In order to present that theory in a general manner, he had to reject the numerous geometric simplifications that had presented themselves to him. The goal that I shall set in this paper is precisely that of making Gauss's method known, along with the simplifications that it admits that will render it simpler than the ones that have been proposed up to now, if I am not mistaken.

After giving a new proof of the results that Gauss obtained, I will endeavor to apply his method to some very simple questions in order to be able to compare the experiments with the results that analysis implies. The following theorems, which will be *rigorously* true, if the theory is exact, seem to me to fulfill that condition:

1. If a capillary tube us immersed in a liquid and the column of liquid that is raised in it is separated into several parts by air bubbles that are introduced artificially, then the total mass of the raised liquid will depend upon neither the number of those bubbles, nor their volume.

2. When a column of liquid is suspended in a capillary tube that is open at both ends and placed vertically in free air, the total volume of that column is *at most* equal to the product of the volume that is raised in the tube when it is immersed in a vessel that is full of the same liquid with the sum $\left(1+\frac{1}{\cos i}\right)$, in which i is the angle at which the capillary surface that is formed by that liquid cuts the walls of the vessel.

3. If several liquids are superimposed in the same capillary tube and that tube is immersed in a vessel of the same nature as the lower liquid then the sum of the weights of the raised liquids will depend upon only the nature of the tube and that of the lower liquid.

4. Upon calling the volume of a drop of mercury V, letting b denote the area of the base of that drop, letting L denote the length of the contour of that base, and finally, letting h denote the height to which the liquid will be raised in a very large vessel that communicates with the drop by a full tube of mercury then one will have the relation:

$$V = bh + \alpha^2 L \sin i,$$

in which α is a constant and i is the angle that was defined above.

I.

Consider a liquid to be composed of material molecules $m, m', m'', \ldots$ that act upon each other according to a function of their mutual distance and proportionally to the product of their masses, while supposing that the liquid is contained in a fixed tube whose various molecules have masses $M, M', M'', \ldots$ that attract $m, m', m'', \ldots$ according to another function of distance. If one adopts the notation $\overline{mm'}$ in order to represent the distance between the points m and m', and one represents the forces that are exerted between m, m' and m, M by $mm' f(\overline{mm'})$ and $mM f(\overline{mM})$, resp., then the principle of virtual velocities will teach us that for all of the displacements that leave the total volume invariant, one must have:

$$(1) \qquad 0 = \sum m \begin{Bmatrix} -g\,dz - m'\,f(m,m')\,d(m,m') - m''\,f(m,m'')\,d(m,m'')\cdots \\ -M\,F(m,M)\,d(m,M) - M'\,F(m,M')\,d(m,M')\cdots \end{Bmatrix},$$

in which the variations d (m, m'), d (m, M) refer to the virtual displacement of the point m.

Replace the functions f and F with their integrals φ and Φ, or, in other words, set:

$$\int_r^\infty f(r)\,dr = -\varphi(r), \qquad \int_r^\infty F(r)\,dr = -\Phi(r),$$

so equation (1) will become:

$$\sum m \begin{Bmatrix} -g\,dz + m'\,d\varphi(m,m') + m''\,d\varphi(m,m'') + \cdots \\ +M\;d\varphi(m,M) + M'\,d\varphi(m,M') + \cdots \end{Bmatrix},$$

in which the differentiations again refer to only the displacement of the point m. However, it is obvious that each of those partial differentials can be combined with another differential that will complete it and define the total variation of the function together with it. For example, upon summing relative to the point m, we will have the term:

$$m\,m'\,d\varphi(m, m'),$$

and in the sum that relates to m' :

$$m'\,m\,d\varphi(m, m') .$$

The first variation refers to the displacement of m and the second one, to that of m'. The sum of those terms can be written:

$$m\,m'\,d\varphi(m, m'),$$

in which the symbol d expresses the total variation of the function φ.

From that, the sum of the virtual moments of the forces that act upon the system can be considered to be the total variation of the expression:

$$\Omega = \sum m \begin{Bmatrix} -g\,z + \tfrac{1}{2}m'\,\varphi(m,m') + \tfrac{1}{2}m''\,\varphi(m,m'') + \cdots \\ +M\;\Phi(m,M) + M'\,\Phi(m,M') + \cdots \end{Bmatrix};$$

the terms $m'\ \varphi(m, m')$ and $m''\ \varphi(m, m'')$ are divided by 2, because each of them will be duplicated in the summation.

If we now suppose that the molecules m, m', …, M, M', M'', … form two continuous masses then, upon calling the volume that is occupied by the liquid v and its density ρ, while v' and ρ' are the volume and density, resp., of the solid matter that the tube is composed of, then the preceding expression will become:

$$\Omega = -g\rho\int z\,dv + \tfrac{1}{2}\rho^2\iint dv\,dv_1\,\varphi(dv, dv_1) + \rho\rho'\iint dv\,dv'\,\Phi(dv, dv');$$

here, dv and dv_1 denote two arbitrary elements of the liquid volume, in such a way that the second term of the value of Ω represents a sextuple integral, in reality; the same thing is true for the third one.

