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CHAPTER IV

Fundamental formulas from the theory of surfaces

Translated by D. H. Delphenich

The two fundamental quadratic forms $\begin{cases} E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ D\,du^2 + 2D'\,du\,dv + D''\,dv^2 \end{cases}$. – Formulas that give the second derivatives of x, y, z and the first derivatives of X, Y, Z. – Equation of Gauss and Mainardi-Codazzi between the coefficients E, F, G, D, D', D'' of the two fundamental forms. – Existence and uniqueness of the surfaces that correspond to two given fundamental forms for which the equations of Gauss and Codazzi are satisfied. – Lines of curvature. – Radii of first curvature of lines that are traced on a surface. – Meunier's theorem. – Euler's formula. – Dupin indicatrix. – Total curvature and mean curvature. – Conjugate systems. – Asymptotic lines. – Calculating of differential parameters. – Elements of a surface in Cartesian coordinates.

§ 54.

The second fundamental quadratic form.

We saw just one differential form intervene in the properties that were studied in the preceding chapter, and it gave the line element of the surface:

$$f = ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

namely, the *first fundamental form.* However, when one studies the properties that are inherent to the actual form that the surface has in space, a second fundamental form will intervene, in addition to the preceding one, and as we shall soon see: *The theory of surfaces, when considered from our viewpoint, reduces essentially to the study of two simultaneous quadratic differential forms.*

In order to introduce the second differential form that was just mentioned, we begin by fixing the cosines of the positive direction of the normal to the surface, which shall always be denoted by:

$$X, Y, Z.$$

As in § 42, we establish that the positive face of the tangent plane is the one on which the positive direction of the tangent to the line u lies to the left of that of the line v ([1]).

The positive direction of the normal will be the one around which the positive face of the tangent plane revolves. From known formulas of analytical geometry, one will then have:

$$X = \frac{1}{\sin\omega}\begin{vmatrix} \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u} & \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u} \\ \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v} & \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v} \end{vmatrix}, \qquad Y = \frac{1}{\sin\omega}\begin{vmatrix} \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u} & \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u} \\ \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v} & \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v} \end{vmatrix},$$

$$Z = \frac{1}{\sin\omega}\begin{vmatrix} \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u} & \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u} \\ \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v} & \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v} \end{vmatrix},$$

in which ω is the angle between the coordinate lines that were defined in § 41. It will then result from (6^*) of that § (pp. 88) that:

$$(1) \qquad X = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \qquad Y = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix},$$

$$Z = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The second differential form that one introduces will be:

$$\varphi = -(dx\,dX + dy\,dY + dz\,dZ),$$

for which one always adopts the notation ([2]):

$$(2) \qquad \varphi = -\sum dx\,dX = D\,du^2 + 2D'\,du\,dv + D''\,dv^2.$$

([1]) We always agree upon the convention that the positive direction Oy lies to the left of Ox on the positive face of the xy-plane.

([2]) Here and in what follows, the summation symbol Σ denotes a sum of three terms that can be deduced from the first one by changing x, X into y, Y, z, Z, resp.

One observes immediately the various forms that the coefficients D, D', D'' of φ can take. If one differentiates the identities:

$$\sum X \frac{\partial x}{\partial u} = 0, \qquad \sum X \frac{\partial x}{\partial v} = 0,$$

with respect to u, v then some other ones will follow:

$$\begin{cases} \sum X \dfrac{\partial^2 x}{\partial u^2} = -\sum \dfrac{\partial X}{\partial u}\dfrac{\partial X}{\partial v}, \\ \sum X \dfrac{\partial^2 x}{\partial u\,\partial v} = -\sum \dfrac{\partial X}{\partial v}\dfrac{\partial x}{\partial u} = -\sum \dfrac{\partial X}{\partial u}\dfrac{\partial x}{\partial v}, \\ \sum X \dfrac{\partial^2 x}{\partial v^2} = -\sum \dfrac{\partial X}{\partial v}\dfrac{\partial x}{\partial v}. \end{cases}$$

One will then have:

$$(3) \qquad \begin{cases} D = \sum X \dfrac{\partial^2 x}{\partial u^2} = -\sum \dfrac{\partial x}{\partial u}\dfrac{\partial X}{\partial u}, \\ D' = \sum X \dfrac{\partial^2 x}{\partial u\,\partial v} = -\sum \dfrac{\partial x}{\partial u}\dfrac{\partial X}{\partial v} = -\sum \dfrac{\partial x}{\partial v}\dfrac{\partial X}{\partial u}, \\ D'' = \sum X \dfrac{\partial^2 x}{\partial v^2} = -\sum \dfrac{\partial x}{\partial v}\dfrac{\partial X}{\partial v}. \end{cases}$$

From (1), one can also writes D, D', D'' in the form of determinants:

$$(3^*) \qquad D = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial^2 y}{\partial u^2} & \dfrac{\partial^2 z}{\partial u^2} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}, \qquad D' = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \dfrac{\partial^2 x}{\partial u\,\partial v} & \dfrac{\partial^2 y}{\partial u\,\partial v} & \dfrac{\partial^2 z}{\partial u\,\partial v} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix},$$

$$D'' = \frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} \dfrac{\partial^2 x}{\partial v^2} & \dfrac{\partial^2 y}{\partial v^2} & \dfrac{\partial^2 z}{\partial v^2} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

The two quadratic differential forms:

$$f = \sum dx^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$
$$\varphi = -\sum dx\,dX = D\,du^2 + 2D'\,du\,dv + D''\,dv^2$$

are called the first and second fundamental forms of the surface S.

It is clear that when one switches the variables u, v, they will be transformed into new fundamental forms.

§ 55.

Fundamental equations.

In this paragraph, we shall establish the *fundamental equations* of our theory. First, let us make the following observation: If A, B, C are three arbitrary functions of u, v then we can determine three unknown coefficients α, β, γ in such a manner that we will have:

$$(a) \qquad \begin{cases} A = \alpha \dfrac{\partial x}{\partial u} + \beta \dfrac{\partial x}{\partial v} + \gamma X, \\ B = \alpha \dfrac{\partial y}{\partial u} + \beta \dfrac{\partial y}{\partial v} + \gamma Y, \\ C = \alpha \dfrac{\partial z}{\partial u} + \beta \dfrac{\partial z}{\partial v} + \gamma Z, \end{cases}$$

so the determinant:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & X \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & Y \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & Z \end{vmatrix} = \sqrt{EG - F^2}$$

will not be zero.

Having said that, recall for the moment the index notation, and take:

$$u = u_1, \qquad v = u_2,$$
$$E = a_{11}, \qquad F = a_{12}, \qquad G = a_{22},$$
$$D = b_{11}, \qquad D' = b_{12}, \qquad D'' = b_{22}.$$

Since:

$$a_{rs} = \sum \frac{\partial x}{\partial u_r} \frac{\partial x}{\partial u_s},$$

it will then follow that:

$$\sum \frac{\partial x}{\partial u_t} \frac{\partial^2 x}{\partial u_r\,\partial u_s} = \begin{bmatrix} r\,s \\ t \end{bmatrix}.$$

If one sets:

$$A = \frac{\partial^2 x}{\partial u_r\,\partial u_s}, \quad B = \frac{\partial^2 y}{\partial u_r\,\partial u_s}, \quad C = \frac{\partial^2 z}{\partial u_r\,\partial u_s}$$

in (a) then if one first multiplies by $\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, \frac{\partial z}{\partial u_1}$, in sequence, then multiplies by $\frac{\partial x}{\partial u_2}, \frac{\partial y}{\partial u_2}, \frac{\partial z}{\partial u_2}$, and finally multiplies by X, Y, Z, and then sums, the result will be:

$$\begin{cases} a_{11}\alpha + a_{12}\beta = \begin{bmatrix} r\,s \\ 1 \end{bmatrix}, \\ a_{12}\alpha + a_{22}\beta = \begin{bmatrix} r\,s \\ 2 \end{bmatrix}, \end{cases}$$

$$\gamma = b_{rs},$$

so:

$$\alpha = A_{11}\begin{bmatrix} r\,s \\ 1 \end{bmatrix} + A_{12}\begin{bmatrix} r\,s \\ 2 \end{bmatrix} = \begin{Bmatrix} r\,s \\ 1 \end{Bmatrix},$$

$$\beta = A_{21}\begin{bmatrix} r\,s \\ 1 \end{bmatrix} + A_{22}\begin{bmatrix} r\,s \\ 2 \end{bmatrix} = \begin{Bmatrix} r\,s \\ 2 \end{Bmatrix},$$

and therefore:

$$\frac{\partial^2 x}{\partial u_r\,\partial u_s} = \begin{Bmatrix} r\,s \\ 1 \end{Bmatrix}\frac{\partial x}{\partial u_1} + \begin{Bmatrix} r\,s \\ 2 \end{Bmatrix}\frac{\partial x}{\partial u_2} + b_{rs}\,X,$$

or, more briefly, with the notation of the second covariant derivative (§ 32).

$$x_{rs} = b_{rs}\,X.$$

When written in this way in the old notation, one will get the first group of fundamental equations:

$$\text{(I)} \qquad \begin{cases} \dfrac{\partial^2 x}{\partial u^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}\dfrac{\partial x}{\partial u} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\dfrac{\partial x}{\partial v} + D\,X, \\[2ex] \dfrac{\partial^2 x}{\partial u\,\partial v} = \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}\dfrac{\partial x}{\partial u} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}\dfrac{\partial x}{\partial v} + D'X, \\[2ex] \dfrac{\partial^2 x}{\partial v^2} = \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}\dfrac{\partial x}{\partial u} + \begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix}\dfrac{\partial x}{\partial v} + D''X, \end{cases}$$

in which we have omitted the expressions for y, z, which are perfectly to similar to these and are deduced from them by changing X into Y, Z, respectively.

