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Chapter X

Systems of ∞^2 rays or rectilinear congruences

Rectilinear congruences. – Limit points and principal surfaces. – Ribaucour's isotropic congruences. – Foci and developables of a congruence. – Normal congruences. – Beltrami's theorem. – The Malus-Dupin theorem. – Congruences with assigned spherical images of their principal surfaces. – Formulas that relate to two focal surfaces. – Pseudo-spherical congruences. – Guichard congruences. – Guichard-Voss surfaces.

§ 137.

The fundamental form of a congruence.

The theory that we shall develop in the present chapter is concerned with doubly-infinite systems of lines that are distributed in space in such a manner that one (or a finite number) of lines of the system will pass through any point in space (or a convenient region of space). Such a system of ∞^2 lines (i.e., rays) is also called a *rectilinear congruence*, or simply, a *congruence*. The totality of normals to a surface is not a particular case of those systems.

That theory, which was born in geometrical optics, has taken on an increasing importance for the theory of surfaces, and there seems to be no doubt that it should contribute increasingly to progress in geometry from now on.

In this and the following chapter, we shall establish the fundamentals, which are taken from the classic paper of Kummer ([1]), especially, and exhibit some of the principal applications.

To begin with, we shall address the analytical definition of a congruence. In order to do that, we cut the entire system of lines with a surface *S*, and we regard the point (or one of the points) where any ray of the system meets *S* as a starting point. We refer the surface *S* to a system of curvilinear coordinates (u, v), and define the congruence analytically by expressing the coordinates:

$$x, y, z$$

of the starting point and the direction cosines of the ray, which we denote by:

([1]) "Allgemeine Theorie der geradlinigen Strahlensysteme," Crelle's Journal, v. 57.

$$X, Y, Z,$$

as functions of u, v.

In regard to the functions x, y, z, X, Y, Z of u, v, we suppose that they are finite and continuous, along with all of their partial derivatives.

Draw the ray through the center of the sphere:

$$x^2 + y^2 + z^2 = 1$$

that is parallel to the positive direction of a ray of the congruence, so the coordinates of its extreme point M_1 will be X, Y, Z; one regards that point as the spherical image of the line (u, v) of the congruence. If one varies the line (u, v) of the system then the point M_1 will describe the *spherical image* of the congruence.

Observe that the coordinates ξ, η, ζ of any point P on the ray (u, v) are given in the form:

$$\xi = x + t\,X, \qquad \eta = y + t\,Y, \qquad \zeta = z + t\,Z, \tag{1}$$

in which t is the *abscissa* of the point P on the ray that contacts the starting point $P_0 \equiv (x, y, z)$ as its origin.

With Kummer, we introduce the following fundamental formulas:

$$\sum\left(\frac{\partial X}{\partial u}\right)^2 = E, \qquad \sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v} = F, \qquad \sum\left(\frac{\partial X}{\partial v}\right)^2 = G, \tag{2}$$

$$\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} = e, \qquad \sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial v} = f, \qquad \sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u} = f', \qquad \sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial v} = g, \tag{3}$$

by which, one expresses the two quadratic differential forms:

$$ds_1^2 = \sum dX^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{4}$$

$$\sum dx\,dX = e\,du^2 + (f + f')\,du\,dv + g\,dv^2, \tag{5}$$

which are called the two *fundamental forms*. The first one represents the square of the line element of the spherical representation; one will observe that ds_1 also measures the infinitesimal angle between two successive generators (u, v), $(u + du, v + dv)$.

Let dp denote the infinitesimal length of the minimum distance from the ray (u, v) to the infinitely-close ray, while $\cos a$, $\cos b$, $\cos b$ denote the direction cosines of that minimum distance, and finally r denotes the value of the abscissa t at the foot of dp on the ray (u, v).

We will have:

$$\cos a : \cos b : \cos c = (Y\,dZ - Z\,dY) : (Z\,dX - X\,dZ) : (X\,dY - Y\,dX),$$

$$\cos a : \cos b : \cos c = \left\{\left(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\right)du + \left(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\right)dv\right\}$$
$$: \left\{\left(Z\frac{\partial X}{\partial u} - X\frac{\partial Z}{\partial u}\right)du + \left(Z\frac{\partial X}{\partial v} - X\frac{\partial Z}{\partial v}\right)dv\right\}$$
$$: \left\{\left(X\frac{\partial Y}{\partial u} - Y\frac{\partial X}{\partial u}\right)du + \left(X\frac{\partial Y}{\partial v} - Y\frac{\partial X}{\partial v}\right)dv\right\}.$$

From the identity that was observed in § 77, page 162 (footnote), one can write:

$$\cos a : \cos b : \cos c = \left\{\left(E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}\right)du - \left(G\frac{\partial X}{\partial u} - F\frac{\partial X}{\partial v}\right)dv\right\}$$
$$: \left\{\left(E\frac{\partial Y}{\partial v} - F\frac{\partial Y}{\partial u}\right)du - \left(G\frac{\partial Y}{\partial u} - F\frac{\partial Y}{\partial v}\right)dv\right\}$$
$$: \left\{\left(E\frac{\partial Z}{\partial v} - F\frac{\partial Z}{\partial u}\right)du - \left(G\frac{\partial Z}{\partial u} - F\frac{\partial Z}{\partial v}\right)dv\right\},$$

and it will then result that:

$$(6)\qquad \left\{\begin{aligned} \cos a &= \frac{\left(E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}\right)du + \left(F\frac{\partial X}{\partial v} - G\frac{\partial X}{\partial u}\right)dv}{\sqrt{EG-F^2}\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}}, \\ \cos b &= \frac{\left(E\frac{\partial Y}{\partial v} - F\frac{\partial Y}{\partial u}\right)du + \left(F\frac{\partial Y}{\partial v} - G\frac{\partial Y}{\partial u}\right)dv}{\sqrt{EG-F^2}\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}}, \\ \cos c &= \frac{\left(E\frac{\partial Z}{\partial v} - F\frac{\partial Z}{\partial u}\right)du + \left(F\frac{\partial Z}{\partial v} - G\frac{\partial Z}{\partial u}\right)dv}{\sqrt{EG-F^2}\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}}. \end{aligned}\right.$$

Now, one has:

$$dp = \sum \cos a\, dx,$$

or, from the preceding:

$$(7)\qquad dp = \frac{1}{\sqrt{EG-F^2}\,ds_1}\begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv \\ e\,du + f\,dv & e'\,du + f'\,dv \end{vmatrix}.$$

If one lets r be the abscissa of the foot of dp on the ray (u, v), and lets t be that of the point where it meets the ray $(u + du, v + dv)$ then one will have:

$$x + r\,X + dp\cos a = x + dx + t\,(X + dX),$$

with analogous expressions for y, z, or:

$$\begin{cases} rX + dp\cos a = dx + t(X + dX), \\ rY + dp\cos b = dy + t(Y + dY), \\ rZ + dp\cos c = dz + t(Z + dZ). \end{cases}$$

If one multiplies these equations by X, Y, Z in succession and sums them then that will give:

$$t = r - \sum X\,dx\,;$$

i.e., t will differ from r only infinitely little, which is natural. On the contrary, if one multiplies by dX, dY, dZ in succession and sums then one will get:

$$\sum dx\,dX + (r - \sum X\,dx)\cdot \sum dX^2 = 0,$$

or, if one neglects higher-order infinitesimals:

$$r = -\frac{\sum dx\,dX}{\sum dX^2}\,;$$

i.e.:

(8) $$r = -\frac{e\,du^2 + (f + f')\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}\,.$$

§ 138.

Limit points and Hamilton's formula.

The formulas that were established lead to some noteworthy conclusions, which will serve as a convenient transformation of the curvilinear coordinates (u, v) when they are defined in the simplest way. For that, we initially exclude the case in which the two fundamental forms (4), (5) have proportional coefficients; i.e., in which one has the proportions:

$$E : F : G = e : \frac{f + f'}{2} : g\,.$$

One can then simultaneously make:

$$F = 0, \quad f + f' = 0$$

by a *well-defined* real transformation of the coordinates u, v.

