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On spaces with an arbitrary number of dimensions

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1.

Let $z_1, z_2, \ldots, z_n$ be n variables that can take on all real values from $-\infty$ to $+\infty$. The n-fold infinite domain of systems of values of those variables is called an *n-dimensional space* and will be denoted by S_n . A system $(z_1^0, z_2^0, \ldots, z_n^0)$ will determine a point L_0 of that space, and z_1^0, z_2^0, …, z_n^0 will be called the *coordinates* of that point.

A system of m equations will determine a domain of systems of values of $n - m$ independent variables that will be a space S_{n-m} of many dimensions that is contained in S_n . A space of only one dimension that forms a simple continuum will be called a *line*.

Let:

$$F(z_1, z_2, \ldots, z_n) = 0 \tag{1}$$

be the equation of an $(n - 1)$-dimensional space S_{n-1} . If the function F is continuous and has just one value for all real values of the coordinate then the space S_{n-1} will generally separate S_n into two regions, in one of which one will have $F < 0$, and in the other $F > 0$. Moreover, one cannot continuously vary the system of values of the coordinates of a point in the first region and pass to a system of values of the coordinates of a point in the other region without passing through a system of values that satisfies equation (1). The two regions will be two n-dimensional spaces that are bounded by the space S_{n-1} . If one can always pass from the system of values of the coordinates of an arbitrary point in one of the two regions to the system of values of the coordinates of any other point in the same region by a continuous variation without passing through the values of a point in S_{n-1} then one will say that the region is a *connected* space.

Let:

$$\left.\begin{aligned} z_1 &= z_1(u_1, u_2, \ldots, u_{n-1}), \\ z_2 &= z_2(u_1, u_2, \ldots, u_{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ z_n &= z_n(u_1, u_2, \ldots, u_{n-1}) \end{aligned}\right\} \tag{2}$$

be such continuous, single-valued functions that will satisfy equation (1) identically when they are substituted. The space S_{n-1} can be regarded as the domain of the systems of values of the $n-1$ real variables $u_1, u_2, \ldots, u_{n-1}$.

It is obvious that S_{n-1} will satisfy the $n-1$ equations:

$$\frac{dF}{dz_1}\frac{dz_1}{du_m}+\frac{dF}{dz_2}\frac{dz_2}{du_m}+\cdots+\frac{dF}{dz_n}\frac{dz_n}{du_m}=0\,, \tag{3}$$

which are obtained by taking m to be the numbers 1, 2, …, $n-1$, in succession.

Let $A_1, A_2, \ldots, A_n$ denote arbitrary indeterminate quantities and set:

$$\Delta = \begin{vmatrix} A_1 & \frac{dz_1}{du_1} & \frac{dz_1}{du_2} & \cdots & \frac{dz_1}{du_n} \\ A_2 & \frac{dz_2}{du_1} & \frac{dz_2}{du_2} & \cdots & \frac{dz_2}{du_{n-1}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_n & \frac{dz_n}{du_1} & \frac{dz_n}{du_2} & \cdots & \frac{dz_n}{du_{n-1}} \end{vmatrix}, \tag{4}$$

$$\mu^2 = \sum_{m=1}^{n}\left(\frac{dF}{dz_m}\right)^2, \tag{5}$$

$$M^2 = \sum_{m=1}^{n}\left(\frac{d\Delta}{dz_m}\right)^2. \tag{6}$$

One will then get from equations (3) that:

$$\frac{dF}{dz_m} = \frac{\mu}{M}\frac{d\Delta}{dA_m}\,. \tag{7}$$

Now let L be a line that is determined by the equations:

$$z_1 = l_1(t), \qquad z_2 = l_2(t), \qquad \ldots, \qquad z_n = l_n(t)\,. \tag{8}$$

If equation (1) is satisfied by only a finite number of real values of t when those values are substituted then the line L will intersect the space S_{n-1} at only a finite number of points. Let T_0 be one of those points of intersection that corresponds to $t = t_0$. One will have:

$$F(l_1(t_0), l_2(t_0), \ldots, l_n(t_0)) = 0 .$$

Now consider the two points of L that correspond to:

$$t = t_0 + \delta t_0 ,$$
$$t = t_0 - \delta t_0 ,$$

in which δt_0 is an infinitesimal.

For the first of those values of t, the function F will become:

$$\delta F = \delta t_0 \sum_{m=1}^{n} \frac{dF}{dz_m} \frac{dl_m}{dt_0} ,$$

and for the second one:

$$\delta' F = -\delta t_0 \sum_{m=1}^{n} \frac{dF}{dz_m} \frac{dl_m}{dt_0} .$$

Recalling equations (7), one will get:

$$\left.\begin{aligned} \delta F &= \frac{\mu D}{M} \delta t_0 , \\ \delta' F &= -\frac{\mu D}{M} \delta t_0 , \end{aligned}\right\} \tag{9}$$

in which D is the determinant Δ when the quantities dl_m / dt_0 are substituted for the A_m .

