"Über die Erhaltungssätze der Elektrodynamik," Math. Ann. **84** (1921), 258-276.

On the conservation laws of electrodynamics

By

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On the occasion of a colloquium that Herr Geheimrat F. Klein convened in the Winter semester of 1920 on mathematical problems in the relativistic theories of physics, he expressed the desire to apply the theorems on invariant variational problems ([1]) that Emmy Noether posed about two years ago to Maxwell's equations. The content of those theorems can be stated briefly by saying that the invariance of a variational problem under a continuous transformation group will imply a number of relations that are fulfilled identically by means of the differential equations of the problem and are represented by first integrals of those equations in the case of *one* independent variable. In the case of a finite group, those relations have the form of the physicist's so-called "conservation laws."

Now, it is generally known that Maxwell's equations are invariant under a finite ten-parameter group, namely, the so-called Lorentz group, which consists of the real "motions" of four-dimensional x, y, z, t-space, whose metric is based upon the part of the form:

$$x^2 + y^2 + z^2 - c^2\, t^2 = 0$$

that lies at infinity. In the year 1909, H. Bateman discovered ([2]) that Maxwell's equations are invariant under a much more comprehensive group of transformations, namely, the group of all of them that leave the equation:

$$dx^2 + dy^2 + dz^2 - c^2\, dt^2 = 0$$

unchanged and do not change the sense of the directions in the four-dimensional figures ([3]). If one writes x_1, x_2, x_3, x_4, instead of x, y, z, ict then except for the reality of the parameters that group will coincide with the largest subgroup that is contained in the fifteen-parameter group of transformations by reciprocal radii, namely, the so-called

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([1]) Göttingen Nachrichten (1918), pp. 235, *et seq.*, in what follows, it will be denoted briefly by E. Noether. See also Felix Klein, *Gesammelte mathematische Abhandlungen*, **1**, Berlin, 1921, pp. 585.

([2]) Proc. London Math. Soc. (2) **8** (1909), pp. 228, *et seq.* In the same volume of that journal, as well as in the preceding one, one will find more investigations of Bateman and Cunningham into the meaning of our $\mathfrak{G}_{15}$ for physics. See also F. Klein, *Ges. math. Abhandl.* **1**, pp. 552.

([3]) Bateman called them "spherical wave transformations."

conformal group ([4]) in R_4 . Now since, as J. L. Larmor has remarked ([5]), Maxwell's equations can be obtained from a variational problem, and since it is also invariant under the aforementioned $\mathfrak{G}_{15}$, as will be shown below, the theorems of E. Noether must yield fifteen linearly-independent electrodynamical conservation laws. The goal of the present note is to actually show that.

The first seven of them [cf., formulas (27 a_r , a_z , and b_r)] are nothing but the well-known theorems in the conservation of energy, impulse, and angular impulse ([6]); I therefore should not need to go any further into their interpretation. The following three (27 b_z) define a precise analogue of the second center-of-mass theorems in classical mechanics, and to my knowledge, they were obtained for the first time for *electrodynamical* phenomena by A. Einstein ([7]) by formal integration of Maxwell's equations. Einstein asserted (*loc. cit.*) their validity only in the first approximation, since the adaptation of dynamics to the relativity theory of the Lorentz group was still unknown to him at the time. G. Herglotz ([8]) exhibited the corresponding formulas for the *mechanics of continua* in the sense of relativity theory (and in precisely the same way that will come about here), and also interpreted them expressly as the center-of-mass theorems. The five remaining formulas (27 c, d_r , and d_z) are new, to my knowledge. Only the future can decide the extent to which they can be of service for the purposes of physics.

§ 1.

E. Noether's theorems.

I will next present the two theorems of E. Noether, and indeed in a somewhat more general context than they were given in the cited note. I would like to thank Emmy Noether herself for verbally communicating them to me. Let us assume that an integral in given:

(1) $$I_x = \int \cdots \int f\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \cdots\right) dx ,$$

which is extended over an arbitrary real domain of the variables $x_1, x_2, \ldots, x_n$. In this integral, u, $\partial u / \partial x$, $\partial^2 u / \partial x^2$, … are abbreviations for μ real functions of the $x_1, x_2, \ldots, x_n$, and their partial derivatives ([9]), while dx briefly stands for $dx_1\, dx_2 \ldots dx_n$. Under a single-valued and uniquely-invertible transformation of variables:

([4]) More details on the conformal group can be found in S. Lie and G. Scheffers, *Geometrie der Berührungstransformationen*, Leipzig, 1896, **1**, chap. 10, §§ 1 and 2, pp. 441, *et seq.*

([5]) *Aether and matter*, Cambridge, 1909, § 50, pp. 83, *et seq.* See also F. Klein, *Seminarvorträge über die Enwicklun der Mathematik in neunzehnten Jahrhundert*, chap. X, v. II, § 4 (1917). (These contributions have been worked out in transcripts of numerous university mathematical institutes.)

([6]) See, perhaps, M. v. Laue, *Die Relativitätstheorie*, **1**, 4th ed., Braunschweig, 1921, § 15 b-e.

([7]) Ann. Phys. (Leipzig) (4) **20** (1906), pp. 627, *et seq.*

([8]) Ann. Phys. (Leipzig) (4) **36** (1911), pp. 493, *et seq.* Cf. esp., formulas (96′) on pp. 513.

([9]) On the admissibility of complex numbers, cf., E. Noether, pp. 237, footnote 3.

$$
(2) \qquad \begin{cases} y_i = A_i\left(x, u, \dfrac{\partial u}{\partial x}, \cdots\right) & [i = 1, 2, \cdots, n], \\ v_\rho(y) = B_\rho\left(x, u, \dfrac{\partial u}{\partial x}, \cdots\right) & [\rho = 1, 2, \cdots, \mu], \end{cases}
$$

and its extension to a transformation of the derivatives $\partial v / \partial y$, $\partial^2 v / \partial y^2$, …, (1) will go to:

$$
I_y = \int \cdots \int \bar{f}\left(y, v, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \cdots\right) dy,
$$

in which the integral is taken over the y-domain that corresponds to the x-domain in (1). In particular, if the function $\bar{f}$ is identical with the function f then I is said to be *invariant* under the transformation (2).