The second group of fundamental formulas will be the ones that express the first partial derivatives of X, Y, Z in terms of $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, X, etc. If one sets:

$$
\begin{aligned}
A &= \frac{\partial X}{\partial u}, \qquad & B &= \frac{\partial Y}{\partial u}, \qquad & C &= \frac{\partial Z}{\partial u}, \\
A &= \frac{\partial X}{\partial v}, \qquad & B &= \frac{\partial Y}{\partial v}, \qquad & C &= \frac{\partial Z}{\partial v},
\end{aligned}
$$

in (a) in sequence then one will find that the formulas in question are:

$$
\text{(II)} \qquad \left\{
\begin{aligned}
\frac{\partial X}{\partial u} &= \frac{FD' - GD}{EG - F^2}\frac{\partial x}{\partial u} + \frac{FD - ED'}{EG - F^2}\frac{\partial x}{\partial v}, \\
\frac{\partial X}{\partial v} &= \frac{FD'' - GD'}{EG - F^2}\frac{\partial x}{\partial u} + \frac{FD' - ED''}{EG - F^2}\frac{\partial x}{\partial v},
\end{aligned}
\right.
$$

in which we have again suppressed the analogous equations for Y, Z.

As one sees, the coefficients of the right-hand sides of formulas (I), (II) are composed of nothing but the coefficients of the two fundamental forms f, φ ([1]).

§ 56.

Equations of Gauss and Codazzi.

The six coefficients:

$$E, F, G; \qquad D, D', D'',$$

of the two fundamental forms are not mutually independent, but rather they are coupled by three important relations that we shall now establish. For that, we write the integrability conditions of the system (I):

$$
\begin{aligned}
\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u^2}\right) - \frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial u\,\partial v}\right) &= 0, \\
\frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial v^2}\right) - \frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u\,\partial v}\right) &= 0,
\end{aligned}
$$

i.e.:

([1]) In particular, one must always recall that the Christoffel symbols $\left\{ \begin{matrix} r\ s \\ t \end{matrix} \right\}$ that appear in (I) are constructed *from the first fundamental form f.*

$$(b)\qquad \begin{cases} \dfrac{\partial}{\partial v}\left[\begin{Bmatrix}11\\1\end{Bmatrix}\dfrac{\partial x}{\partial u}+\begin{Bmatrix}11\\2\end{Bmatrix}\dfrac{\partial x}{\partial v}+DX\right]-\dfrac{\partial}{\partial u}\left[\begin{Bmatrix}12\\1\end{Bmatrix}\dfrac{\partial x}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}\dfrac{\partial x}{\partial v}+D'X\right]=0, \\ \dfrac{\partial}{\partial u}\left[\begin{Bmatrix}22\\1\end{Bmatrix}\dfrac{\partial x}{\partial u}+\begin{Bmatrix}22\\2\end{Bmatrix}\dfrac{\partial x}{\partial v}+D''X\right]-\dfrac{\partial}{\partial v}\left[\begin{Bmatrix}12\\1\end{Bmatrix}\dfrac{\partial x}{\partial u}+\begin{Bmatrix}12\\2\end{Bmatrix}\dfrac{\partial x}{\partial v}+D'X\right]=0. \end{cases}$$

It is clear that if one makes use of the fundamental formulas (I), (II) in this then the left-hand sides of (b) can be put into the forms:

$$\alpha\,\frac{\partial x}{\partial u}+\beta\,\frac{\partial x}{\partial u}+\gamma\,X,$$
$$\alpha'\frac{\partial x}{\partial u}+\beta'\frac{\partial x}{\partial u}+\gamma' X$$

identically, while the equations:

$$\begin{cases} \alpha\dfrac{\partial x}{\partial u}+\beta\dfrac{\partial x}{\partial v}+\gamma X=0, \\ \alpha\dfrac{\partial y}{\partial u}+\beta\dfrac{\partial y}{\partial v}+\gamma Y=0, \\ \alpha\dfrac{\partial z}{\partial u}+\beta\dfrac{\partial z}{\partial v}+\gamma Z=0, \end{cases}\qquad \begin{cases} \alpha'\dfrac{\partial x}{\partial u}+\beta'\dfrac{\partial x}{\partial v}+\gamma' X=0, \\ \alpha'\dfrac{\partial y}{\partial u}+\beta'\dfrac{\partial y}{\partial v}+\gamma' Y=0, \\ \alpha'\dfrac{\partial z}{\partial u}+\beta'\dfrac{\partial z}{\partial v}+\gamma' Z=0 \end{cases}$$

must also persist, and one will get the integrability conditions:

$$\alpha=0,\qquad \beta=0,\qquad \gamma=0,$$
$$\alpha'=0,\qquad \beta'=0,\qquad \gamma'=0.$$

The four conditions:

$$\alpha=0,\qquad \beta=0,\qquad \alpha'=0,\qquad \beta'=0$$

can be written in terms of the four Christoffel indices (§ 34, page. 72) as:

$$\frac{DD''-D'^2}{EG-F^2}E=\{12,12\},$$
$$\frac{DD''-D'^2}{EG-F^2}F=\{11,21\},$$
$$\frac{DD''-D'^2}{EG-F^2}F=\{22,12\},$$
$$\frac{DD''-D'^2}{EG-F^2}G=\{21,21\}.$$

If one lets K denote the curvature of the first fundamental form then they will give uniquely [§ 37, formula (IV)]:

$$\text{(III)} \qquad \frac{DD'' - D'^2}{EG - F^2} = K.$$

In words, that says: *The quotient of the discriminants of the two fundamental forms φ, f is equal to the curvature K of the first fundamental form f.*

As for the other two conditions, viz.:

$$\gamma = 0, \quad \gamma' = 0,$$

when they are developed they will become:

$$\text{(IV)} \qquad \left\{\begin{aligned} &\frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} \right) D' + \begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix} D'' = 0, \\ &\frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial v} + \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix} D + \left(\begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} \right) D' + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} D'' = 0, \end{aligned}\right.$$

and according to § 38 (page 81), they express the idea that *the trilinear covariant form* (f, φ) *that is constructed from the second fundamental form φ with respect to the first one f is identically zero.*

Equation (III) was given by Gauss in his *Disquisitiones*, *etc.*, where he already found all of its elements by deducing them from (IV). The last one is more commonly referred to by the name of the *Codazzi formula*, since it is precisely equivalent to the equations that were given by that geometer ([1]); it was, however, given for the very first time in another form by *Mainardi* (1856) ([2]).