If one now takes the radicals that give positive signs to μ and M and conveniently fixes the ordering of $z_1, z_2, \ldots, z_m$ while traversing the line L by continuously increasing t then one can deduce from equations (9) that if $D > 0$ when L intersects S_{n-1} at the point T_0 then one will leave the region in which $F < 0$ and enter the one in which $F > 0$, and conversely, if $D < 0$ then one will leave the region in which $F > 0$ and enter the one in which $F < 0$.

If one sets:

$$ds_n^2 = dz_1^2 + dz_2^2 + \cdots + dz_n^2 ,$$

and ds_n is the *line element* on S_n (in which case, RIEMANN called the space S_n *planar*) then the line element ds_{n-1} in the space S_{n-1} will be given by the formula:

$$ds_{n-1}^2 = \sum\sum E_{rs} \, du_r \, du_s ,$$

in which:

$$E_{rs} = \sum \frac{dz_m}{du_r} \frac{dz_m}{du_s} ,$$

and from a known property of determinants, one will have:

$$M^2 = \sum \left(\frac{d\Delta}{dA_m} \right)^2 = \begin{vmatrix} E_{11} & E_{12} & \cdots & E_{1,n-1} \\ E_{12} & E_{22} & \cdots & E_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ E_{1,n-1} & E_{2,n-1} & \cdots & E_{n-1,n-1} \end{vmatrix} . \tag{10}$$

If:

$$dS_n = dz_1\, dz_2 \ldots dz_m$$

is the *spatial element* of S_n then the element of S_{n-1} will be:

$$dS_{n-1} = M\, du_1\, du_2 \ldots du_{n-1} . \tag{11}$$

Now let:

$$F_1 (u_1 , u_2 , \ldots, u_{n-1}) = 0$$

be the equation of a space S_{n-2} that is contained in S_{n-1} . If F_1 is a continuous function and has just one value then S_{n-2} will generally separate S_{n-1} into two regions, one of which will have $F_1 < 0$ and the other of which will have $F_1 > 0$. One can regard S_{n-2} as the domain of $n - 2$ real variables, and one can repeat everything that was said for S_{n-2} . The line element will always be a homogeneous form of degree 2, but the coefficients will have a different form, as well as the coefficient M by means of which one obtains the spatial element of S_{n-2} . Analogous observations will be true for spaces with a lower number of dimensions.

2.

One says that an $(n - m)$-dimensional space S_{n-m} is *linearly connected* if one can take any two points in it and pass a continuous line that goes from one of those points to the other without leaving S_{n-m} . Say that a space S_{n-m} is *closed* if one can divide S_n into two linearly-connected spaces in such a way that when one is given one of them, one cannot pass a continuous line from any point to another that does not intersect S_{n-1} . One says that a linearly-connected space is *closed* if it divides a closed space S_{n-1} into two regions, each of which is linearly connected and is such that one cannot pass a continuous line that is entirely contained in S_{n-1} from one of them to any point in the other that does not intersect S_{n-2} , and so on.

However, instead of considering just one, now consider an arbitrary number of inequalities:

$$F_1 < 0 , \quad F_2 < 0 , \quad \ldots, \quad F_m < 0 ,$$

which will determine a subset R of a space S_t that can be linearly connected. The totality of those $(t - 1)$-dimensional spaces:

$$F_1 = 0 , \quad F_2 = 0 , \quad \ldots, \quad F_m = 0$$

that bound R in such a way that one can pass a continuous line from any point of R to a point outside of R that does not intersect any of those spaces can be called the *contour* of R.

One says that a space is *finite* if the coordinates of all of its points have finite values.

A space is either finite and linearly connected or closed or it will have a contour.

3.

A finite space has properties that are independent of the magnitude of its dimensions and the form of its elements. Those properties that refer to only the way that its parts are connected were considered by LISTING for ordinary spaces in a treatise that was entitled "Der Census räumlicher Complexe" and were determined for surfaces by RIEMANN.

Other than the linear connections, which are the only ones that present themselves in surfaces, I have observed that one can consider other types of connections in spaces with a number of dimensions that is greater than two.

If any closed m-dimensional space (with $m < n$) in an n-dimensional space R that is bounded by one or more $(n - 1)$-dimensional spaces is the contour of a subset of an $(m + 1)$-dimensional linearly-connected space that is completely contained in R then it will be $(m + 1)$-dimensional connected, and one will simply say that R has a connection of the m^{th} kind that is *simple*. If all of the connections in a space R are simple then one says that it is *simply connected.* However, if one can imagine a number p_m of closed m-dimensional spaces in R that do not form the contour of a linearly-connected subset of an $(m + 1)$-dimensional space that is completely contained in R and is such that any other closed m-dimensional space forms, by itself or with all or part of the contour, the contour of a linear-connected subset of an $(m + 1)$-dimensional line that is contained completely in R then one will say that R has a $(p_m + 1)^{\text{th}}$-*order connection of the* m^{th} *kind*.