We now consider a continuous group of transformations (2) and assume that the parameters ε_1, ε_2, …, ε_n, are chosen in such a way that the identity transformation corresponds to the values $\varepsilon = 0$ [the functions $p(x) \equiv 0$, $\partial p(x) / \partial x \equiv 0$, …, resp.] in the case of a finite group $\mathfrak{G}_r$ [the arbitrary functions $p^{(1)}(x)$, $p^{(2)}(x)$, …, $p^{(\rho)}(x)$, resp., in the case of an infinite group $\mathfrak{G}_{\infty\rho}$]. The transformation formulas (2) then take on the form:

$$
(3) \qquad \begin{cases} y_i = x_i + \Delta x_i + \cdots & [i = 1, 2, \ldots, n], \\ v_\rho(y) = u_\rho + \Delta u_\rho + \cdots & [\rho = 1, 2, \ldots, \mu], \end{cases}
$$

and we assume that it is permissible to assume that the Δx_i, Δu_ρ are in linear the ε [p, resp.] and their derivatives ([10]). If we truncate the right-hand side of (3) after these linear terms then, according to Lie, the resulting transformations are known to be infinitesimal. The invariance of the integral I under an infinitesimal transformation correspondingly means that $\bar{f}$ differs from f only by terms that have at least second order in the ε [p, $\partial p / \partial x$, …, resp.].

Let us understand a divergence to mean an expression of the form:

$$
\operatorname{Div} A = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_n}{\partial x_n},
$$

in which A_i are functions of x, u, $\partial u / \partial x$, … The differentiations with respect to x are taken to be total differentiations; i.e., when the u, $\partial u / \partial x$, … are considered to be functions of x.

I shall now call I "invariant up to a divergence" under an infinitesimal transformation when:

$$
(4) \qquad \bar{f} = f + \operatorname{Div} C + \text{higher terms},
$$

([10]) Cf., E. Noether, pp. 244 at the bottom and pp. 246, beginning of § 4.

in which the expression C is *linear* in the ε [p, $\partial u / \partial x$, …, resp.]. The case in which C is identically zero is occasionally included in this manner of speaking ([11]). In the introduction of this concept lies the generalization of Emmy Noether's original publication that was mentioned at the beginning of the paragraph.

Moreover, we can express the theorems of Emmy Noether as follows:

If the integral I is invariant over the infinitesimal transformation of a finite group $\mathfrak{G}_\rho$ *up to a divergence then there will be precisely* ρ *linearly-independent connections between the Lagrangian expressions and divergences.*

Namely, one sets:

$$\delta u_i = v_i(x) - u_i(x) = \Delta u_i - \sum_\lambda \frac{\partial u_i}{\partial x_\lambda} \Delta x_\lambda \,, \tag{5}$$

and defines $A_1, \ldots, A_n$ by the identity:

$$\sum \psi_i \, \delta u_i = \delta f + \text{Div}\, A,$$

in which the ψ_i means the Lagrangian expressions of the function f, and $B_1, \ldots, B_n$ are defined by the equations:

$$B_i = C_i + A_i - f \, \Delta x_i \,. \tag{6}$$

One then splits δu and B according to the individual ε:

$$\begin{aligned} \delta u_i &= \varepsilon_1 \, \delta^{(1)} u_i + \ldots + \varepsilon_\rho \, \delta^{(\rho)} u_i \,, \\ B_i &= \varepsilon_1 B_i^{(1)} \quad + \ldots + \varepsilon_\rho^i B_i^{(\rho)} \,, \end{aligned}$$

and the desired divergence relations will be ([12]):

$$\sum \psi_i \, \delta^{(1)} u_i = \text{Div}\, B^{(1)}, \qquad \ldots, \qquad \sum \psi_i \, \delta^{(\rho)} u_i = \text{Div}\, B^{(\rho)}. \tag{7}$$

Conversely, if it is known from the Lagrangian expressions that for suitable functions δu and B there exist precisely ρ linearly-independent relations (7) then one can exhibit ([13]) ρ linearly-independent infinitesimal transformations under which I is invariant up to a divergence. Since the splitting of B into C and $A - f \, \Delta x$ is possible in many ways, one can also arrive at many systems of such infinitesimal transformations. One easily convinces oneself of the validity of the remark that the aforementioned splitting can be performed if and only if the resulting transformations are free of $\partial u / \partial x$, $\partial^2 u / \partial x^2$, … when the

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([11]) Whereas complete invariance under an infinitesimal transformation T will imply complete invariance under a one-parameter group that is generated by T with no further assumptions, the corresponding statement about the invariance up to a divergence is not the case, in general. For that reason, the definition of the concept must necessarily be linked with the infinitesimal transformations.

([12]) Cf., E. Noether, § 2, pp. 242.

([13]) As E. Noether did in § 2, pp. 242.

functions δu are free of $\partial^2 u / \partial x^2$, $\partial^3 u / \partial x^3$, …, and either likewise free of the $\partial u / \partial x$ or linearly-dependent in a very special way ([14]). If that condition is fulfilled then it can be proved that the ρ infinitesimal transformation to which one arrives generate precisely a ρ-parameter group.

For the sake of later applications, I shall point out the expression for B_i in the case for which f depends upon only the first derivatives $\partial u / \partial x$:

$$(8) \qquad B_i = C_i - \sum_k \frac{\partial f}{\partial \frac{\partial u_k}{\partial x_i}} \Delta x_i + \sum_\lambda \Delta x_\lambda \left(\sum_k \frac{\partial f}{\partial \frac{\partial u_k}{\partial x_i}} \frac{\partial u_k}{\partial x_\lambda} - \delta_{\lambda i} f \right),$$

$$\delta_{\lambda i} = \begin{cases} 0 & \text{when } \lambda \neq i, \\ 1 & \text{when } \lambda = i. \end{cases}$$

The second theorem relates to an infinite continuous group $\mathfrak{G}_{\infty\rho}$ and says that:

The invariance of I up to a divergence under the infinitesimal transformations of $\mathfrak{G}_{\infty\rho}$ has the existence of ρ linearly-independent dependencies between the ψ_i and their total derivatives with respect to x, and that conversely the existence of ρ such linearly-independent dependencies will imply the invariance of I up to a divergence under a certain set of ρ infinitesimal transformation with ρ arbitrary functions.