II.

The two sextuples that enter into the values of Ω can be considered to provide one solution and the other to the following problem:

If two bounded spaces are given then take the sum of the products that are obtained by multiplying an arbitrary element of the first one with an arbitrary element of the second one and with a function of the distance between those two elements.

The first of our two integrals refers to the case in which the two spaces are identical, and the second one refers to the one in which one of them is the volume that is occupied by the liquid and the other is the volume that is occupied by the vessel, while those two spaces are entirely distinct. We consider, in a general manner, the reduction of the sextuple integral that expresses the solution of the problem upon supposing that the two volumes in question are completely arbitrary in form, as well as in relative position.

Upon calling the elements of those two volumes dv, dv', one will be dealing with the sextuple integral:

$$\iint dv\, dv'\, \varphi(dv, dv').$$

We shall see that, in any case, one can come down to a quadruple integral.

Let μ be an element of the volume v' (which might or might not belong to the volume v), and first consider the triple integral:

$$\int dv\, \varphi(\mu, dv),$$

which we extend over the entire volume v. Imagine a sphere of radius 1 that is described about the point μ as its center and is divided into infinitely-small elements; let $d\,\Pi$ be one of those elements. Consider $d\,\Pi$ to be the base of a cone that has one of the points of $d\mu$ for its summit, and let p', p'', p''', … be the points where that cone cuts the surface s that bounds the volume v, the number of those points will obviously be odd or even according to whether μ does or does not belong to v, resp. Let dt', dt'', dt''' denote the elements that the cone cuts out from the surface s, and let q', q'', q''', … denote the angles that are formed by the generators with the exterior normals to those elements, and finally let υ', υ'', υ''', … denote the distances $\mu p'$, $\mu p''$, $\mu p'''$, …, resp. We will obviously have:

$$d\,\Pi = \pm\frac{dt'\cos\varphi'}{\upsilon'^2} = \mp\frac{dt''\cos\varphi''}{\upsilon''^2} = \pm\frac{dt'''\cos\varphi'''}{\upsilon'''^2}, \ldots,$$

in which the upper signs refer to the case where the point μ is external to v and the lower signs refer to the contrary case.

If one now forms the portion of the triple integral that relates to the elements of v that are situated inside the cone considered then that portion will obviously represent:

1. If the point μ is external to v:

$$d\Pi\left[\int_{r'}^{r''}\varphi(r)\,r^2\,dr+\int_{r'''}^{r^{iv}}\varphi(r)\,r^2\,dr+\cdots\right].$$

2. If the point μ belongs to the volume v:

$$d\Pi\left[\int_{0}^{r'}\varphi(r)\,r^2\,dr+\int_{r''}^{r'''}\varphi(r)\,r^2\,dr+\cdots\right],$$

in such a way that if one sets:

$$\int_0^\infty \varphi(r)\,r^2\,dr = -\,\psi(r)$$

then that integral will become:

$$d\,\Pi\,[\psi(r') - \psi(r'') + \psi(r''')\ \ldots] = \frac{dt'\cos q'\,\psi(r)}{r'^2}+\frac{dt''\cos q''\,dt''}{\upsilon''^2} + \ldots$$

in the former case and:

$$d\,\Pi\ \psi(0) + \frac{dt'\cos q'}{\upsilon'^2}\psi(r')+\frac{dt''\cos q''}{r''^2}dt'' + \ldots$$

in the latter. If one now sums those results for all possible positions of the element $d\,\Pi$ then one will get:

$$\int\frac{dt\cos q}{\upsilon^2}\psi(r)$$

in the former case and:

$$4\,\Pi\ \psi(0) + \int\frac{dt\cos q}{\upsilon^2}\psi(r)$$

in the latter, where the two integrals must be extended over the entire surface that bounds the volume v.

When one denotes the volume that both v and v' belong to by s, one will easily deduce from the preceding results that the sextuple integral that we would like to evaluate is equal to:

$$4\,\Pi\,\sigma\,\psi(0) + \iint\frac{dt\,dv\cos q\,\psi(dt,dv)}{(dt,dv)^2},$$

in such a way that in order to calculate this, it will suffice to define a quintuple integral in which one successively considers the volume elements v, when they are combined with those of the surface t.

In order to reduce that quintuple integral, consider a fixed element dt of the surface t and first define the triple integral that refers to the combination of that element with all of the volume elements dv ; that integral will become:

$$\int \frac{dv \cos q\, \psi(dt, dv)}{(dt, dv)^2},$$

in which q denotes the angle that the line (dv, dt) makes with the exterior normal to dt.