Suppose that this transformation has been performed, so (8) becomes:

$$(8^*) \qquad r = -\frac{e\,du^2 + g\,dv^2}{E\,du^2 + G\,dv^2}.$$

If one denotes the value of r that corresponds to $dv = 0$ by r_1 and the one that corresponds to $du = 0$ by r_2 then one will have:

$$r_1 = -\frac{e}{E}, \qquad r_2 = -\frac{g}{G},$$

in which the case of $r_1 = r_2$ is still excluded by the hypotheses that were made. (8^*) will then be written:

$$(9) \qquad r = \frac{Er_1\,du^2 + Gr_2\,dv^2}{E\,du^2 + G\,dv^2},$$

and if supposes that, e.g., $r_2 > r_1$ then one will have:

$$r = r_1 + \frac{G(r_2 - r_1)dv^2}{E\,du^2 + G\,dv^2} = r_2 - \frac{E(r_2 - r_1)du^2}{E\,du^2 + G\,dv^2},$$

hence:

$$r_1 \leq r \leq r_2 .$$

Let L_1, L_2 be the feet of the minimum distances from the ray (u, v) to the two infinitely-close rays $(u + du, v)$, $(u, v + dv)$, respectively; their abscissas are r_1, r_2. From the preceding, the foot of the minimum distance from the ray (u, v) to any other infinitely-close ray $(u + du, v + dv)$ falls on the segment $L_1 L_2$; the extremes L_1, L_2 of that segment will then be called *limit points.*

If one denotes the values of $\cos a$, $\cos b$, $\cos c$ at the limit points L_1, L_2 by:

$$\begin{matrix} \cos a_1, & \cos b_1, & \cos c_1, \\ \cos a_2, & \cos b_2, & \cos c_2, \end{matrix}$$

respectively, then from (6), one will have:

$$\cos a_1 = \frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}, \qquad \cos b_1 = \frac{1}{\sqrt{G}}\frac{\partial Y}{\partial v}, \qquad \cos c_1 = \frac{1}{\sqrt{G}}\frac{\partial Z}{\partial v},$$

$$\cos a_2 = \frac{1}{\sqrt{G}}\frac{\partial X}{\partial u}, \qquad \cos b_2 = \frac{1}{\sqrt{G}}\frac{\partial Y}{\partial u}, \qquad \cos c_2 = \frac{1}{\sqrt{G}}\frac{\partial Z}{\partial u},$$

so:

$$\cos a_1 \cos a_2 + \cos b_1 \cos b_2 + \cos c_1 \cos c_2 = 0.$$

One then has the theorem:

The directions of the minimum distances from the ray (u, v) *to the two rays of the congruences for which the feet of the distance falls between the limit points* L_1, L_2 *are mutually orthogonal.*

We call the planes that go through the ray (u, v) normally to the ray of minimum distance the *principal planes* of the former ray; the preceding result can then be stated:

The two principal planes of any ray are mutually orthogonal.

One can now write (9) in another way by introducing the angle ω that the minimum distance dp from the ray (u, v) to the ray ($u + du$, $v + dv$) forms with dp_1 relative to the limit point L_1. In fact, one has:

$$\cos\omega = \sum \cos a \cos a_1 = \frac{\sqrt{E}\,du}{\sqrt{E\,du^2 + G\,dv^2}},$$

$$\cos^2\omega = \frac{E\,du^2}{E\,du^2 + G\,dv^2}, \qquad \sin^2\omega = \frac{G\,du^2}{E\,du^2 + G\,dv^2},$$

so (9) will give *Hamilton's formula*:

$$r = r_1 \cos^2\omega + r_2 \sin^2\omega. \tag{10}$$

§ 139.

Isotropic congruences.

We now examine the excluded case:

$$e : \frac{f + f'}{2} : g = E : F : G.$$

The considerations of the preceding number still remain applicable, with the difference that the transformations that are performed can now be accomplished in an infinitude of ways. It then results that $r_1 = r_2$, the limit points L_1, L_2 coincide at just one point on any ray, and the feet of all the minimum distances to the infinitely-close rays will fall upon that point. These singular congruences were considered for the first time by Ribaucour, who gave them the name of *isotropic congruences.* Their study is of great interest in regard to the relationship between these congruences and surfaces of minimal area, which we will establish shortly.

We make the following observations, here: An equation:

$$\varphi(u, v) = 0$$

between the coordinates u, v of a ray of any congruence represents a ruled surface, whose generators are rays of the congruence, or, as one says more briefly, a *ruling of the congruence.* For any ruling of an isotropic congruence, it is clear that the line of striction will coincide with the locus of limit points of its rays. However, for a general congruence, that will happen only for the two series of ruled surfaces:

$$u = \text{constant}, \quad v = \text{constant},$$

in which the variable u, v are the ones that were introduced in the preceding number. For any surface v = constant, the line of striction will be the locus of limit points L_1 on the corresponding rays, and similarly for any u = constant, it will be the locus of limit points L_2. The ruled surfaces of those two series are then called the *principal surfaces* of the congruence. For isotropic congruences (and only for them), any ruling of the congruence will be a principal surface.

If one chooses an orthogonal system on the sphere for the line (u, v) in an isotropic congruence, and takes the starting surface to be the locus of limit points (which is called the *middle surface* of the congruence) then one have:

$$r_1 = r_2 = 0,$$

so

$$e = 0, \qquad f + f' = 0, \qquad g = 0;$$

i.e., one will have:

$$dx\,dX + dy\,dY + dz\,dZ = 0,$$

identically.

If one then represents the middle surface S on the sphere as Gauss did, but directing the ray of the sphere parallel to the direction of the ray of the isotropic congruence, then the preceding formula will teach us that any line element of S will be perpendicular to the corresponding one on the sphere. One then has Ribaucour's theorem:

The middle surface of an isotropic congruence S corresponds to the elements of the sphere by orthogonality.

Conversely, one sees immediately that if a surface S corresponds to the elements of the sphere by orthogonality then if one draws the rays through the points of S that are parallel to the rays that go to the corresponding points of the sphere then one will describe an isotropic congruence.

Finally, observe that if one starts on the middle surface S then if one goes along any ray through a constant distance t then the coordinates of the extreme point will be:

$$\xi = x + t\,X, \qquad \xi = x + t\,X, \qquad \xi = x + t\,X,$$

and the line element of the surface that is the locus of extreme points will be given by:

$$d\xi^2 + d\eta^2 + d\zeta^2 = dx^2 + dy^2 + dz^2 + t^2\,(dX^2 + dY^2 + dZ^2),$$

and will not vary when one changes t into $-t$. The two surfaces S_1, S_2 that one forms by going a constant segment t on one side or the other can be mapped to each other, corresponding to the points on the same ray, and the distance between the two corresponding points is constant and equal to $2t$. Conversely, it is clear that:

If the distance between corresponding points of a pair of surfaces that can be mapped to each other is constant then connecting the corresponding points will define an isotropic congruence.

§ 140.

Abscissas of the limit points.

We now return to the general results of § 138, which were obtained by introducing a particular system of variables, i.e., the ones that gave the principal surfaces of the congruence when they were equated to constants. Now suppose that the variables u, v are arbitrary, so we wish to establish the fundamental formula that gives the abscissas r_1, r_2 of the limit points. The differential equation of the principal surfaces is obtained by equating the Jacobian of the two fundamental forms (4), (5) to zero; i.e., the determinant:

$$\begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv \\ e\,du + \dfrac{f+f'}{2}dv & \dfrac{f+f'}{2}du + g\,dv \end{vmatrix}.$$

That determinant can then be written:

$$\left\{\frac{f+f'}{2}E - eF\right\}du^2 + \{gE - eG\}\,du\,dv + \left\{gF - \frac{f+f'}{2}G\right\}dv^2 = 0. \tag{A}$$

For the values of dv / du that satisfy the latter equation, equation (8), namely:

$$r = -\frac{\left(e\,du + \dfrac{f+f'}{2}dv\right)du + \left(\dfrac{f+f'}{2}du + g\,dv\right)dv}{(E\,du + F\,dv)\,du + (F\,du + G\,dv)\,dv},$$

can be written as:

$$r = -\frac{e\,du + \dfrac{f+f'}{2}dv}{E\,du + F\,dv} = -\frac{\dfrac{f+f'}{2}du + g\,dv}{F\,du + G\,dv},$$

and one will then have:

$$\begin{cases} (Er+e)\,du + \left(Fr + \dfrac{f+f'}{2}\right)dv = 0, \\[2mm] \left(Fr + \dfrac{f+f'}{2}\right)du + (Gr+g)dv = 0. \end{cases}$$

If one eliminates the ratio $du : dv$ from this then one will obtain the following equation, which is of degree two in r:

$$(B) \qquad (EG - F^2)\, r^2 + \{g\,E - (f+f')\,F + e\,G\}r + eg - \left(\frac{f+f'}{2}\right)^2 = 0,$$

whose roots are the abscissas of the limit points.

§ 141.

