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Theory of capillarity

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1.

Forces of cohesion and adhesion.

Each element of a fluid is subject to the action of the elements that surround it and are in its vicinity, which manifests itself in liquids by the resistance that they present when one wishes to reduce them to separate parts, and in aeroform fluids when one removes the obstacles to their expansion. An element of a fluid that is found in the neighborhood of a solid is also subject to the action of that solid that manifests itself by the resistance that is encountered when one tries to separate the fluid from the solid. Without investigating the regions of those actions, one can always regard them as the product of the forces that the fluid elements exert upon each other and that the solid elements exert upon the fluids, and reciprocally, and which depend upon only the relative positions of the elements when the temperature is invariable. The forces that act between the elements of such a fluid are called *forces of cohesion*, and the ones that act between the elements of the different fluids or between a fluid and a solid are called *forces of adhesion*.

Both types of force are exerted only at a small distance, because their actions are independent of the masses of the fluids or solids that are found at a distance from the point where the action is exerted that is not small.

Those forces have another property that is deduced from the fundamental principle of modern physics, viz., the principle of the *conservation of forces*, and that consists of them having a potential function.

Indeed, let a fluid system A be in contact with a solid system B, and let X, Y, Z be the components along the three axes of the forces of cohesion and adhesion on a point (x, y, z) of the fluid. If the density of the fluid varies continuously from one point to another then X, Y, Z will be continuous functions of the points of the space that is occupied by the fluid. Now suppose that the forces of cohesion and adhesion do not have a force potential, i.e., that the trinomial:

$$X\,dx + Y\,dy + Z\,dz$$

is not an exact differential of a function that would be called a potential function if it were to exist. The three quantities:

$$\xi = \frac{dY}{dz} - \frac{dZ}{dy},$$
$$\eta = \frac{dZ}{dx} - \frac{dX}{dz},$$
$$\zeta = \frac{dX}{dy} - \frac{dY}{dx}$$

will not be zero, except for special values of x, y, and z.

If we let v denote the velocity of the element dm of the fluid and let v_0 denote the initial velocity then we will have the known *vis viva* equation:

$$\tfrac{1}{2}\int v^2\, dm - \tfrac{1}{2}\int v_0^2\, dm = \int dm \int_0^s \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} \right) ds,$$

in which the integrals that relate to dm are triple integrals that must be extended over all of the space that is occupied by the fluid, and the integrals that relate to ds are extended over all the lines that traverse each point during the passage from the initial state to the state in which it is animated with the velocity v.

If one now imagines that the system returns to its original state after an arbitrary motion when the forces that act upon it have a potential function ϕ then one will have:

$$X\, dx + Y\, dy + Z\, dz = d\phi,$$

and therefore:

$$\tfrac{1}{2}\int v^2\, dm - \tfrac{1}{2}\int v_0^2\, dm = \int (\phi - \phi_0)\, dm.$$

Moreover, if the forces depend upon only the relative positions of the elements of the elements of the system then when it returns to that state, it is clear that the potential function must return to the same value, so $\phi = \phi_0$, and one will have:

$$\int v^2\, dm = \int v_0^2\, dm,$$

i.e., when the system returns to the original state, regardless of the motions that it passes through, the *vis viva* will neither be increased or decreased. However, when a potential function does not exist – i.e., when ξ, η, ζ are non-zero – the integral:

$$I = \int (X\, dx + Y\, dy + Z\, dz),$$

when extended along the entire line that is traversed by the elements that comprise it (even when that line is closed) and the element around the precise point of departure, will no longer be equal to zero. Indeed, if c is the closed curve that is traversed by the point (x, y, z), and one imagines that the curve lies in a continuous surface S that does not extend to infinity and has only one sheet then one will have:

$$p = \frac{dz}{dx}, \qquad q = \frac{dz}{dy},$$

in which the derivatives dz / dx, dz / dy are obtained from the equation of the surface S. Since the integral I must be taken along the line c that is found on the surface S, one will have:

$$dz = p\,dx + q\,dy\,,$$

and therefore:

$$I = \int_0^l \left[(X + Z\,p)\frac{dx}{ds} + (Y + Z\,q)\frac{dy}{ds} \right] ds$$

$$= \iint \left[\frac{d(X + Z\,p)}{dy} - \frac{d(Y + Z\,p)}{dx} \right] dx\,dy\,,$$

in which l is the length of the line c, and the double integral must be extended over all of the projection of the surface S onto the xy-plane.

When one performs the differentiations, one will have:

$$I = \iint (\zeta - p\,\xi - q\,\eta)\,dx\,dy\,.$$

Set:

$$\xi^2 + \eta^2 + \zeta^2 = \rho^2\,,$$

$$\xi = \rho \cos \lambda\,,$$
$$\eta = \rho \cos \mu\,,$$
$$\zeta = \rho \cos \nu\,,$$

and let α, β, γ denote the angles that the normal to the surface S makes with the three axes. One will have:

$$I = \iint \rho\,(\cos\lambda\cos\alpha + \cos\mu\cos\beta + \cos\nu\cos\gamma)\frac{dx\,dy}{\cos\gamma}\,,$$

so:

$$I = \int \rho \cos(\rho, n)\,dS\,,$$

in which (ρ, n) is the angle that the normal makes with the line whose direction makes the angles λ, μ, ν with the axes.

Now one can take the line c and the surface S in such a way that $\cos(\rho, n)$ always keeps the same sign, and therefore I will be non-zero, and since traversing the line c in the opposite sense will change the sign of I, one can always have a positive value for I, so:

$$\int I\, dm$$

will have nothing but positive elements and will then be non-zero. Therefore, one can give motions to the system such that when it returns to its original state, one will have that:

$$\tfrac{1}{2}\int v^2\, dm - \tfrac{1}{2}\int v_0^2\, dm$$

is non-zero, and therefore one will have a variation of the *vis viva* of the system without the system being altered and without any external action. That would contradict the principle of the conservation of force.

A third property of the forces of cohesion and adhesion is deduced from the principle of the equivalence of action and reaction, and it is that when one has two systems B and A that upon each other, the potential of B with respect to A is equal to the potential of A with respect to B.

Those three properties are the fundamentals of the theory of capillarity, which I shall proceed to discuss.

2.

Potential of a system of fluids in contact with each other and with solid bodies.

Suppose that one is given some fluids $A_1, A_2, \ldots, A_n$ in contact with each other and with some solid bodies $B_1, B_2, \ldots, B_n$. Let:

$\rho_1, \rho_2, \ldots, \rho_n$	be the respective densities of the fluid,
$V_1, V_2, \ldots, V_n$	be the spaces that they occupy,
$S_1, S_2, \ldots, S_n$	be their free surfaces,
$S_{tt'}$	be the surface that separates A_t from $A_{t'}$,
$S'_{tt'}$	be the surface that separates A_t from $B_{t'}$,
$\phi_1, \phi_2, \ldots, \phi_n$	be the potentials of the fluids over their elements,
$\psi_1, \psi_2, \ldots, \psi_n$	be the potentials of all the solids B with respect to the fluids,
$\theta_{tt'}$	be the potential function of the fluid A_t with respect to the fluid $A_{t'}$.

The potential of the system of n fluids will be:

$$P = \tfrac{1}{2}\sum_{t=1}^{n}\int_{V_t}(\phi_t + 2\psi_t + 2\sum\theta_{t't} + 2g\,z)\,\rho_t\,dv\,.$$

The equilibrium of the system is obtained by setting equal to zero the first variation of P that results from varying the densities of the fluids and the displacements of the points that do not change the densities under the condition that the total mass of each fluid should remain constant.

One can separately consider the variations that are due to the changes in densities and the ones that are derived from the displacements that do not change the densities.