Examples. – In the ordinary space, the space that is found between two concentric spheres has a second-order connection of the second kind and a simple one of the first kind.

The space inside an annulus has a simple connection of the second kind and a second order connection of the first kind.

Both connections in the space between a sphere and an annulus have second order.

In order to justify the definition that was given for the various kinds of connections, it is necessary to prove that the number p_m is well-defined for any bounded space R – i.e., that no matter how one defines the m-dimensional spaces that possess the stated property, the number of them is always the same. We will base that upon the following lemma, as RIEMANN did in order to prove the corresponding theorem that relates to surfaces:

If a system of closed m-dimensional spaces A, along with another system C, forms the contour of a linearly-connected $(m + 1)$*-dimensional space* S_{m+1} *that is contained completely with R, and if another system of closed m-dimensional spaces B, along with the system C, forms the contour of a linearly-connected space* S'_{m+1} *that is contained completely in R then the system A, together with the system B, will form the contour of a linearly-connected* $(m + 1)$*-dimensional space that is contained completely in R.*

Indeed, the two spaces S_{m+1} and S'_{m+1} will either be on opposite sides of the contour C or on the same side. In the former case, the contour of the space that is composed of S_{m+1} and S'_{m+1} will be the system A, along with the system B. In the latter case, if one removes S'_{m+1} from S_{m+1} then what will remain will be a space whose contour is the system A, along with the system B.

If t closed m-dimensional spaces $A_1, A_2, \ldots, A_t$, *by themselves or with any other closed m-dimensional space, cannot form the contour of a linearly-connected* $(m + 1)$*-dimensional space that is contained completely in R, and if another system of t′ closed m-dimensional spaces* $B_1, B_2, \ldots, B_t$ *possesses that property then one will have* $t = t'$.

Indeed, suppose that $t' > t$. If C is any closed m-dimensional space then either the system $(A_1, A_2, \ldots, A_t, C)$ or the system $(A_1, A_2, \ldots, A_t, B_1)$ will form the contour of a linearly-connected m-dimensional space that is contained completely in R. Therefore, either the system $(A_2, A_3, \ldots, A_t, C)$ or the system $(A_2, A_3, \ldots, A_t, B_1)$, together with A_1, will form the contour of a linearly-connected $(m + 1)$-dimensional space that is contained completely in R, and as a consequence, from the preceding lemma, the system $(A_2, A_3, \ldots, A_t, C)$, along with the system $(A_2, A_3, \ldots, A_t, B_1)$ – i.e., the system $(B_1, A_2, A_3, \ldots, A_t, C)$ – will form the contour of an $(m + 1)$-dimensional space that is contained completely in R. Therefore, when the system $(B_1, A_1, A_2, \ldots, A_t)$ is united with any closed space C, it will form the contour of a linearly-connected $(m + 1)$-dimensional space, and if one now continues to substitute one of the spaces B for one of the spaces A, in succession, then one will finally have that the system $(B_1, B_2, \ldots, B_t)$, along with any closed space, and therefore also with B_{t+1}, will form the contour of a linearly-connected $(m + 1)$-dimensional space that is contained completely in R, but that will contradict what we have supposed when $t' > t$. One likewise proves that one cannot have $t > t'$. Therefore, $t = t'$, which was to be proved.

4.

When one supposes that the connection of a bounded space R is routed along a space with a lower number of dimensions that has its contour on the contour of R, one says that one has made a *transverse section* in R.

If one is given an m-dimensional space and one separates from it an infinitesimal part with an infinitesimal $(m – 1)$-dimensional space for its contour then one says that one has made a *point section*.

If a bounded space R can be reduced to another one R' without making any transverse section, and only by means of continuous enlargements and reductions of its parts, then one will say that R can be reduced to R' by *continuous transformation*.

Two bounded spaces R and R' that can be reduced to each other by means of continuous transformation will have equal orders for their connections of each kind. Now, a point is simply connected, so any space that can be reduced to a point by continuous transformation will be simply connected.

One can always make a space with a contour lose one dimension by continuous transformation.

Indeed, let R be that space, let m be its number of dimensions, let C be its contour, let S_m be the space that it belongs to, and let $u_1, u_2, u_3, \ldots, u_m$ denote a system of coordinates in S_m . Imagine an $(m - 1)$-fold infinite system of lines that occupies all of S_m continuously. For example, take the lines whose equations are:

$$u_2 = a_2 , \quad u_3 = a_3 , \quad \ldots, \quad u_{m-1} = a_{m-1} ,$$

where $a_2 , a_3 , \ldots, a_{m-1}$ take all values from $-\infty$ to $+\infty$, and consider only that part of the system that contains the lines that meet the contour C of R. When each of those lines is continued by increasing u_1 until it meets C, where it enters R many times and leaves it just as many times, one can approach each entry point indefinitely with the following exit point and thus make R lose one dimension, as we wished to show.