Foci and developables of a congruence.

We now search among the ruled surfaces of the congruence for the ones that are developables. For such a surface:

$$(11) \qquad \varphi(u, v) = 0,$$

one must have $dp = 0$; i.e., from (7):

$$\begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv \\ e\,du + f\,dv & f'\,du + g\,dv \end{vmatrix} = 0,$$

or, upon developing this:

$$(C) \qquad (f'E - e\,F)\,du^2 + \{g\,E + (f' - f)\,F - e\,G\}\,du\,dv + (g\,F - f\,G)\,dv^2 = 0.$$

Therefore:

The rays of a congruence can be associated with two series (real or imaginary) of developable surfaces.

One can arrive at the same differential equation (C) for the developables of the congruence in the following way, which will provide another important element, as well. Suppose that (11) is the equation for a developable of the congruence, let ρ denote the abscissa of the point F where the ray (u, v) meets the edge of regression of (11); the coordinates of F will be:

$$x_1 = x + \rho X, \quad y_1 = y + \rho Y, \quad z_1 = z + \rho Z.$$

If one differentiates these formulas, while u, v are still related by (11), then since dx_1, dy_1, dz_1 are proportional to X, Y, Z, resp., by hypothesis, one will have that:

$$dx + \rho\, dX = \lambda\, X, \qquad dy + \rho\, dY = \lambda\, Y, \qquad dz + \rho\, dZ = \lambda\, Z,$$

in which λ is an (infinitesimal) proportionality factor. If one multiplies these three equations, first by $\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}$, and secondly by $\frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}$, and then sums them then one will get:

$$e\, du + f\, dv + \rho\,(E\, du + F\, dv) = 0,$$
$$f'\, du + g\, dv + \rho\,(F\, du + G\, dv) = 0.$$

If one eliminates ρ then one will get the differential equation (C) of the developables of the congruence precisely. If one eliminates the ratio $du : dv$ instead then one will get the following second-degree equation for ρ:

$$(D) \qquad (EG - F^2)\,\rho^2 + \{g\,E - (f + f')\,F + e\,G\}\,\rho + e\,g - f\,f' = 0.$$

Its roots ρ_1, ρ_2 are obviously the abscissas of the two points F_1, F_2 along the ray (u, v) that touch the edge of regression of one or the other developable of the two series that pass through that ray. Those two points are called *foci* of the ray (u, v), and they can also be considered to be the two points at which the ray (u, v) is met by the two infinitely close rays that belong to one or the other of the developables ([1]). They will be real or imaginary according to whether the developables of the congruence are real or imaginary.

It results from a comparison with (B), (D) that:

$$\rho_1 + \rho_2 = r_1 + r_2,$$

so:

The midpoint of the limit points coincides with the midpoint of the foci.

That point is then called the *midpoint* of the ray, and the surface that is the locus of midpoints is called the *middle surface.* It then results from (B), (D) that one also has:

$$\rho_1\, \rho_2 = r_1\, r_2 + \frac{(f - f')^2}{4(EG - F^2)},$$

so

$$(r_1 - r_2)^2 - (\rho_1 - \rho_2)^2 = \frac{(f - f')^2}{EG - F^2}.$$

If one then lets $2d$ denote the distance between the limit points and lets 2δ denote the distance between the foci then one will have:

([1]) That contact is meaningful only up to higher-order infinitesimals; i.e., dp is an infinitesimal of order higher than one in F_1, F_2 .

(12) $$d^2 - \delta^2 = \frac{(f - f')^2}{4(EG - F^2)}.$$

When the two foci are real, they will lie along the line segment of the limit points, as would also follow from § 138.

For simplicity, take the starting surface to be the middle surface. Then take:

$$r_1 = d, \qquad r_2 = -d,$$

in Hamilton's formula (10) (page 6), so it can be written:

$$r = d \cos 2\omega,$$

and one will see that when the foot of the minimum distance from the ray (u, v) to an infinitely-close ray traverses the line segment of the limit points from $+ d$ to $- d$, the angle ω will increase from 0 to $\pi / 2$ and assume the value $\pi / 4$ at the midpoint of the ray. Denote its values at the foci by ω_1, ω_2, so:

$$\rho_1 = \delta, \quad \rho_2 = -\delta,$$

and one will have:

$$\cos 2\omega_1 = \frac{\delta}{d}, \qquad \cos 2\omega_2 = -\frac{\delta}{d},$$

and therefore:

$$\omega_1 + \omega_2 = \frac{\pi}{2}.$$

One calls the planes that go through the ray and two infinitely-close rays that meet it the *focal planes.* It then results that: *The focal planes have the same bisecting planes as the principal planes.*

If one lets:

$$g = \omega_2 - \omega_1 = \frac{\pi}{2} - 2\omega_1$$

denote the angle between the two focal planes then, from the preceding, one will have the formulas:

(13) $$\sin \gamma = \frac{\delta}{d}, \qquad \cos \gamma = \frac{\sqrt{d^2 - \delta^2}}{d}.$$

§ 142.

Focal surfaces.

We must consider five surfaces that relate to a given congruence, namely, the *middle surface*, which is the locus of midpoints, the two *limit surfaces*, which are the loci of limit points, and finally, the two *focal surfaces*, which are loci of the foci ([1]). The first three are all real, while the last two are real only for congruences that have real developables. The congruence is then composed of the tangents that are common to the two sheets S_1, S_2 of the focal surface. The rays of the congruence envelop a system of ∞^1 curves on S_1 that are edges of regression Γ_1 of the developable of one of the two systems, and analogously for S_2. One will see immediately that the osculating plane to the curve Γ_1 (which passes through it) at F_1 is also the tangent plane to S_2 at F_2. The two series of developables of the congruence cut each of the focal surfaces along a conjugate system.

Can focal surfaces coincide? In that case, the enveloping line of the focal surfaces of the rays of the congruence will coincide with that of the conjugate system; i.e., it will be the asymptote of one system. Moreover, one easily proves that the distance $2d$ between the limit points is then given by:

$$2d = \frac{1}{\sqrt{-K}},$$

in which K is the curvature of the focal surface.

Indeed, take the starting surface to be the focal surface, the coordinate line to be the asymptotic line v, its the orthogonal trajectory to be u, and let:

$$ds^2 = E'\,du^2 + G'\,dv^2$$

be the linear element of the surface. We will have:

$$D = 0, \quad \frac{D'^2}{E'G'} = -K$$

for the coefficients of the second fundamental form.

Set:

$$X_1 = \frac{1}{\sqrt{E'}}\frac{\partial x}{\partial u}, \qquad Y_1 = \frac{1}{\sqrt{E'}}\frac{\partial y}{\partial u}, \qquad Z_1 = \frac{1}{\sqrt{E'}}\frac{\partial z}{\partial u},$$

$$X_2 = \frac{1}{\sqrt{G'}}\frac{\partial x}{\partial v}, \qquad Y_2 = \frac{1}{\sqrt{G'}}\frac{\partial y}{\partial v}, \qquad Z_2 = \frac{1}{\sqrt{G'}}\frac{\partial z}{\partial v},$$

and from the fundamental formulas (I), (II), pages 116-117, we deduce that:

([1]) In many studies, it proves to be useful to consider a sixth surface that Ribaucour called the *middle envelope*, which is the envelope of the normal planes to the rays at their midpoints (viz., the middle planes).

$$\frac{\partial X_1}{\partial u} = -\frac{1}{\sqrt{G'}}\frac{\partial\sqrt{E'}}{\partial v}X_2\,, \qquad \frac{\partial X_1}{\partial u} = \frac{1}{\sqrt{E'}}\frac{\partial\sqrt{G'}}{\partial u}X_2 + \frac{D'}{\sqrt{E'}}X\,. \text{ (?)}$$

Since X_1, Y_1, Z_1 are precisely the direction cosines of the ray (u, v) of the congruence, one will find that the fundamental quantities (2), (3) on page 2 are:

$$E = \left(\frac{1}{\sqrt{G'}}\frac{\partial\sqrt{E'}}{\partial v}\right)^2, \quad F = -\frac{1}{\sqrt{E'G'}}\frac{\partial\sqrt{E'}}{\partial v}\frac{\partial\sqrt{G'}}{\partial u}, \quad G = \left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{G'}}{\partial u}\right)^2 + \frac{D'^2}{E'},$$

$$e = 0, \quad f = \frac{\partial\sqrt{E'}}{\partial v}, \qquad f' = 0, \quad g = \sqrt{\frac{G'}{E'}}\frac{\partial\sqrt{G'}}{\partial u},$$

so

$$EG - F^2 = \frac{D'^2}{E'G'}\cdot\left(\frac{\partial\sqrt{E'}}{\partial v}\right)^2,$$

$$eg - (f + f')^2 = -\frac{1}{4}\cdot\left(\frac{\partial\sqrt{E'}}{\partial v}\right)^2.$$

From (B), since the middle term is zero, that will give:

$$\frac{1}{4r^2} = \frac{D'^2}{E'G'} = -K. \qquad \text{Q. E. D.}$$

§ 143.