In order to account for the variability of mass, one can equate the variation of the function:

$$W = \tfrac{1}{2}\sum_{t=1}^{n}\int_{V_t}(\phi_t + 2\psi_t + 2\sum\theta_{t't} + 2g\,z + 2k_t)\,\rho_t\,dv$$

to zero, in which k_t is a constant when the mass of the entire liquid A_t is invariable. Therefore, if the particles of the liquid are not perfectly mobile and one finds an impediment to the passage from one layer of the medium to the other in which the mass of each of those layers is invariable then k_t must be set to a constant in each of those layers, but vary from one to the other.

Varying ρ_t will vary only the functions:

$$\phi_t\,,\quad \phi_{t,1}\,,\quad \phi_{t,2}\,,\ \ldots,\quad \phi_{t,\,t-1}\,,\ \phi_{t,\,t+1}\,,\ \ldots,\quad \phi_{t,n}\,.$$

Therefore:

$$\begin{aligned}\delta W = \tfrac{1}{2}\sum_{t=1}^{n}\Bigg[&\int_{V_t}(\phi_t + 2\psi_t + 2\sum\theta_{t't} + 2g\,z + 2k_t)\,\delta\rho_t\,dv\\ &+ 2\int_{V_1}\rho_1\,\delta\theta_{t,1}\,dv + 2\int_{V_2}\rho_2\,\delta\theta_{t,2}\,dv + \ldots\\ &+ 2\int_{V_{t-1}}\rho_{t-1}\,\delta\theta_{t,t-1}\,dv + 2\int_{V_{t+1}}\rho_{t+1}\,\delta\theta_{t,t+1}\,dv + \ldots\\ &+ \int_{V_t}\delta\phi_t\,\rho_t\,dv\Bigg]\,.\end{aligned}$$

Now if Φ_1 is the potential function of a system A_2 with respect to the points of a system A_1 and Φ_2 is the potential function of the system A_1 with respect to the points of the system A_2 , while ρ_1 is the density of A_1 and ρ_2 is the density of A_2 , and V_1 and V_2 are the spaces that are occupied by A_1 and A_2, respectively, then from the third property of the forces of adhesion and cohesion, one will have:

$$\int_{V_1}\Phi_1\,\rho_1\,dv = \int_{V_2}\Phi_2\,\rho_2\,dv\,,$$

and that equivalence will persist even when the spaces V_1 and V_2 coincide wholly or partially. That theorem is the generalization of one that **Gauss** gave for the forces that act according to **Newton**'s law ([1]).

Meanwhile, one has:

$$\int_{V_t} \rho_t\, \delta\phi_t\, dv = \int_{V_t} \phi_t\, \delta\rho_t\, dv\,,$$

$$\int_{V_{t'}} \rho_{t'}\, \delta\theta_{tt'}\, dv = \int_{V_t} \theta_{t't}\, \delta\rho_t\, dv\,,$$

and therefore:

$$\delta W = \int_{V_t} (\phi_t + \psi_t + 2\theta_{1,t} + 2\theta_{2,t} + \cdots + 2\theta_{t-1,t} + 2\theta_{t+1,t} + \cdots + 2\theta_{n,t} + gz + k_t)\, \delta\rho_t\, dv\,,$$

and that variation must be equal to zero for any variations $\delta\rho_t$ at the different points in the space V_t . We then have:

$$(1) \qquad \phi_t + \psi_t + 2\sum \theta_{t't} + g\,z + k_t = 0\,.$$

If one uses those equations to reduce to the value of W then one will have:

$$(2) \qquad W = \int_{V_t} (k_t + \psi_t + gz)\, \rho_t\, dv\,.$$

At the points that are found inside the fluid A_t far from the surface that bounds the space that it occupies at distances that are greater than the distance over which the action of the force of adhesion extends, the functions ψ_t and $\theta_{t't}$ will be constants, and they can be set equal to zero, because any potential function contains an arbitrary constant plus the variable part whose derivative gives the components of the action. Therefore, one deduces from (1) that for those points:

$$(3) \qquad \phi_t + g\,z + k_t = 0\,,$$

and since that internal mass is invariable, so k_t is also constant, ϕ_t will vary by a quantity of order $g\,dz$, which is very small compared to the forces of cohesion, or the function ϕ_t will vary only by the weight of the liquid that is above any point. Now the value of ϕ_t at any point of A_t with an invariable temperature depends uniquely upon the distribution of the density that it has around that point. Therefore, the densities at the points of A_t that are at a greater distance from the surface than the extent of the forces of adhesion can be regarded as constant, and $\rho_{t'}$ will denote their values.

([1]) See Journal de Liouville (1842), pp. 301.

Now consider the points of A_t whose distance from S_t is less than the extent of the action of the forces of cohesion and whose distance from the surfaces $S_{tt'}$ and $S'_{tt'}$ is greater than the extent of the action of the forces of adhesion.

Divide the layer that is compressed between the surface S_t and a surface that is parallel and far from the radius of activity of the forces of cohesion into a very large number of parallel layers in each of which one can regard ϕ_t , k_t , and ρ_t as constant with no appreciable error. However, those quantities cannot all preserve the same values when passing from one layer to the other, and there can then be three cases:

1. ρ_t can be constant, as **Laplace** and **Gauss** supposed. Since an innermost layer will receive the same action as the internal part of the fluid A_t and a greater action than the external part of the fluid in a layer that is less internal, ϕ_t will necessarily vary from layer to layer, and therefore from equation (3), the same thing will also be true for k_t .

2. k_t can be constant. One then deduces from equation (3) that ϕ_t can also be regarded as constant, not only in any partial layer, but in all of the layer that is as thick as the radius of activity of the forces of cohesion, and therefore ρ_t will vary with distance from the surface in that layer, while ϕ_t , k_t can be set equal to zero.

3. ϕ_t , k_t , and therefore also ρ_t will vary with distance from the surface.

The first case can take place only if the forces of cohesion are not capable of varying with the density of the liquid. The second one leads to the consequence that there can be no action of any force of cohesion on the surface S_t, and therefore the liquid will be subject to only the force of gravity at the points at a greater distance from the surfaces $S_{tt'}$ and $S'_{tt'}$ than the radius of activity of the forces of adhesion, and therefore the surface S_t must be planar, which contradicts experience. All that remains is the last case.

The same thing can be said for the layers that adhere to the surfaces $S_{tt'}$ and $S'_{tt'}$.

The variability of k_t from one surface layer to another leads to the consequence that in each of those layers, the mass of the liquid will be invariable, as we have pointed out before, and that the particles of the liquid are therefore not perfectly mobile in the vicinity of the surface, but there will be an impediment to their passage from one layer to another. If k_t varies with time then that means that the passage can take place but will require a certain amount of time to complete.

We shall not go into the explanation for the impediments that make it possible for both of the quantities ρ_t and ϕ_t to vary from layer to layer, but simply take the medium to be an experimental fact. We should observe only that the internal equilibrium does not exist in the bodies and that the notions that the facts of thermodynamics have brought to science allow us to believe that the apparent equilibrium is nothing but a permanent state of rapid motion. Hence, if one treats a fluid as if it were composed of points that are in equilibrium under the action of forces that act between the media then one can probably substitute an ideal state for the real one, and in order to obtain results that conform to experience, one needs to take the data that is necessary to establish the equivalence between the supposed state of equilibrium and the permanence of the motions from the facts that have been observed so far that reveal their existence, but not their nature.

Let η_t denote the thickness of the layer near the free surface in which k_t and the density ρ_t vary. Let $\eta_{tt'}$ be the thickness of the layer of variable density near the surface $S_{tt'}$, and let $\eta'_{tt'}$ be the thickness of the layer of variable density near the surface $S'_{tt'}$.