One can always make a closed space lose one dimension by continuous transformation after one has made a point section.

Indeed, after making a point section, the space will acquire a contour, and therefore, from the previous theorem, it can always lose one dimension by continuous transformation.

If one makes only one point section in a closed m-dimensional space R then that will not change the orders of its connections. However, if one makes s + 1 point sections then the order of the $(m-1)^{\text{th}}$ *kind will increase by unity, while the higher orders of connections will not change.*

Indeed, let $\alpha + 1$ be the order of the connection of the $(m - 1)^{\text{th}}$ kind in a closed m-dimensional space R. One can imagine a system A of α closed $(m - 1)$-dimensional spaces in R that form the contour of a subset of R, not by themselves, but with any other closed $(m - 1)$-dimensional space C. Since R is closed, the system A, together with C, will divide it into two separate regions, R' and R'' that both have the same contour – i.e., the system A, along with C. Now if one makes a point section in R then it will be in one of the two regions; suppose that it is in R'. It is then clear that the system A, along with C, will no longer form the entire contour of R', but rather, it will always comprise the entire contour of R''. Therefore, the order of the connection of the $(m - 1)^{\text{th}}$ kind of R will not change under just one point section. However, if one makes two point sections in R then one can always take C in such a way that one of those points is in R' and the other is in R'', so the system A, along with C, will no longer form the contour of part of R, and it will be necessary to add another closed $(m - 1)$-dimensional space in order to get the entire contour from part of R. Therefore, one can increase the order of the connection of the $(m - 1)^{\text{th}}$ kind of R with two point sections. One proves analogously that that order will increase by 2, 3, …, s units with 3, 4, …, s + 1 point sections, resp.

Now let $\beta + 1$ be the order of the connection of the $(m - t - 1)^{\text{th}}$ kind of R, where $0 < t < m$. One can imagine a system of closed $(m - t - 1)$-dimensional spaces in R that does not form the contour of an $(m - t)$-dimensional space T that is contained completely in R by itself. Let as many point sections as one desires be made in R, but still a finite number of them. With the given contour, one can always arrange that T does not pass through any of those point sections. Therefore, an arbitrary finite number of point sections will not change the orders of the connections of the kinds that are lower than the $(m - 1)^{\text{th}}$.

Since one does not change the orders of the connections of a closed space R by making just one point section, in order to determine those orders, it will make no difference whether one regards R as closed or as having an infinitesimal contour. Thus, one can always regard a finite space as being bounded by a contour, and therefore one can always make it lose one dimension by continuous transformation without changing the orders of its connections.

5.

In order to make a finite space n-dimensional space R simply connected by means of simply-connected transverse sections, it is necessary and sufficient to make p_{n-1} *linear sections,* p_{n-2} *two-dimensional ones,* p_{n-3} *three-dimensional ones, …, and* p_1 $(n - 1)$*-dimensional ones, if the orders of its connection of the* $1^{\text{st}}, 2^{\text{nd}}, \ldots, (n - 1)^{\text{th}}$ *kind are* $p_1 + 1, p_2 + 1, \ldots, p_{n-1} + 1$*, respectively.*

Indeed, let $p_{n-1} + 1$ be the order of the connection of the $(n - 1)^{\text{th}}$ kind of R. One can imagine a system A of closed $(n - 1)$-dimensional spaces in it that do not form the contour of part of R, but they will form it with any other closed $(n - 1)$-dimensional space. One will then have more bounded regions, each of which will have a contour that is all or part of the system A and part of the contour of R. Hence, when one makes that region lose one dimension by a continuous transformation, it will reduce to the system A, which is connected along $(n - 2)$-dimensional spaces. Thus, R can reduce to an $(n - 1)$-dimensional space R_1 by continuous transformation, which is a space that is composed of p_{n-1} closed $(n - 1)$-dimensional spaces A that are connected with each other by $(n - 2)$-dimensional spaces, and R_1 will have the same orders of its connections of the $(n - 2)^{\text{th}}, (n - 3)^{\text{th}}, \ldots, 1^{\text{st}}$ kind as R. Now, without changing the orders of the connections of R_1, one can make as many point sections as the number of closed spaces that comprise it – i.e., p_{n-1} . Let R_1' be the space R_1 in which those point sections are made.

If one reduces R_1' to R by continuous transformation then the point sections will acquire one dimension and become continuous lines that go from a point of the contour of R to another point of the same contour, i.e., they will become transverse linear sections, and therefore the orders of the connections of kinds lower than the $(n - 1)^{\text{th}}$ will again remain the same. Hence, one can make only a number p_{n-1} of transverse sections in R that do not change the orders of its connections of kinds less than the $(n - 1)^{\text{th}}$.