Imagine a sphere of radius 1 that is described about dt as its center. Once more, let $d\Pi$ be an element of the surface of that sphere. Consider the cone that has its summit at the point of dt and $d\Pi$ for its base. Suppose that this cone cuts the surface T that bounds the volume v at the points P', P'', P''', … Let R', R'', R''', …, resp., be the distances from those various points to the point μ, let dT', dT'', dT''', resp., be the portions of the surface T that are cut out by that infinitely-small cone, and finally, let Q', Q'', Q''', resp., be the angles between the cone and the exterior normal to those elements. We will have:

$$d\,\Pi = \pm \frac{dT' \cos Q'}{R'^2} = \mp \frac{dT'' \cos Q''}{R''^2} = \pm \frac{dT''' \cos Q'''}{R'''^2},$$

in which the upper or lower signs must be adopted according to whether dt is exterior or interior to the volume v, resp. If we now integrate over the entire extent of the infinitely-small cone that was considered above then q must be considered to be constant over the entire extent of the integration, and if we set:

$$\int_r^\infty \psi(r)\, dr = -\,\theta(r)$$

then we will see, as in the preceding case, that when the integral:

$$\int \frac{dv \cos q\, \psi(v, dt)}{(dv, dt)^2}$$

is extended over all of the elements of the infinitely-small cone, it will be equal to:

$$\cos q \left[\frac{dT' \cos Q'}{R'^2} \theta(R') + \frac{dT'' \cos Q''}{R''^2} \theta(R'') + \cdots \right]$$

in the former case and to:

$$d\,\Pi \cos q\; \theta(0) + \cos q \left(\frac{dT' \cos Q'}{R'^2} \theta(R') + \cdots \right)$$

in the latter.

If we now integrate over $d\,\Pi$ then we will get:

1.

$$\int \frac{dv \cos q\, \psi(v, dt)}{(dv, dt)^2} = \int \frac{dT \cos q\, Q\, \theta(v, dt)}{R^2}$$

in the case where the element dt is exterior to v, in which dT denotes any of the elements of the surface that bounds v.

2. In the case where dt is located inside of v, one must add the term:

$$q\,(0) \int d\Pi \cos q$$

to the preceding expression.

It is easy to see that if that integral is extended over all elements of the sphere of radius 1 then the parts that it is comprised of will cancel pair-wise, and its total value will be 0. However, if the element dt belongs to the surface that bounds v; i.e., if the spaces v and v' are partly bounded by a common surface, and that if dt denotes the element of that surface then the integral:

$$\theta\,(0) \int d\Pi \cos q$$

must be extended over only those elements of the spherical surface for which the line (dt, $d\Pi$) is found interior to the volume v in the neighborhood of dt; i.e., for all of the elements that are found on the same side of the tangent plane to dt and for which the angle q is acute, if the volumes v and v' are situated on different sides of the surface that is common to them and obtuse in the contrary case. One will very easily find the integral:

$$\int d\Pi \cos q = +\,\Pi$$

in the first case and:

$$\int d\Pi \cos q = -\,\Pi$$

in the second one. It will result from the preceding considerations that the sextuple integral:

$$\iint dv\, dv'\, \varphi(dv, dv')$$

can reduce to the following forms:

1. If the volumes v and v' have a common part σ, while their surfaces are entirely distinct:

$$\iint dv\, dv'\, \varphi(dv, dv') = 4\,\Pi\,\sigma\,\psi(0) + \iint \frac{dt\, dT \cos\varphi \cos Q}{(dt, dT)^2}\,\theta\,(dt,\, d\Gamma).$$

2. If the surfaces t, T that bound the volumes v and v', resp., have a common part σ, which we call S, then:

$$\iint dv\,dv'\,\varphi(dv, dv') = 4\,\Pi\,\sigma\,\psi(0) \mp \Pi\,\rho\,S\,\theta(0) + \iint \frac{dt\,dT\cos\varphi\cos Q\,\theta(dt, dT)}{(dt, dT)^2},$$

in which the upper sign refers to the case in which v and v' are on the same side of their separating surface, and the lower sign refers to the contrary case.

III.

From the reduction formula to which we have arrived, the quantity Ω, whose variation represents the sum of the virtual moments of the forces that act upon the system, can be put into the following form:

$$\Omega = -g\rho\int z\,dv + \tfrac{1}{2}\rho^2 v\,\psi(0) - \tfrac{1}{2}\Pi\rho^2 t\,\theta(0) + \Pi\,\rho\rho' T\,\Theta(0)$$

$$+ \iint \frac{dt\,dt'\cos q\cos q'\,\theta(r)}{v^2} + \iint \frac{dt\,dT\cos q\cos Q\,\theta(r)}{v^2},$$

in which ρ denotes the density of the liquid, as we said above, and ρ' denotes that of the glass. v is the volume of the liquid, t is the area of the surface that bounds that volume, and T is the area of the portion of that surface that is in contact with the walls of the tube or with those of the vessel. The functions ψ, θ, and Θ are deduced from the functions φ and Φ by way of the following equations:

$$\int_r^\infty r^2\varphi(r)\,dr = -\,\psi(r), \qquad \int_r^\infty \psi(r)\,dr = -\,\theta(r),$$

$$\int_r^\infty r^2\,\Phi(r)\,dr = -\,\Psi(r), \qquad \int_r^\infty \Psi(r)\,dr = -\,\Theta(r).$$

The functions φ and Φ are totally unknown, so the same thing will be true for θ and Θ, which are deduced from them. One can nevertheless assume that those two functions, just like φ and ψ, are annulled for all of the meaningful values of the variable. In order to do that, it will suffice to remark that the action of the molecules that are located at an appreciable distance has no influence on the phenomena, so it will not change anything if one assumes that φ and Φ are rigorously zero for finite values of the variable, which will obviously imply the same condition for the functions θ and Θ.

