"Théorie nouvelle des principes de la mécanique," Journal des Savants (1895), 471-482.

BOOK REVIEW

Die Prinzipien der Mechanik, in neuem Zusammenhange dargestellt, by Heinrich Hertz, Johann Ambrosius Barth, Leipzig, 1894.

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This strange book presents a very different interest than what one might hope from the title. The question that is being discussed by a scholar who is rapidly becoming famous has great significance. While presenting the principles of the science of motion in a new order, Hertz exhibits the desire, if not the pretense, of changing its basis. In a highly-developed introduction, that eminent author first points out the difficulties that have tormented him in the study of classical dynamics. I hope that none of the objections that he raises will shake the faith of those who regard mechanics as the most perfect of the physico-mathematical sciences. Moreover, Heinrich Hertz neither hopes nor desires to do that:

> "Some say that the objections presented to emphatically against the classical exposition of the principles of mechanics can make one believe that I pretend to attack and negate them, but that is neither my intention nor my conviction."

By showing only that the accepted theories are not absolutely certain, he earns the right to envision some other ones. The new doctrine that he timidly presents indeed seems to be the vision of a brilliant thinker.

The success of Heinrich Hertz gives him the right to be heard. An unknown author who embarks upon such a program to begin with will undoubtedly find no listeners. Let us first recall the main features of a brilliant and too-short career.

Heinrich Hertz, the eldest son of a doctor Hertz, who then became a lawyer, and much later a senator in Hamburg, was born in that city in 1857. Schooled at the Johaneum, the high school of the city, he was less interested in theory than in practice and learned to work at the lathe and workbench. He studied geometric drawing without a teacher and spent his Sundays at the School of Arts and Crafts. He wished to become an engineer and began his technical studies in Munich. However, the mechanical exercises in which he had excelled before were getting neglected in favor of pure science, which soon had much more of an attraction for him than the lectures that were intended for engineering students. Everything that was being said persuaded him that it was more mediocre to be an engineer than it was to be a scholar, and he devoted all of his time to science. When sharing that irresistible attraction with his father, he added: "My mind is made up. I do not ask for your advice but your permission to change careers." That permission was given, and Hertz left for Berlin, where Helmholtz opened up his laboratory to Hertz. By the way that he handled the corks and wires and perfectly mounted the simplest devices, the teacher immediately recognized that he had an extraordinarily gifted student. On the contrary, Hertz waited impatiently for the chance to show his knowledge and complained that he had too little time left for filling in the gaps in his theoretical studies. Meanwhile, the pleasure of seeing and producing the phenomena in that laboratory compelled him to study his teacher's books:

> "It is a greater thrill for me [he wrote to his father] to question nature directly, for others, as well as for myself, than to borrow from others what they have done without me. Reading great books has inspired me with a sense of my own uselessness."

The question that was posed to the students at the University for a physics prize was a problem in electrodynamics. Helmholtz, who had posed it, deemed his young student to be capable of solving it. In less than fifteen days, he had surmounted the main difficulties and confided to his father that he was almost certain to achieve that goal.

Meanwhile, the difficulties were followed by more difficulties. Hertz rose to the challenge. Not only did he get the prize, but for him, the praise that was heaped upon his work exceeded the value of the gold medal that he preferred to the silver that he could have chosen. That was the first step in his brilliant career. He was successively given the titles of Adjunct Head of the laboratory at the University of Berlin, professor at Kiel, Karlsruhe, and the University of Bonn, and soon became famous. His experiments on electric oscillations made his name immortal. Some celebrated academies conferred him their highest honors. In 1888, he obtained the Medal of Honor from the Italian Scientific Society, in 1889, the Lacaze Prize from the Paris Academy of Science and the Baumgarten Prize at the Vienna Academy. In 1890, it was the Rumford Medal from the Royal Society of London, and in 1891, it was the Bressa Prize from the Royal Society of Turin. He died on 1 January 1894.

The book that we review is the fruit of his final meditations. In a preface that was written by Helmholtz (who had himself died by the time that the book was published), Helmholtz said that one can see the true magnitude of Hertz's genius in the book:

> "The extent to which Hertz's musings were directed towards the most general viewpoint on science once more shows itself to be the ultimate monument to his earthly existence."

That celebrated teacher of Heinrich Hertz summarized his assessment of the book as follows:

> "The presentation of the principles of mechanics by Hertz is a source of great pleasure to any reader whose is interested in a system of dynamics whose mathematical conception is ingenious and complete."