In order to exhibit the aforementioned dependencies, one writes down equation (5) in the developed form:

$$(9) \qquad \delta u_i = \sum_{\lambda=1}^{\rho} \left\{ a_i^{(\lambda)}(x, u, \ldots)\, p^{(\lambda)}(x) + b_i^{(\lambda)}(x, u, \ldots) \frac{\partial p^{(\lambda)}}{\partial x} + \cdots + c_i^{(\lambda)}(x, u, \ldots) \frac{\partial^\sigma p^{(\lambda)}}{\partial x^\sigma} \right\}.$$

The dependencies then read simply ([15]):

$$(10) \qquad \sum_i \left\{ (a_i^{(\lambda)} \psi_i) - \frac{\partial}{\partial x}(b_i^{(\lambda)} \psi_i) + \cdots + (-1)^\nu \frac{\partial^\sigma}{\partial x^\sigma}(c_i^{(\lambda)} \psi_i) \right\} = 0 \qquad [\lambda = 1, 2, \ldots, \rho].$$

([14]) Namely:

$$\delta u_i = \alpha_i\,(x, u) + \sum_\lambda \beta_\lambda (x, u) \frac{\partial u_i}{\partial x_\lambda}.$$

One can then arrive at:

$$\Delta x_i = -\beta_i\,(x, u), \quad \Delta u_i = \alpha_i\,(x, u), \qquad C_i = \frac{A_i - B_i}{f} + \beta_i\,(x, u).$$

([15]) E. Noether, § 2, pp. 243.

§ 2.

Application to the *n*-body problem.

The derivation of the ten known integrals of the n-body problem will serve as a first example of how convenient it is to apply E. Noether's theorems. Although the basic ideas, like the detailed implementation, will require nothing new ([16]), I would like to carry out the brief calculations completely by a formal analogy with the electrodynamical conservation laws that will be presented later. The differential equations of the n-body problem are obtained by the variational problem:

$$\delta \int_{\bar{}}^{\bar{}} L\,dt = 0,$$

in which the lines might suggest that the variation is performed with fixed limits. The Lagrangian function L has the following meaning in this:

$$L = T - U,$$

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i(\dot{x}_{i1}^2 + \dot{x}_{i2}^2 + \dot{x}_{i3}^2),$$

$$U = -\sum \frac{\kappa\, m_i\, m_k}{r_{ik}}, \qquad 1 \le i < k \le n,$$

$$r_{ik} = \sqrt{(x_{i1} - x_{k1})^2 + (x_{i2} - x_{k2})^2 + (x_{i3} - x_{k3})^2},$$

κ = gravitational constant,

and the x_{ik} are determined as functions of t by the variational problem. Here, one is therefore dealing with a *simple* integral, where t enters in place of the quantity that was previously denoted by x, and the x_{ik} enter in place of what was previously u.

It is known that the equations of motion of the n-body problem are invariant under a finite, ten-parameter group, namely, the so-called "Galilei-Newton group." This invariance manifests itself in the variational problem in such a way that L is invariant under the infinitesimal transformations of the group, in part completely and in part up to a divergence. In fact, that invariance reads:

([16]) F. Engel [Gött. Nachr. (1916), pp. 270, *et seq.*] also treated this question by Lie's method, but without the use of the advantageous fact that the differential equations arose from a variational problem. (A comparison will show the advantage of the variational theorem quite clearly.) For the historical development of our understanding of the meaning of the ten integrals and their connection to the equations of motion, confer the relevant places in Jacobi's *Vorlesungen über Dynamik*, 1897, pp. 110, *et seq.*, as well as the interesting note of J. R. Schutz in Gött. Nachr. (1897), pp. 110, *et seq.* and the summary presentation of F. Klein in *Die Entwicklung der Mathematik in neunzehnten Jahrhundert*, chap. 10, A, § 2 and C, § 4, 1917.

(11)

$$
\begin{array}{llll}
\text{a)} & \Delta t = \tau, & \Delta x_{ik} = 0, & \\
\text{b)} & \Delta t = 0, & \Delta x_{ik} = \alpha_k\,, & \\
\text{c)} & \Delta t = 0, & \Delta x_{ik} = \sum_{\rho=1}^{3} \beta_{k\rho} x_{i\rho} & \left(\begin{matrix}\beta_{kk} = 0, \\ \beta_{k\rho} = -\beta_{\rho k}\end{matrix}\right), \\
\text{d)} & \Delta t = 0, & \Delta x_{ik} = \gamma_k\, t & [k = 1, 2, 3],
\end{array}
$$

and one sees with no effort that $\Delta L = 0$ for a), b), c), while for d), one will have:

$$\Delta L = \frac{d}{dt}\left(\sum_{i=1}^{n}\sum_{k=1}^{3} m_i\, \gamma_k\, x_{ik}\right) = \frac{d}{dt} C = \operatorname{Div} C.$$

Moreover, formulas (5) and (8) imply that:

$$\delta x_{ik} = \Delta x_{ik} - \dot{x}_{ik}\, \Delta t,$$

$$B = C - \sum_{i=1}^{n}\sum_{k=1}^{3} m_i\, \dot{x}_{ik}\, \Delta x_{ik} + \Delta t\,(T + U),$$

and E. Noether's divergence relations will assume the form:

(12)

$$
\begin{array}{llll}
\text{a)} & -\sum_{i=1}^{n}\sum_{k=1}^{3} \dot{x}_{ik}\, \psi_{ik} & = \frac{d}{dt}(T + U), & \\
\text{b)} & \sum_{i=1}^{n} \psi_{ik} & = -\frac{d}{dt}\sum_{i=1}^{n} m_i\, \dot{x}_{ik} & [k = 1, 2, 3], \\
\text{c)} & \sum_{i=1}^{n} (x_{i\mu}\, \psi_{i\nu} - x_{i\nu}\, \psi_{i\mu}) & = -\frac{d}{dt}\sum_{i=1}^{n} m_i (x_{i\mu}\, \dot{x}_{i\nu} - x_{i\nu}\, \dot{x}_{i\mu}) & [(\mu, \nu), (2, 3), (3, 1), (1, 2)], \\
\text{d)} & \sum_{i=1}^{n} t\, \psi_{ik} & = \frac{d}{dt}\left\{\sum_{i=1}^{n} m_i\, x_{ik} - t\sum_{i=1}^{n} m_i\, \dot{x}_{ik})\right\} & [k = 1, 2, 3].
\end{array}
$$

So far, we are dealing with only purely *formal* identities that can be subsequently verified directly quite easily when one sets:

$$\psi_{ik} = \frac{\partial L}{\partial x_{ik}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{ik}}\right) = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n}{}' \frac{\kappa\, m_i\, m_\nu}{r_{i\nu}^3}(x_{\nu k} - x_{ik}) - \frac{d}{dt}(m_i\, \dot{x}_{ik}).$$

Up till now, no use has been made of the demand that $\delta \int_{\bar{}}^{\bar{}} L\, dt = 0$. However, if we now consider the *differential equations* of the *n*-body problem then we will have to set the Lagrangian expressions ψ_i equal to zero, and equations (12), whose left-hand sides will then vanish, will yield the ten known first integrals of the problem, namely:

(12a) is the law of energy,
(12b) are the three first center-of-mass laws (also called laws of impulse),
(12 c) are the three area laws, and
(12 d) are the three second center-of-mass laws.

The form that the latter assume deviates somewhat from what is customary; in order to arrive at the latter, one must only observe that from (12b), one will have $\sum_i m_i \dot{x}_{ik} = c_k$, from which, it will follow that:

$$\sum_{i=1}^{n} m_i x_{ik} = c_k t + c'_k \qquad [k = 1, 2, 3] \qquad (c_k,\ c'_k = \text{constants}). \tag{13}$$

However, only the form (12 d) will meaningful for us because, first of all, it shows that the second center-of-mass laws are arranged completely in the sequence of the remaining conservation laws, and secondly, it gives us the key to our interpretation of the analogous electrodynamical relations (28).

§ 3.

Overview of the notations that will be used in what follows.

Before getting into a treatment of the electrodynamical equations, I shall first give an overview of the notations that will be used in the following. In general, I shall use the symbolism that M. v. Laue used in his book *Die Relativitätstheorie*, Bd. I, and I also follow v. Laue in his symbolism for three-dimensional and four-dimensional vector and tensor calculus (even if it is rather ugly), such that the reader can refer to v. Laue for any symbol that is perhaps unknown to him. The system of measurement is the Lorentzian one ([17]) in CGS units.

([17]) See Encycl. d. math. Wissenschaften **5**, art. 13, 7d.

Table of notations ([18])

In four-dimensional notation	In three-dimensional notation
x_1, x_2, x_3, x_4	$x, y, z, i\,ct$
Electromagnetic six-vector tensor:	$\mathfrak{r}$ = Vector from the coordinate origin to a fixed spatial point, but not to a moving particle
$f: f_{23}, f_{31}, f_{12}\ ;\ f_{14}, f_{14}, f_{34}$ $f_{ik} = -f_{ki}$	$\mathfrak{H}_x, \mathfrak{H}_y, \mathfrak{H}_z;\quad -i\,\mathfrak{E}_x, -i\,\mathfrak{E}_y, -i\,\mathfrak{E}_z$
The six-vector that is dual to it: $f^*:\ f^*_{12} = f_{34},\ f^*_{13} = f_{42},\ f^*_{14} = f_{23}$ $f^*_{23} = f_{14},\ f^*_{24} = f_{31},\ f^*_{34} = f_{12}$	
Four-potential: $\varphi: \varphi_1, \varphi_2, \varphi_3, \varphi_4$	
Analogue of the Lagrangian function: $\Lambda = \frac{1}{4}\sum_{i=1}^{4}\sum_{k=1}^{4} f_{ik}^2 = \frac{1}{2}\sum_{1\le i<k\le 4} f_{ik}^2$	$\frac{1}{2}(\mathfrak{H}^2 - \mathfrak{E}^2)$
Electromagnetic energy-impulse tensor: $S_{ik} = S_{ki} = \sum_{\lambda=1}^{4} f_{i\lambda} f_{\lambda k} + \delta_{ik}\Lambda$ $\delta_{ik} = \begin{cases} 0 & \text{when } i \neq k \\ 1 & \text{when } i = k \end{cases}$	$\begin{pmatrix} p_{exx} & p_{exy} & p_{exz} & \frac{i}{c}\mathfrak{S}_{ex} \\ p_{eyx} & p_{eyy} & p_{eyz} & \frac{i}{c}\mathfrak{S}_{ey} \\ p_{ezx} & p_{ezy} & p_{ezz} & \frac{i}{c}\mathfrak{S}_{ez} \\ \frac{i}{c}\mathfrak{S}_{ex} & \frac{i}{c}\mathfrak{S}_{ey} & \frac{i}{c}\mathfrak{S}_{ez} & -W_e \end{pmatrix}$
	$\boldsymbol{p}_\varepsilon$ = Maxwellian stress densities

([18]) Not all quantities are listed by juxtaposition in both columns, in order to not lead to confusion by further increasing the list symbols with ones that will not be used.