From that remark, one will effortlessly see that the two quadruple integrals that enter into the value of Ω are both negligible.

Indeed, consider the first of those two integrals:

$$\iint \frac{dt\,dt'\cos q\cos q'}{\upsilon^2}\theta(r).$$

We can write it in the following manner:

$$\int dt \int \frac{dt'\cos q\cos q'}{\upsilon^2}\theta(r).$$

However, upon letting $d\,\Pi$ denote the element of the spherical surface of radius 1 that is described about dt as its center, one can set:

$$\frac{dt'\cos q}{\upsilon^2} = d\,\Pi,$$

which will reduce the integral in question to:

$$\int dt \int d\Pi \cos q \cdot \theta(r).$$

Now, in that form, it is obvious that:

$$\int d\Pi \cos q \cdot \theta(r)$$

will have a negligible effect, because if r is not very small then $\theta(r)$ will be negligible, and if r is very small then the line (dt, dt') will be very close to the tangent plane, and cos q will differ only slightly from zero. An entirely similar argument will show that one can neglect the second quadruple integral and consequently take the following expression for Ω:

$$\Omega = -\,g\rho\int z\,dv + \tfrac{1}{2}\rho^2 v\,\psi(0) - \tfrac{1}{2}\Pi\,\rho^2 t\,\theta(0) + \Pi\,\rho\rho' T\,\Theta(0).$$

That sum must be zero in order for there to be equilibrium. Now, it is obvious that one can suppress the term $\tfrac{1}{2}\rho^2 v\,\psi(0)$, which is constant. If one divides by $g\rho$, in addition, and one changes the sign of Ω, after setting:

$$\frac{\Pi\,\rho\,\theta(0)}{2g} = \alpha^2,$$

$$\frac{\Pi\,\rho'\,\Theta(0)}{2g} = \beta^2,$$

and that function K must be a minimum.

If one lets U denote the area of the free surface of the liquid then one will have:

$$t = U + T,$$

in such a way that the expression that one must minimize will take the form:

$$\int z\,dv + (\alpha^2 - \beta^2)\,T + \alpha^2\,U = K,$$

in which U is the area of the free surface of the liquid, and T is the area of the portion of that surface that is in contact with the walls of the tube or those of the vessel.

IV.

The preceding result was obtained by Gauss, and all of the preceding was extracted from his paper. However, instead of solving the problem of the minimum to which we will be led with the calculus of variations, as he did, we shall deduce the differential equation for the surface U that renders K a minimum from very simple geometric considerations, as well as the conditions that must be fulfilled on the boundary.

In order for K to be a minimum, while the volume v remains constant, it is necessary that the variation of the sum $K + \lambda v$ must be zero, where λ denotes a constant that must be ultimately determined.

First suppose that one subjects the free surface U to an infinitely-small variation that preserves the same contour; i.e., one leaves invariable the portion of the surface of the tube that is wet by the liquid, which has been denoted by T_1 .

In my paper on orthogonal isothermal surfaces, I showed that if one considers an infinitely-small rectangle $d\omega$ on the surface U that is composed of four lines of curvature then the normals that are drawn through the various points of the contour of that element will cut out a corresponding infinitely-small element from the neighboring surface that is equal to:

$$d\omega\left(\frac{1}{R} + \frac{1}{R'}\right)\varepsilon + d\omega,$$

in which ε is the infinitely-small distance between the two surfaces that one compares.

From that theorem, whose geometric proof is quite simple, the variation of $\alpha^2\,U$ will be equal to:

$$\alpha^2 \int d\omega\left(\frac{1}{R} + \frac{1}{R'}\right)\varepsilon\,.$$

As for $\int z\,dv$, it is obvious that its variation will be:

$$\int z\,\varepsilon\,d\omega,$$

and finally, the variation of v will be equal to:

$$\int \varepsilon\,d\omega\,.$$

One must then have:

$$\int \varepsilon\, d\omega \left[\alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right) + z + \lambda \right] = 0,$$

and since that result must be true for any ε, one can conclude that:

$$\alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right) + z + \lambda = 0,$$

which is indeed the known differential equation for the capillary surface.