One can given another useful form to formulas (IV) when one observes that if one avails oneself of formulas (20), § 31:

$$\left\{\begin{aligned} \frac{\partial \log \sqrt{EG - F^2}}{\partial u} &= \begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}, \\ \frac{\partial \log \sqrt{EG - F^2}}{\partial v} &= \begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}; \end{aligned}\right.$$

that result is, in fact, equivalent to the following system:

([1]) Annali di mat. **2** (1868), pp. 273.
([2]) Giornale dell'Istituto Lombardo, t. IX, pp. 395.

$$(\mathrm{IV}^*)\quad \begin{cases} \dfrac{\partial}{\partial v}\left(\dfrac{D}{\sqrt{EG-F^2}}\right)-\dfrac{\partial}{\partial u}\left(\dfrac{D'}{\sqrt{EG-F^2}}\right)+\begin{Bmatrix}2\,2\\2\end{Bmatrix}\dfrac{D}{\sqrt{EG-F^2}}-2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\dfrac{D'}{\sqrt{EG-F^2}} \\ \qquad\qquad +\begin{Bmatrix}1\,1\\2\end{Bmatrix}\dfrac{D''}{\sqrt{EG-F^2}}=0, \\ \dfrac{\partial}{\partial u}\left(\dfrac{D''}{\sqrt{EG-F^2}}\right)-\dfrac{\partial}{\partial v}\left(\dfrac{D'}{\sqrt{EG-F^2}}\right)+\begin{Bmatrix}2\,2\\1\end{Bmatrix}\dfrac{D}{\sqrt{EG-F^2}}-2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\dfrac{D'}{\sqrt{EG-F^2}} \\ \qquad\qquad +\begin{Bmatrix}1\,1\\1\end{Bmatrix}\dfrac{D''}{\sqrt{EG-F^2}}=0. \end{cases}$$

The relations (III), (IV) that exist between the coefficients of the two fundamental forms give the necessary and sufficient conditions that must be satisfied. We state that property in the more precise form as the following fundamental theorem:

If one is given two quadratic differential forms:

$$f = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$
$$\varphi = D\,du^2 + 2D'\,du\,dv + D''\,dv^2,$$

the first of which is definite, then for there to exist a surface that admits these forms as its first and second fundamental forms, it is necessary and sufficient that the relations (III), (IV) *must be verified. If those conditions are verified then the corresponding surface will be unique and determinate, up to motions in space.*

From the proof of that theorem, which we will now carry out, the terms "fundamental forms" that are given to f, φ will be justified, and we intend that all of the properties that are inherent to the form of the surface can depend upon only the six coefficients of the fundamental forms. In analogy with the name of "intrinsic equation" for a curve (chap. I, § 8), one can say, in summation, that the equations:

$$f = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$
$$\varphi = D\,du^2 + 2D'\,du\,dv + D''\,dv^2$$

are the *intrinsic equations* for the surface.

§ 57.

Integration of the intrinsic equations.

From the invariant character of the fundamental equations (III), (IV), one can, moreover, conveniently introduce the independent variables u, v into the proof of the stated theorem, and then, utilizing the result in § 38, assume that the variables u, v reduce them simultaneously to:

$$F = 0, \quad D' = 0.$$

As we saw in the cited number, except for the case in which the proportions:

$$D : D' : D'' = E : F : G$$

are valid (which are true only in the case of a spherical (or planar) surface ([1]), as one easily sees], these new variables u, v will be completely determinate. When one equates

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([1]) And indeed, one will have:

$$D = \lambda E, \qquad D' = \lambda F, \qquad D'' = \lambda G$$

in that case. However, if substitutes that in (IV), while recalling (§ 38) that one has:

$$\frac{\partial E}{\partial v} - \frac{\partial F}{\partial u} - \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} E + \left(\begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} \right) F + \begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix} G = 0,$$

$$\frac{\partial G}{\partial u} - \frac{\partial F}{\partial v} - \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix} E + \left(\begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} \right) F + \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} G = 0$$

identically, then it will result that:

$$E \frac{\partial \lambda}{\partial v} - F \frac{\partial \lambda}{\partial u} = 0,$$

$$F \frac{\partial \lambda}{\partial v} - G \frac{\partial \lambda}{\partial u} = 0,$$

and therefore:

$$\lambda = \text{constant}.$$

If one takes:

$$\lambda = -\frac{1}{R} \qquad (R \text{ constant})$$

then (II), page. 117, will give:

$$\begin{cases} \dfrac{\partial u}{\partial x} = R \dfrac{\partial X}{\partial u}, \\[2mm] \dfrac{\partial x}{\partial v} = R \dfrac{\partial X}{\partial v}, \end{cases} \qquad \begin{cases} \dfrac{\partial y}{\partial u} = R \dfrac{\partial Y}{\partial u}, \\[2mm] \dfrac{\partial y}{\partial v} = R \dfrac{\partial Y}{\partial v}, \end{cases} \qquad \begin{cases} \dfrac{\partial z}{\partial u} = R \dfrac{\partial Z}{\partial u}, \\[2mm] \dfrac{\partial z}{\partial v} = R \dfrac{\partial Z}{\partial v}, \end{cases}$$

which will give:

$$x = RX + a, \qquad y = RY + b, \qquad z = RZ + c$$

when they are integrated, with a, b, c, and therefore:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2,$$

which is the equation of a sphere of radius R. In the case $\lambda = 0$, it then results that X, Y, Z are constants; i.e., the surface is a plane. In fact, with no loss of generality, one can then suppose that:

$$X = 0, \qquad Y = 0, \qquad Z = 1,$$

them to constants, that will give what one calls the *lines of curvature* of the surface (cf., § 60).

When one replaces the symbols in the last of the fundamental equations (III), (IV*) with their actual values [Table (*A*), page 92] and takes the value for *K* that was given in (18) on page 93, those equations will become:

$$
\text{(V)} \qquad \begin{cases}
\dfrac{DD''}{\sqrt{EG}} + \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}\right) + \dfrac{\partial}{\partial v}\left(\dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}\right) = 0, \\[2ex]
\dfrac{\partial}{\partial v}\left(\dfrac{D}{\sqrt{E}}\right) - \dfrac{D''}{G}\dfrac{\partial\sqrt{E}}{\partial v} = 0, \\[2ex]
\dfrac{\partial}{\partial u}\left(\dfrac{D''}{\sqrt{G}}\right) - \dfrac{D}{G}\dfrac{\partial\sqrt{G}}{\partial u} = 0.
\end{cases}
$$

For the surface whose existence and uniqueness we would like to prove [under the hypothesis that (V) are verified], consider a tri-rectangular trihedron at any point, which one calls the *principal trihedron*, that is composed of the positive directions of the tangent to the line *v*, the tangent to the line *u*, and the normal to the surface. If (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) denote the cosines of those three directions, respectively, then we will have:

$$
\begin{cases}
X_1 = \dfrac{1}{\sqrt{E}}\dfrac{\partial x}{\partial u}, & Y_1 = \dfrac{1}{\sqrt{E}}\dfrac{\partial y}{\partial u}, & Z_1 = \dfrac{1}{\sqrt{E}}\dfrac{\partial z}{\partial u}, \\[2ex]
X_2 = \dfrac{1}{\sqrt{G}}\dfrac{\partial x}{\partial v}, & Y_2 = \dfrac{1}{\sqrt{G}}\dfrac{\partial y}{\partial v}, & Z_2 = \dfrac{1}{\sqrt{G}}\dfrac{\partial z}{\partial v}, \\[2ex]
X_3 = X, & Y_3 = Y, & Z_3 = Z.
\end{cases}
$$

From the fundamental formulas (I), (II), page 116-117, we replace the Christoffel symbols with their present effective values, and deduce the following formulas:

$$
\begin{cases}
\dfrac{\partial X_1}{\partial u} = -\dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}X_2 + \dfrac{D}{\sqrt{E}}X_3, \\[2ex]
\dfrac{\partial X_1}{\partial v} = \dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}X_1,
\end{cases}
$$

---

and from (1) on page 113, it will then result that:

$$\frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = 0;$$

i.e., z = constant.

$$\begin{cases} \dfrac{\partial X_2}{\partial u} = \dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}X_1, \\ \dfrac{\partial X_2}{\partial v} = -\dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}X_1 + \dfrac{D''}{\sqrt{E}}X_3, \end{cases}$$

$$\begin{cases} \dfrac{\partial X_3}{\partial u} = \dfrac{D}{\sqrt{E}}X_1, \\ \dfrac{\partial X_3}{\partial v} = -\dfrac{D''}{\sqrt{G}}X_2. \end{cases}$$

The unknown functions X_1, X_2, X_3 must then satisfy the three homogeneous linear equations in total differentials:

$$(4) \qquad \begin{cases} dX_1 = \left\{-\dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}X_2 + \dfrac{D}{\sqrt{E}}X_{33}\right\}du + \dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}X_2 dv, \\ dX_2 = \dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}X_2\,du + \left\{-\dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}X_1 + \dfrac{D''}{\sqrt{G}}X_3\right\}dv, \\ dX_3 = -\dfrac{D}{\sqrt{E}}X_1\,du - \dfrac{D''}{\sqrt{G}}X_2\,dv. \end{cases}$$

(Y_1, Y_2, Y_3), (Z_1, Z_2, Z_3) must also satisfy system (4).

Now, the system (4) is an *unlimited integrable* system, since the integrability conditions will reduce to precisely three relations (V) that one assumes to be satisfied.

§ 58.

Existence and uniqueness.