In four-dimensional notation	In three-dimensional notation
	$\mathfrak{S}_e =$ $c\,[\mathfrak{E}, \mathfrak{H}]$ = Poynting vector of electromagnetic energy-flux
	$W_e =$ $= \frac{1}{2}(\mathfrak{E}^2 + \mathfrak{H}^2)$ = density of electromagnetic field energy
Density of four-current: P_1, P_2, P_3, P_4	$\frac{\rho\mathfrak{q}_x}{c}, \frac{\rho\mathfrak{q}_y}{c}, \frac{\rho\mathfrak{q}_z}{c}, i\rho$
	$\rho =$ Spatial density of electric charge
	$\mathfrak{q} =$ Velocity vector of electric charge (its material carrier, resp.)
Density of electric four-force: F_1, F_2, F_3, F_4	Density of the force that the field exerts on the charges $\mathfrak{F} = \rho\,(\mathfrak{E} + \frac{1}{c}[\mathfrak{q}, \mathfrak{H}])$
$F_i = \sum_k f_{ik} P_k$	Power density of that force: $(\mathfrak{F}\,\mathfrak{q}) = \rho\,(\mathfrak{E}\,\mathfrak{q})$
Mechanical energy-impulse tensor: $R_{ik} = R_{ki}$	$\left(\begin{array}{c\|c} [[\mathfrak{g}_m, \mathfrak{q}]] & ic\,\mathfrak{g}_m \\ \hline \frac{i}{c}\mathfrak{S}_m & -W_m \end{array}\right)$
	$\mathfrak{g}_m =$ Density of mechanical impulse $= \frac{k_0\,\mathfrak{q}}{\sqrt{1-\frac{\mathfrak{q}^2}{c^2}}}$
	$k_0 =$ Mass density
	$W_m =$ Kinetic energy density of the moving matter

Four-dimensional notation	Three-dimensional notation
	$\mathfrak{S}_m = \dfrac{k_0\,\mathfrak{q}}{\sqrt{1-\dfrac{\mathfrak{q}^2}{c^2}}} = \mathfrak{q}\,W_m$ = Density of the energy-flux that is mediated by the motion of the matter
Total energy-impulse tensor: $T_{ik} = T_{ki} = R_{ik} + S_{ik}$	$\left(\begin{array}{c\|c} \mathbf{p} & ic\,\mathfrak{g} \\ \hline \frac{i}{c}\mathfrak{S} & -W \end{array}\right)$
	$\mathbf{p} = \mathbf{p}_e + [[\mathfrak{g}_m, \mathfrak{q}]]$ = Total stress tensor
	$\mathfrak{g} = \mathfrak{g}_e + \mathfrak{g}_m$ = Total impulse density
	$\mathfrak{S} = \mathfrak{S}_e + \mathfrak{S}_m$ = Total energy-flux
	$W = W_e + W_m$ = Total energy density

§ 4.

The invariance of Maxwell's equations under the conformal group.

Of the two systems of Maxwell's equations for the free ether:

(14) $$\text{I.}\qquad \Delta\text{iv}\, f^* = 0, \qquad \text{II.}\qquad \Delta\text{iv}\, f = 0,$$

the first one will be satisfied identically by the Ansatz:

(15) $$f = \mathfrak{Rot}\,\varphi\,.$$

If one introduces this into II then the left-hand side of II will become precisely the Lagrangian expression ψ_i for the variational problem:

$$\delta \iiiint \Lambda\, dx_1\, dx_2\, dx_3\, dx_4 = 0,$$

in which x_1, x_2, x_3, x_4 are considered to be independent variables, and φ_1, φ_2, φ_3, φ_4 are the desired functions of them, and the variation is performed with fixed boundary and fixed boundary values for the φ, which is suggested by horizontal lines. Now, the integral that appears here will remain unchanged when one subjects the $x_1, \ldots, x_4$ to an arbitrary transformation of the 15-parameter conformal group of R_4 and at the same time transforms the components φ_1, φ_2, φ_3, φ_4 of the four-potential contragrediently to the differentials dx_1, dx_2, dx_3, dx_4. Since we are dealing with only purely formal operations here, we do not need to concern ourselves with the reality conditions on the group parameters (which are necessary from the standpoint of the physicists). One sees that aforementioned invariance easily when one keeps in mind the fact that the quantities $f_{ik} = \dfrac{\partial\varphi_k}{\partial x_i} - \dfrac{\partial\varphi_i}{\partial x_k}$ are converted contragrediently to the quantities $dx_i\ dx_k$ and calculates the expression that arises from the expression:

$$\left(\sum_{i,k} f_{ik}^2\right) dx_1\, dx_2\, dx_3\, dx_4$$

by an arbitrary linear transformation of the dx. One then finds that it will keep the same form in the new variables:

$$\left(\sum_{i,k} \bar{f}_{ik}^2\right) d\bar{x}_1\, d\bar{x}_2\, d\bar{x}_3\, d\bar{x}_4$$

that it had in the old ones if and only if the transformation takes the equation $\sum_i dx_i^2 = 0$ to the corresponding one $\sum_i d\bar{x}_i^2 = 0$. However, the totality of these transformations is precisely the conformal group.

Along with this finite continuous group, the variational problem also obviously admits an infinite group that includes the first derivatives of the arbitrary function:

$$\bar{x}_i = x_i\,, \qquad \bar{\varphi}_i = \varphi_i + \frac{\partial p}{\partial x_i} \qquad [i = 1, 2, 3, 4],$$

since [cf., (15)] the rotation of a gradient will vanish identically.

§ 5.

Presentation of the formal identities.

On the basis of E. Noether's theorems, 15 linearly-independent linear couplings of the Lagrangian expressions:

(16) $$\psi_i = \sum_k \frac{\partial f_{ik}}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k}\left(\frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k}\right)$$

must be identical to divergences now, and in addition, a dependency between the ψ_i and their first derivatives must be fulfilled identically.