In order to determine λ, one should point out that $R = \infty$, $R' = \infty$ for the points that belong to the surface of the liquid that is outside of the tube, so $z + \lambda = 0$. If one then measures z by starting from the level of the liquid in the vessel then $\lambda = 0$, and the equation that one finds will be:

$$z + \alpha^2 \left(\frac{1}{R} + \frac{1}{R'} \right) = 0.$$

In order to get the condition that relates to the surface U, we shall suppose that one varies the figure of the liquid without preserving the same contour for U. The difference between U and the *corresponding* portion of the infinitely-close surface that replaces it will always be:

$$\int \varepsilon\, d\omega \left(\frac{1}{R} + \frac{1}{R'} \right).$$

However, one must further add the portion of the new surface that does not correspond to any point of U; i.e., the small zone that is found between the tube and the curve along which the normals to the different points of the contour to U cut the surface infinitely closely. Now, one effortlessly sees that when one lets dP denote the contour that terminates U and lets i denote the angle between the tangent plane to the tube and that of U, the small zone will have the expression:

$$\int dP\, \varepsilon \cot i\,;$$

the variation of the volume will be composed of the term that was written out before:

$$\int d\omega \varepsilon\,,$$

which represents the volume that is found between U and the corresponding portion of the neighboring surface, and another term that expresses the second-order infinitesimal volume that is found between the walls of the tube and the skew surface that is the locus of the normals to U that go through the various points of its contour. However, the latter term can be neglected as infinitely small compared to the preceding ones. The same thing will then be true for the analogous term that is provided by the variation of $\int z\, dv$.

As for the area *T*, which did not vary in the preceding case, one effortlessly sees that it is increased by the area of the portion of the surface of the tube that found between the curve that bounds *U* and the new contour that replaces it; i.e., the integral:

$$\int \frac{\varepsilon\, dP}{\sin i},$$

so the terms that are due to the variation of the boundary are:

$$\int \varepsilon\, dP \left(\frac{\alpha^2 - 2\beta^2}{\sin i} + \alpha^2 \cot i \right).$$

In order for this to vanish for any ε, it is necessary that one must have:

$$\frac{\alpha^2 - 2\beta^2}{\sin i} + \alpha^2 \cot i = 0 ;$$

i.e., that:

$$\cos i = \frac{\alpha^2 - 2\beta^2}{\alpha^2},$$

which proves that the angle *i* must have a constant value that depends upon the nature of the liquid and that of the tube.

If $\frac{2\beta^2 - \alpha^2}{\alpha^2}$ is greater than 1 (i.e., if β^2 exceeds α^2) then the preceding formula will give an imaginary value for the angle *i*. One must then conclude that in that case, the hypotheses that were made up to now would be inadmissible, and that the liquid would have to form an extremely thin layer that wets the walls well above the surface that bounds the mass of the raised liquid. In that case, the two quadruple integrals that we have neglected could not have a very small value. One would then suppose that everything happens as if the tube were closed by the very thin liquid layer that wets the walls, in which case, one would have $\beta^2 = \alpha^2$, and as a result $\cos i = 1$.

V.

We have obtained two results that permit us to reduce the solution of any problem that relates to capillary phenomena to a question of analysis. The differential equation of the capillary surface was obtained by Laplace in a slightly simpler way, but the argument by means of which he proved that the angle that is denoted by *i* is constant was much less satisfactory, as Gauss remarked. One of the more remarkable results that Laplace deduced from his formulas was the rigorous expression for the total volume of the raise liquid in the case of a cylindrical tube with vertical walls whose section could be arbitrary, moreover. I believe that the following proof has the advantage of simplicity over that of Laplace:

Upon taking the *xy*-plane to be the level of the external liquid, the differential equation of the capillary surface will be:

$$z = -\alpha^2\left(\frac{1}{R}+\frac{1}{R'}\right).$$

Multiply the two sides of that equation by $dx\,dy$ and integrate over the entire extent of the surface that serves as the base of the right cylinder in which the liquid is raised. The left-hand side is obviously the total volume of the raised liquid, and in order to understand the value in the right-hand side, it will suffice to perform the integration:

$$\iint dx\,dy\left(\frac{1}{R}+\frac{1}{R'}\right).$$

Now, one can obviously regard that integral as the vertical component of a system of forces that will have an intensity on each element $d\omega$ that is equal to $d\omega\left(\frac{1}{R}+\frac{1}{R'}\right)$ when they are exerted upon any liquid surface. However, such a system can be replaced with two other systems that are much simpler in the following manner: Imagine a surface that is parallel to that of the liquid and located at an infinitely-small distance ε; by that, I mean a surface that is obtained by moving along each normal by a constant length ε. Suppose that each element $d\omega'$ of that surface is acted upon by a force $\frac{1}{\varepsilon}d\omega'$, and that each element $d\omega$ of the first surface is acted upon in the opposite sense by a force $\frac{1}{\varepsilon}d\omega$. If $d\omega$ and $d\omega'$ are two corresponding elements then from a theorem that was cited before in this paper, one will have:

$$d\omega' - d\omega = d\omega\left(\frac{1}{R}+\frac{1}{R'}\right)\varepsilon,$$

in such a way that the difference between the two forces is equal to precisely:

$$d\omega\left(\frac{1}{R}+\frac{1}{R'}\right).$$

Instead of composing the proposed forces in order to look for their vertical component, one can then look for the component that is provided by each of the two systems that we spoke of separately. Now, one knows that an arbitrary surface is subject to a constant normal pressure, so the resultant of the forces that act upon it will give a vertical component that is equal to the product of the pressure per unit area with the horizontal projection of the area considered. If we call the projections of the areas of the liquid and the parallel surface P_1 and P_2, resp., then we will have:

$$\iint dx\,dy\left(\frac{1}{R}+\frac{1}{R'}\right) = (P_1 - P_2)\,\frac{1}{\varepsilon}.$$