That promise of great inspiration encouraged more than one reader to pursue up to the last page a discourse that presented him with some serious reasons to object to it in the beginning. One is initially tempted to declare the program to be unrealizable. Nonetheless, the author devotes so much skill and care to its details, so much rigor in proving the formulas, so much precision, and finally in the definition of the bold concepts that he timidly proposed, that without leaning in favor of the new theory or sharing Hertz's ideas at any point, one can take great pleasure in them. If one forgets that one is dealing with physics then one can unreservedly admire the methodical and ingenious geometric spirit of the presentation.

The introduction begins with a critical summary of the science that was created by Galilei, Huygens, Newton, d'Alembert, and Lagrange. Hertz does not ignore the confidence that it inspired in the great minds and the innumerable verifications that have dispelled all doubts:

> "It is difficult [he said] for one to imagine that one might doubt the logical admissibility of such a set of principles and find imperfections in a system that had been advanced by innumerable thinkers and the greatest of them. However, before one abstains from examining it, one must ask whether, in reality, all of those minds were satisfied. More than one difficulty can perplex a serious and incisive mind."

Hertz wished to make that more precise, and cited an example:

> "Throw a stone [he said] that is attached to a string, while giving it a rotational motion. We certainly communicate a force to that stone by that motion. That force will make the stone constantly deviate from the straight line. If we modify the mass of the stone and the length of the string then we will find that the motion takes place according to Newton's second law. However, the third one demands that there must be a reaction to the force that our hand exerts on the stone. One responds to that necessity of a reaction by saying that the stone reacts to the hand by virtue of centrifugal force, and that centrifugal force is equal and opposite to the force that we produce. Is such a manner of expression admissible? Does the force that we call the *ponderomotive force* or the *centrifugal force* differ from the inertia of the stone? Can we account for the action of the force of inertia twice, once as mass and once as force without diminishing the clarity of our thinking on the subject? According to our laws of motion, the cause of the motion is the force that precedes it. Do we have the right to suddenly speak of forces that will be the *result* of the motion without harming our argument? Might we not give the illusion that we are confusing forces of different types by referring to them with the same name? The term "force" is then improper when it is applied to centrifugal force. One must accept the terminology, like that of *vis viva*, because of historical tradition. One can excuse the preservation of the term on the grounds of utility, but one cannot justify its use."

Hertz found another objection in the confusion that all who had wished to present the principles of mechanics had exhibited, according to him. That always gave rise to the need to excuse it here and there and to glaze over it in the beginning. Newton himself exhibited that confusion when he defined mass to be the product of volume with density. Lagrange was likewise suspected of being sometimes hesitant, and as a final proof, Hertz alleged that the proofs of the parallelogram of forces and the principle of virtual velocities were accepted by some great minds and condemned by others. He demanded to know "How could one encounter such dissidence in a perfectly-logical science?"

If the reader decides not to read the book after that detailed presentation of such criticisms then it would be excusable.

The objection that relates to centrifugal force, one might dare to say, reveals a most complete ignorance of the theories that are taught to all schoolboys. Heinrich Hertz must not have studied them. One throws a stone attached to one extremity of a string whose other extremity is held by the hand of an observer. Since the stone cannot go further away and it follows the tangent, it is pulled by the string. The force that it develops is *centripetal*. Since the reaction is equal to the action, the stone will exert an equal and opposite force on the string, which is the centrifugal force. The string that is held by the hand cannot yield to that force. It pulls at the hand that resists it with an intensity that is equal to precisely the centrifugal force.

One recognizes none of the contradictions that Hertz pointed out in that series of actions and reactions. Does the centrifugal force differ from the inertia of the stone, as he said? Not only does it differ, but it does not even have the slightest relationship to it. The inertia of the stone is a reality, while the force of inertia is a fiction that plays no role in the explanation for the phenomenon. Due to that inertia, in order to prevent the stone from describing a straight line, one must exert a *centripetal* force on it. The string must deal with that force under penalty of being broken. That centripetal force, along with the acquired velocity, is the cause of the motion. The centrifugal force does not act upon the stone. It is exerted on the string and on the hand that holds it by its intermediary. No one would dream of giving the motion as the cause of it, it is only one aspect of it. It is nothing less than a real force, and it is sufficiently real that it will break the string or rip off the finger that does not let go of it when the velocity is sufficiently large.

A schoolboy whose answers suggested that the objections that Hertz believed that he could raise against the classical theories were possible would be blackballed by all of the European universities, not for daring to doubt them, but for having failed to grasp them. As for the disagreement between geometers on the rigor of certain proofs, one can respond that Euclidian geometry offers some analogous examples. The principle of virtual velocities is difficult to prove rigorously. Laplace gave a proof that was worthless. No one ignored that fact, but no one recognized an argument against science in it, either.