Normal congruences.

A system of rays will be called a *normal* system or congruence if there exists a surface that is normal to all of the rays, and therefore (§ 133) a series of such surfaces.

If a congruence is normal then it must be possible to assume in (1), page 1, that t is a function of u, v such that the surface that is the locus of points (ξ, η, ζ) is normal to the rays. The differentials $d\xi$, $d\eta$, $d\zeta$ must then satisfy the condition:

$$X\,d\xi + Y\,d\eta + Z\,d\zeta = 0.$$

Now, one has:

$$d\xi = dx + dt\cdot X + t\cdot dX, \qquad d\eta = dy + dt\cdot Y + t\cdot dY, \qquad d\zeta = dz + dt\cdot Z + t\cdot dZ,$$

and therefore the required condition will become:

$$dt + \sum X\,dx = 0\,.$$

If one sets:

$$(14) \qquad U = \sum X \frac{\partial x}{\partial u}, \qquad V = \sum X \frac{\partial x}{\partial v}$$

then one will get the relation:

$$dt = -(U\,du + V\,dv)$$

for the determination of t, so the required condition will translate into the equation:

$$(15) \qquad \frac{\partial U}{\partial v} = \frac{\partial V}{\partial u},$$

which can also be written, from (3):

$$(15^*) \qquad f = f'.$$

Suppose that (15) or (15*) is satisfied, so there will exist a series of (parallel) surfaces that are orthogonal to the congruence, which are given by the formula:

$$(16) \qquad t = C - \int (U\,du + V\,dv).$$

Since $f = f'$, one will have:

$$\delta = d, \quad \gamma = \frac{\pi}{2},$$

and conversely from one or the other of these, it will ultimately follow that $f = f'$. Therefore:

The necessary and sufficient condition for a congruence to be normal is that the foci must coincide with the limit points, or that the focal planes must be mutually perpendicular ([1]).

The two focal surfaces for a congruence obviously coincide with the two sheets of the evolute of the surfaces that are orthogonal to the rays.

§ 144.

Malus-Dupin theorem.

Put (15) into another form, and introduce the angles α, β that the ray (u, v) makes with the coordinate lines v, u of the starting surface S. If:

$$ds^2 = E'du^2 + 2F'du\,dv + G'dv^2$$

([1]) This theorem will also result immediately from the geometric considerations of § 142.

is the line element of that surface then we will have:

$$\cos\alpha = \sum X \frac{1}{\sqrt{E'}}\frac{\partial x}{\partial u} = \frac{U}{\sqrt{E'}}, \qquad \cos\beta = \sum X \frac{1}{\sqrt{G'}}\frac{\partial x}{\partial v} = \frac{V}{\sqrt{G'}},$$

so (15) can be written:

(17) $$\frac{\partial(\sqrt{E'}\cos\alpha)}{\partial v} = \frac{\partial(\sqrt{G'}\cos\beta)}{\partial u},$$

and if we suppose that this is satisfied then (16) will become:

(18) $$t = C - \int(\sqrt{E'}\cos\alpha\, du + \sqrt{G'}\cos\beta\, dv).$$

Only the angles α, β and the coefficients of the line element of the starting surface figure in these formulas. Beltrami has deduced the following interesting consequences: Suppose that (17) is satisfied, and imagine that *S* is deformed in such a way that it carries along the system of rays that is invariably linked with the surfaces without changing the angles α, β. (17) will always remain satisfied, and the values (18) of *t* will not vary under the deformation. One then has Beltrami's theorem:

If the rays of a normal congruence that emanate from the points of a surface S are imagined to terminate on one of the orthogonal surfaces Σ *then any deformation by flexion of S that carries along the rays of the congruence that are invariably connected with the surface that is the locus of their extremes will be an orthogonal surface to the rays* ([1]).

In addition, one easily deduces the Malus-Dupin theorem from formula (17):

If a normal congruence of light rays is subjected to an arbitrary number of reflections or refractions then it will always remain a normal congruence.

Take the starting surface *S* to be the reflecting or refracting surface, take the coordinate lines *u* on *S* to be the enveloping lines of the orthogonal projections of the rays onto the tangent planes to *S*, and take the lines *v* to be the orthogonal trajectories. One will have:

$$\alpha = , \quad \beta = \frac{\pi}{2} - \gamma,$$

in which γ is the angle between the ray and the normal to *S*. (17) will then become:

([1]) One can observe that since only E', G' appear in (17) and (18), one can also assume that the flexible surface *S* is only *partially inextensible*; i.e., that Beltrami's theorem can persist along the coordinate lines *u*, *v*, and also under those deformations, more generally.

$$\frac{\partial(\sqrt{G'}\sin\gamma)}{\partial u} = 0,$$

and if it is satisfied then it will continue to be true if one changes γ into γ' by means of the condition:

$$\sin \gamma' = n \sin \gamma' \qquad (n \text{ constant}),$$

which proves the theorem.

§ 145.

Congruences that have assigned spherical images of the principal surfaces.

We now return to the general congruences in order to treat two problems in sequence that can be considered to be generalizations of the problem of finding a surface whose lines of curvature have assigned spherical images, i.e., of finding a normal congruence whose developables (§ 83) have assigned spherical images. The developables of a congruence will coincide with the principal surfaces when it is a normal congruence, while the two systems are distinct in the case of a general congruence. One then agrees that one must address two problems in turn:

1. *Determine the congruence when the spherical images of its principal surfaces are assigned.*

2. *Determine the congruence when the spherical images of its developables are assigned.*

In this number, we shall address the first problem, which always has a real significance, regardless of whether the developables are real or imaginary.

The spherical system (u, v) that is the image of the principal surfaces, must be an orthogonal system (§ 138), and one lets:

$$ds'^2 = E\,du^2 + G\,dv^2$$

be the line element of the spherical representation. Take the starting surface to be the middle surface, so the unknowns in our problem will be the coordinates x, y, z of the midpoint of the ray (u, v). By hypothesis, one must have:

$$F = 0, \qquad f + f' = 0, \qquad e\,G + g\,E = 0,$$

and if one denotes the distance between the limit points by $2r$ then one will have:

$$(19) \qquad \sum \frac{\partial x}{\partial u}\frac{\partial X}{\partial u} = r\,E, \qquad \sum \frac{\partial x}{\partial v}\frac{\partial X}{\partial v} = -\,r\,G, \qquad \sum \frac{\partial x}{\partial v}\frac{\partial X}{\partial u} + \sum \frac{\partial x}{\partial u}\frac{\partial X}{\partial v} = 0.$$

Upon introducing a new unknown function φ, take:

$$(20) \qquad f = \sum \frac{\partial x}{\partial v}\frac{\partial X}{\partial u} = \varphi\sqrt{EG}, \qquad f' = \sum \frac{\partial x}{\partial u}\frac{\partial X}{\partial v} = -\varphi\sqrt{EG}.$$

The geometrical significance of φ results immediately from (D) (page 10) since if 2ρ denotes the distance between the foci then one will have:

$$(21) \qquad \varphi^2 = r^2 - \rho^2.$$

Calculate $\frac{\partial(\varphi\sqrt{EG})}{\partial u}$ from the first of (20), and observe that from the fundamental formulas of chap. V (page 152), one will have:

$$\frac{\partial^2 X}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}\frac{\partial X}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\partial X}{\partial v} - E\,X,$$

and in addition, from the first of (19):

$$\sum \frac{\partial^2 x}{\partial u\,\partial v}\frac{\partial X}{\partial u} = \frac{\partial(r\,E)}{\partial v} - \sum \frac{\partial x}{\partial u}\left(\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}\frac{\partial X}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\frac{\partial X}{\partial v}\right).$$

If one observes (19) and (20) in this then it will result that:

$$\frac{\partial(\varphi\sqrt{EG})}{\partial u} = \varphi\sqrt{EG}\left(\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\right) - r\left(E\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + G\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\right) + \frac{\partial(r\,E)}{\partial v} - E\sum X\frac{\partial x}{\partial v}.$$