It is clear that when V'_t denotes the space inside of A_t in which the density of the liquid A_t can be regarded as constant, one will have:

$$
\begin{aligned}
W = \sum_{t=1}^{n} \Big\{ & \rho'_t k_t \int_{V'_t} dv + \int_{V_t} z\, dv \\
& + \int_{S_t} ds_t \int_0^{\eta_t} \alpha_t \, dp_t \left[(\phi_t + \psi_t + k_t)\, \rho_t + (\rho_t - \rho_{t'})\, g\, z \right] \\
& + \int_{S_{tt'}} ds_{tt'} \int_0^{\eta_{tt'}} \alpha_{tt'} \, dp_{tt'} \left[(\phi_t + \psi_t + k_t)\, \rho_t + (\rho_t - \rho_{t'})\, g\, z \right] \\
& + \int_{S'_{tt'}} ds'_{tt'} \int_0^{\eta'_{tt'}} \alpha'_{tt'} \, dp'_{tt'} \left[(\phi_t + \psi_t + k_t)\, \rho_t + (\rho_t - \rho_{t'})\, g\, z \right] \Big\} .
\end{aligned}
$$

In that expression, $\alpha_t \, dp_t \, ds_t$, $\alpha_{tt'} \, dp_{tt'} \, ds_{tt'}$, $\alpha'_{tt'} \, dp'_{tt'} \, ds'_{tt'}$ are the elements of the layers that adhere to the surfaces S_t, $S_{tt'}$, $S'_{tt'}$, respectively, and therefore α_t , $\alpha_{tt'}$, $\alpha'_{tt'}$ are functions of the distances between those surfaces and the points of the media.

Set:

$$
\begin{aligned}
\int_0^{\eta_t} \alpha_t \, dp_t \, (k_t + \psi_t)\, \rho_t &= a_t , \\
\int_0^{\eta_{tt'}} \alpha_{tt'} \, dp_{tt'} \, (k_t + \psi_t)\, \rho_t &= a_{tt'} , \\
\int_0^{\eta'_{tt'}} \alpha'_{tt'} \, dp'_{tt'} \, (k_t + \psi_t) &= b_{tt'} , \\
\int_0^{\eta'_{tt'}} \alpha'_{tt'} \, dp'_{tt'} \, (\rho_t - \rho'_t) &= \mu_{tt'} .
\end{aligned}
$$

Ignoring the weight that results from the condensation on the surfaces S_t and $S_{tt'}$, regarding z as constant under the variation of the normal $p'_{tt'}$, and adding gives:

$$
\sum_{t=1}^{n} c_t \int_{V_t} dv
$$

for expressing the condition of the invariability of the mass of each fluid. The function W whose variation that results from the displacements that do not change the densities will give equilibrium when it is set equal to zero will then be:

$$W = \sum_{t=1}^{n} \left\{ \rho'_t \, k_t \, V'_t + \int_{S_t} a_t \, ds + \int_{S_{tt'}} a_{tt'} \, ds + \int_{S'_{tt'}} b_{tt'} \, ds + g \int_{S'_{tt'}} \mu_{tt'} \, z \, ds + \int_{V_t} (\rho'_t \, g \, z + c_t) \, dv \right\} . \tag{4}$$

3.

Variation of the potential.

If one sets:

$$p = \frac{dz}{dx}, \qquad q = \frac{dz}{dy},$$

$$P = \sqrt{1 + p^2 + q^2} \; ,$$

in which the derivatives of z are deduced from the equation of the surface S, then one will have:

$$\int_S a \, ds = \iint a \, P \, dx \, dy \, ,$$

in which the double integral is extended over the entire projection of the surface S onto the xy-plane.

Now observe that the a that appear in the integral in formula (4) have the following form:

$$a = \int a \, dp \, (k + \psi) \, \rho \, ,$$

in which ψ has a constant value at distances from the contour of the surface S that are not very small, and if that contour is above the surface of a solid, it will vary rapidly and continuously in the vicinity of the contour. It can be said to have a density ρ, which will also vary in the vicinity of the contour, even when it is found above other fluids. Therefore, a can be regarded as constant at a distance from the contour, but rapidly-varying in the vicinity of it, and on the contour, it will have a value that is appreciably different from the one that it has at the other points of the surface and that depends upon nature of the liquids and the solids that intersect over that contour, and it can also depend upon the greater or lesser curvatures of the surfaces of the solids over which one finds that contour.

One can repeat that argument for the b that appears in the integrals in formula (4), and it can be regarded as constant for all of the surface except for the points that are close to the contour, where it will have values that depend upon the nature of the fluids that terminate there.

In addition, observe that if the quantities a and b vary appreciably from one point to another on the surface, even at a distance from the contour, then they could be considered to be independent of the variations of the form of the surface, assuming that the variations of the distributions of density in the surface layers require a certain time to complete, as the experiments of **Quinke** on the mutability of the capillary surface of mercury ([1]) would tend to show and which we would tend to believe would explain the variability of k_t that was noted above.

Meanwhile, if $\delta_1 z$ denotes the variation of z under the deformation of the surface and δx, δy, δz denote the variations of the coordinates that are due to the displacement of the points of the surface for which:

$$\delta z = \delta_1 z + p\ \delta x + q\ \delta y\,,$$

then for an arbitrary displacement of the points of the surface S, one will have the variation:

$$\delta \int_S a\,ds = \iint dx\,dy \left[\frac{d(a\,P\,\delta x]}{dx} + \frac{d(a\,P\,\delta y]}{dy} + a\left(\frac{dP}{dp}\delta p + \frac{dP}{dq}\delta q \right) \right]$$

$$= \int_0^l a\,d\sigma \left[\left(P\,\delta y + \frac{p}{P}\delta_0 z \right)\frac{dx}{d\sigma} - \left(P\,\delta x + \frac{p}{P}\delta_0 z \right)\frac{dy}{d\sigma} \right] - \iint \delta_1 z\,dx\,dy \left(\frac{d\,\frac{ap}{P}}{dx} + \frac{d\,\frac{aq}{P}}{dy} \right),$$

in which l expresses the length of the contour σ.

Now let α, β, γ be the cosines of the angles that the normal N to the surface S makes with the three axes, while α', β', γ' are those of the angles for the tangent T to the contour σ, and α'', β'', γ'' are the cosines of the angles that the line T' that is normal to T and N make with the axes. One has:

$$\alpha = -\frac{p}{P}, \qquad \beta = -\frac{q}{P}, \qquad \gamma = -\frac{1}{P},$$

$$\alpha' = \frac{dx}{d\sigma}, \qquad \beta' = \frac{dy}{d\sigma}, \qquad \gamma' = \frac{dz}{d\sigma},$$

$$\alpha'' = \beta\,\gamma' - \beta'\,\gamma, \qquad \beta'' = \gamma\,\alpha' - \alpha\,\gamma', \qquad \gamma'' = \alpha\,\beta' - \alpha'\,\beta.$$

Thus:

$$\delta_1 z = \frac{\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z}{\gamma},$$

$$\left(P\,\delta y + \frac{p}{P}\delta_1 z \right)\frac{dx}{d\sigma} - \left(P\,\delta x + \frac{p}{P}\delta_1 z \right)\frac{dy}{d\sigma}$$

([1]) See Poggendorff’s Ann. der Physik und Chemie, Bd. 105.

$$= \frac{\alpha'\,\delta y - \beta'\,\delta x + (\alpha\,\beta' - \beta\,\alpha')(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)}{\gamma}$$

$$= \alpha''\,\delta x + \beta''\,\delta y + \gamma''\,\delta z \,.$$

If δr denotes the displacement whose projections onto the three axes are δx, δy, δz then one will have:

$$\delta x = \delta r \cos(x, r) \,,$$
$$\delta y = \delta r \cos(y, r) \,,$$
$$\delta z = \delta r \cos(z, r) \,,$$

so

$$\delta_1 z = \delta r \frac{\cos(N, r)}{\gamma} \,.$$

Let δu, δv, δw denote the projections of the displacement δr of the points of the contour s onto two arbitrary orthogonal directions u and v in the plane that is normal to the contour σ and above the tangent to σ; one will have:

$$\delta x = \delta u \cos(u, x) + \delta v \cos(v, x) + \alpha'\,\delta w,$$
$$\delta y = \delta u \cos(u, y) + \delta v \cos(v, y) + \beta'\,\delta w,$$
$$\delta z = \delta u \cos(u, z) + \delta v \cos(v, z) + \gamma'\,\delta w \,.$$

Now observe that one has:

$$\alpha'\,\alpha'' + \beta'\,\beta'' + \alpha\gamma\,\gamma'' = 0 \,,$$

so one gets:

$$\alpha''\,\delta x + \beta''\,\delta y + \gamma''\,\delta z = \delta u \cos(T', u) + \delta v \cos(T', v) \,.$$

In addition, one has:

$$\frac{d\,\dfrac{p}{P}}{dx} + \frac{d\,\dfrac{q}{P}}{dy} = -\left(\frac{1}{R} + \frac{1}{R'}\right) \,,$$

in which R and R' are the radii of the maximum and minimum curvature of the surface.