Now, $P_1 - P_2$ is obviously the projection of the area of the skew surface that is composed of the normals of length ε that are drawn through the points of the contour, which is a projection that is, as one easily sees (due to the constant inclination of those normals), equal to the product of the perimeter of the cross-section with $\varepsilon \cos i$, where i is the angle that the normal to the surface U makes with the normal to the cylindrical surface. Upon calling the length of the contour of the cross-section of the tube L and the volume of the raised liquid V, we will finally have:

$$V = \alpha^2 L \cos i,$$

which is Laplace's result, up to notations.

VI.

Gauss's method supposed that no other external force acted upon the liquid besides weight. If one assumes, for example, that the atmospheric pressure is not the same inside the tube and outside of it then one must draw one's attention to the forces of pressure in the evaluation of the virtual moments of the forces of the system, and consequently, in place of equating the variation of the function that was denoted by Ω above to zero, it is convenient to write that it is equal and opposite to the sum of the virtual moments of the forces of pressure. Suppose that the pressure that is exerted upon the external level of the liquid is P, and let P' denote the pressure that acts upon the raised liquid in the tube. Recall the expression for Ω that was calculated above:

$$\Omega = -g\rho\int z\,dv + \tfrac{1}{2}\rho^2 v\,\psi(0) - \tfrac{1}{2}\Pi\rho^2 t\,\theta(0) + \Pi\,\rho\rho' T\,\theta(0).$$

Upon introducing the conventions:

$$\alpha^2 = \frac{\Pi\,\rho\,\theta(0)}{2g},$$

$$\beta^2 = \frac{\Pi\,\rho'\,\Theta(0)}{2g},$$

$$\Omega = -g\rho\int z\,dv + \tfrac{1}{2}\rho^2 v\,\psi(0) - \rho^2\alpha^2 t + 2\rho g\,\alpha^2 T,$$

or, upon setting $t = T + U$:

$$\Omega = -g\rho\int z\,dv + \tfrac{1}{2}\rho^2 v\,\psi(0) - \rho g\alpha^2 U + \rho g\,T(2\beta^2 - \alpha^2).$$

In order for there to be equilibrium, the variation $\delta\Omega$ must be equal and opposite to the sum of the virtual moments of the forces of pressure. Suppose that the tube is a vertical right cylinder and write down that this condition is fulfilled for a virtual displacement that consists of lowering all of the points of the portion of the surface *U* that corresponds to the liquid that is found inside of the tube by the same quantity *dh* in the sense of the vertical, and at the same time, raise all of the points of the external surface by another quantity *dh′*. The ratio of *dh* to *dh′* is calculated in such a manner that the total volume will remain invariable, so one will effortlessly see that the surface *U* will not change for such a displacement, and if one calls the length of the contour of the interior section of the tube *L* and that of the section of the vessel that is also supposed to be cylindrical *L′* then the variation of Ω will be:

$$-g\rho\int z\,dv + g\rho(\alpha^2 - 2\beta^2)(-L\,dh + L'\,dh')\,.$$

Now, $\delta\int z\,dv$ is the sum of the moments of the various truncated cylinders of height *dh* or *dh′* whose liquid volume is diminished or augmented. Any one of those cylinders will have a measure that is equal to the product of *dh* with its cross-section, which one can represent by *dx dy*, one will have:

$$\delta\int z\,dv = -dh\int z\,dx\,dy + dh'\int z\,dx\,dy\,.$$

The first integral extends over the portion of the liquid that is inside of the tube, so it can represent the volume of the raised liquid if *z* is measured by starting from the level of the external liquid, which is a hypothesis that will annul the second integral.