Now each of those p_{n-1} transverse linear sections will traverse one of the p_{n-1} closed spaces A, which can at most be arranged in R such that it will not comprise the contour of a portion of R by itself, but it will when one adds another one of dimension $n - 1$ to it. Therefore, after having made that transverse section, each of the spaces A will no longer be closed, and thus any closed $(n - 1)$-dimensional space will become the contour of a portion of R, and the connection of R of the $(n-1)^{\text{th}}$ kind will be rendered simple.

Therefore, in order to make the connection of the $(n-1)^{\text{th}}$ kind of R simple by means of simply-connected transverse sections without changing the orders of the connections of the lower kinds, it is necessary and sufficient to make p_{n-1} transverse linear sections.

The $(n-1)$-dimensional space R_1' to which the space R in which one makes the p_{n-1} transverse linear sections is reduced by continuous transformation that has a point section in any of the closed spaces that comprise it and an order of connection of the $(n-2)^{\text{th}}$ kind that is equal to p_{n-2} can lose one dimension by continuous transformation and reduce to a space R_2 that is composed of p_{n-2} closed $(n-2)$-dimensional spaces that are connected along $(n-3)$-dimensional spaces. Now one can make at most p_{n-2} point sections without changing the orders of the connections of R_2 . Let R_2' denote the space R_2 in which those point sections are made. When one reduces R_2' to R_2 , the point sections of R_2' will acquire two dimensions and become two-dimensional spaces that have their contours on the contour of R and will be simply connected since they are reducible to a point by continuous transformation, and will therefore be two-dimensional transverse sections. Let R'' denote the space R in which one makes the transverse sections in one and two dimensions. The two-dimensional sections will make the connection of the $(n-2)^{\text{th}}$ kind simple. Hence, in order to reduce R to a space R'' that has connections of the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ kind that are simple by continuous transformation without changing the orders of the connections of the lower kinds, it is necessary and sufficient to make p_{n-1} transverse linear sections and p_{n-2} two-dimensional ones. One then continues to do that for the connections of the lower kinds.

When a finite space R is reduced to a simply-connected one by means of simply-connected transverse sections, any closed m-dimensional space that is in R that is composed of as many closed m-dimensional spaces as the number of transverse $(n-m)$*-dimensional sections that it meets, the contour of an* $(m+1)$*-dimensional space will be contained completely in R.*

Indeed, if $p_m + 1$ is the order of the connection of the m^{th} kind of R then any closed m-dimensional space C, along with a system of p_m closed m-dimensional spaces, will form the contour of an $(m+1)$-dimensional space S that is contained completely in R. Now, each of the spaces A will be intersected by one and only one transverse $(n-m)$-dimensional section that makes up part of the one that makes R simply connected, and therefore since each of those sections has a contour that is on the contour of R, if C forms the contour of S, along with s of the spaces A, then it must intersect precisely those s transverse sections that intersect those s closed spaces in the system A.

For greater clarity, let us make some applications of that to some ordinary spaces.

The space between two concentric spheres is made simply connected by means of just one transverse linear section that goes from a point of the outer spherical surface to any point of the inner one.

The space inside an annular surface is made simply connected by means of just one transverse surface section that is made along the meridian of the surface.

The space between two annular surfaces is made simply connected by means of just one transverse linear section that goes from a point of the outer annular surface to a point on the inner one and by means of two transverse surface sections that both go through the linear section, one of which is made along the meridian of the surface and one of them is made along the equator.

The space between a sphere and a ring is made simply connected by means of a transverse linear section that goes from the surface of the sphere to that of the ring, and the other surface section that terminates at the linear section also goes from the surface of the ring to that of the sphere.

6.

Suppose that one is given an n-dimensional space R that is bounded by an arbitrary number of closed $(n - 1)$-dimensional ones. Let S'_{n-1}, S''_{n-1}, …, $S^{(t)}_{n-1}$ be the ones whose equations are:

$$F_1 = 0 , \qquad F_2 = 0 , \qquad \ldots, \qquad F_t = 0 ,$$

and let R be determined from the inequalities:

$$F_1 < 0 , \qquad F_2 < 0 , \qquad \ldots, \qquad F_t < 0 .$$

Let X_1, X_2, …, X_n be n functions of the points of R that are finite and continuous. Take under consideration the n-fold integral:

$$\Omega_n = \int_n \left(\frac{dX_1}{dz_1} + \frac{dX_2}{dz_2} + \cdots + \frac{dX_n}{dz_n} \right) dz_1\, dz_2 \cdots dz_n ,$$

which is extended over all of the space R.