We now appeal to the known theorem that there always exists an integral system for an unlimited integrable system of total differential equations such that for the initial values:

$$u = u_0, \qquad v = v_0$$

the variables will reduce to arbitrarily-given initial values, which can easily lead to the conclusion of our proof. For that, it is further convenient to observe that if (X_1, X_2, X_3), (X_1', X_2', X_3') are two (distinct or coincident) integral systems of equations (4) then due to the special form of those equations, one must have:

$$X_1 X_1' + X_2 X_2' + X_3 X_3' = \text{constant},$$

since the total differential of the left-hand side proves to be zero identically as a result of equations (4) and the analogous ones for X_1', X_2', X_3'.

Having said that, let (X_1, X_2, X_3), (Y_1, Y_2, Y_3), (Z_1, Z_2, Z_3) be three integral systems of (4) that reduce to the nine coefficients of an orthogonal substitution:

$$\begin{matrix} X_1^{(0)} & X_2^{(0)} & X_3^{(0)} \\ Y_1^{(0)} & Y_2^{(0)} & Y_3^{(0)} \\ Z_1^{(0)} & Z_2^{(0)} & Z_3^{(0)} \end{matrix}$$

for $u = u_0$, $v = v_0$. It results from the preceding observation that for any values of u, v:

$$\begin{matrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{matrix}$$

will be the coefficients of an orthogonal substitution; in particular, one will have:

$$X_1^2 + Y_1^2 + Z_1^2 = 1,$$
$$X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 = 0, \qquad \text{etc.}$$

Now, from (4) itself, the three expressions:

$$\sqrt{E}\, X_1\, du + \sqrt{G}\, X_2\, dv, \qquad \sqrt{E}\, Y_1\, du + \sqrt{G}\, Y_2\, dv, \qquad \sqrt{E}\, Z_1\, du + \sqrt{G}\, Z_2\, dv$$

will be exact differentials, and if one takes:

$$x = \int (\sqrt{E}\, X_1\, du + \sqrt{G}\, X_2\, dv), \qquad y = \int (\sqrt{E}\, Y_1\, du + \sqrt{G}\, Y_2\, dv),$$
$$z = \int (\sqrt{E}\, Z_1\, du + \sqrt{G}\, Z_2\, dv),$$

in which one regards x, y, z as the current coordinates of a point on a surface, then one will verify that this surface has the two assigned forms for its fundamental forms.

Finally, as for the part of the fundamental theorem that refers to uniqueness, that will result from either the linear form of (4) or by repeating the argument that was made already for the curve in § 8.

Observation: In the proof of the stated theorem, one refers, for simplicity, to a particular system of line coordinates (viz., the lines of curvature). However, one should observe that one can also choose the independent variables to be completely general and introduce a principal trihedron that is, e.g., the one that is composed of the bisectors to the tangents to the line coordinates and the normal at any point of the surface. The nine cosines of those three directions will again define a linear system of total differential equations, and like system (4), it will be unlimited integrable by virtue of the fundamental equations (III), (IV). Moreover, as in § 9, one can reduce the problem of the determination of the surface to the integration of one (total differential) equation of Ricatti type, from which, it results that:

In order to actually find the surface that corresponds to two given fundamental forms, one must integrate an equation of Ricatti type.

§ 59.

Lines of curvature.

If one considers an arbitrary line L on a surface S, and one follows the normals to the surface along it, then that will generally define a *non-developable ruled* surface. In the particular case in which the ruled surface is developable – i.e., the normals to S along L are the tangents to a curve in space (or pass through one of its points) – the line L will be called a *line of curvature* of the surface.

Observe immediately that according to that definition, any line that is traced on a plane or sphere will be considered to be a line of curvature, since the ruled surface of the corresponding normals is a cylinder or a cone.

For any other surface, as we shall now prove, there exists only a simple infinitude of lines of curvature that form a doubly orthogonal system of lines that are always real.

In the first place, we shall note some properties of lines of curvature that follow from their definition itself and theorems (A), (B) on the evolute that were given in § 18 (page. 38).

If the intersection C of two surfaces is a line of curvature for both of them then the angle by which the surfaces intersect along C will be constant. Conversely, if two surfaces meet at a constant angle and their intersection is the line of curvature for one of the surfaces then that will also be true for the other one.

Furthermore, since any line on a plane or a sphere is a line of curvature, one will have as a corollary:

If a plane or a sphere cuts a surface S along a line of curvature then it will cut S at a constant angle. Conversely, if a plane or a sphere cuts S at a constant angle then the intersection will be a line of curvature on S.

Hence, e.g., the meridians and parallels on a surface of revolution will be lines of curvature.

We look for the analytic conditions that will characterize a line of curvature L. u, v; x, y, z; X, Y, Z are regarded as functions of just one variables along it; e.g., the arc length s along L. If $M \equiv (x, y, z)$ is a point of L, and $M_1 \equiv (x_1, y_1, z_1)$ is the contact point of the normal to M along the edge of regression C_1 of the developable that is generated by the normals to S along L then we will have:

(5) $$x_1 = x - r\,X, \quad y_1 = y - r\,Y, \quad z_1 = z - r\,Z,$$

in which r denotes the algebraic value of the line segment M_1M (in which r will then be positive or negative according to whether the direction from M_1 to M coincides with the positive direction of the normal or its opposite).

If one differentiates (5) with respect to s and observes that:

$$\frac{dx_1}{ds}, \frac{dz_1}{ds}, \frac{dz_1}{ds}$$

are proportional to X, Y, Z, by hypothesis, then one will have:

$$\begin{cases} \lambda X = \dfrac{dx}{ds} - r\dfrac{dX}{ds} - X\dfrac{dr}{ds}, \\ \lambda Y = \dfrac{dy}{ds} - r\dfrac{dY}{ds} - Y\dfrac{dr}{ds}, \\ \lambda Z = \dfrac{dz}{ds} - r\dfrac{dZ}{ds} - Z\dfrac{dr}{ds}. \end{cases}$$

If one multiplies these by X, Y, Z in succession and then sums then the result will be:

$$\lambda = -\frac{dr}{ds},$$

so

$$\frac{dx}{ds} = r\,\frac{dX}{ds}, \quad \frac{dy}{ds} = r\,\frac{dY}{ds}, \quad \frac{dz}{ds} = r\,\frac{dZ}{ds},$$

or: *When one moves along the line of curvature L, the proportions:*

(6) $$dx : dy : dz = dX : dY : dZ$$

must remain valid.

Conversely, if the proportions (6) are valid along L, and r denotes the common value of the three ratios:

$$\frac{dx}{dX} = \frac{dy}{dY} = \frac{dz}{dZ}$$

then one will see immediately that (5) defines a curve C_1 whose tangents will be the normals to S along L. Hence: *The proportion* (6) *is characteristic of the lines of curvature.*

We exclude the case in which the curve C_1 reduces to a point from this; one will then have simply:

$$dx_1 = dy_1 = dz_1 = 0,$$

so $dr = 0$; i.e., r = constant.

§ 60.

Lines of curvature in curvilinear coordinates.

We now transform the equations:

$$dx = r\,dX, \qquad dy = r\,dY, \qquad dz = r\,dZ,$$

which are characteristic of a line of curvature, into curvilinear coordinates. For that, we write:

$$\left\{\begin{aligned} \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv &= r\left(\frac{\partial X}{\partial u}du + \frac{\partial X}{\partial v}dv\right), \\ \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv &= r\left(\frac{\partial Y}{\partial u}du + \frac{\partial Y}{\partial v}dv\right), \\ \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv &= r\left(\frac{\partial Z}{\partial u}du + \frac{\partial Z}{\partial v}dv\right), \end{aligned}\right.$$

which can be replaced with the equivalent system that one will obtain upon first multiplying by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$, resp., then by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$, resp., and finally by X, Y, Z, resp., and summing them.

One will obtain an identity the last time, and then find the equations (cf., § 54):

$$(7) \qquad \left\{\begin{aligned} E\,du + F\,dv &= -r(D\,du + D'\,dv), \\ F\,du + G\,dv &= -r(D'\,du + D''\,dv). \end{aligned}\right.$$

If one eliminates r from these two then one will obtain:

$$(8) \qquad \begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv \\ D\,du + D'\,dv & D'\,du + D''\,dv \end{vmatrix} = 0$$

as the differential equation for the lines of curvature.

The determinant that was just written is precisely the Jacobian of the two fundamental forms. If one therefore excludes the case:

$$D : D' : D'' = E : F : G,$$

in which the surface is a sphere or a plane ([1]), and recalls the results of § 39, page 83, then one will have the theorem:

([1]) It is easy to add a simple geometric proof of this fact to the analytical proof that was given in the footnote in § 57. In the case of the proportion:

$$D : D' : D'' = E : F : G,$$

There exists a doubly-orthogonal system of lines of curvature on any surface that are always real. One will have indeterminacy only for the sphere and the plane, for which any line will be a line of curvature.