One system of fifteen linearly-independent infinitesimal transformations of our $\mathfrak{G}_{15}$ is the following one ([19]):

(17)

a) $$\Delta x_k = \alpha_k ,$$

b) $$\Delta x_k = \sum_\rho \beta_{k\rho} x_\rho \qquad \begin{pmatrix} \beta_{kk} = 0 \\ \beta_{k\rho} = -\beta_{\rho k} \end{pmatrix},$$

c) $$\Delta x_k = \gamma x_k ,$$

d) $$\Delta x_k = 2x_k \sum_\rho \varepsilon_\rho x_\rho - \varepsilon_k \sum_\rho x_\rho^2 \qquad [k = 1, 2, 3, 4].$$

Of those transformations, the ones in a) for which $k = 4$ correspond to the transformations (11 a) of the Galilei-Newton group, while the other three (17 a) refer to the spatial translations (11 b); the "spatial rotations" of (17 b) that belong to the parameters β_{23}, β_{31}, β_{12}, correspond to the rotations (11 c) of the Galilei-Newton group, while the remaining three "temporal rotations" that belong to β_{14}, β_{24}, β_{34} correspond to the introduction of a another t-axis in a different direction for fixed x, y, z-space into the Galilei-Newton group (11 d). Formulas (17 c) and (17 d) correspond to the composition of two transformations by reciprocal radii. The transformation of φ that is contragredient to the transformation of the dx is given simply by the formula:

(18) $$\Delta \varphi_k = -\sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k} \varphi_\lambda \qquad [k = 1, 2, 3, 4],$$

and the infinitesimal transformation of the infinite continuous group is given by:

$$\Delta x_k = 0, \qquad \Delta \varphi_k = \frac{\partial p}{\partial x_k}.$$

We next consider the dependency that corresponds to the last one. From (5), we will have:

$$\delta \varphi_k = \frac{\partial p}{\partial x_k},$$

([19]) See, e.g., S. Lie and F. Engel, *Theorie der Transformationsgruppen* **3**, Leipzig, 1893, pp. 281, 234, 347-351.

and then a comparison of (9) and (10) will yield:

$$\sum_i \frac{\partial}{\partial x_i}\psi_i = 0. \tag{19}$$

In order to exhibit the divergence relations, we use (8) to construct:

$$B_i = -\sum_k f_{ik}\,\Delta\varphi_k + \sum_\lambda \Delta x_k \left\{\sum_k f_{ik}\,\frac{\partial\varphi_k}{\partial x_\lambda} - \delta_{\lambda i}\Lambda\right\},$$

and on the basis of the definition of the electromagnetic energy-impulse tensor S_{ik} that will go to:

$$B_i = -\sum_k f_{ik}\,\Delta\varphi_k + \sum_\lambda \Delta x_\lambda \left\{-S_{\lambda i} + \sum_k f_{ik}\,\frac{\partial\varphi_k}{\partial x_\lambda}\right\}. \tag{20}$$

Now, if we were to substitute the expressions in formulas (17) and (18) for Δx, $\Delta\varphi$ then we would arrive at divergence relations that would be very long and not very intuitive to understand ([20]), and above all, they would suffer from the essential flaw that the four-potential components φ in them, which are only mathematical tools here and have no autonomous real physical meaning, appear explicitly and not only in the couplings f_{ik} , which are the only things that are physical meaningful. However, that flaw can be remedied by a gimmick. One can isolate finite groups within the infinite continuous group in many ways when one takes the function p to be not just arbitrary, but dependent upon only finitely-many parameters. Now, if we add the expression $\partial p / \partial x_k$ to the transformation (18) when p has been specialized in a suitable way then we will get precisely those infinitesimal transformations that lead to divergence relations in which the φ will appear only in the combinations f_{ik} . How we must specialize p will be shown in the course of calculations.

If we accordingly substitute the expression for $\Delta\varphi_k$:

$$-\sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k}\varphi_\lambda + \frac{\partial p}{\partial x_k} \tag{21}$$

in (20) then we will get:

$$B_i = \sum_k f_{ik}\left\{\sum_\lambda \frac{\partial \Delta x_\lambda}{\partial x_k}\varphi_\lambda - \frac{\partial p}{\partial x_k} + \sum_\lambda \Delta x_\lambda\,\frac{\partial\varphi_\lambda}{\partial x_k}\right\} - \sum_\lambda S_{\lambda i}\,\Delta x_\lambda$$

([20]) As an example, I shall give the formula that flows from (17 d):

$$\sum_i \frac{\partial}{\partial x_i}\left\{2 f_{ik}\sum_r x_r\varphi_r - 2x_i x_k\Lambda + \sum_r f_{ir}\left[2(x_k\varphi_r - x_r\varphi_k) + 2x_k\sum_s x_s\frac{\partial\varphi_r}{\partial x_s} - \frac{\partial\varphi_r}{\partial x_k}\sum_s x_s^2\right]\right\} + \frac{\partial}{\partial x_k}\left(\Lambda\sum_s x_s^2\right)$$
$$= -\sum_i \psi_i\left\{2(x_k\varphi_r - x_r\varphi_k) + 2x_k\sum_s x_s\frac{\partial\varphi_i}{\partial x_s} - \frac{\partial\varphi_i}{\partial x_k}\sum_s x_s^2\right\} - 2\psi_k\sum_r x_r\varphi_r \qquad [k = 1, 2, 3, 4].$$

$$= \sum_k f_{ik} \frac{\partial}{\partial x_k}\left(\sum_\lambda \varphi_\lambda - p\right) - \sum_\lambda S_{\lambda i}\, \Delta x_\lambda ,$$

and we now see that the correct specialization of p is:

$$p = \sum_\lambda \varphi_\lambda\, \Delta x_\lambda .$$

From (5) and (21), we will now have:

$$\delta\varphi_i = \sum_\lambda f_{i\lambda}\, \Delta x_\lambda ,$$

such that the divergence relations will assume the form:

$$\sum_i \sum_\lambda \psi_i f_{i\lambda}\, \Delta x_\lambda = \sum_i \frac{\partial}{\partial x_i}\left(\sum_\lambda S_{i\lambda}\, \Delta x_\lambda\right). \tag{22}$$

If we now insert the expressions (17) then we will get:

(23) a) $$\sum_i \frac{\partial}{\partial x_i} S_{\lambda i} = \sum_i \psi_i f_{i\lambda} \qquad [\lambda = 1, 2, 3, 4],$$

b) $$\sum_i \frac{\partial}{\partial x_i}(x_\mu S_{\nu i} - x_\nu S_{\mu i}) = \sum_i \psi_i (x_\mu f_{i\lambda} - x_\nu f_{i\mu})$$
$$[(\mu, \nu) = (1, 2), (1, 3), (1, 4), (2, 4), (3, 4)],$$

c) $$\sum_i \frac{\partial}{\partial x_i}\left(\sum_\rho x_\rho S_{\rho i}\right) = \sum_i \psi_i \left(\sum_\rho x_\rho f_{i\rho}\right),$$

d) $$\sum_i \frac{\partial}{\partial x_i}\left\{2x_\lambda \sum_\rho x_\rho S_{\rho i} - S_{\lambda i} \sum_\rho x_\rho^2\right\} = \sum_i \psi_i \left\{2x_\lambda \sum_\rho x_\rho f_{i\rho} - f_{i\lambda} \sum_\rho x_\rho^2\right\}$$
$$[\lambda = 1, 2, 3, 4].$$

§ 6.