A second conception of how to present mechanical phenomena and relate them to each other that is of much more recent origin was proposed:

> "Towards the middle of this century [said Heinrich Hertz], one attempted to explain the phenomena of nature by innumerable forces that are exerted between atoms. The physicists of today give preference to a different conception of things. The discovery of the conservation of energy led them to consider phenomena to be transformations of energy. Their goal has become that of connecting phenomena to that law, which dominates all others. Upon applying that concept to the theory of motion, one is led to a new way of presenting mechanics in which the notion of

> force is replaced with that of energy. It is that second method to which we must direct our attention."

The theory of which Hertz spoke, and which he proposes as a second conception of the science of motion, has never been presented in detail, as he himself declared:

> "There exists [he said] no treatise on mechanics that takes its point of departure to be the theory of energy, when it is made independent of the notion of force. The originators of the theory of energy thought that it would be possible to create such a program. One remarks that one can avoid the notion of forces and the difficulties that it presents. Many arguments already lend themselves to that way of looking at things. We are also able to give a broad sketch of that vista at present. We can indicate the general plan of a theory of mechanics, thus-composed. In that conception of things, we accept four independent fundamental notions. The first two, *space* and *time*, have a mathematical character. The other two, *mass* and *energy*, appear to be physical realities that nothing can destroy or create."

Unfortunately, Hertz hastens to declare:

> "One must accept, as a *consequence of experience*, that energy is divided into two parts, one of which depends upon the mutual positions of points, and the other upon their velocity. The first one is potential energy, and the other is kinetic energy."

How can one hope that such a division will never become a result of experience? Hertz gave no explanation regarding that difficult subject. If the distinction between the two types of energy is accepted and understood then a great source of confusion will persist:

> "It is the task of physics to find the formula that expresses the potential energy of every body that is due to its ambient conditions *in terms of prior experiments*. Up to now, only three elements – viz., space, mass, and energy – are coupled with each other for the determination of the relations between the four fundamental quantities, as well as the succession of phenomena as functions of time. We then appeal to one of the known integrals of conventional mechanics, and it does not matter which."

Having made that choice, Hertz proposed Hamilton's principle, since he deemed that law to be the only one that could be verified by experiment:

> "Any system of masses moves as if its mission were to achieve a given configuration at a well-defined time in such a way that the difference between the kinetic and potential energy would be a minimum."

The realization of this second program demands that one must previously determine potential energy in each case. The difficulty in doing that would be great.

Upon renouncing that second way of presenting and organizing the principles, Hertz revealed the true goal of his book, namely, to point to the new method that he proposed to be a source of hope, without daring to accept it as a source of guidance at that point in time.

Once more, one must transliterate:

> "A third arrangement of the principles of mechanics that will be developed in the main part of the book is distinguished from the other two, insofar as it proceeds from only three independent fundamental ideas: the idea of time, the idea of space, and the idea of mass. One considers how one might strive to establish the natural relationships between the three ideas. A fourth notion like that of force or energy no longer exists as a fundamental concept."
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> "When we try to comprehend the motions of that bodies that surround us and to refer them to formulas that are clear and simple, while considering only the things that we have before our eyes, our experimentation will generally lack purpose. We soon perceive that the totality of what we can see and comprehend does not form a legitimate world in which identical states are associated with identical consequences. We remain convinced that the real world is more vast than the visible world. We sense the presence of invisible entities. Behind the ones that we see, we must look beyond the limits of our senses for other hidden actors. We have recognized those deeper influences in the first two pictures of science, and we have represented then as entities of a particular type. A different path opens up to us: We can assume that there is some hidden collaboration without attributing its cause to a new category of entities. It is permissible to assume that this occult entity is, in turn, in motion and massive, i.e., it has a motion and a mass that is distinguished from the visible motions and masses only in regard to our way of perceiving them. That is our hypothesis. We assume that it is possible to combine the visible masses in the universe with other ones that follow the same laws without there having to exist any other cause for the phenomena. That which one customarily refers to by the names of *force* and *energy* will become nothing but the effects of masses and motions that are not always perceptible. That is how one relates the forces of heat to the motion of tangible masses. Thanks to Maxwell, one assumes that electrodynamical forces show us the effects of the motion of hidden masses. Lord Kelvin had a predilection towards considering the possibility of placing dynamical explanations at the forefront of his conception of the physical world. In his theory on the rotatory nature of atoms, he tried to give a description of the universe that would conform to that way of looking at things. Helmholtz treated the same question in his examination of systems of vortices: It is thanks to him that the expressions *hidden mass* and *hidden motion* (*verborgene Masse*, *verborgene Bewegung*) have become accepted technical expressions."