Two lines of curvature L_1, L_2 pass through any point M of the surface S that will meet at a right angle there. The normal at M touches the edge of regression of the developable, which is generated by the normals to S along L_1, at a point that shall be denoted by M_1. That point is called the *center of curvature* of the surface at M relative to the line of curvature L_1. Similarly, one has a second center of curvature M_2 relative to L_2 on the normal at M, and the segments ([1]):

$$r_1 = \overline{M_1M}, \qquad r_2 = \overline{M_2M}$$

bear the names of *principal radii of curvature* of the surface at M for a reason that we shall now see.

If we eliminate the ratio $du : dv$ from our equations (7) then we will obviously get the following result:

The principal radii of curvature r_1, r_2 of the surface at any point are given at any point by the roots of the second-degree equation in r:

$$(9) \qquad (D\,D'' - D'^2)\,r^2 + (E\,D'' + G\,D - 2F\,D')\,r + E\,G - F^2 = 0.$$

§ 61.

Curvature of the normal sections.

We now pass on to the examination of the relations that exist between the radii of (first) curvature of the infinitude of lines that are traced on a surface through the same point M.

Let C be one such curve, along which, u, v; x, y, z are functions of the arc length s of C. Keep the notations of chap. I for C, so one will have, first of all, the direction cosines of its tangent:

$$(10) \qquad \alpha = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds}, \quad \beta = \frac{\partial y}{\partial u}\frac{du}{ds} + \frac{\partial y}{\partial v}\frac{dv}{ds}, \quad \alpha\gamma = \frac{\partial z}{\partial u}\frac{du}{ds} + \frac{\partial z}{\partial v}\frac{dv}{ds}.$$

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any line that is traced on the surface S will be, from (8), a line of curvature. It will then follow that if M, M' are two arbitrary points of S then the normals at M, M' will lie in a plane. Indeed, one can pass a plane through the normals at M and M' that cuts S along the curve C. The normals to S along C form a developable – i.e., they are tangents to an evolute of C – and since the normal at M lies in the plane at C, any other normal along C (in particular, the one at M') will lie in that plane. Therefore, all of the normals to S intersect pair-wise, and thus they cannot lie in a plane that passes through the same point O. If O is at a finite distance then S will consequently be a sphere (whose center is at O), while if O is at infinity then S will be a plane.

([1]) Recall that r_1, r_2 are regarded as positive or negative according to whether the direction from M_1 to M (or from M_2 to M) coincides with the positive direction of the normal or its opposite, respectively.

If one lets σ denote the angle from 0 to π that is defined by the positive directions of the principal normal to X and the normal to the surface then, from Frenet's formula, one will have:

$$\sum X \frac{d\alpha}{ds} = \frac{\cos\sigma}{\rho},$$

so, from (10):

$$\frac{\cos\sigma}{\rho} = \frac{D\,du^2 + 2D'\,du\,dv + D''\,dv^2}{ds^2},$$

or:

$$\frac{\cos\sigma}{\rho} = \frac{D\,du^2 + 2D'\,du\,dv + D''\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}. \tag{11}$$

One can pass a plane through the normal to the surface at M and the tangent to C at M; it will produce a section Γ of the surface that is called the *normal section tangent* to C. The first curvature $1 / R$ of Γ at M will be given by the same formula (11), in which one sets:

$$\cos\sigma = \pm 1$$

according to whether the concavity of Γ rises up towards the positive or negative direction of the normal, respectively. At the same, the formula:

$$\rho = \pm R \cos\sigma$$

will result; i.e., Meunier's theorem:

The radius of first curvature of a curve C that is traced on a surface S is equal at any point M to the radius of curvature of the normal section that is tangent to the curve C at M, multiplied by the cosine of the angle that the plane of the section makes with the osculating plane of the curve.

We can thus limit our studies to a study of normal sections.

Formula (11) will become:

$$\frac{1}{R} = \pm \frac{D\,du^2 + 2D'\,du\,dv + D''\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2},$$

in which the choice of upper or lower sign is linked with the situation that was described above. With that choice (according to the convention that was made in the theory of curves of always giving the first curvature a positive value), the actual sign of the right-hand side will prove to be positive in any case.

However, since all along the length R (for the infinitude of normal sections) one contacts the same line here – viz., the normal at M (on which one has already established a positive sign) – it is better to also attribute a sign to R. Moreover, we agree to count R as positive when the direction that goes from the center of curvature of the normal section to the foot M of the normal coincides with its positive sense and negative in the contrary case. (Cf., the preceding number.) *With this new convention*, one will certainly have:

$$\frac{1}{R} = -\frac{D\,du^2 + 2D'\,du\,dv + D''\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \tag{12}$$

in any case.

§ 62.

Euler's formula.

We now assume that the coordinate lines are lines of curvature and if r_1, r_2 denote the quantities that were introduced in § 60 then as we move along the line of curvature u, we will have:

$$dx = r_1\,dX, \qquad dy = r_1\,dY, \qquad dz = r_1\,dZ,$$

and along v:

$$dx = r_2\,dX, \qquad dy = r_2\,dY, \qquad dz = r_2\,dZ\,;$$

i.e. ([1]):

$$\left\{\begin{array}{lll} \dfrac{\partial x}{\partial u} = r_2\dfrac{\partial X}{\partial u}, & \dfrac{\partial y}{\partial u} = r_2\dfrac{\partial Y}{\partial u}, & \dfrac{\partial z}{\partial u} = r_2\dfrac{\partial Z}{\partial u}, \\ \dfrac{\partial x}{\partial v} = r_1\dfrac{\partial X}{\partial v}, & \dfrac{\partial y}{\partial v} = r_1\dfrac{\partial Y}{\partial v}, & \dfrac{\partial z}{\partial v} = r_1\dfrac{\partial Z}{\partial v}, \end{array}\right. \tag{13}$$

so:

$$D = -\frac{E}{r_2}, \qquad D' = 0, \qquad D'' = -\frac{G}{r_1}, \tag{14}$$

so, from (12):

$$\frac{1}{R} = \frac{\dfrac{E}{r_2}du^2 + \dfrac{G}{r_1}dv^2}{E\,du^2 + G\,dv^2} = \frac{E}{r_2}\left(\frac{du}{ds}\right)^2 + \frac{G}{r_1}\left(\frac{dv}{ds}\right)^2.$$

If θ denotes the angle that the normal section considered makes with the line v then that will give *Euler's formula:*

$$\frac{1}{R} = \frac{\cos^2\theta}{r_2} + \frac{\sin^2\theta}{r_1}. \tag{15}$$

Meanwhile it results from this that: r_1, r_2 *are the radii of curvature of the normal sections that are tangent to the lines of curvature.* Those sections are called *principal sections*, and r_1, r_2 are then called the *principal radii of curvature*, as we said above. The

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([1]) These are the formulas that are commonly referred to by the name of Rodrigues's formulas.

centers of curvature of the principal sections are the two points M_1, M_2 that were considered at the end of § 60, which one calls the *centers of curvature of the surface* at M.

We now examine how the radius of curvature R of the normal section varies when one rotates the plane of the section. In order to imagine the manner of variation that one obtains more clearly, we make use of the following considerations:

1. Suppose that r_1, r_2 have the same sign (e.g., positive) at the point under consideration. Establish a system of orthogonal Cartesian axes ξ, η in the tangent plane to M that coincide with the tangents to lines of curvature u, v, respectively, and consider the ellipse that has the equation:

$$\frac{\xi^2}{r_1}+\frac{\eta^2}{r_2} = 1. \tag{16}$$

A semi-diameter of that ellipse, when it is inclined by angle θ on the η-axis (tangent to v) has a length ρ that is given by the formula:

$$\frac{1}{\rho^2} = \frac{\cos^2\theta}{r_2}+\frac{\sin^2\theta}{r_1},$$

so one will have, from (15):

$$\rho^2 = R.$$

Therefore: *The square of any semi-diameter of the ellipse* (16) *is equal to the radius of curvature of the normal section whose plane goes through that diameter.*

For that reason, the ellipse (16) is called the *indicatrix ellipse.*

One should note that if $r_1 = r_2$ then the indicatrix ellipse becomes a circle and all of the normal sections through M will have the same radius of curvature. The point M is then called a *circular* or *umbilic* point, and the only surface that is circular at every point is the sphere ([1]).