Let even indices distinguish the values of X_r at the points where the line Z_r that has the equations:

$$z_1 = a_1 , \qquad z_2 = a_2 , \qquad \ldots, \qquad z_{r-1} = a_{r-1} , \qquad \ldots, \qquad z_n = a_n \tag{2}$$

when z_r increases cross one of the spaces S_{n-1} and enter into the space R in which all of the inequalities (1) are satisfied. Let odd indices distinguish the values of X_r at the points where the line Z_r crosses one of the spaces S_{n-1} and leaves the space R. One will have:

$$\int_{n-1} \frac{dX_r}{dz_r} dz_r = X_r^0 - X'_r + X''_r - X'''_r + \cdots$$

Thus:

$$\Omega_n = \sum \int_{n-1} \left(X_r^0 - X'_r + X''_r - X'''_r + \cdots \right) dz_1\, dz_2 \cdots dz_{r-1}\, dz_{r+1} \cdots dz_n .$$

Now the number of points at which the line Z_r meets each of the closed spaces S_{n-1} is even, and Z_r will enter into R as many times as it leaves it. For example, the space S'_{n-1} will be met by Z_r at the points 0, $2l_1 + 1$, $2l_2$, $2l_3 + 1$, …, and the part of the integral Ω_n that refers to the points of S'_{n-1} will be:

$$\Omega'_n = \int_{n-1} \left(X_r^0 - X_r^{(2l_2+1)} + X_r^{(2l_2)} - X_r^{(2l_3+1)}{}_r + \cdots \right) dz_1\, dz_2 \cdots dz_{r-1}\, dz_{r+1} \cdots dz_n \ .$$

Consider S'_{n-1} to be the domain of the $n - 1$ real variables u'_1, u'_2, …, u'_{n-1}. One has:

$$dz_1\, dz_2 \ldots dz_{r-1}\, dz_{r+1} \ldots dz_n = \pm \frac{d\Delta}{dA_r} du'_1\, du'_2 \ldots du'_{n-1} \ ,$$

which will take the + or – sign according to whether $d\Delta / dA_r$ is > 0 or < 0, resp.

Now, as was proved in the first section, it results that one can always take the ordering of the $z_1, z_2, \ldots, z_m$ in such a way that the sign of $d\Delta / dA_r$ is equal to that of dF_1 / dz_r, if the equation of S'_{n-1} is $F_1 = 0$.

Now, $dF_1 / dz_r < 0$ at the point of S'_{n-1} where Z_r enters R, and $dF_1 / dz_r > 0$ at the point where it leaves R. One then has:

$$\int_{n-1} X_r^0\, dz_1\, dz_2 \cdots dz_{r-1}\, dz_{r+1} \cdots dz_n = - \int_{n-1} X_r^0 \frac{d\Delta}{dA_r} du'_1\, du'_2 \cdots du'_{n-1} \ ,$$

$$\int_{n-1} X_r^{(2l_1+1)}\, dz_1\, dz_2 \cdots dz_{r-1}\, dz_{r+1} \cdots dz_n = \int_{n-1} X_r^{(2l_1+1)} \frac{d\Delta}{dA_r} du'_1\, du'_2 \cdots du'_{n-1} \ ,$$

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Thus:

$$\Omega'_n = - \int_{n-1} \left(X_r^0 \frac{d\Delta}{dA_r} + X_r^{(2l_1+1)} \frac{d\Delta}{dA_r} + \cdots \right) du'_1\, du'_2 \cdots du'_{n-1} \ ,$$

or

$$\Omega'_n = - \int_{n-1} X_r \frac{d\Delta}{dA_r} du'_1\, du'_2 \cdots du'_{n-1} \ ,$$

which is extended over all of the space S'_{n-1} .

One can make analogous reductions of the other spaces, and one will get:

$$\Omega_n = - \sum \int_{n-1} X_r \frac{d\Delta}{dA_r} du'_1\, du'_2 \cdots du'_{n-1} - \sum \int_{n-1} X_r \frac{d\Delta}{dA_r} du''_1\, du''_2 \cdots du''_{n-1} - \ldots$$

However:

$$\sum X_r \frac{d\Delta}{dA_r} = \begin{vmatrix} X_1 & \frac{dz_1}{du_1} & \cdots & \frac{dz_1}{du_{n-1}} \\ X_2 & \frac{dz_2}{du_1} & \cdots & \frac{dz_2}{du_{n-1}} \\ \vdots & \vdots & \cdots & \vdots \\ X_n & \frac{dz_n}{du_1} & \cdots & \frac{dz_n}{du_{n-1}} \end{vmatrix}.$$

Hence:

$$\Omega_n = -\sum_t \int \begin{vmatrix} X_1 & \frac{dz_1}{du_1^{(t)}} & \cdots & \frac{dz_1}{du_{n-1}^{(t)}} \\ X_2 & \frac{dz_2}{du_1^{(t)}} & \cdots & \frac{dz_2}{du_{n-1}^{(t)}} \\ \vdots & \vdots & \cdots & \vdots \\ X_n & \frac{dz_n}{du_1^{(t)}} & \cdots & \frac{dz_n}{du_{n-1}^{(t)}} \end{vmatrix} du_1^{(t)}\, du_2^{(t)} \dots du_{n-1}^{(t)}\, .$$