If one substitutes those values in the variation of the integral and also takes $P\,ds$, instead of $dy\;dy$, then one will get:

(5) $$\delta \int_S a\,ds$$

$$= \int_0^l a\,d\sigma \left[\delta u \cos(T', u) + \delta v \cos(T', v)\right] + \int_S ds\,\delta r \cos(N, r) \left[a\left(\frac{1}{R} + \frac{1}{R'}\right) + \alpha\,\frac{da}{dx} + \beta\,\frac{da}{dy} \right].$$

Now since a can be regarded as constant over the surface S at points that are distant from the contour and have a value a^0 on the points of the contour that is different from a, one will have:

(6) $$\delta\int_S a\,ds = \int_0^l a^0\,d\sigma\left[\delta u\cos(T',u)+\delta v\cos(T',v)\right] + \int_S ds\,\delta r\cos(N,r)\left(\frac{1}{R}+\frac{1}{R'}\right).$$

If the contour of S is on the surface S' of a solid then the fluid will be obliged to move over the surface of that solid. Therefore, take v to be the direction of the normal T'' to the tangent to σ in the plane that is tangent to the surface S', while δu is equal to zero. Take:

$$\cos(T', v) = \cos(T', T'') = \cos\omega,$$

in which ω denotes the angle between the tangent planes to the surfaces S and S' along their line of intersection. Therefore:

(7) $$\delta\int_S a\,ds = \int_0^l a^0\,d\sigma\cos\omega\,\delta T'' + \int_S ds\,\delta r\cos(N,r)\left(\frac{1}{R}+\frac{1}{R'}\right).$$

If the surface S then has an invariable form, since it is the surface of a solid, as in the integrals:

$$\int_{S'_{tt'}} b_{tt'}\,ds$$

that appear in formula (4), then one will have:

$$\delta_1 z = \delta r\cos(N, r) = 0,$$

and if one takes v to have the direction of T'' then one will have:

$$\delta u = 0,$$

because the fluid does not move normally to the surface and $\cos(T', v) = \cos(T', T'') = 1$. Thus:

(8) $$\delta\int_S b\,ds = \int_0^l b^0\,\delta T'\,d\sigma.$$

The triple integrals that appear in formula (4) have the form:

$$\int_V a\,dv,$$

in which a is a function of the coordinates x, y, z. If one varies the form of the surface S that bounds the space V and displaces the elements in any way then one will have:

$$\delta \int_V a\,ds = \delta \iiint a\,dx\,dy\,dz = \iiint \left(\frac{d(a\,\delta x)}{dx} + \frac{d(a\,\delta y)}{dy} + \frac{d(a\,\delta z)}{dz} \right) dx\,dy\,dz$$

$$= \int_S a\,(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)\,ds = \int_S a\,\gamma\,ds\,\delta_1 z\,,$$

in which α, β, γ denote the cosines of the angles that the normal to S makes with the axes.

For the portions of the surface that close the space V that belongs to a solid body and which have an invariable form, one has:

$$\delta_1 z = 0\,,$$

and if:

$$\gamma\,\delta_1 z = \delta r \cos(N, r)$$

then one will get:

(9)
$$\delta \int_V a\,ds = \int_S a\,ds\,\delta r \cos(N, r)\,,$$

in which the integral on the right-hand side must be extended over just the part of the surface that closes the space V that is free or that is in contact with another fluid.

With formulas (6), (7), (8), and (9), we have the variations of each of the terms in the value of W that is given in formula (4).

4.

Surfaces of capillarity.

In order to determine the free surface S_t of the fluid A_t or the surface $S_{tt'}$ that separates the fluid A'_t, it is enough to set the first variation of the potential W that is derived from deforming one or the other surface equal to zero. Consider just the surface $S_{tt'}$, because the surface S_t can be regarded as the surface $S_{tt'}$ for which the fluid A'_t has a density equal to zero.

Deforming the surface $S_{tt'}$ will produce variations in only the following part of the potential W:

$$\int_{S'_{tt'}} a'_{tt'}\,ds + \int_{V_t} (\rho_t\,g\,z + c_t)\,dv + \int_{V'_t} (\rho_t\,g\,z + c'_t)\,dv\,,$$

when one does not give any displacement to the points that are found on the intersection of $S_{tt'}$ with the other fluids and the solids.

From formula (5), one can set:

$$\delta u = 0\,, \qquad \delta v = 0\,,$$

because the points of the contour are supposed to be immobile, and:

$$\frac{da}{dx} = 0 , \qquad \frac{da}{dy} = 0 .$$

One gets:

$$\delta \int_{S_{tt'}} a'_{tt'} \, ds = \int_{S_{tt'}} a'_{tt'} \left(\frac{1}{R} + \frac{1}{R'} \right) ds \, \delta r \cos (N, r) ,$$

and from formula (9):

$$\delta \int_{V_t} (\rho_t \, g \, z + c_t) \, dv = \int_{S_{tt'}} (\rho_t \, g \, z + c_t) \, ds \, \delta r \cos (N, r) ,$$

$$\delta \int_{V_{t'}} (\rho'_t \, g \, z + c'_t) \, dv = - \int_{S_{tt'}} (\rho'_t \, g \, z + c'_t) \, ds \, \delta r \cos (N, r) ,$$

because the normals to the spaces V_t and $V_{t'}$ point in opposite directions.

Meanwhile, the first variation that is due to the deformation of $S_{tt'}$ is:

$$\int_{S_{tt'}} ds \, \delta r \cos (N, r) \left[c_t - c'_t + g (\rho_t - \rho'_t) \, z + a'_{tt'} \left(\frac{1}{R} + \frac{1}{R'} \right) \right],$$

which must be annulled for any values of δr at each point of the surface, which will give:

(10) $$c_t - c'_t + g (\rho_t - \rho'_t) \, z + a'_{tt'} \left(\frac{1}{R} + \frac{1}{R'} \right) = 0 ,$$

and that will be the equation of the separating surface of the two liquids A_t and A'_t.

For the free surface S_t , one must set:

$$\rho'_t = 0 ,$$

and have:

(11) $$c_t + g \; \rho_t \; z + a'_{tt'} \left(\frac{1}{R} + \frac{1}{R'} \right) = 0 ,$$

Equation (10) contains two indeterminate constants c_t and c'_t , and equation (11) contains one indeterminate constant c_t . They are determined by means of the values of z at the points where the surface is planar, and where one has:

$$R = \infty , \qquad R' = \infty$$

as a consequence, and the values of z at those points are obtained from the known laws of hydrostatics.

However, since the sum of the inverses of the radii of curvature contain the second derivatives of z, neither equations (10) nor (11) are sufficient to determine the respective surface of capillarity, and one must also know other some boundary conditions.

5.

Angles between surfaces of capillarity and each other, as well as with solids.