Now, one will have:

$$E\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + G\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} = 0, \qquad \frac{\partial \log\sqrt{EG}}{\partial u} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}$$

in this case, so:

$$(a) \qquad \sum X\frac{\partial x}{\partial v} = \frac{1}{E}\frac{\partial(r\,E)}{\partial v} - \sqrt{\frac{G}{E}}\frac{\partial\varphi}{\partial u}.$$

Similarly, if one differentiates the second of (20) with respect to v then one will find that:

$$(b) \qquad \sum X\frac{\partial x}{\partial u} = -\frac{1}{G}\frac{\partial(r\,G)}{\partial u} + \sqrt{\frac{E}{G}}\frac{\partial\varphi}{\partial v}.$$

It is enough to associate (*a*), (*b*) with (19), (20), and if solves them for $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$; $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$ then one will get:

$$(22)\qquad \begin{cases} \dfrac{\partial x}{\partial u} = \quad r\dfrac{\partial X}{\partial u} - \sqrt{\dfrac{E}{G}}\varphi\dfrac{\partial X}{\partial u} + \left\{\sqrt{\dfrac{E}{G}}\dfrac{\partial \varphi}{\partial v} - \dfrac{1}{G}\dfrac{\partial (r\,G)}{\partial u}\right\}X, \\ \dfrac{\partial x}{\partial v} = -r\dfrac{\partial X}{\partial v} + \sqrt{\dfrac{G}{E}}\varphi\dfrac{\partial X}{\partial u} + \left\{\dfrac{1}{E}\dfrac{\partial (r\,E)}{\partial v} - \sqrt{\dfrac{G}{E}}\dfrac{\partial \varphi}{\partial u}\right\}X, \end{cases}$$

with analogous expressions for *y*, *z*.

Conversely, if *r*, φ are two functions of *u*, *v* such that the integrability conditions for (22) are satisfied then they will define a congruence by quadratures that has assigned spherical images for its principal surfaces. Now, if one actually calculates the integrability conditions for (22), while taking into account the fundamental equations that give the second derivatives of *X*, *Y*, *Z* (§ 72, page 152), then one will find that they reduce to the single condition between *r* and φ:

$$(23)\qquad 2\frac{\partial^2 r}{\partial u\,\partial v} + \frac{\partial r}{\partial u}\frac{\partial \log E}{\partial v} + \frac{\partial r}{\partial v}\frac{\partial \log G}{\partial u} + r\frac{\partial^2 \log(EG)}{\partial u\,\partial v} = \sqrt{EG}\;(\Delta_2\varphi + 2\varphi),$$

in which $\Delta_2\varphi$ is the second parametric differential of φ:

$$\Delta_2\varphi = \frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\sqrt{\frac{G}{E}}\frac{\partial \varphi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\frac{\partial \varphi}{\partial v}\right)\right\}.$$

One then sees that the problem that was posed will admit considerable arbitrariness in its solutions, since one can take *r* and φ arbitrarily and successively determine φ or *r* from the partial differential equations (23).

In particular, if the congruence is normal then one will have $\varphi = 0$ and the equation for *r* will become:

$$(24)\qquad \frac{\partial^2 r}{\partial u\,\partial v} + \frac{\partial \log\sqrt{E}}{\partial v}\frac{\partial r}{\partial u} + \frac{\partial \log\sqrt{G}}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial^2 \log\sqrt{EG}}{\partial u\,\partial v}r = 0,$$

which is precisely the *adjoint* equation ([1]) to the other one:

$$(25)\qquad \frac{\partial^2 W}{\partial u\,\partial v} - \frac{\partial \log\sqrt{E}}{\partial v}\frac{\partial W}{\partial u} - \frac{\partial \log\sqrt{G}}{\partial u}\frac{\partial W}{\partial v} = 0,$$

([1]) See Darboux, t. II, pp. 71, *et seq.*

which we saw in § 83 depends upon the same problem. One notes that the integration of equation (24) and that of its adjoint (25) are analytically-equivalent problems.

§ 146.

Middle envelope of an isotropic congruence.

Here, we confine ourselves to applying (23) to the case of an isotropic congruence, for which one has $r = 0$. From (23), the search for isotropic congruences will depend upon the equation:

$$\Delta_2\varphi + 2\varphi = 0,$$

which, from Weingarten's formulas that relate to the tangential coordinates (cf., § 81), one can also interpret as the tangential equations of a surface of minimal area.

It was precisely in the theory of isotropic congruences that Ribaucour proposed the following fundamental theorem to relate them to minimal surfaces:

The middle envelope ([1]) *of an isotropic congruence is a surface of minimal area.*

That theorem follows easily from our general formula (22), in which one can assume that the orthogonal lines (u, v) on the representative sphere are arbitrary when one is dealing with an isotropic congruence, for which the principal surfaces are indeterminate, and we suppose that they are isothermal by taking:

$$E = G = \lambda, \qquad r = 0.$$

(22) will then become:

$$\begin{cases} \dfrac{\partial x}{\partial u} = X\dfrac{\partial \varphi}{\partial v} - \varphi\dfrac{\partial X}{\partial v}, \\[2ex] \dfrac{\partial x}{\partial v} = -X\dfrac{\partial \varphi}{\partial u} + \varphi\dfrac{\partial X}{\partial u}, \end{cases}$$

and if one denotes the distance from the middle plane to the origin by W then one will have:

$$W = \sum X\,x,$$

so:

$$\begin{cases} \dfrac{\partial W}{\partial u} = \dfrac{\partial \varphi}{\partial v} + \sum x\dfrac{\partial X}{\partial u}, \\[2ex] \dfrac{\partial W}{\partial v} = -\dfrac{\partial \varphi}{\partial u} + \sum x\dfrac{\partial X}{\partial v}. \end{cases}$$

It then follows that:

([1]) Cf., the footnote in § 142 (page 12).

$$\frac{\partial^2 W}{\partial u^2}+\frac{\partial^2 W}{\partial v^2} = \sum x\left(\frac{\partial^2 X}{\partial u^2}+\frac{\partial^2 X}{\partial v^2}\right) = -2\lambda\sum x\,X\,;$$

i.e.:

$$\frac{\partial^2 W}{\partial u^2}+\frac{\partial^2 W}{\partial v^2}+2\lambda\,W=0,$$

which proves Ribaucour's theorem.

§ 147.

Congruences with assigned spherical images for their developables.

We now go on to the second question that was posed in § 145, which involves much less arbitrariness in its solution, as we have seen. The important results that we shall establish are due to Guichard, who proved them in the following way ([1]):

Let:

$$ds'^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

be the assigned spherical line element, in which (u, v) are the images of the developables of the congruences. Also take the starting surface to be the middle surface of the congruence, while assuming that the unknowns are the coordinates x, y, z of the midpoint of the ray. If one denotes the distance between the foci by 2ρ then:

$$x+\rho X, \qquad y+\rho Y, \qquad z+\rho Z$$

will be coordinates of one focus, while:

$$x-\rho X, \qquad y-\rho Y, \qquad z-\rho Z$$

will be those of the other. Suppose that the former corresponds to the line v = constant, while the latter corresponds to the line u = constant. One must then have:

$$\frac{\partial(x+\rho X)}{\partial u}=h\,X, \qquad \frac{\partial(y+\rho Y)}{\partial u}=h\,Y, \qquad \frac{\partial(z+\rho Z)}{\partial u}=h\,Z,$$

$$\frac{\partial(x-\rho X)}{\partial v}=l\,X, \qquad \frac{\partial(y-\rho Y)}{\partial v}=l\,Y, \qquad \frac{\partial(z-\rho Z)}{\partial v}=l\,Z,$$

in which h, l are convenient proportionality factors. If one writes these equations as:

([1]) "Surfaces rapportées à leurs lignes asymptotiques et congruences rapportées à leurs développables," Annales scientifiques de l'École Normale Supérieure, t. VI, 3rd series.