If one replaces $\delta\int z\,dv$ with that values and remarks that the ratio of *dh* to *dh′* must be the inverse of the ratio of the area of the section of the cylinder to that of the vessel in which is it immersed, while calling those areas *b* and *B*, then one will have:

$$\delta\Omega = \left[-Vg\rho + g\rho(2\beta^2 - \alpha^2)\left(L - \frac{bL}{B}\right)\right] dh.$$

The sum of the virtual moments that are due to the forces of pressure can be easily calculated. Indeed, if one considers an element $d\omega$ of the surface *U* then the pressure that it supports will be $P\,d\omega$ or $P'\,d\omega$, according to whether it belongs to the portion of the interior surface of the tube or the external level. The virtual displacement of the point of application will be the product of *dh* with the cosine of the angle that the element under pressure makes with the vertical, which will give the product of *P dh* or *P′ dh′* by the projection of the element $d\omega$, and consequently, *P b dh* or – *P′ B dh′* for its integral, according to whether one is dealing with the liquid that is inside of the tube or outside of it. Upon remarking that *b dh* = *B dh′*, the sum of those two integrals will be (*P* – *P′*) *b dh*. Finally, we will then have:

$$-Vg\rho + g\rho\,(2\beta^2 - \alpha^2)\left(L - \frac{bL}{B}\right) = (P - P')\,b$$

for the equilibrium equation. If we let h denote the height of a liquid column of volume V that has b for its base – or, in other words, the mean height of the raised liquid – then we will infer from that equation that:

$$h = (\alpha^2 - 2\beta^2)\left(\frac{l}{b} - \frac{L}{B}\right) + \frac{(P - P')}{g\rho}.$$

In that result, one can neglect L / B with respect to l / b, and if one simultaneously replaces $\alpha^2 - 2\beta^2$, with its new value above (viz., $\alpha^2 \cos i$, in which i is the angle that the liquid forms with the capillary surface) then one will have:

$$h = \alpha^2 \frac{l}{b}\cos i + \frac{P - P'}{g\rho}.$$

That proves that the height h is composed of two parts, one of which is precisely equal to the elevation that was calculated above for the case of $P = P'$, and the other of which is equal to the difference in the level that is *due* to the excess of external pressure over the internal pressure.

If, instead of considering, as in the preceding argument, the case of a tube that is immersed in a liquid, we suppose that a liquid column is suspended in a tube and supports different pressures over its two surfaces, then we will see that upon giving a common vertical motion to all point, in the sense of the vertical, δU and δT will be zero, and $\delta\Omega$ will reduce to $-g\rho V$, and upon equating that virtual moment to $(P' - P)\,b$, we will see that the weight of the column will be simply proportional to the pressure difference, and that capillarity will have no influence on the phenomenon. If we suppose that $P = P'$, as a special case, then we will have $V = 0$. When the facts of reality are contrary to that result, one can explain that by the influence of friction.

One will immediately deduce the preceding results from the first of the theorems that were stated at the beginning of this paper. If the liquid column, which is situated in a capillary tube that is open at both ends, is separated into several parts by air bubbles that are interposed in the liquid then the total weight of the raised liquid will remain the same, no matter what the density or number of those air bubbles would be.

VII.

I shall now consider a well-known phenomenon that seems, on first glance, to contradict the results of the preceding analysis.

One knows that a capillary tube that is open at both ends can contain a column that is almost double in height to the one that rises up in it when one immerses it in a liquid mass. As one knows, it will suffice that the liquid column should occupy the lower part of the tube and form a meniscus whose curvature will explain that increase in height. In order to reconcile that fact with the theory that was proved above, one must remark that in general in the application of the principle of virtual velocities, the sum of the virtual moments that correspond to a certain displacement of the system must be zero only when a displacement that is equal and opposite is possible and provides a sum of moments that has precisely the opposite sign to the one that corresponds to the former case. When that condition is not fulfilled, in order for there to be equilibrium, it will suffice that the sum of the virtual moments, while not being zero, can never become positive. Now, in the case that we are addressing, if we give the liquid a virtual motion that consists of raising all of the molecules that are contained inside the tube by the same quantity in such a way that one does not change the surface that is denoted by U and T, in order to apply the argument of the preceding paragraph, then an equal and opposite displacement, which is a displacement for which the constraints will not oppose anything, will imply a change in the value of U and T, because the tube no longer extends above the present contour of the surface U, so one cannot lower that surface without supposing that the liquid forms a small cylinder below the lower liquid in the tube that is completely external to it, where the convex surface must be considered to belong to U. The variation of T will cease to be zero, because the reduction that the surface experiences upward in the volume will no longer be compensated by an increase that is equal to the lower part.

From those remarks, upon calling the virtual motion that is give to the system dh and letting L and b denote the length of the contour and the area of the cross-section, resp., of the tube, as usual, one will find that:

$$\delta\Omega = g\rho\, V\, dh - gr\, \alpha^2\, L\, dh - \rho g\, (2\beta^2 - \alpha^2)\, L\, dh,$$

and since $\delta\Omega$ must be negative, one will have:

$$V < 2\beta^2 L.$$

Now, one found above that the volume V' that is raised by immersing the tube into an indefinite mass of liquid was:

$$V' = \alpha^2\, L \cos i.$$

One will then have:

$$\frac{V}{V'} < \frac{2\beta^2}{\alpha^2 \cos i},$$

or, upon remarking that $2\beta^2 / \alpha^2 = 1 + \cos i$:

$$\frac{V}{V'} < 1 + \frac{1}{\cos i},$$

which is precisely the result that was stated at the beginning of this paper.