The contour of the surface $S_{tt'}$ can be composed of just one closed curve if the surface is simply connected or also more than one if the surface is multiply connected. However, one can always consider each of the closed curves separately when there are more than one. One must examine two cases:

1^0. The closed curve that makes up part of the contour is on a solid $B_{t''}$, and therefore it is the intersection of three surfaces $S_{tt'}$, $S'_{tt'}$, $S'_{t't''}$.

2^0. The closed curve is on another fluid $A_{t''}$ in such a way that it is the intersection of three surfaces $S_{tt'}$, $S'_{t't''}$, $S_{t''t}$.

In the first case, the displacements of the points of the contour of $S_{tt'}$ will produce variations of only the following part of the potential W:

$$\int_{S_{tt'}} a'_{tt'}\,ds + \int_{S'_{tt'}} b_{tt'}\,ds + \int_{S'_{tt''}} b_{tt''}\,ds\,.$$

From equation (7), one can set:

$$\delta r = 0$$

and get:

$$\delta \int_{S_{tt'}} a'_{tt'}\,ds = \int_0^l a^0_{tt'}\,d\sigma \cos\omega\,\delta T\,.$$

One deduces from equation (8) that:

$$\delta \int_{S'_{tt''}} b_{tt''}\,ds = \int_0^l b^0_{tt''}\,d\sigma\,\delta T\,,$$

$$\delta \int_{S'_{t't''}} b_{t't''}\,ds = -\int_0^l b^0_{t't''}\,d\sigma\,\delta T\,,$$

since dT has the opposite sign with respect to the two surfaces $S'_{t't''}$ and $S'_{tt''}$ as T is the perpendicular to the tangent to the contour s and to the normal to the surface of the solid that points towards the part that is external to the part of the surface considered. Thus, one must have:

$$\int_0^l ds\,\delta T\,(a^0_{tt'}\cos\omega + b^0_{tt''} - b^0_{t't''})$$

for equilibrium, and therefore:

(12) $$a^0_{tt''}\cos\,\omega = b^0_{t't''} - b^0_{tt''}\,,$$

in which the values of a and b near the contour are distinguished by a superscript 0.

If the quantity $a^0_{tt''}$ depends upon only the nature of the fluids A_t and $A_{t'}$, and the quantity $b^0_{tt'}$ depends upon only the nature of the fluid A_t and the solid $B_{t'}$ then we will have the following theorem:

The angle at which a capillary surface meets a solid is constant for any form of the solid and the space that is occupied by the liquid, and it depends upon only the nature of the liquid and the solid.

However, the experiments of **Wertheim** ([1]) and **Wilhelmy** ([2]) show that the angle also varies with the curvature of the surface of the solid, and those of **Quinke** ([3]) prove that the angle varies appreciably with time, as well as without producing any alteration in the nature of the liquid when that liquid is mercury.

What we have previously discussed about the nature of the quantities a and b explains the results of those experiments and indicates the limitations of posing the stated theorem.

In the second case, i.e., when the contour of the surface $S_{tt'}$ is a closed curve that is the intersection of three surfaces:

$$S_{tt'}\,,\qquad S_{t't''}\,,\qquad S_{t''t}\,,$$

the displacements of the points of the contour will produce variations in only the following part of the potential W:

$$\int_{S_{tt'}} a_{tt'}\,ds + \int_{S_{t't''}} a_{t't''}\,ds + \int_{S_{t''t}} a_{t''t}\,ds\,.$$

However, in that case of equation (7), if one sets:

([1]) Annales de Phy. et Ch., t. 63.
([2]) Poggendorff's Ann. der Ph. und Ch., Bd. 119.
([3]) Poggendorff's Ann. der Ph. und Ch., Bd. 105.

$$\delta \int_{S_{tt'}} a_{tt'}\, ds = \int_0^l a_{tt'}^0\, d\sigma \left[\delta u \cos(T'', u) + \delta v \cos(T'', v)\right],$$

$$\delta \int_{S_{t't''}} a_{t't''}\, ds = \int_0^l a_{t't''}^0\, d\sigma \left[\delta u \cos(T, u) + \delta v \cos(T, v)\right],$$

$$\delta \int_{S_{t''t}} a_{t''t}\, ds = \int_0^l a_{t''t}^0\, d\sigma \left[\delta u \cos(T', u) + \delta v \cos(T', v)\right],$$

so:

$$\int_0^l d\sigma\ \delta u \left[a_{tt'}^0 \cos(T'', u) + a_{t't''}^0 \cos(T, u) + a_{t''t}^0 \cos(T', u)\right]$$
$$+ \int_0^l d\sigma\ \delta v \left[a_{tt'}^0 \cos(T'', v) + a_{t't''}^0 \cos(T, v) + a_{t''t}^0 \cos(T', v)\right] = 0,$$

and therefore:

$$a_{tt'}^0 \cos(T'', u) + a_{t't''}^0 \cos(T, u) + a_{t''t}^0 \cos(T', u) = 0\ ,$$

$$a_{tt'}^0 \cos(T'', v) + a_{t't''}^0 \cos(T, v) + a_{t''t}^0 \cos(T', v) = 0\ .$$

The mutually-orthogonal directions u and v are arbitrary in the plane that is normal to the contour σ, and one also finds the three directions T, T', T'' in that plane. One can take one of those directions – T'', for example – to be the direction v, and the let ω, ω', ω'' denote the angles that are subtended between T' and T'', T'' and T, and T and T', resp., or the angles between the planes that are tangent to the surfaces $S_{tt''}$ and $S_{tt'}$, $S_{tt'}$ and $S_{t't''}$, and $S_{t''t}$ and $S_{t't''}$, resp., and one will have:

$$\cos(T'', v) = \cos(T'', T'') = 1\ ,$$

$$\cos(T', v) = \cos(T, T'') = \cos \omega'\ ,$$

$$\cos(T'', v) = \cos(T', T'') = \cos \omega\ ,$$

$$\cos(T'', u) = 0\ ,$$

$$\cos(T, u) = \cos\left(\frac{\pi}{2} + (T, T'')\right) = -\sin \omega'\ ,$$

$$\cos(T', u) = \cos\left(\frac{\pi}{2} + (T', T'')\right) = -\sin \omega\ ,$$

so

$$a^0_{t't''} \sin \omega' + a^0_{t''t} \sin \omega = 0 ,$$

$$a^0_{t't''} \cos \omega' + a^0_{t''t} \cos \omega + a^0_{tt'} = 0 ,$$

from which, if one observes that one has:

$$\omega + \omega' + \omega'' = 2\pi ,$$

one will deduce the three relations:

$$(13) \qquad \left\{ \begin{aligned} (a^0_{t't''})^2 &= (a^0_{tt'})^2 + (a^0_{t''t})^2 + 2\, a^0_{tt'}\, a^0_{tt''} \cos \omega , \\ (a^0_{t''t})^2 &= (a^0_{tt'})^2 + (a^0_{tt''})^2 + 2\, a^0_{tt'}\, a^0_{t't''} \cos \omega' , \\ (a^0_{tt'})^2 &= (a^0_{t't''})^2 + (a^0_{t''t})^2 + 2\, a^0_{t't''}\, a^0_{t''t} \cos \omega'' . \end{aligned} \right.$$

If the quantity $a_{tt'}$ has the same values on the contour that it has at the other points of the surface $S_{tt'}$ (which is a value that depends upon only the nature of the liquids A_t and $A_{t'}$) then one will have the following theorem, which was communicated by **Paul Du Bois-Reymond** to Prof. **I. Neumann** ([1]):

The angles between the separating surfaces of three fluids that intersect along a closed line are the supplements to the angles of a triangle whose sides are proportional to three quantities, each of which depends upon only the nature of those two fluids.

As a consequence of the considerations in regard to the coefficients $a_{tt'}$ that were presented above, only the last part of that theorem must be modified. That is, the three sides of the triangle that determine the angles between the three surfaces are proportional to three quantities that each depend upon the nature of all three of the fluids.