Introduction of the physical Ansätze.

Up to this point, we have been dealing with only purely formal identities that can be verified by way of the replacements of (16) and the values of the S_{ik} as functions of the φ that follow from the table on pages 9-11. Only now do we introduce the physical Ansatz when we set the Lagrangian expressions ψ_i for the free ether in (14 II) equal to zero and do the same thing for the components of the four-current P_i for the region that is filled with ponderable matter ([21]). Furthermore, physical laws that one cares to refer to as "conservation laws" will follow from the identities (19) and (23) in that way. However, in order to highlight the physical content completely, in the absence of ponderable matter, we cannot restrict ourselves to the phenomena in the electromagnetic field that the Ansatz $\psi_i = P_i$ will imply by itself, but we must consider the interaction of the field and ponderable masses that is given by the expression for the four-force:

$$F_k = \sum_i f_{ki} P_i \, .$$

This force density (power density, resp.) is, in its own right, related to the impulse and energy densities of the moving mass by relativistic dynamics:

$$\mathfrak{F} = \frac{\partial \mathfrak{g}_m}{\partial t} + \mathfrak{div}\, [[\mathfrak{g}_m , \mathfrak{q}]],$$

$$(\mathfrak{F}\, \mathfrak{q}) = \frac{\partial W_m}{\partial t} + \operatorname{div} \mathfrak{q}\, W_m \, ,$$

or, when written four-dimensionally:

$$F_k = \sum_i \frac{\partial}{\partial x_i} R_{ki} \, .$$

With that, the conservation laws will take on the form:

(24) a) $$\sum_i \frac{\partial}{\partial x_i} T_{\lambda i} = 0 \qquad [\lambda = 1, 2, 3, 4],$$

b) $$\sum_i \frac{\partial}{\partial x_i} (x_\mu T_{\nu i} - x_\nu T_{\mu i}) = 0 \qquad [(\mu, \nu) = (1, 2), \ldots, (3, 4)],$$

c) $$\sum_i \frac{\partial}{\partial x_i} \left(\sum_\rho x_\rho T_{\rho i} \right) = \sum_i R_{ii} \, ,$$

d) $$\sum_i \frac{\partial}{\partial x_i} \left\{ 2x_\lambda \sum_\rho x_\rho T_{\rho i} - T_{\rho i} \sum_\rho x_\rho^2 \right\} = 2x_\lambda \sum_i R_{ii} \qquad [\lambda = 1, 2, 3, 4],$$

in which we have written $S_{\lambda i} + R_{\lambda i} = T_{\lambda i}$ and made use of the symmetry $R_{ik} = R_{ki}$.

([21]) For the sake of simplicity, I shall restrict myself to the fundamental equations of the theory of electrons; i.e., to the limiting case $\varepsilon = 1$, $\mu = 1$, $\sigma = 0$ of the equations for ponderable matter.

§ 7.

The physical interpretation of the results.

In order to ascertain the physical meaning of the laws (24), we must unavoidably split them into their spatial and temporal parts, although the beautiful symmetry of the formulas will be spoiled most cruelly in that way. If we write K as an abbreviation for the sum $\sum R_{ii}$ then rewriting them in three-dimensional vector analysis ([22]) will give:

(25) a$_r$) $$\frac{\partial}{\partial t}\mathfrak{g} + \mathfrak{div}\,\mathbf{p} = 0,$$

a$_z$) $$\frac{\partial}{\partial t} W + \operatorname{div}\mathfrak{S} = 0,$$

b$_r$) $$\frac{\partial}{\partial t}[\mathfrak{r}, \mathfrak{g}] + \mathfrak{div}\,[\mathfrak{r}\times\mathbf{p}] = 0 \quad ([23]),$$

c) $$\frac{\partial}{\partial t}\{(\mathfrak{r}\,\mathfrak{g}) - W\,t\} + \operatorname{div}\{[\mathfrak{r}, \mathbf{p}]] - \mathfrak{S}\,t\} = K,$$

d$_r$) $$\frac{\partial}{\partial t}\{\mathfrak{r}\,(\mathfrak{r}\,\mathfrak{g}) + [\mathfrak{r}, [\mathfrak{r}, \mathbf{p}]] - 2\mathfrak{r}\,t\,W + c^2\,t^2\,\mathfrak{g}\}$$
$$+ \mathfrak{div}\,\{\mathbf{[}[\mathfrak{r}, [\mathfrak{r}, \mathbf{p}]]\mathbf{]} + [\mathfrak{r}\times[\mathfrak{r}\times\mathbf{p}] - 2t\,\mathbf{[}[\mathfrak{r}\times\mathfrak{S}]\mathbf{]} + c^2\,t^2\,\mathbf{p}\} = 2\mathfrak{r}\,K,$$

d$_z$) $$\frac{\partial}{\partial t}\left\{2t(\mathfrak{r}\,\mathfrak{g}) - \frac{W}{c^2}(\mathfrak{r}^2 + c^2t^2)\right\} + \operatorname{div}\{2t\,[\mathfrak{r}, \mathbf{p}] - \mathfrak{g}\,(\mathfrak{r}^2 + c^2\,t^2)\} = 2t\,K.$$

The following so-called continuity equation for electricity, which follows from (19), must be added to these equations:

(26) $$\operatorname{div}(\rho\,\mathfrak{q}) + \frac{\partial\rho}{\partial t} = 0.$$

([22]) The index r corresponds to the spatial components, while the index z corresponds to the temporal ones. [Trans.: From the German: r = *räumlichen*, z = *zeitlichen*.]