(26)
$$\begin{cases} \dfrac{\partial x}{\partial u} = \left(h - \dfrac{\partial \rho}{\partial u}\right) X - \rho \dfrac{\partial X}{\partial u}, \\[2ex] \dfrac{\partial x}{\partial v} = \left(l + \dfrac{\partial \rho}{\partial u}\right) X + \rho \dfrac{\partial X}{\partial v}, \end{cases}$$

with analogous expressions in y, z, and forms the integrability conditions from them:

$$\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u}\right) - \frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\right) = 0,$$

while observing that:

$$\frac{\partial^2 X}{\partial u \, \partial v} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial X}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial X}{\partial v} - F\, X,$$

then one will find that:

(α)
$$\frac{\partial h}{\partial v} - \frac{\partial l}{\partial u} - 2 \frac{\partial^2 \rho}{\partial u \, \partial v} + 2\rho F = 0,$$

(β)
$$\begin{cases} l = -2\left[\dfrac{\partial \rho}{\partial v} + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \rho\right], \\[2ex] h = 2\left[\dfrac{\partial \rho}{\partial v} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \rho\right]. \end{cases}$$

(26) will then become:

(27)
$$\begin{cases} \dfrac{\partial x}{\partial u} = \left[\dfrac{\partial \rho}{\partial u} + 2\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \rho\right] X - \rho \dfrac{\partial X}{\partial u}, \\[2ex] \dfrac{\partial x}{\partial v} = -\left[\dfrac{\partial \rho}{\partial v} + 2\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \rho\right] X + \rho \dfrac{\partial X}{\partial v}, \end{cases}$$

and when one substitutes the values of l, h from (β) in (α), that will give the following equation in ρ:

(28)
$$\frac{\partial^2 \rho}{\partial u \, \partial v} + \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial \rho}{\partial u} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial \rho}{\partial v} + \left[\frac{\partial}{\partial u}\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \frac{\partial}{\partial v}\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + F\right] \rho = 0.$$

Conversely, if ρ is a solution of that equation then (27) will give a corresponding congruence by quadrature that has assigned images for its developables.

Observe that the Laplace equation (28), upon which the problem depends, is the adjoint of the other one:

$$\frac{\partial^2 W}{\partial u \, \partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial W}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial W}{\partial v} + F\, W = 0,$$

which we saw in § 82 depends upon the problem of finding the surface that the system (u, v) for the spherical image of a conjugate system. The two problems are then equivalent.

§ 148.

General formulas that relate to spheres.

We would like to express the elements that relate to the two sheets of the focal surface, so we agree to establish a system of formulas that will be useful later on in other research.

Consider the tri-rectangular trihedron at any point (u, v) of the sphere that is defined by the normal to the sphere and the bisecting directions of the coordinate lines (u, v). The cosines of the last two directions will be denoted by:

$$\begin{array}{lll} X_1, & Y_1, & Z_1, \\ X_2, & Y_2, & Z_2, \end{array}$$

respectively, and if one let Ω denote the angle between the spherical lines (u, v), which is defined by the formulas:

$$\cos\Omega = \frac{F}{\sqrt{EG}}, \qquad \sin\Omega = \frac{\sqrt{EG - F^2}}{\sqrt{EG}},$$

then one will find immediately that:

$$(29) \qquad \left\{ \begin{aligned} X_1 &= \frac{1}{2\sin\frac{\Omega}{2}} \left\{ \frac{1}{\sqrt{E}}\frac{\partial X}{\partial u} - \frac{1}{\sqrt{G}}\frac{\partial X}{\partial v} \right\}, \\ X_2 &= \frac{1}{2\cos\frac{\Omega}{2}} \left\{ \frac{1}{\sqrt{E}}\frac{\partial X}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial X}{\partial v} \right\}, \end{aligned} \right.$$

with analogous formulas for Y, Z.

The formulas that must be established express the partial derivatives of the nine direction cosines:

$$\begin{array}{lll} X, & X_1, & X_2, \\ Y, & Y_1, & Y_2, \\ Z, & Z_1, & Z_2 \end{array}$$

linearly in terms of the cosines themselves and the coefficients of the spherical line element.

Meanwhile, from (29), one has:

$$\begin{cases} \dfrac{\partial X}{\partial u} = \sqrt{E}\sin\dfrac{\Omega}{2}X_1 + \sqrt{E}\cos\dfrac{\Omega}{2}X_2, \\ \dfrac{\partial X}{\partial v} = -\sqrt{G}\sin\dfrac{\Omega}{2}X_1 + \sqrt{G}\cos\dfrac{\Omega}{2}X_2, \end{cases}$$

and therefore:

$$\begin{cases} \sum X\dfrac{\partial X_1}{\partial u} = -\sum X_1\dfrac{\partial X}{\partial u} = -\sqrt{E}\sin\dfrac{\Omega}{2}, \\ \sum X\dfrac{\partial X_1}{\partial v} = -\sum X_1\dfrac{\partial X}{\partial v} = \sqrt{G}\sin\dfrac{\Omega}{2}, \\ \sum X\dfrac{\partial X_2}{\partial u} = -\sum X_2\dfrac{\partial X}{\partial u} = -\sqrt{E}\cos\dfrac{\Omega}{2}, \\ \sum X\dfrac{\partial X_2}{\partial v} = -\sum X_2\dfrac{\partial X}{\partial v} = -\sqrt{G}\cos\dfrac{\Omega}{2}. \end{cases}$$

Now, calculate the two sums:

$$\sum X_2\frac{\partial X_1}{\partial u} = -\sum X_1\frac{\partial X_2}{\partial u},$$

$$\sum X_2\frac{\partial X_1}{\partial v} = -\sum X_1\frac{\partial X_2}{\partial v}.$$

From (29), one has:

$$\sum X_2\frac{\partial X_1}{\partial u} = \frac{1}{2\sin\Omega}\sum\left\{\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial X}{\partial u}\right) - \frac{1}{\sqrt{E}}\frac{\partial X}{\partial u}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}\right)\right\},$$

and since:

$$\cos\Omega = \sum\frac{1}{\sqrt{E}}\frac{\partial X}{\partial u}\cdot\sum\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v},$$

differentiating with respect to u will yield:

$$\sum\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial X}{\partial u}\right) = -\sin\Omega\frac{\partial\Omega}{\partial u} - \sum\frac{1}{\sqrt{E}}\frac{\partial X}{\partial u}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}\right),$$

such that the preceding can be written:

$$\sum X_2\frac{\partial X_1}{\partial u} = -\frac{1}{2\sin\Omega}\left[\sin\Omega\frac{\partial\Omega}{\partial u} + \frac{2}{\sqrt{E}}\sum\frac{\partial X}{\partial u}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial X}{\partial v}\right)\right].$$

If one develops this, while observing the formula:

$$\frac{\partial^2 X}{\partial u\,\partial v} = \begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial X}{\partial u} + \begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial X}{\partial v} - F\,X,$$

then it will result that:

$$\sum X_2 \frac{\partial X_1}{\partial u} = -\frac{1}{2\sin\Omega}\left[\sin\Omega\frac{\partial\Omega}{\partial u} + \frac{2}{\sqrt{EG}}\left\{E\begin{Bmatrix}12\\1\end{Bmatrix} + F\begin{Bmatrix}12\\2\end{Bmatrix} - \frac{F}{2G}\frac{\partial G}{\partial u}\right\}\right].$$

Now, one has:

$$\frac{\partial G}{\partial u} = 2\begin{bmatrix}12\\2\end{bmatrix} = 2F\begin{Bmatrix}12\\1\end{Bmatrix} + 2G\begin{Bmatrix}12\\2\end{Bmatrix},$$

so

$$E\begin{Bmatrix}12\\1\end{Bmatrix} + F\begin{Bmatrix}12\\2\end{Bmatrix} - \frac{F}{2G}\frac{\partial G}{\partial u} = \frac{EG-F^2}{G}\begin{Bmatrix}12\\1\end{Bmatrix} = E\sin^2\Omega\begin{Bmatrix}12\\1\end{Bmatrix},$$

from which, one will get:

$$\sum X_2 \frac{\partial X_1}{\partial u} = -\sum X_1 \frac{\partial X_2}{\partial u} = -\frac{1}{2}\frac{\partial\Omega}{\partial u} - \sqrt{\frac{E}{G}}\begin{Bmatrix}12\\1\end{Bmatrix}\sin\Omega.$$

Similarly:

$$\sum X_2 \frac{\partial X_1}{\partial v} = -\sum X_1 \frac{\partial X_2}{\partial v} = \frac{1}{2}\frac{\partial\Omega}{\partial v} + \sqrt{\frac{G}{E}}\begin{Bmatrix}12\\2\end{Bmatrix}\sin\Omega.$$