(*to be continued*)

([1]) Du Bois-Reymond, *De aequilibrio fluidorum*, Dissertatio inauguralis.

“Teoria della capillarita” Il Nuovo Cimento **25** (1867), 225-237.

Theory of capillarity ([1])

By Prof. ENRICO BETTI

Translated by D. H. Delphenich

6.

Equilibrium of fluids in communicating vessels.

Consider a vessel that is composed of three parts: Two cylindrical arms B and B_1 that are united below by a transverse arm C of arbitrary form. Let S and S_1 be sections of the cylinders that are made normal to their generators. The cylinder B contains the fluids $A_1, A_2, \ldots, A_m$, and the order in which they are arranged from top to bottom is given by the respective indices. The cylinder B contains the fluids $A_m, A_{m+1}, \ldots, A_{n-1}, A_n$, and the order from top to bottom is the same as the order in which they are written. The transverse arm C is filled with only the fluid A_m . Finally, let $\rho_1, \rho_2, \ldots, \rho_n$ be the respective densities of the fluids, let S_1, S_n be the free surfaces of the fluids A_1 and A_n, and let $S_{t,t+1}$ be the separating surface of the two fluids A_t, A_{t+1} .

The equations of the surfaces:

$$S_1\,,\quad S_{12}\,,\quad S_{12}\,,\quad \ldots,\quad S_{n,n-1}\,,\quad S_n\,,$$

which are deduced from formulas (10) and (11), when they are added to the terms that result from the variability of the coefficients:

$$a_1\,,\quad a_{12}\,,\quad a_{12}\,,\quad \ldots,\quad a_{n,n-1}\,,\quad a_n$$

in the vicinity of the walls, will be:

([1]) *Continuation*, see page 81.

$$
(14)\qquad \left\{ \begin{aligned}
& c_1 + g\,\rho_1\, z_1 - \frac{d(a_1\,\alpha_1)}{dx} - \frac{d(a_1\,\beta_1)}{dy} = 0, \\
& c_2 - c_1 + g\,(\rho_2 - \rho_1)\, z_1 - \frac{d(a_{12}\,\alpha_{12})}{dx} - \frac{d(a_{12}\,\beta_{12})}{dy} = 0, \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& c_m - c_{m-1} + g\,(\rho_m - \rho_{m-1})\, z_m - \frac{d(a_{m,m-1}\,\alpha_{m,m-1})}{dx} - \frac{d(a_{m,m-1}\,\beta_{m,m-1})}{dy} = 0,
\end{aligned} \right.
$$

$$
(15)\qquad \left\{ \begin{aligned}
& c_m - c_{m+1} + g\,(\rho_m - \rho_{m+1})\, z_{m+1} - \frac{d(a_{m.m+1}\,\alpha_{m,m+1})}{dx} - \frac{d(a_{m,m-1}\,\beta_{m,m-1})}{dy} = 0, \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& c_{n-1} - c_n + g\,(\rho_{n-1} - \rho_n)\, z_n - \frac{d(a_{n-1,n}\,\alpha_{n-1,n})}{dx} - \frac{d(a_{n-1,n}\,\beta_{n-1,n})}{dy} = 0, \\
& c_n - c_{n-1} + g\,\rho_n\, z_{n+1} - \frac{d(a_n\,\alpha_n)}{dx} - \frac{d(a_n\,\beta_n)}{dy} = 0.
\end{aligned} \right.
$$

If one sums equations (14) and (15) separately then one will have:

$$
(16)\qquad \left\{ \begin{aligned}
c_m + g\left(\sum_{s=1}^{m-1} \rho_s (z_s - z_{s+1}) + \rho_m\, z_m\right) &= \sum_{s=1}^{m}\left(\frac{d(\alpha_{s-1,s} a_{s-1,s})}{dx} + \frac{d(a_{s-1,s}\beta_{s-1,s})}{dy}\right), \\
c_m - g\left(\sum_{s=m+1}^{n+1} \rho_s (z_s - z_{s+1}) + \rho_m\, z_{m+1}\right) &= \sum_{s=m+1}^{n+1}\left(\frac{d(\alpha_{s-1,s} a_{s-1,s})}{dx} + \frac{d(a_{s-1,s}\beta_{s-1,s})}{dy}\right),
\end{aligned} \right.
$$

in which one needs to set:

$$
a_{01} = a_1, \qquad \alpha_{01} = \alpha_1, \qquad \beta_{01} = \beta_1,
$$

$$
a_{n,n+1} = a_n, \qquad \alpha_{n,n+1} = \alpha_n, \qquad \beta_{n,n+1} = \beta_n.
$$

Now observe that in the functions $a\alpha$, $a\beta$, one must take the derivatives with respect to x and y after one supposes that the coordinate z in them has been replaced with its value that is determined as a function of x and y from the equation of the surface. Therefore:

$$
\frac{d(a\alpha)}{dx} + \frac{d(a\beta)}{dy} = a\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + p\,\frac{d\alpha}{dz} + q\,\frac{d\beta}{dy}\right) + \alpha\,\frac{da}{dx} + \beta\,\frac{da}{dy} + \frac{da}{dz}\,(p\alpha + q\beta),
$$

and if:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 ,$$

so:

$$\alpha \frac{d\alpha}{dz} + \beta \frac{d\beta}{dz} + \gamma \frac{d\gamma}{dz} = 0 ,$$

then one will get:

$$p \frac{d\alpha}{dz} + q \frac{d\beta}{dz} = \frac{d\gamma}{dz} .$$

In addition, one has:

$$\gamma = p\,\alpha + q\,\beta .$$

Therefore, substitute:

$$\frac{d(a\alpha)}{dx} + \frac{d(a\beta)}{dy} = a\left(\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}\right) + \alpha \frac{da}{dx} + \beta \frac{da}{dy} + \gamma \frac{da}{dz} .$$

As **Cauchy** has proved ([1]), that will not change under orthogonal transformations of the coordinates. Therefore, if one takes the origin of the coordinates to be the point where the horizontal *xy*-plane meets the line that is parallel to the generators of the cylinder and passes through the centers of gravity of the sections *S*, and one orthogonally transforms the coordinates so that the new z'-axis is the line that passes through the centers of gravity of the sections *S* and lets λ, μ, ν denote the cosines of the angles that the vertical (or original *z*-axis) makes with the new axes then one will have:

$$z = \lambda\, x' + \mu\, y' + \nu\, z' ,$$

and the first of equations (16) will become:

$$c_m + g \sum_{s=1}^{m-1} \rho_s \left[\lambda (x'_s - x'_{s+1}) + \mu (y'_s - y'_{s+1}) + \nu (z'_s - z'_{s+1})\right] + g\,\rho_m (\lambda\, x'_m + \mu\, y'_m + \nu\, z'_m)$$
$$= \sum_{s=1}^{m} \left(\frac{d(a_{s-1,s}\,\alpha'_{s-1,s}}{dx'} - \frac{d(a_{s-1,s}\,\beta'_{s-1,s}}{dy'}\right) .$$

Multiply that by $dx'\,dy'$ and integrate it, while extending the integral over the entire area of the section *S*. If one lets the origin be the center of gravity of the section *S* then one will have:

$$\iint y'_s\, dx'_s\, dy'_s = 0 , \qquad \iint x'_s\, dx'_s\, dy'_s = 0 ,$$

and therefore, if one lets $P_1, P_2, \ldots, P_{m-1}$ denote the weights of the fluid masses $A_1, A_2, \ldots, A_{m-1}$, resp., and lets P_m denote the weight of the fluid A_m that is contained in the cylinder *B*, when prolonged to the original horizontal *xy*-plane, then one will have:

([1]) Cauchy, *Exercises d'Analyse et de Physique Mathématique*, t. 1, pp. 102.

$$c_m\, S + \nu \sum_{s=1}^{m-1} P_s + \nu P_m \;=\; l \sum_{s=1}^{m} a^0_{s-1,s} \cos\,\omega_{s-1,s}\,,$$

where $\omega_{s-1,s}$ denotes the angle at which the surface $S_{s-1,s}$ meets the vessel, and l is the length of the perimeter of the section S.