2. Now, let r_1, r_2 have contrary signs, and to fix ideas, suppose that r_1 is positive and r_2 is negative. Consider the two conjugate hyperbolas in the tangent plane:

$$\left\{\begin{aligned} \frac{\xi^2}{r_1}-\frac{\eta^2}{-r_2}&=1,\\ -\frac{\xi^2}{r_1}+\frac{\eta^2}{-r_2}&=1, \end{aligned}\right. \tag{17}$$

and one will have the geometric representation of that system of two hyperbolas, which will, in fact, be provided by the ellipse (16).

([1]) In fact, one will then have:

$$D : D' : D'' = E : F : G .$$

In the first case, the ellipse (16) [and the system of two hyperbolas (17), in the second one] constitutes what one calls the *Dupin indicatrix*, after the name of the geometer that first gave the geometric interpretation above for Euler's formula.

One should observe that, while in the first case, the surface in the vicinity of M all lies on part of the tangent plane (the normal sections all rotate around the same part of the normal to its concavity), in the second case, the surface will lie on one side or the other of the tangent plane ([1]), and it is precisely the planes of those normal sections that will meet the first hyperbola (17) at real points and will all rotate around part of its concavity, while the remaining ones (whose planes meet the conjugate hyperbola at real points) will rotate around the contrary part. The passage from one to the other type of section will be valid when the normal plane passes through one or the other asymptote of the hyperbola (17), and then one will have:

$$\frac{1}{R} = 0$$

for the corresponding section, which points to an *inflection* in the corresponding section. These two special directions that emanate from M in the tangent plane at that point then take on the name of *asymptotic directions.* They divide the surface in the neighborhood of M into four sectors that pass from one part of the tangent plane to the other.

§ 63.

Mean curvature and total curvature.

The way in which a surface S is curves in the neighborhood of one of its points M depends essentially upon the values of the *principal radii of curvature* r_1, r_2, as we will now see. Instead of r_1, r_2, one can define that manner of curvature by giving two combinations of r_1, r_2 whose values can be inferred inversely from those of r_1, r_2. The most important functions of r_1, r_2 that come under consideration are the products and sums of the two principal curvatures $1 / r_1$, $1 / r_2$. One denotes them by:

([1]) One will arrive at the same result more briefly as follows:

Consider the tangent plane at the point (u, v) of the surface and calculate the distance δ to the infinitely-close point $(u + h, v + k)$ (in which h, k are regarded as first-order infinitesimals) of that plane; one will find:

$$\delta = \tfrac{1}{2}(D\,h^2 + 2D'\,h\,k + D''\,k^2) + \eta,$$

in which η is a third-order infinitesimal. The sign of δ then depends upon that of:

$$(\alpha) \qquad D\,h^2 + 2D'\,h\,k + D''\,k^2.$$

Now, if $D\,D'' - D'^2 > 0$ – i.e., if the point is elliptic – then the form (α) will be definite, and δ will always keep the same sign; if $D\,D'' - D'^2 < 0$ (viz., hyperbolic point) then the form (α), and therefore δ, will assume positive and negative values.

$$K = \frac{1}{r_1 r_2}, \qquad H = \frac{1}{r_1} + \frac{1}{r_2}.$$

The first one bears the name of *total* (or Gaussian) *curvature* for the surface, while the second one bears the name of *mean curvature.* If one recalls that in arbitrary curvilinear coordinates the principal radii of curvature are the roots of the second-degree equation (9), page 17, then one will indeed get the general values of K and H:

$$\text{(18)} \qquad \left\{ \begin{aligned} K &= \frac{DD'' - D'^2}{EG - F^2}, \\ H &= \frac{2FD' - ED'' - GD}{EG - F^2}, \end{aligned} \right.$$

in which the right-hand sides are *absolute invariants* of the two fundamental forms (cf., § 39)([1]).

However, from the results of § 56, one will further have the most important theorem: *The total curvature of a surface is equal to the curvature of the first fundamental form.*

That property of the Gaussian curvature (viz., that it depends upon only the coefficients of the form that represents the line element) is the one that gives the paramount importance to that curvature in geometric applications (as we will see later in the chapter on its applications). For that reason, it is often endowed with simply the name of *curvature.*

The curvature K is positive at the points of the elliptic indicatrix and negative at those of the hyperbolic indicatrix. The former are called *elliptic* points of the surface, and the latter are called *hyperbolic* points.

In general, there will exist a region of elliptic points and a region of hyperbolic points on a surface that are bounded by a line of *parabolic* points, at which the curvature K is zero.

As a complement to these observations, we shall prove the theorem: *A surface that has zero curvature at all of its points will be a developable surface.*

The fact that developables all have zero curvature results immediately from the observation that, from the theorems on the evolute of the curve (§ 18), the lines of curvature of a developable are the generators of its orthogonal trajectories; the two principal curvatures that relate to the generators are always zero.

Conversely, if the surface S has zero curvature K then one will have:

$$D\,D'' - D'^2 = 0,$$

and if one takes the coordinate lines u, v to be lines of curvature then one will have:

$$D' = 0,$$

([1]) This corresponds to the fact that total and mean curvature of a surface have a significance that is entirely independent of the chosen curvilinear coordinates on the surface.

so one will also have the vanishing of either D or D''. One then sets:

$$D = 0, \quad D' = 0.$$

From the fundamental formulas (II), page 6, one will then have:

$$\frac{\partial X}{\partial u} = 0, \qquad \frac{\partial Y}{\partial u} = 0, \qquad \frac{\partial Z}{\partial u} = 0;$$

i.e., X, Y, Z will be functions of only v. However, from the formulas:

$$\begin{cases} \dfrac{1}{\sqrt{E}}\dfrac{\partial x}{\partial u} X + \dfrac{1}{\sqrt{E}}\dfrac{\partial y}{\partial u} Y + \dfrac{1}{\sqrt{E}}\dfrac{\partial z}{\partial u} Z = 0, \\ \dfrac{1}{\sqrt{E}}\dfrac{\partial x}{\partial u}\dfrac{\partial X}{\partial u} + \dfrac{1}{\sqrt{E}}\dfrac{\partial y}{\partial u}\dfrac{\partial Y}{\partial u} + \dfrac{1}{\sqrt{E}}\dfrac{\partial z}{\partial u}\dfrac{\partial Z}{\partial u} = 0, \end{cases}$$

the second of which implies that $D' = 0$, it will result that the direction cosines:

$$\frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}, \quad \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u}, \quad \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u}$$

of the tangent to the line of curvature v are functions of only v, and thus constant along any individual line v. The lines of curvature v are then straight, and from the theorem that was stated on the evolute, S will then be developable.

§ 64.

Conjugate tangents.

Two tangents to a surface that emanate from one of its points M are called (by Dupin) *conjugate* when they are conjugate with respect to the indicatrix.

Refer u, v to the lines of curvature, and let θ, θ' denote the inclinations of the two conjugate tangents with respect to the line v. From their definitions, one will have:

$$\tan\theta \tan\theta' = -\frac{r_1}{r_2}.$$

On the other hand, if the symbol d denotes the increments of the curvilinear coordinates along the first direction, and δ denotes the ones along the conjugate direction then one will have:

$$\tan\theta = \sqrt{\frac{G}{E}}\frac{dv}{du}, \qquad \tan\theta' = \sqrt{\frac{G}{E}}\frac{\delta v}{\delta u},$$

and therefore:

$$\frac{E}{r_2}\,du\,\delta u + \frac{G}{r_1}\,dv\,\delta v = 0. \tag{19}$$

Considering the conjugate directions on the surface will also lead one to the following observation: Let C be an arbitrary curve that is traced on the surface S, which is referred to an arbitrary system of curvilinear coordinates (u, v). The tangent planes to S along C envelop a developable that circumscribes S along C. We prove that *the tangent to C at any point of that curve is the generator of the developable that is circumscribed by the conjugate tangents.* ([1]).

Hence, write down the equation of the tangent plane to S at a point (x, y, z) of C:

$$(\xi - x)\,X + (\eta - y)\,Y + (\zeta - z)\,Z = 0, \tag{20}$$

in which ξ, η, ζ denote the current coordinates. Displace (x, y, z) along C, where x, y, z, as well as X, Y, Z, are functions of the arc length s along C, and differentiate (20) with respect to s. The equation that results from this:

$$(\xi - x)\frac{dX}{ds} + (\eta - y)\frac{dY}{ds} + (\zeta - z)\frac{dZ}{ds} = 0, \tag{21}$$

which is associated with (20), gives the generator G of the indicated developable that emanates from (x, y, z). Let the symbol δ denote the increases in x, y, z when one displaces it on the surface in the direction G, and observe that the direction cosines of G are proportional to:

$$Y\frac{dZ}{ds} - Z\frac{dY}{ds}, \qquad Z\frac{dX}{ds} - X\frac{dZ}{ds}, \qquad X\frac{dY}{ds} - Y\frac{dX}{ds},$$

as are $\delta x, \delta y, \delta z$. One will get:

$$\delta x\,dX + \delta y\,dY + \delta z\,dZ = 0,$$

or, if expresses x, y, z, X, Y, Z in terms of u, v:

$$D\,du\,\delta u + D'\,(du\,\delta v + dv\,\delta v) + D''\,dv\,\delta v = 0. \tag{22}$$

If one takes the lines u, v to be the lines of curvature then, from (14), this last equation will coincide precisely with (19), and will prove the state property.