These two formulas, which are associated with (*a*) and the identities:

$$\sum X_1 \frac{\partial X_1}{\partial u} = 0, \qquad \sum X_1 \frac{\partial X_1}{\partial v} = 0, \qquad \text{etc.},$$

immediately give the group of requested formulas:

$$(30)\quad \left\{\begin{aligned} \frac{\partial X}{\partial u} &= \sqrt{E}\sin\frac{\Omega}{2}X_1 + \sqrt{E}\cos\frac{\Omega}{2}X_2, & \frac{\partial X}{\partial v} &= -\sqrt{G}\sin\frac{\Omega}{2}X_1 + \sqrt{G}\cos\frac{\Omega}{2}X_2, \\ \frac{\partial X_1}{\partial u} &= -AX_2 - \sqrt{E}\sin\frac{\Omega}{2}X, & \frac{\partial X_1}{\partial v} &= BX_2 + \sqrt{G}\sin\frac{\Omega}{2}X, \\ \frac{\partial X_2}{\partial u} &= AX_1 - \sqrt{E}\cos\frac{\Omega}{2}X, & \frac{\partial X_2}{\partial v} &= -BX_1 - \sqrt{G}\cos\frac{\Omega}{2}X, \end{aligned}\right.$$

in which one sets:

$$(31)\qquad A = \frac{1}{2}\frac{\partial\Omega}{\partial u} + \sqrt{\frac{E}{G}}\begin{Bmatrix}12\\1\end{Bmatrix}\sin\Omega, \qquad B = \frac{1}{2}\frac{\partial\Omega}{\partial v} + \sqrt{\frac{G}{E}}\begin{Bmatrix}12\\2\end{Bmatrix}\sin\Omega,$$

to abbreviate.

One should note that from the formula that was developed in § 86 (page. 184), one can also express A, B for the geodetic curvature $1 / \rho_u$, $1 / \rho_v$ of the coordinate lines in the following way:

$$A = -\frac{\sqrt{E}}{\rho_v} - \frac{1}{2}\frac{\partial\Omega}{\partial u}, \qquad B = -\frac{\sqrt{G}}{\rho_u} - \frac{1}{2}\frac{\partial\Omega}{\partial v}. \tag{31*}$$

When one is given the spherical line element, formulas (30) will give the system of total differential equations for X, X_1, X_2 that was mentioned already in § 58, which is infinitely integrable; its integration depends upon a Ricatti equation.

§ 149.

Elements of the two sheets of a focal surface.

We now return to Guichard's problem and formulas, in which the angle Ω between the spherical lines (u, v) presently represents the angle between the focal planes, as well ([1]). (27) can be written:

$$(32) \quad \left\{ \begin{aligned} \frac{\partial x}{\partial u} &= \left[\frac{\partial \rho}{\partial u} + 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\rho\right] X - \sqrt{E}\sin\frac{\Omega}{2}\rho X_1 - \sqrt{E}\cos\frac{\Omega}{2}\rho X_2, \\ \frac{\partial x}{\partial v} &= -\left[\frac{\partial \rho}{\partial v} + 2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\rho\right] X - \sqrt{G}\sin\frac{\Omega}{2}\rho X_1 + \sqrt{G}\cos\frac{\Omega}{2}\rho X_2. \end{aligned}\right.$$

Let S_1, S_2 denote two focal surfaces, and let x_1, y_1, z_1; x_2, y_2, z_2 denote the coordinates of their foci F_1, F_2, respectively, such that one has:

$$x_1 = x + \rho X, \quad y_1 = y + \rho Y, \quad z_1 = z + \rho Z,$$
$$x_2 = x - \rho X, \quad y_2 = y - \rho Y, \quad z_2 = z - \rho Z;$$

let:

$$E_1, F_1, G_1; \qquad D_1, D_1', D_1'',$$
$$E_2, F_2, G_2; \qquad D_2, D_2', D_2'',$$

([1]) The spherical lines u, v are the indicatrices of the tangents to the edges of regression of the developables u, v of the congruence, from which our assertion will result immediately. Analytically, one can arrive at the same result by observing that one has:

$$e = -\rho E, \quad f = \rho F, \quad f = -\rho F, \quad g = \rho G,$$

so (B), page 9, will give:

$$\frac{\rho^2}{r^2} = \frac{EG - F^2}{EG} = \sin^2\Omega.$$

denote the coefficients of the two fundamental forms of S_1, S_2, respectively. One finds from (30), (31) that:

$$\frac{\partial x_1}{\partial u} = 2\left[\frac{\partial \rho}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\right]X, \quad \frac{\partial x_1}{\partial v} = 2\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho X - 2\sqrt{G}\sin\frac{\Omega}{2}\rho X_1 + 2\sqrt{G}\cos\frac{\Omega}{2}\rho X_2,$$

$$\frac{\partial x_2}{\partial u} = 2\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho X - 2\sqrt{E}\sin\frac{\Omega}{2}\rho X_1 - 2\sqrt{E}\cos\frac{\Omega}{2}\rho X_2, \quad \frac{\partial x_1}{\partial v} = -2\left[\frac{\partial \rho}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\right]X.$$

Meanwhile, one gets the formulas:

$$(33)\begin{cases} E_1 = 4\left[\dfrac{\partial \rho}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\right]^2, \ F_1 = -4\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\rho\left[\dfrac{\partial \rho}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\right], \ G_1 = 4\rho^2\left[\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}^2 + G\right], \\ E_1G_1 - F_1^2 = 16G\rho^2\left[\dfrac{\partial \rho}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\right]^2, \end{cases}$$

and analogously:

$$(33^*)\begin{cases} E_2 = 4\rho^2\left[\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}^2 + E\right], \ F_2 = -4\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\rho\left[\dfrac{\partial \rho}{\partial v} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\rho\right], \ G_2 = 4\left[\dfrac{\partial \rho}{\partial v} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\rho\right]^2, \\ E_2G_2 - F_2^2 = 16E\rho^2\left[\dfrac{\partial \rho}{\partial v} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\rho\right]^2. \end{cases}$$

Now, let ξ_1, η_1, ζ_1 denote the direction cosines of the normal to S_1 and let ξ_2, η_2, ζ_2 denote those of the normal to S_2; we have:

$$\xi_1 = \cos\frac{\Omega}{2}X_1 + \sin\frac{\Omega}{2}X_2,$$
$$\xi_2 = \cos\frac{\Omega}{2}X_1 - \sin\frac{\Omega}{2}X_2.$$

Calculate:

$$D_1 = -\sum\frac{\partial \xi_1}{\partial u}\frac{\partial x_1}{\partial u}, \qquad D_1' = -\sum\frac{\partial \xi_1}{\partial v}\frac{\partial x_1}{\partial u}, \qquad D_1'' = -\sum\frac{\partial \xi_1}{\partial v}\frac{\partial x_1}{\partial v},$$

$$D_2 = -\sum\frac{\partial \xi_2}{\partial u}\frac{\partial x_2}{\partial u}, \qquad D_2' = -\sum\frac{\partial \xi_2}{\partial u}\frac{\partial x_2}{\partial v}, \qquad D_2'' = -\sum\frac{\partial \xi_2}{\partial v}\frac{\partial x_2}{\partial v},$$

and find that:

$$(34)\begin{cases} D_1 = 2\sqrt{E}\sin\Omega\left[\dfrac{\partial\rho}{\partial u}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\rho\right], \qquad D_1' = 0,\ D_1'' = -2\sqrt{G}\rho\left[\dfrac{\partial\Omega}{\partial v}+\sqrt{\dfrac{G}{E}}\begin{Bmatrix}1\,2\\2\end{Bmatrix}\sin\Omega\right], \\ D_2 = 2\sqrt{E}\rho\left[\dfrac{\partial\Omega}{\partial u}+\sqrt{\dfrac{E}{G}}\begin{Bmatrix}1\,2\\1\end{Bmatrix}\sin\Omega\right],\ D_2' = 0,\ D_2'' = -2\sqrt{G}\sin\Omega\left[\dfrac{\partial\rho}{\partial v}+\begin{Bmatrix}1\,2\\1\end{Bmatrix}\rho\right]. \end{cases}$$

so the curvatures K_1, K_2 of the two sheets will be given by the formulas:

$$(53)\qquad\begin{cases} K_1 = -\dfrac{\sqrt{\dfrac{E}{G}}\sin\Omega\left[\dfrac{\partial\Omega}{\partial v}+\sqrt{\dfrac{G}{E}}\begin{Bmatrix}1\,2\\2\end{Bmatrix}\sin\Omega\right]}{4\rho\left[\dfrac{\partial\rho}{\partial u}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\rho\right]}, \\ K_2 = -\dfrac{\sqrt{\dfrac{G}{E}}\sin\Omega\left[\dfrac{\partial\Omega}{\partial u}+\sqrt{\dfrac{E}{G}}\begin{Bmatrix}1\,2\\1\end{Bmatrix}\sin\Omega\right]}{4\rho\left[\dfrac{\partial\rho}{\partial v}+\begin{Bmatrix}1\,2\\1\end{Bmatrix}\rho\right]}. \end{cases}$$

§ 150.