Analogously to the second of equations (16), one will get:

$$c_m\, S' + \nu' \sum_{m+1=1}^{n} P_s + \nu' P'_m \;=\; l' \sum_{m+1=1}^{n+1} a^0_{s-1,s} \cos\,\omega_{s-1,s}\,.$$

If one eliminates c_m then one will finally get:

$$\text{(17)} \qquad \frac{\nu}{S} \sum_{s=1}^{m} P_s - \frac{\nu'}{S'} \sum_{s=m}^{n} P'_s \;=\; \frac{l}{S} \sum_{s=1}^{m} a^0_{s-1,s} \cos\,\omega_{s-1,s} - \frac{l'}{S'} \sum_{s=m+1}^{n+1} a^0_{s-1,s} \cos\,\omega_{s-1,s}\;.$$

From equations (12), one has:

$$a^0_{s-1,s} \cos\omega_{s-1,s} + b^0_{s-1} - b^0_s \;=\; 0\,,$$
$$a^0_{s,s+1} \cos\omega_{s,s+1} + b^0_s - b^0_{s+1} \;=\; 0\,,$$

since the nature of the solid in contact with all of the liquids is the same. Nonetheless, observe that the value of b^0_s is not the same those two equations, because it depends upon not only the nature of the solid and the liquid A_s , but also upon that of the other liquid that passes through the same contour. Therefore, one agrees to adopt different notations and write:

$$a^0_{s-1,s} \cos\omega_{s-1,s} + b^s_{s-1} - b^{s-1}_s \;=\; 0\,,$$

and equation (17) will become:

$$\text{(18)} \qquad \frac{\nu}{S} \sum_{s=1}^{m} P_s - \frac{\nu'}{S'} \sum_{s=m}^{n} P'_s \;=\; \frac{l}{S} \sum_{s=1}^{m} (b^{s-1}_s - b^s_{s-1}) - \frac{l'}{S'} \sum_{s=m+1}^{n+1} (b^{s-1}_s - b^s_{s-1})\;.$$

If both of the sections S and S are very large then l / S and l' / S' will both be negligible quantities, and therefore:

$$\frac{\nu}{S} \sum_{s=1}^{m} P_s \;=\; \frac{\nu'}{S'} \sum_{s=m}^{n} P'_s\,,$$

which is the equation of ordinary hydrostatic equilibrium.

If one has:

(19) $$b_s^{s-1} = b_s^{s+1},$$

as in the theory of **Laplace** and **Poisson**, then will have:

$$\frac{\nu}{S}\sum_{s=1}^{m} P_s - \frac{\nu'}{S'}\sum_{s=m}^{n} P'_s = \frac{l'}{S'}\,b_m^{m+1} - \frac{l}{S}\,b_m^{m-1} = b_m\left(\frac{l'}{S'} - \frac{l}{S}\right),$$

i.e., the difference between hydrostatic equilibrium and the equilibrium that takes into account the forces of cohesion and adhesion will depend upon only the lower fluid, i.e., the correction that is due to capillarity will depend upon only the fluid that is found below all of the other ones. It is known that **Young** was the first to observe that this result does not correspond to reality, and he inferred from that an objection of the theory of **Laplace** and **Mosotti** ([1]), and in order to make the theory agree with the experiments, he rejected equation (19). We have already pointed out the conditions under which we cannot apply that equation.

Now suppose that the two arms B and B_1 are vertical, the sections S and S' are circular, and they have radii r and r', respectively. One has:

$$\nu = 1, \qquad \nu' = 1, \qquad l = 2\pi r, \qquad S = \pi r^2, \qquad l' = 2\pi r', \qquad S' = \pi r^2.$$

Thus:

$$\frac{1}{r^2}\sum_{s=1}^{m} P_s - \frac{1}{r'^2}\sum_{s=m}^{n} P'_s = \frac{2\pi}{r}\sum_{s=1}^{m}(b_{s-1}^{s} - b_s^{s-1}) - \frac{2\pi}{r'}\sum_{s=m+1}^{n+1}(b_{s-1}^{s} - b_s^{s-1}).$$

Let h_s denote the mean height of the fluid A_s in the arm B, and let h'_s denote that of the fluid A_s in the arm B'; one will have:

$$P_s = \pi\, r^2\, h_s\, \rho_s\, g\,,$$
$$P'_s = \pi\, r'^2\, h'_s\, \rho_s\, g\,,$$

so

$$\sum_{s=1}^{m} h_s\,\rho_s - \sum_{s=m}^{n} h'_s\,\rho_s = \frac{2}{g\,r}\sum_{s=1}^{m}(b_{s-1}^{s} - b_s^{s-1}) - \frac{2}{g\,r'}\sum_{s=m+1}^{n+1}(b_{s-1}^{s} - b_s^{s-1}).$$

If r' is very large compared to r then one will have:

$$\sum_{s=1}^{m} h_s\,\rho_s - \sum_{s=m}^{n} h'_s\,\rho_s = \frac{2}{g\,r}\sum_{s=1}^{m}(b_{s-1}^{s} - b_s^{s-1}).$$

If one has only one liquid then one will have:

([1]) R. Taylor, *Scientific Memoirs*, vol. III.

$$h_1 - h_1' = -\frac{2}{g\,\rho\, r}\, b_1^0 ,$$

or the difference between the levels will be inversely proportional to the diameter of the tube.

7.

Equilibrium for a floating body.

Suppose that a body K of arbitrary form floats between two fluids A_1 and A_2 that are contained in a vessel B in such a way that the lower part of K is immersed in the fluid A_2, while the upper part is in the fluid A_1 .

Let t_1 be the surface of K that is in contact with A_1, while t_2 is the surface of K that is in contact with A_2 . Let S be the separation surface between A_1 and A_2 , while V, V_1 , V_2 are the spaces that are occupied by K, A_1 , A_2 , resp., and ρ, ρ_1 , ρ_2 are the respective densities of those bodies.

For equilibrium, it is sufficient to annul the variation of the potential, which will be:

$$g\int_V \rho\, z\, dv + \int_{V_1} (g\,\rho\, z_1 + c_1)\, dv + \int_{V_2} (g\,\rho\, z_2 + c_2)\, dv + \int_S a\, ds + \int_{t_1} b_1\, ds + \int_{\tau_1} c_1\, ds + \int_{\tau_2} c_2\, ds \tag{20}$$

in this case, where τ_1 and τ_2 denote the parts of the walls of the vessel that are in contact with A_1 and A_2 .

Since the body K is moving, the variations to be considered at the points of the two fluids are partially arbitrary, but vary from one point to the other, and partially derived from the motion of the solid body K, and therefore the variations of the coordinates will have the form that is given the mechanics of rigid bodies ([1]):

$$\begin{aligned} \delta x &= \delta \varepsilon_1 + [(z - \zeta)\,\lambda_2 - (y - \eta)\,\lambda_3]\,\delta\phi , \\ \delta y &= \delta \varepsilon_2 + [(x - \xi)\,\lambda_3 - (z - \zeta)\,\lambda_1]\,\delta\phi , \\ \delta z &= \delta \varepsilon_3 + [(y - \eta)\,\lambda_1 - (x - \xi)\,\lambda_2]\,\delta\phi . \end{aligned} \tag{21}$$

If the walls of the vessel are far enough away from the floating body then variations at all points of the surface S, except for the ones on the line of intersection with the surface K, will be arbitrary. Along that line and on the surface of K, they are partially arbitrary and partially of the form (21). If one sets the first part of the variation equal to zero (i.e., the arbitrary one) then one will get, as in numbers (5) and (6), the equation of the surface S and the angles that it must make with the walls of the vessel and the floating body. All that remains for one to consider is the second

---

([1]) See Mossotti, *Lezioni di Meccanica razionale*, Lez. 23.

part, in which the variations of the coordinates have the form (21), and which results from only the first four integrals of the potentials (20).