There is nothing new in that concept. It is a return to Cartesian ideas. Subtle matter is a hidden mass and vortices are invisible motions. The idea has remained just as vague for two centuries. The author does not indicate a single application of it. The motions and masses that must replace the forces remain completely unknown, as well as their effects. That is an immense oversight that Hertz does not deem to be useful to address, despite the fact that it is so obvious.

The applications of the new method, if one wishes to attempt them, would undoubtedly be no different from the deductions of the ordinary theories. For example, if one wishes to calculate the motion of a pendulum without introducing gravity as a force then one might introduce invisible vorticial masses around the oscillating body in order to exert unknown actions on it. Upon consulting experiments, one will see that those actions impress upon each mass element (when it is free) an acceleration that points vertically and is measured by the number g = 9.860 m/sec^2. When the body is not free, those effects (no matter what name one wishes to given them) will combine, and when they are added together, they will be associated with the action of constraints in absolutely the same way as forces. Any rule that is truly new will lead to inexact results.

There is one case in which we see velocities and masses that act upon our eyes by impacts without having to introduce the invisible world. It is the one in which a machine composed of organs in contact with each other mutually repel each other, such as the teeth in gears, for example. That general case is treated in a chapter entitled *Systeme durch Kraft beeinflust* (systems that are influenced by forces). The word "Kraft," which we are allowed to translate into "force," thus reappears in science.

The somewhat-confusing definition that the author proposes for those forces gives them neither a point of application, nor a direction, nor a well-defined magnitude. Nonetheless, one calculates with them as if they were multipliers that are introduced algebraically into the equations. The author is then led to represent them by *vectors*, to compose them, and finally to apply the rule of the *parallelogram of forces*, which therefore persists, and even when one has abandoned the idea of force, one has still eternalized the name.

While excluding the forces that we call *external* from his theory, Hertz (to appeal to a language that is known to all of us) can treat only free systems, i.e., a system of masses that are subject to mutual constraints and for which no force perturbs the motion. The theory, which is very cleverly and skillfully composed, is presented with a true talent for pedagogy. The definitions, propositions, theorems, and corollaries, the algebraic translation of the hypotheses, the ingenious and knowledgeable transformations, imprint upon each page the stamp of a powerful mind that was prepared by much study. That this mind is one of an experimenter is in no doubt throughout all of Hertz's work, but no reader can sense it in this treatise on mechanics.

The first book, which is entitled *The Geometry and Kinematics of Material Systems*, beings with this declaration:

> "Experience is entirely foreign to the studies that are contained in this book. Everything belongs to judgments that are *a priori* in the sense that Kant attached to the term. Everything rests upon the laws of our mind and has no other connection with experience than that those laws themselves."

After pointing out the confusion the Newton showed in his definition of mass, Hertz, in turn, seemed to be even more confused. A mass-element [he said] is an attribute (*Merkmal*) by which we distinguish a well-defined point in space at a given instant from another point that is considered at another instant.

> "A mass-particle is an attribute by which a well-defined point in space at a given time is assigned a unique well-defined point in space at every other time."

Unfortunately, that definition plays no role, and when the word *mass* appears, the reader must be content to give it the familiar meaning.

Hertz gave the very general meaning to the word *system* that Lagrange had adopted. In order to study motion, he introduced the very carefully defined notion of a certain magnitude to which he assigned a direction, but without associating that direction with the direction of a well-defined straight line. Indeed, one defines the angle that two different changes that were performed subtend by starting from a given initial state. When the angle is zero, the displacements will be *parallel*, and the *direction* of an arbitrary displacements is, *by definition*, the one that is common to all displacements that make an angle of zero with it.

When a system is displaced continuously, the difference between two infinitely-close displacements is called the *curvature* of the route being traversed. If the points are free and independent then each of them will describe a straight line with a constant velocity. The *curvature* is zero in that case, and that is the only case for which that is true. When constraints exist, the *curvature* of the displacement cannot be zero. Hertz stated this very elegant theorem without proof:

Any free system remains in a state of rest or in a state of motion while following the straightest path.

That fundamental theorem translates into a form that intentionally recalls one of Newton's laws (†):

"Systema omne liberum perseverare in statu suo quiescendi vel movendi uniformiter in *directissimam*."