Observe that (22), which expresses the idea that the two line elements that correspond to the increases d, δ are conjugate, is constructed from the second fundamental form in the same way that the orthogonality condition (11), § 42, page 91:

([1]) It follows from this, in particular, that: *On the circumscribed developable to a surface S along a line of curvature C, the curve C will be the orthogonal trajectory of the generator.* That is a characteristic property of the lines of curvature, which can also serve to define them.

$$E\,du\,\delta u + F\,(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0$$

is constructed from the coefficients of the first fundamental form.

A double system of lines that is traced on a surface is called a *conjugate system* when the directions of the lines of the two systems that pass through any point of it are conjugate.

It is clear that one of the two systems can be taken arbitrarily, and if its equation, when solved for the arbitrary constant, is:

$$\varphi\,(u, v) = c$$

then the lines of the conjugate system will be the lines that are integrals of the first-order differential equation (cf., § 42):

$$\left(D\frac{\partial\rho}{\partial v} - D'\frac{\partial\rho}{\partial u}\right)du + \left(D'\frac{\partial\varphi}{\partial v} - D''\frac{\partial\varphi}{\partial u}\right)dv = 0.$$

In particular, observe that: *The necessary and sufficient condition for the coordinate lines u, v to form a conjugate system is that one must have* $D' = 0$.

The double system of the lines of curvature is collectively an orthogonal conjugate system, and it is the only one that is endowed with those two properties.

§ 65.

Asymptotic lines.

A line that is traced on a surface is called *asymptotic* when the tangent to the line coincides with its own conjugate at any of its points. If follows from (22) that the condition:

(23) $$D\,du^2 + 2D'\,du\,dv + D''\,dv^2 = 0$$

must be satisfied along an asymptote, and conversely, if a line of the surface satisfies the differential equation (23) then it will be an asymptote. Like the lines of curvature, the asymptotes, which have (23) for their differential equation, define a double system (which is not orthogonal, in general), and the directions of the two asymptotes that pass through any point of the surface will coincide with the asymptotes of the Dupin indicatrix.

Naturally, the asymptotes will be real only when $D\,D'' - D'^2 < 0$ (i.e., in the region of hyperbolic points) and imaginary in the region of elliptic points. It is only for the developables (§ 63) that it can happen that the two systems of asymptotic lines will coincide (with the generators of the developable).

Thus, observe that one will get the following theorem from the definition of asymptotic lines itself:

The osculating plane of an asymptotic line A at any of its points will coincides with the tangent plane to the surface. Conversely, if a line A has that property then it will be an asymptote.

In fact, the circumscribed developable to the surface along the asymptote A will have the tangent to A for its generator, which is the edge of regression.

Conversely, if the circumscribed developable to the surface along A has that line for its edge of regression then it will be an asymptote.

The property that was just observed also follows immediately and analytically from formula (11), page 18, since it results that if one has $D\,du^2 + 2D'\,du\,dv + D''\,dv^2 = 0$ along a curve then one will also have $\frac{\cos\sigma}{\rho} = 0$, and therefore either $\cos\sigma = 0$ or $1 / \rho = 0$; i.e., either the osculating plane of the line coincides with the tangent plane to the surface or the line is straight. However, in the latter case, the osculating plane will be indeterminate, so one can also regard it as coincident with the tangent plane.

§ 66.

Properties of conjugate systems.

Following Darboux ([1]), we now proceed to give some important properties of conjugate systems and asymptotic lines.

Suppose that the formulas:

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

define a surface that is referred to a conjugate system (u, v). The equation that one connects with the fundamental equations (I) of § 55, page 5 will then be $D' = 0$, which gives the theorem:

The Cartesian coordinates x, y, z of a point that moves on the surface are solutions of the same Laplace equation, which takes the form:

$$\frac{\partial^2\theta}{\partial u\,\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} \qquad \left(a = \begin{Bmatrix}1\,2\\1\end{Bmatrix}, b = \begin{Bmatrix}1\,2\\2\end{Bmatrix}\right). \tag{24}$$

Conversely, one has the theorem: *If* $x(u, v)$, $y(u, v)$, $z(u, v)$ *are solutions of the same Laplace equation* (24) *then the lines* (u, v) *on the surface:*

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

will indicate a conjugate system

([1]) V. I, pp. 127, *et seq.*

Moreover, one will then have, in fact:

$$\begin{vmatrix} \dfrac{\partial^2 x}{\partial u\,\partial v} & \dfrac{\partial^2 y}{\partial u\,\partial v} & \dfrac{\partial^2 z}{\partial u\,\partial v} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0;$$

i.e., $D' = 0$.

Now, suppose that the lines u, v are asymptotes. In such a case, one will have, from (23):

$$D = 0, \quad D'' = 0,$$

and equations (I), § 55 will give the theorem:

The coordinates x, y, z of a point that moves on a surface, when expressed as functions of the parameters u, v of the asymptotic lines, simultaneously satisfy two equations of the form:

$$(25)\qquad \left\{\begin{aligned} \frac{\partial^2\theta}{\partial u^2} &= \alpha\frac{\partial\theta}{\partial u} + \beta\frac{\partial\theta}{\partial v}, \qquad \alpha = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}, \qquad \beta = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}, \\ \frac{\partial^2\theta}{\partial v^2} &= \gamma\frac{\partial\theta}{\partial u} + \delta\frac{\partial\theta}{\partial v}, \qquad \gamma = \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}, \qquad \delta = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}. \end{aligned}\right.$$

Conversely, if two simultaneous equations (25) admit three common linearly-independent solutions x, y, z ([1]) then the formulas:

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v)$$

will define a surface that is referred to its asymptotic lines.

This property can serve to give the analytic proof of the theorem:

Projective transformations preserve the conjugate systems and asymptotic lines of a surface ([2]).

A projective transformation is given by the formulas:

$$x' = \frac{\alpha}{\delta}, \qquad y' = \frac{\beta}{\delta}, \qquad z' = \frac{\gamma}{\delta},$$

([1]) The system (25) must then constitute an unlimited integrable system.

([2]) Geometrically, this results immediately from the fact that the circumscribed developable to a surface along a curve will change into the circumscribed developable to a transformed surface along the transformed curved under a projective transformation.

in which α, β, γ are complete linear expressions in x, y, z, and will therefore be solutions of (24) if (u, v) is a conjugate system or of the system (25) if u, v are asymptotic. However, if one sets:

$$\theta' = \frac{\theta}{\delta}$$

then (24) will be transformed in an analogous equation for θ', and similarly, the system (25) will be transformed into a system of the same form, which proves that asserted property. In the next chapter, which treats tangential coordinates, one will similarly see that the dualistic – or spatial reciprocal – transformations also possess that property (see § 82).

One also has systems of lines of curvature for the conjugate systems (u, v). One can then ask what special properties belong to equation (24) that x, y, z satisfy. In that case, one can see, with Darboux, that:

$$x^2 + y^2 + z^2$$

will also be a solution of (24). In fact, set:

$$\rho = x^2 + y^2 + z^2,$$

so it will result from (I), § 55 that:

$$\frac{\partial^2 \rho}{\partial u \, \partial v} - \begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix} \frac{\partial \rho}{\partial u} - \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix} \frac{\partial \rho}{\partial v} = 2F,$$

and then ρ will be a solution of (24) if *and only if* $F = 0$.

With that observation, Darboux gave an elegant proof of the theorem: *The inversion by reciprocal radius vectors preserves the lines of curvature.* The known formulas for that inversion are, in their simplest form:

$$x' = \frac{R^2 x}{x^2 + y^2 + z^2}, \qquad y' = \frac{R^2 y}{x^2 + y^2 + z^2}, \qquad z' = \frac{R^2 z}{x^2 + y^2 + z^2}.$$

Now, since:

$$\rho = x^2 + y^2 + z^2$$

is a solution of (24) in the present case, the transformation:

$$\theta' = \frac{R^2 \theta}{\rho}$$

will change (24) into an equation of the same type that obviously is satisfied by:

$$x', y', z',$$

and also $x'^2 + y'^2 + z'^2 = R^4 / \rho$, since $\theta = R^2$ is a solution of (24) ([1]). From the preceding observation, the lines (u, v) will also be lines of curvature on the surface S' that is the locus of the point (x', y', z').