Set:

$$\delta r = \delta x^2 + \delta y^2 + \delta z^2 ,$$

so one will have:

$$(22) \qquad \begin{aligned} & g \int_{t_1+t_2} \rho\, z\, ds\, \delta r \cos(r, N) + \int_{t_1} (g\, \rho_1\, z_1 + c_1)\, ds\, \delta r \cos(r, N) \\ & + \int_{t_2} (g\, \rho_2\, z_2 + c_2)\, ds\, \delta r \cos(r, N) + \int_0^l a^0\, d\sigma\, \delta r \cos(r, T) = 0 , \end{aligned}$$

in which l denotes the length of the line σ that is the intersection of the surface S with the surface K, N denotes the direction of the normal to the surface of K, and T denotes the direction that is perpendicular to the tangent to the contour σ and to the normal to the surface S at the point considered.

Now imagine that the lower part of the inside of the floating body is filled with the liquid A_2 up to the contour σ and the upper part is filled with the liquid A_1 , and suppose that the separation surface between the two liquids, which is then conceived of as being inside of K, is the capillary surface, i.e., it is determined by the equation:

$$(23) \qquad c_2 - c_1 + g\,(\rho_2 - \rho_1)\, z - \frac{d(a\,\alpha)}{dx} - \frac{d(a\,\beta)}{dy} = 0$$

and the condition that the angle that the contour σ makes with the surface of K is equal to the one that the surface S makes with that surface, i.e., suppose that it is the continuation of S inside of K.

If one lets S' denote that surface then it will be clear that the displacements of the form (21) that are given to all points of K will annul the variation of the integral:

$$\int_{S'} a\, ds .$$

Therefore, from formula (7):

$$\int_0^l a^0\, \delta r \cos(r, T)\, d\sigma = \int_{S'} \delta r \cos(r, N)\, ds \left(\frac{d(a\,\alpha)}{dx} + \frac{d(a\,\beta)}{dy} \right) ,$$

and due to equation (23):

$$\int_0^l a^0\, \delta r \cos(r, T)\, d\sigma = \int_{S'} [c_2 - c_1 + g(\rho_2 - \rho_1)\, z]\, ds\, \delta r \cos(r, N) .$$

Substituting that in equation (22) will give:

$$(24)\quad g\int_{t_1+t_2}\rho\, z\, ds\,\delta r\cos(r,N) - \int_{S'+t_1}(g\,\rho_1\,z_1+c_1)\,ds\,\delta r\cos(r,N) + \int_{S'+t_2}(g\,\rho_2\,z_2+c_2)\,ds\,\delta r\cos(r,N) = 0\,.$$

Now suppose that $\delta\phi = 0$, and therefore δr is constant, let V_1' and V_2' denote the spaces inside of K that are occupied by the two fluids that we have imagined, and observe that we have:

$$\int_{S'+t_1} ds\cos(r,N) = 0\,,$$

$$\int_{S'+t_2} ds\cos(r,N) = 0\,.$$

We will have:

$$g\,\rho\,V = g\,\rho_1\,V_1' + g\,\rho_2\,V_2'\,.$$

We then have the following theorem:

If a floating body is at the boundary of two fluids, and their separation surface (no matter what form it might have) is imagined to be continued inside of the floating body using the same law that shapes it outside then the weight of the floating body will be equal to the weight of the volume of that body that is situated above and below the separation surface, which is supposed to be filled with the fluid in which the body is immersed.

That generalization of Archimedes' principle is due to **Paul Du Bois-Reymond**.

If one now no longer takes $\delta\phi = 0$ in equations (21), but rather just:

$$\delta\,\varepsilon_1 = \delta\,\varepsilon_1 = \delta\,\varepsilon_1 = 0\,,$$

then one will have:

$$\delta r\cos(r,N) = \alpha\,\delta x + \rho\,\delta y + \gamma\,\delta z$$

$$(25)\qquad = \delta\phi\,\lambda_1\,[\gamma\,(y-\eta) - \beta\,(z-\zeta)] + \lambda_2\,[\alpha\,(z-\zeta) - \gamma\,(x-\xi)] + \lambda_3\,[\beta\,(x-\xi) - \alpha\,(y-\eta)]\,.$$

If one lets X, Y, Z denote the coordinates of the center of gravity of the volume V, while X_1, Y_1, Z_1 are those of the center of gravity of V_1, and X_2, Y_2, Z_2 are those of the center of gravity of V_1' then one will have:

$$
(26)\quad \left\{\begin{array}{llll}
V\,X = \int\limits_{t_1+t_2} z\,x\,\gamma\,ds, & V\,Y = \int\limits_{t_1+t_2} z\,y\,\gamma\,ds, & 0 = \int\limits_{t_1+t_2} z^2\,\alpha\,ds, & 0 = \int\limits_{t_1+t_2} z^2\,\beta\,ds, \\
V_1'X_1 = \int\limits_{S_1+t_2} z\,x\,\gamma\,ds, & V_1'Y_1 = \int\limits_{S_1+t_2} z\,y\,\gamma\,ds, & 0 = \int\limits_{S_1+t_2} z^2\,\alpha\,ds, & 0 = \int\limits_{S_1+t_2} z^2\,\beta\,ds, \\
V_2'X_2 = \int\limits_{S'+t_2} z\,x\,\gamma\,ds, & V_2'Y_2 = \int\limits_{S'+t_2} z\,y\,\gamma\,ds, & 0 = \int\limits_{S'+t_2} z^2\,\alpha\,ds, & 0 = \int\limits_{S'+t_2} z^2\,\beta\,ds.
\end{array}\right.
$$

If one substitutes the value (25) in equation (24), sets the coefficients of ξ, η, ζ and of λ_1, λ_2, λ_3 equal to zero separately and reduces using formulas (26) then one will get:

$$
\begin{aligned}
\rho\,X\,V &= \rho_1\,X_1\,V_1' + \rho_2\,X_2\,V_2' \;, \\
\rho\,Y\,V &= \rho_1\,Y_1\,V_1' + \rho_2\,Y_2\,V_2' \;,
\end{aligned}
$$

and one will have:

$$\rho\,V = \rho_1\,V_1' + \rho_2\,V_2' \;,$$

in addition. One will also have:

$$
\begin{aligned}
X\,V &= X_1\,V_1' + X_2\,V_2' \;, \\
Y\,V &= Y_1\,V_1' + Y_2\,V_2' \;, \\
V &= V_1' + V_2' \;.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\rho_1\,V_1'(X - X_1) + \rho_2\,V_2'(X - X_2) &= 0 \;, \\
V_1'(X - X_1) + V_2'(X - X_2) &= 0 \;, \\
\rho_1\,V_1'(Y - Y_1) + \rho_2\,V_2'(Y - Y_2) &= 0 \;, \\
V_1'(Y - Y_1) + V_2'(Y - Y_2) &= 0 \;,
\end{aligned}
$$

and therefore:

$$
\begin{aligned}
X &= X_1 = X_2 \;, \\
Y &= Y_1 = Y_2 \;.
\end{aligned}
$$

One then has the following theorem, which is due to **Paul Du Bois-Reymond**:

The centers of gravity of the three volumes V, V_1', and V_2' lie along the same vertical.

As we have seen, both of those theorems on floating bodies are independent of the supposition that was made in the theory of **Laplace** and **Poisson** in regard to the quantities that are denoted by *a* and *b*.

(*to be continued* [†])

[†] Translator: Apparently, that statement was not true; no continuation of the article seems to exist.