The expression "the straightest path," which translates into *directissima* and *geradeste Bahn*, has been explained and algebraically translated into several forms that are studied quite conventionally. After having thus prepared the reader for understanding the fundamental law – the *Grundsetz* – the author confines himself to declaring that, *based upon experience*, that law is necessary and sufficient for solving all of the problems in mechanics. Our mechanics, said that bold innovator, knows no other cause for motion than the one that results from our principle. Not even the motions of living beings are excepted, although Hertz deferred addressing that question. In that case, the principle would take on the character of a provisional hypothesis.

(†) Translator: "Every system is free to persist in its state of rest or to move uniformly in the *straightest* direction."

One undoubtedly asks, out of curiosity, "What are the proofs that this unique basis for science is based upon?" Heinrich Hertz added, without proposing any:

> "We consider the fundamental principle to be the most-probable result of the most-ordinary experiments. Rigorously-speaking, the fundamental principle is a hypothesis that agrees with a large number of experiments and is not refuted by any. In time, it might gain certain proofs that are implied by precise experiments."

Hertz undoubtedly wished to say that after having studied the mathematical consequences of his principle, one can subject them to experimental proof. That is because the principle itself, in the form that he gave to it, does not presume to any direct verification.

The proof was carried out a long time ago. It is strange that Hertz did not know that. The classical laws of the science of motion, namely, the ones that Hertz summarized by the name of the *first picture*, while associating them with some objections that gave him the right to contest their logical perfection, are in nothing less than perfect concordance with the facts, as everyone assumes. The new principle can then be regarded as sufficiently verified by experiments if it can be proved that in each case it leads to the same mathematical deductions as the formulas of classical science. Now, the consequences of Hertz's fundamental law in the case of a free system to which no external force is applied (which is the only one that we wished to study) are no different from those of a famous theorem of Gauss, which Hertz knew quite well, because he cited it while comparing it to that of d'Alembert in one of the pages of his book.

Gauss's admirable theorem is well known. The works of that great geometer occupy a place of honor in the entire bibliography of science, although none of its professors teach that theorem. Its application to the particular cases does not seem to be very simple, and furthermore, it does not appear in any of the official programs of study that are nonetheless overburdened with details. It would be strange (but not at all impossible) if it were to appear, as if in a dream, in a form that is very different and much less general in the mind of a young physicist who was to die after a brilliant debut, and if it were to finally enjoy the celebrity and importance that it is due after more than a half century, and to which one can given only the name of Gauss.

Permit us to recall its statement:

Let A be an arbitrary point of a mechanical system, and let m be the mass of that point. Let A' be the position that the point A takes by virtue of the velocity that it has acquired and *the influence of the forces that act upon it directly* if all of the constraints were eliminated during an infinitely-small time interval. Let B be the position to which the point A must actually arrive after the time interval dt. The sum $\sum m\,\overline{BA'}^2$ is a minimum, i.e., it has a value that is less than all of the ones that it could take if the point B is replaced with any other one that is compatible with the constraints.

In the case for which Hertz's fundamental law was proposed, the forces that act directly upon the points of the system do not exist. In that case, Gauss's theorem will then translate into an analytical formula.

If the coordinates of the point A are x, y, z then those of the point A' will be $x + \frac{dx}{dt}dt$, $y + \frac{dy}{dt}dt$, $z + \frac{dz}{dt}dt$, since no forces are applied to it. The position B that it actually occupies has the coordinates be $x + \frac{dx}{dt}dt + \frac{1}{2}\frac{d^2x}{dt^2}dt^2$, $y + \frac{dy}{dt}dt + \frac{1}{2}\frac{d^2y}{dt^2}dt^2$, $z + \frac{dz}{dt}dt + \frac{1}{2}\frac{d^2z}{dt^2}dt^2$. Gauss's theorem can then be stated by saying that when the sum:

$$\sum dm\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right]$$

is extended over all points of the system, it will be a minimum, i.e., smaller than if the points B were replaced with other ones that are compatible with the constraints.

If we now search through Hertz's book for the expression for curvature that his fundamental law declares to be a minimum then we will that it is expressed on page 85, formula (106), by:

$$\sum_{\gamma=1}^{3n} m_\gamma \, \alpha_\gamma''^{\,2} .$$

$\alpha_\gamma''^{\,2}$ denotes the second derivative of the coordinate x with respect to what one calls the *element of displacement* (*Bahnelement*), ds. The only difference between the analytical translations of the two theorems is then produced by replacing the differential ds with the differential of time dt. Now, upon studying Hertz's presentation, one sees that he does not point out that if one calls M the total mass of the system then $M\left(\frac{ds}{dt}\right)^2$ will be the *vis viva*. That *vis viva* is constant since no force is applied to the system. ds will then be proportional to dt, and the formulas will be equivalent.

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