If $(z_1^0, z_2^0, \dots, z_n^0)$ is a point of $S_{n-1}^{(t)}$ then a line that passes through that point and has the equations:

$$z_1 - z_1^0 = \frac{dF_t}{dz_1^0}\frac{\rho}{\mu},$$

$$z_2 - z_2^0 = \frac{dF_t}{dz_2^0}\frac{\rho}{\mu},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$z_n - z_n^0 = \frac{dF_t}{dz_n^0}\frac{\rho}{\mu}$$

is called the *normal* to the space $S_{n-1}^{(t)}$, and since one has:

$$\rho^2 = (z_1 - z_1^0)^2 + (z_2 - z_2^0)^2 + \cdots + (z_n - z_n^0)^2,$$

ρ is called the *distance* from the point $(z_1, z_2, \dots, z_n)$ to $(z_1^0, z_2^0, \dots, z_n^0)$. If ρ is infinitesimal and equal to dp_t then one will have:

$$\frac{dz_r}{dp_t} = \frac{dF_t}{dz_r}\frac{1}{\mu}.$$

However:

$$\frac{dF_t}{dz_r} = \frac{d\Delta}{dA_r}\frac{\mu}{M},$$

so:

$$\frac{dz_r}{dp_t} = \frac{d\Delta}{dA_r}\frac{1}{M}.$$

Thus, when one substitutes:

$$\Omega_n = -\sum_t \int_{n-1} \left(X_1 \frac{dz_1}{dp_t} + X_2 \frac{dz_2}{dp_t} + \cdots + X_n \frac{dz_n}{dp_t} \right) M\, du_1^{(t)}\, du_2^{(t)} \cdots du_{n-1}^{(t)} .$$

However, if $dS_{n-1}^{(t)}$ is the spatial element of $S_{n-1}^{(t)}$ then one will have:

$$dS_{n-1}^{(t)} = M\, du_1^{(t)}\, du_2^{(t)} \cdots du_{n-1}^{(t)},$$

so:

$$\Omega_n = -\sum_t \int_{S_{n-1}^{(t)}} \sum_r X_r \frac{dz_r}{dp_t} dS_{n-1}^{(t)} .$$

Therefore, if:

$$X_r = \frac{dV}{dz_r}$$

then one will have:

$$\Omega_n = \int_R \left(\frac{d^2V}{dz_1^2} + \frac{d^2V}{dz_2^2} + \cdots + \frac{d^2V}{dz_n^2} \right) dR = -\sum_t \int_{S_{n-1}^{(t)}} \frac{dV}{dp_t} dS_{n-1}^{(t)} ,$$

and therefore if:

$$\frac{d^2V}{dz_1^2} + \frac{d^2V}{dz_2^2} + \cdots + \frac{d^2V}{dz_n^2} = 0$$

in all of the space *R* then one will have:

$$\sum_t \int_{S_{n-1}^{(t)}} \frac{dV}{dp_t} dS_{n-1}^{(t)} = 0 .$$

If the space *R* has a simple connection of the $(n-1)^{\text{th}}$ kind then any closed $(n-1)$-dimensional space *C* that enter into it will form the contour of a portion of *R*. Thus, if equation (4) is satisfied for all *R* and *V*, along with its first derivatives, are finite and continuous then one will always have:

$$\int_C \frac{dV}{dp_C} dC = 0 .$$

If the space R has a connection of the $(n-1)^{\text{th}}$ kind of order $p_{n-1}+1$ and one passes p_{n-1} closed $(n-1)$-dimensional spaces $A_1, A_2, \ldots, A_{p_{n-1}}$ through it then any closed space C that is contained in R, together with the system A, will form the contour of part of R, and if $a_1, a_2, \ldots, a_{p_{n-1}}$ are the transverse linear sections that cross the closed spaces $A_1, A_2, \ldots, A_{p_{n-1}}$, respectively, and make the connection of the $(n-1)^{\text{th}}$ kind in the space R simple then C will form the contour of a part of R in which the spaces of the system that are crossed by the sections a will meet C.

Then set:

$$\int_{A_r} \frac{dV}{dp_t} dA_r = M_r ,$$

and one will have:

$$\int_C \frac{dV}{dp_C} dC + \sum M_r = 0 ,$$

in which the sum extends over all values of r that are indices of the transverse sections that meet C, and one will have the following theorem:

If the space R has a connection of the $(n-1)^{\text{th}}$ kind of order p_n+1 and that connection can be made simple by transverse linear sections, and if C is a closed $(n-1)$-dimensional space that is contained in R then the integral:

$$\int_C \frac{dV}{dp_C} dC$$

will differ from zero by as many moduli of periodicity as the number of transverse linear sections that cross the space C.