Applications to pseudo-spherical congruences.

We apply the general formulas of the preceding paragraph to two particular cases. In the first place, we pose the question: *Do there exist congruences in which one simultaneously has the constancy of the distances between the foci and the distances between the limit points?* From § 130, we know that there actually exist normal congruences of that type, and that they are defined by the normals to a surface W for which the radii of curvature r_1, r_2 are coupled by the relation:

$$r_1 - r_2 = \text{constant}.$$

Now, if we treat the general question then we must suppose that:

$$\rho = \text{constant}, \quad \Omega = \text{constant}$$

in the formulas of the preceding number; (35) will then become:

$$K_1 = K_2 = -\frac{\sin^2\Omega}{4\rho^2},$$

and since:

$$\frac{2\rho}{\sin\Omega} = 2r$$

is the distance between the limit points, we will have the theorem:

If the distances between the foci and the distances between the limit points are constant in a rectilinear congruence then the two focal surfaces will be pseudo-spherical surfaces whose radii are equal to the distance between the limit points.

The congruences of this kind (whose existence for all values of ρ and Ω we will prove much later) are called *pseudo-spherical congruences.* Here, we shall deduce some further properties of the correspondence between points on the two sheets of the focal surface under the hypothesis that such congruences do exist. One finds from (34) that the differential equation of the asymptotes of both sheets is:

$$E\,du^2 - G\,dv^2 = 0,$$

so the asymptotes on both sheets will correspond. Moreover, one has:

$$ds_1^2 = 4\rho^2\left[\left(\begin{Bmatrix}1\,2\\2\end{Bmatrix}du - \begin{Bmatrix}1\,2\\2\end{Bmatrix}dv\right)^2 + G\,dv^2\right],$$

$$ds_2^2 = 4\rho^2\left[\left(\begin{Bmatrix}1\,2\\2\end{Bmatrix}du - \begin{Bmatrix}1\,2\\1\end{Bmatrix}dv\right)^2 + E\,dv^2\right],$$

for the line elements ds_1, ds_2, from which it will result that the arcs of corresponding asymptotes will be equal. From (33), (34), one will then find that the differential equation of the lines of curvature on one or the other sheet is:

$$E\begin{Bmatrix}1\,2\\1\end{Bmatrix}\begin{Bmatrix}1\,2\\1\end{Bmatrix}du^2 - \left[EG + E\begin{Bmatrix}1\,2\\1\end{Bmatrix}^2 + G\begin{Bmatrix}1\,2\\2\end{Bmatrix}^2\right]du\,dv + G\begin{Bmatrix}1\,2\\1\end{Bmatrix}\begin{Bmatrix}1\,2\\2\end{Bmatrix}dv^2 = 0.$$

One will then have the theorem:

The lines of curvature and asymptotic lines on the two sheets of the focal surface of a pseudo-spherical congruence correspond, and the corresponding arcs of the asymptotes are equal ([1]).

([1]) It is worth pointing out the consequences that one derives from the formulas in § 145 for the spherical image (u, v) of the principal surfaces of a pseudo-spherical congruence. If one lets r and ρ (and therefore $\varphi = \sqrt{r^2-\rho^2}$) be constants then (23) on page 18 will become:

$$\frac{\partial^2 \log\sqrt{EG}}{\partial u\,\partial v} = \frac{\varphi}{r}\sqrt{EG} = \cos\Omega\sqrt{EG}\,.$$

§ 151.

Guichard congruences.

The second question that we pose is the following one ([1]):

For which congruences does it happen that the developables of the congruence cut the focal surface along the lines of curvature?

One must then have:

$$F_1 = 0, \qquad F_2 = 0,$$

and the necessary and sufficient conditions for the case above to exist then result from (33), (33*) (when one suppose that the focal surfaces do not reduce to curves), namely:

$$\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} = 0, \qquad \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} = 0.$$

That now expresses (§ 76) the idea that the spherical lines u, v are the images of the asymptotes of a pseudo-spherical surface, and one will then have:

Therefore: *When the spherical line element is referred to the lines* (u, v) *that are the images of the principal surfaces of a pseudo-spherical congruence, it will take the form:*

$$(a) \qquad ds'^2 = E\,du^2 + G\,dv^2,$$

in which the product $\sqrt{EG}$ *is a solution of the Liouville equation:*

$$(b) \qquad \frac{\partial^2 \log \sqrt{EG}}{\partial u\,\partial v} = \cos\Omega \sqrt{EG} \qquad (\Omega \text{ constant}).$$

Conversely, it is clear from § 145 that whenever the spherical line element is reduced to the form (a), where (b) is satisfied, there will exist a corresponding pseudo-spherical congruence.

In particular, if the pseudo-spherical congruence is normal then one will have $\Omega = \pi / 2$, $\frac{\partial^2 \log \sqrt{EG}}{\partial u\,\partial v} = 0$, and one can certainly make $\sqrt{EG} = 1$. If one then regards u, v as the orthogonal Cartesian coordinates of a point in a representative plane then one will have a representation of the sphere on the plane that preserves area and in which the doubly-orthogonal system of lines that are parallel to the coordinate axes in the representative plane will correspond to a doubly-orthogonal system on the sphere.

From theorem (C) on page 290, one can arrive at this last result directly by looking for the lines that are the spherical images of the lines of curvature of the surface W for which the difference between the principal radii of curvatures is constant.

The case of $\Omega = 0$ is also noteworthy. The congruence will then be composed of tangents to the asymptotic lines of a system of pseudo-spherical surfaces. Cf., § 142, page 12.

([1]) Cf., GUICHARD, *loc. cit.*

The desired congruences are the ones such that the images of their developables are the images of the asymptotes of a pseudo-spherical surface, and only those congruences.

One can then set (§ 76):

$$E = G = 1, \qquad \text{so} \qquad F = \cos\Omega,$$

so the Laplace equation (28), which defines ρ, will become:

$$\frac{\partial^2\rho}{\partial u\,\partial v} + \rho\cos\Omega = 0. \tag{36}$$

Any solution ρ of this equation will correspond to a congruence of the kind that we now consider; we call such a congruence a *Guichard congruence.*

From (33), one has the simple formulas:

$$\begin{cases} ds_1^2 = 4\left(\dfrac{\partial\rho}{\partial u}\right)^2 du^2 + 4\rho^2 dv^2, \\ ds_2^2 = 4\rho^2 du^2 + 4\left(\dfrac{\partial\rho}{\partial u}\right)^2 dv^2 \end{cases}$$

for the line elements of the two sheets of the focal surface of a Guichard congruence.

In (36), Ω denotes an arbitrary solution of the equation:

$$\frac{\partial^2\Omega}{\partial u\,\partial v} = -\sin\Omega,$$

and one can observe, with Guichard, that $\dfrac{\partial\Omega}{\partial u}$, $\dfrac{\partial\Omega}{\partial v}$ are particular solutions of (36); one of the two focal sheets will be a sphere then.

The lines of curvature v = constant have the rays of the congruence for their tangents on the Guichard surface S_1. Let Γ_1 be the evolute of S_1 with respect to v = constant. The normal to Γ_1 at a point is parallel to the corresponding ray of the Guichard congruence, and since (u, v) are conjugate on Γ_1, Γ_1 will have the property that if one takes their Gaussian spherical representation then the image of the conjugate system (u, v) to Γ_1 will coincide with the image of the asymptotes of a pseudo-spherical surface. It is now enough to refer to formula (25), § 78, page 167, in order to see that if $\begin{Bmatrix} r\,s \\ t \end{Bmatrix}_1$ denote the Christoffel symbols that are constructed from Γ_1 then one will have:

$$\begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}_1 = 0, \qquad \begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix}_1 = 0;$$

i.e., the *u*, *v* will be geodetic on Γ_1. The surfaces of that type – viz., the ones on which there exists a conjugate system that is composed of geodetic lines – were studied for the first time by VOSS ([1]) and are called *Voss surfaces.* Therefore:

Any Guichard surface has a Voss surface for one sheet of its evolute.

Conversely, one will see immediately that:

The evolutes of a Voss surface with respect to one or the other geodetic system of the conjugate system are Guichard surfaces.

We shall return later on to the properties of the surfaces that were considered in this number and their relationships with pseudo-spherical surfaces.

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([1]) Sitzungsberichte der Münchener Akademie der Wissenschaften (March 1888).