([23]) Since v. Laue already employed the symbol $[\mathfrak{r}, \mathbf{p}]$ for the vector with the x-component $x\,p_{xx} + y\,p_{yy} + z\,p_{zz}$, I have allowed myself to use the symbol $[\mathfrak{r}\times\mathbf{p}]$ for the tensor:

$$\begin{pmatrix} y\,p_{zx} - z\,p_{yx} & y\,p_{zy} - z\,p_{yy} & y\,p_{zz} - z\,p_{yz} \\ z\,p_{xx} - x\,p_{zx} & z\,p_{xy} - x\,p_{xy} & z\,p_{xz} - x\,p_{zz} \\ x\,p_{yx} - y\,p_{xx} & x\,p_{yy} - y\,p_{xy} & x\,p_{yz} - y\,p_{xz} \end{pmatrix}$$

here. Moreover, one has $[\mathfrak{r}, \mathfrak{div}\,\mathbf{p}] = \mathfrak{div}\,[\mathfrak{r}\times\mathbf{p}]$.

One often converts equations (25) from the differential to the integral form when one integrates them over a three-dimensional piece of space, which will convert the integral of the divergence terms into outer surface integrals. We would like to assume that we have a closed system of masses and charges that lie at finite points before us, and that the components of the energy-impulse tensor drop off so quickly as one goes away from them that we can neglect the outer surface integral in comparison to the space integral for a sufficiently large domain of integration B that includes masses and charges inside it. Formulas (25 a_r), (25 a_z), and (25 b_r) will then imply the conservation of impulse, energy, and angular impulse for our total system:

$$\text{(27)} \qquad a_r) \quad \iiint_B \mathfrak{g}\, d\tau = \mathfrak{G} = \text{constant vector},$$

$$a_z) \quad \iiint_B W\, d\tau = E = \text{constant},$$

$$b_r) \quad \iiint_B [\mathfrak{r}, \mathfrak{g}]\, d\tau = \mathfrak{L} = \text{constant vector}.$$

By contrast, formula (25 b_z) will initially assume the form:

$$\text{(28)} \qquad \frac{\partial}{\partial t}\left\{\iiint_B \mathfrak{r}\frac{W}{c^2}\, d\tau - t\iiint \mathfrak{g}\, d\tau\right\} = 0,$$

whose perfectly-intuitive analogy with the second center-of-mass theorem (12 d) for the n-body problem we recognize directly. From the standpoint of the theory of relativity, one must indeed consider mass and energy to be identical, and in fact, a mass m must be regarded as energy with a magnitude mc^2. Conversely, it is permissible to regard any energy with a density of W as equivalent to a mass density of magnitude $W / c^2 = k$. In that way, the electromagnetic field in the free ether will also take on a "center of mass," and:

$$\iiint_B \mathfrak{r}\frac{W}{c^2}\, d\tau = \iiint_B \mathfrak{r}\, k\, d\tau$$

will be the radius vector from the coordinate origin to the common center of mass of the electromagnetic field and ponderable matter, multiplied by the total mass E / c^2, so it will correspond completely to the quantity $\sum_i m_i\, x_{ik}$ that appears in (12 d), while the terms in (28) and (12 d) that are multiplied by t both mean the total impulse of the system. It will then follow from formula (28), in conjunction with (27 a_r), that:

$$\text{(27 } b_z\text{)} \qquad \iiint_B \mathfrak{r}\frac{W}{c^2}\, d\tau = \mathfrak{C}_1 + \mathfrak{G}\, t \qquad (\mathfrak{C}_1 = \text{constant vector})$$

which is once more in complete analogy with (13); i.e.:

The common center of mass of the electromagnetic field and the ponderable matter moves in a uniform, rectilinear way.

Due to the appearance of the quantity K, the five remaining laws will no longer have the form of pure conservation laws ($\partial / \partial t$ of space integral = outer surface integral) when moving bodies are present in the domain of integration; therefore, the integration over time cannot be performed explicitly. Nonetheless, the laws naturally do take on a well-defined physical sense. However, in order to make it easier for us to understand it, we would like to restrict ourselves to the case in which we are dealing with only phenomena in the free ether, and no ponderable mass moves through the field. We will then have $K = 0$, and we will again get pure conservation laws, which we can write in the forms:

$$\text{(27)} \qquad \text{c)} \quad \iiint_B (\mathfrak{r}\,\mathfrak{g}_e)\,d\tau = C_1 + E_e\,t,$$

$$\text{d}_r) \quad \iiint_B \{\mathfrak{r}\,(\mathfrak{r}\,\mathfrak{g}_e) + [\mathfrak{r}, [\mathfrak{r}, \mathfrak{g}_e]]\}\,d\tau = \mathfrak{C}_2 + 2c^2\,\mathfrak{C}_1\,t + c^2\,\mathfrak{E}_e\,t^2,$$

$$\text{d}_z) \quad \iiint_B \mathfrak{r}\,\frac{W_e}{c^2}\,d\tau = C_2 + 2c^2\,C_1\,t + E_e\,t^2,$$

C_1, C_2 = constants, $\quad$ C_2 = constant vector.

The easiest to understand of these equations is d$_z$). From the relation W_e / c^2 = "mass density" of the electromagnetic field, the left-hand side of d$_z$) means one-half the sum of the principal moments of inertia of the "electromagnetic mass" of the field relative to the coordinate origin, such that we can say:

The sum of the electromagnetic principal moments of inertia of the field relative to an arbitrary fixed point is a quadratic function of time, and the coefficient of the square of time is twice the total energy of the field.

By contrast, equations (27 c) and d$_r$) seem to have no immediate analogues in mechanics. One must probably introduce the integrands on the left-hand sides as new quantities in physics. The dimension of ($\mathfrak{r}$ $\mathfrak{g}_e$) is that of the density of a quantity of action, and the dimension of $\{\mathfrak{r}\ (\mathfrak{r}\ \mathfrak{g}_e) + [\mathfrak{r}\ [\mathfrak{r}\ \mathfrak{g}_e]]\}$ is that of a moment of an action density.

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