The preceding method does not give the precise value of the ratio V' / V, but only a limit of that ratio: One must point out that an exact determination of its values is, indeed, completely impossible, because if, under the circumstances that we have assumed, a certain liquid column can be maintained in the tube, then *a fortiori* the same thing will be true for a lower column. Nonetheless, in order for the solution to be completely satisfactory, it will be necessary for one to be able to show that the limit that one finds can be attained in reality.

VIII.

Upon applying the method that was presented at the beginning of this paper in the case of two liquids that are superposed in the same tube, one will find, with no difficulty, that upon denoting the area of the surface that bounds the upper liquid by U, the area of the separating surface between the two liquids in the tube by U', the area of the free surface of the external liquid that is contained in the vessel by U'', the areas of the portions that belong to the surface of the tube or to those surfaces of the vessel that are wet by the upper and lower liquid T and T', the densities of the upper and lower liquid by ρ and ρ', and finally, upon denoting the constants that are analogous to the ones that were defined above by α^2, β^2, α'^2, β'^2, α''^2, β''^2, it will be necessary for equilibrium that the sum:

$$K = \rho\int z\,dv + \rho'\int z\,dv' + \alpha^2\,\rho\,U + (\alpha^2 - 2\beta^2)\,\rho\,T$$

$$+ (\alpha^2\rho + \alpha'^2\rho' - 2\,\beta''^2\rho')\,U' + (\alpha'^2\rho' - 2\,\beta'^2\rho')\,T' + \alpha'^2\rho'U'$$

should be a minimum.

Now, upon giving a common virtual displacement to all of the points of the mass that is inside the tube and the inverse displacement to each point of the external mass, one must have:

$$0 = -\rho\,bh - \rho' bh' + (\alpha'^2\rho' - 2\,\beta'^2\rho')\,L - (\alpha'^2 - 2\,\beta'^2)\frac{Lb}{B}\,\rho',$$

so one can infer a value for $\rho h + \rho' h'$ that does not depend upon the nature of the upper liquid at all.

IX.

Among the numerous phenomena that are attached to capillarity, one of the simplest and easiest to study experimentally seems to me to be the formation of drops of mercury on a horizontal glass plane. The preceding principles apply to the study of those phenomena with no difficulty and will lead to some results that one might perhaps take advantage of.

If the drop of mercury rests upon a horizontal plane then the differential equation of the free surface will be the same as that of a liquid that is placed in a tube; i.e.:

$$h - z = \alpha^2\left(\frac{1}{R} + \frac{1}{R'}\right),$$

only the constant h, which was determined by the position of the external level in the preceding problems, will remain unknown here and can be obtained only by equating the volume that is calculated for the drop to the given volume of the liquid that it is composed of. In the particular case in which the drop is very large, R and R' can be considered to be infinite for the points of the upper surface, in such a way that for those points, one must suppose that $z = h$ and that the constant h will then represent the *thickness* of the drop. In the general case, in order to define that constant, one must suppose that the plate upon which the mercury rests is pierced at the very center of the drop in such a way that it will communicate by means of a filled liquid channel with a vessel that is large enough for the liquid in it to be horizontal. h will then denote the elevation of the level of that liquid above the glass plate.

Upon supposing that one has succeeded in realizing the circumstances that I just indicated and has thus determined the value of h by experiment, one can obtain a simple relation between the measurable elements of a drop of mercury whose verification seems important to me.

Consider the equation:

$$h - z = \alpha^2 \left(\frac{1}{R} + \frac{1}{R'}\right).$$

Multiply this by $dx\,dy$ and integrate over the entire extent of the projection of the drop and for all points of its free surface; i.e., one takes the ordinates that can answer to the same value of x and y twice and with opposite signs. Upon calling the volume of the drop V and the area of the base by which it rests upon the glass plane b, one will have:

$$bh - V = \iint \alpha^2 \left(\frac{1}{R} + \frac{1}{R'}\right) dx\,dy.$$

Now, the right-hand side can be considered to represent the vertical component of a system of forces that are exerted normally on each element $d\omega$ of the surface of the drop with an intensity that is equal to $\alpha^2\, d\omega \left(\frac{1}{R} + \frac{1}{R'}\right)$, but we have seen that such a system can be replaced with two other ones in which a pressure $\alpha^2\, d\omega / s$ is exerted upon each element of the surface of the drop and the parallel surface that is drawn at a distance of ε from it. Now, each of those systems of forces will give rise to a vertical component that is equal to the product of $\alpha^2\, d\omega / s$ with the area of the projection of the surface under pressure. One will easily see that the difference between those two projections is $\alpha L \varepsilon \sin i$, in which L is the length of the contour of the drop and i is the constant inclination of its tangent plane over the horizontal plane: The integral that is in the right-hand side will then have the value $-\alpha^2 L \varepsilon \sin i$, and we will have:

$$bh - V = -\alpha^2 L \sin i,$$

so

$$V = bh + \alpha^2 L \sin i,$$

which is a relation that can be verified by experiments.