Since two $(n-1)$-dimensional spaces that have the same contour together form a closed space, one will further have the following theorem:

If one is given a closed $(n-2)$-dimensional space in an n-dimensional space R that has a connection of the $(n-1)^{\text{th}}$ kind of order $p_{n-1}+1$, and the former space can make that connection simple by means of p_{n-1} linear sections then when the previous integral is extended over a space C that has Γ for its contour and meets s transverse linear sections, it will differ from the same integral, when it is extended over a space C' that has the same contour and does not meet any section, by moduli of periodicity that relate to those sections, and therefore if the space R has a simple connection of the $(n-1)^{\text{th}}$ kind then the integral will always have the same value when it is extended over any space C that is contained in R and has Γ for its contour.

7.

In a closed n-dimensional space R that has a connection of the first kind of order $p_1 + 2$, let s_1, s_2, …, s_{p_1} be the simply-connected $(n - 1)$-dimensional transverse sections that render the connection of the first kind in R simple. Let L_1, L_2, …, L_{p_1} be p_1 closed lines that cross the sections s_1, s_2, …, s_{p_1}, respectively, and are such that any other closed line l, together with the line L that crosses the same sections that the they go through, will form the contour of a two-dimensional space that is contained completely in R.

Let:

$$z_1 = z_1(u), \qquad z_2 = z_2(u), \qquad \ldots, \qquad z_n = z_n(u)$$

be the equations of the line l and take under consideration the integral:

$$\Omega_1 = \sum \int X_r \, dz_r = \sum \int X_r \frac{dz_r}{du} du,$$

which extends over the entire line l, in which the X_r are finite and continuous in all of R.

Now, if the line l forms part of the contour of C then if the space C is determined by the equations:

$$z_1 = z_1(v_1, v_2), \qquad z_2 = z_2(v_1, v_2), \ldots, \qquad z_n = z_n(v_1, v_2),$$

one will have:

$$\frac{dz_r}{du} du = \frac{dz_r}{dv_1} dv_1 + \frac{dz_r}{dv_2} dv_2,$$

and therefore:

$$\Omega_1 = \int \sum X_r \frac{dz_r}{dv_1} dv_1 + \int \sum X_r \frac{dz_r}{dv_2} dv_2 .$$

Now, from what was shown in the preceding section:

$$\iint \left[\frac{d}{dv_2} \left(\sum X_r \frac{dz_r}{dv_1} \right) - \frac{d}{dv_1} \left(\sum X_r \frac{dz_r}{dv_2} \right) \right] dv_1 \, dv_2 = \int \sum X_r \frac{dz_r}{dv_1} dv_1 + \int \sum X_r \frac{dz_r}{dv_2} dv_2,$$

in which the double integral extends over all of the space C, and the simple one extends over the entire system of lines l, L_1, L_2, … that form the contour of C.

However, one has:

$$\iint \left[\frac{d}{dv_2} \left(\sum X_r \frac{dz_r}{dv_1} \right) - \frac{d}{dv_1} \left(\sum X_r \frac{dz_r}{dv_2} \right) \right] dv_1 \, dv_2$$

$$= \iint \sum \frac{dX_r}{dz_t} \begin{vmatrix} \frac{dz_r}{dv_1} & \frac{dz_r}{dv_2} \\ \frac{dz_t}{dv_1} & \frac{dz_t}{dv_2} \end{vmatrix} dv_1\, dv_2 = \iint \sum \left(\frac{dX_r}{dz_t} - \frac{dX_t}{dz_r} \right) dv_n\, dv_t \ .$$

Therefore, the double integral will be zero if the equations:

$$\frac{dX_r}{dz_t} - \frac{dX_t}{dz_r} = 0 \tag{2}$$

are also verified in all of R. Therefore, if the X_r satisfy (2) and are finite and continuous in R then the integral:

$$\sum \int X_r\, dz_r$$

will always be zero when it is extended over all of the lines l, L_1, L_2, … that form the contour of a space C, no matter what the closed line l is. One gets the following theorem from that:

If R is an n-dimensional space that has a connection of the first kind of order p_1 *and one has simply-connected transverse* $(n - 1)$*-dimensional sections* s_1, s_2, … s_{p_1} *of it that make the connection of the first kind on R simple, and* L_1, L_2, … L_{p_1} *are* p_1 *closed lines that meet the sections* s_1, s_2, … s_{p_1}, *respectively, and one sets:*

$$M_t = \sum \int_{L_t} X_r\, dz_r$$

then the integral:

$$\sum \int_{Z^0}^{Z^1} X_r\, dz_r \ ,$$

when extended between two points Z^0 *and* Z^1 *along a line that meets some sections s, will differ from the one that is taken along a line that goes from the point* Z^0 *to the point* Z^1 *without meeting any section s by a quantity M that relates to the sections s that are met, which is taken to be positive or negative according to whether they are met by proceeding in one direction or the other. If the connection of the first kind on the space R is simple then the integral will always have the same value when it is taken along any in R that goes from* Z^0 *to* Z^1.

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