§ 67.

Particular cases.

We shall give some applications of the results of the preceding number.

1. Consider the equation ([2]):

$$\frac{\partial^2 \theta}{\partial u \, \partial v} = 0,$$

whose general integral is the sum of two arbitrary functions, one of which is u, and the other of which is v. Consequently, take:

$$x = f_1(u) + \varphi_1(v), \qquad y = f_2(u) + \varphi_2(v), \qquad z = f_3(u) + \varphi_3(v) \tag{26}$$

on a surface for which the lines (u, v) define a conjugate system. That surface is called a *surface of translation* because it is generated by the translatory motion of a curve whose points describe just as many congruence curves by translation. In fact, it is enough to give a translatory motion to the curve:

$$x = f_1(u), \qquad y = f_2(u), \qquad z = f_3(u),$$

in which each of its points describes a curve that is congruent to the curve:

$$x = \varphi_1(v), \qquad y = \varphi_2(v), \qquad z = \varphi_3(v).$$

It is clear that there are two ways of generating that surface; viz., it will arise from translating a curve u or a curve v.

One can, with Lie, consider the surface of translation that is generated in the following way. Take the two curves:

$$\begin{aligned} x &= 2f_1(u), & y &= 2f_2(u), & z &= 2f_3(u), \\ x &= 2\varphi_1(v), & y &= 2\varphi_2(v), & z &= 2\varphi_3(v). \end{aligned}$$

The surface is the locus of points between all of the segments that connect a point of the first curve with a point of the second one.

One observes that the differential equation of the asymptotes for the surface of translation is given by:

([1]) DARBOUX, v. I, page 208.
([2]) DARBOUX, v. I, page 98, *et seq.*

$$\begin{vmatrix} f_1''(u) & f_2''(u) & f_3''(u) \\ f_1'(u) & f_2'(u) & f_3'(u) \\ \varphi_1'(v) & \varphi_2'(v) & \varphi_3'(v) \end{vmatrix} du^2 + \begin{vmatrix} \varphi_1''(v) & \varphi_2''(v) & \varphi_3''(v) \\ f_1'(u) & f_2'(u) & f_3'(u) \\ \varphi_1'(v) & \varphi_2'(v) & \varphi_3'(v) \end{vmatrix} dv^2 = 0,$$

and if one supposes in particular that:

$$f_2 = 0, \qquad \varphi_1 = 0$$

then the variables will separate; i.e.: *The asymptotes of a surface of translation whose generating curves are in perpendicular planes are obtained by quadratures.*

2. In the second place, consider the equation [1]:

$$(27) \qquad (u - v)\frac{\partial^2\theta}{\partial u\,\partial v} = m\,\frac{\partial\theta}{\partial v} - n\frac{\partial\theta}{\partial u}.$$

One sees immediately that:

$$\theta = A\,(u - a)^m\,(v - a)^n$$

will be a solution, no matter what the constants A, a are. Then take:

$$x = A\,(u - a)^m\,(v - a)^n, \qquad y = B\,(u - b)^m\,(v - b)^n, \qquad z = C\,(u - c)^m\,(v - c)^n,$$

and get a surface on which the lines u, v trace out a conjugate system. One finds that the differential equation of the asymptotes of that surface is:

$$\frac{m(m-1)\,du^2}{(u-a)(u-b)(u-c)} = \frac{n(n-1)\,dv^2}{(v-a)(v-b)(v-c)},$$

which is integrated by quadrature with elliptic functions.

If $m = n$ then the equation of the surface will be:

$$\left(\frac{x}{A}\right)^{1/m}(b - c) + \left(\frac{y}{B}\right)^{1/m}(c - a) + \left(\frac{z}{C}\right)^{1/m}(a - b) = (a - b)\,(b - c)\,(a - c),$$

and the integral of the asymptotes will be algebraic in u, v.

In particular, consider the case of:

$$m = n = \tfrac{1}{2},$$

and observe that $u + v$ is then an integral of (27). One sees that if one takes:

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[1] DARBOUX, v. I, pp. 242.

$$A^2 + B^2 + C^2 = 0$$

then the lines u, v will be precisely the lines of curvature of the second-degree surface, since $x^2 + y^2 + z^2$ is a solution of (27). Therefore, for the ellipsoid:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1, \qquad \alpha^2 > \beta^2 > \gamma^2,$$

it is enough to take:

$$x^2 = \frac{\alpha^2(\alpha^2+u)(\alpha^2+v)}{(\alpha^2-\beta^2)(\alpha^2-\gamma^2)}, \qquad y^2 = \frac{\beta^2(\beta^2+u)(\beta^2+v)}{(\beta^2-\gamma^2)(\beta^2-\alpha^2)}, \qquad z^2 = \frac{\gamma^2(\gamma^2+u)(\gamma^2+v)}{(\gamma^2-\alpha^2)(\gamma^2-\beta^2)},$$

in which u varies between $-\gamma^2$ and $-\beta^2$, and v varies between $-\beta^2$ and $-\alpha^2$, and all of the points of the ellipsoid will be real (in elliptic coordinates).

§ 68.

Lines and principal radii of curvature in Cartesian coordinates.

One must often apply the general formulas of the present chapter to the case in which the equation of the surface is given only in the form $z = z(x, y)$ in Cartesian orthogonal coordinates. With the Monge notation, set:

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x\,\partial y}, \qquad t = \frac{\partial^2 z}{\partial x^2};$$

if we intend that $u = x$, $v = y$ then the coefficients E, F, G of the line element will be:

$$(\alpha) \qquad E = 1 + p^2, \qquad F = p\,q, \qquad G = 1 + q^2.$$

It then results that the direction cosines of the normal will be:

$$(\beta) \qquad X = \frac{-p}{\sqrt{1+p^2+q^2}}, \qquad Y = \frac{-q}{\sqrt{1+p^2+q^2}}, \qquad Z = \frac{1}{\sqrt{1+p^2+q^2}}$$

The coefficients D, D', D'' of the second fundamental form will then be:

$$(\gamma) \qquad D = \frac{r}{\sqrt{1+p^2+q^2}}, \qquad D' = \frac{s}{\sqrt{1+p^2+q^2}}, \qquad D'' = \frac{t}{\sqrt{1+p^2+q^2}}.$$

The mean curvature H and the total curvature K are consequently given by the formulas:

$$(\delta) \qquad H = \frac{2pqs - (1+p^2)t - (1+q^2)r}{(1+p^2+q^2)^{3/2}},$$

$$(\delta^*) \qquad K = \frac{rt - s^2}{(1+p^2+q^2)^2}.$$

Finally, note that the differential equation of the asymptotic lines will be:

$$(\varepsilon) \qquad r\,dx^2 + 2s\,dx\,dy + t\,dy^2 = 0,$$

and that of the lines of curvature will be:

$$(\varepsilon^*) \qquad \{(1+p^2)\,s - pqr\}\,dx^2 + \{(1+p^2)\,t - (1+q^2)\,r\}\,dx\,dy + \{pqt - (1+q^2)\,s\}\,dy^2 = 0.$$

§ 69.

Calculating the differential parameters.

We conclude this chapter by giving the very important expressions for the differential parameters of x, y, z, X, Y, Z, and two functions of them:

$$\rho = (x^2 + y^2 + z^2), \qquad W = Xx + Yy + Zz,$$

the first of which represents one-half the square of the distance from the origin to the point (x, y, z) of the surface, and the second of which represents the distance from the origin to the tangent plane.

For these calculations, we give some invariant properties of the differential parameters that refer (when it is appropriate) to the lines of curvature as coordinate lines, and recall that the determinant:

$$\begin{vmatrix} \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u} & \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u} & \frac{1}{\sqrt{E}}\frac{\partial z}{\partial u} \\ \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v} & \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v} & \frac{1}{\sqrt{G}}\frac{\partial z}{\partial v} \\ X & Y & Z \end{vmatrix} = +1$$

is the determinant of an orthogonal substitution, and appeal to the formulas:

$$\frac{\partial x}{\partial u} = r_2\frac{\partial X}{\partial u}, \qquad \frac{\partial x}{\partial v} = r_1\frac{\partial X}{\partial v}$$

in that case.

If one lets:

$$\Delta_1\varphi = \frac{1}{E}\left(\frac{\partial\varphi}{\partial u}\right)^2 + \frac{1}{G}\left(\frac{\partial\varphi}{\partial v}\right)^2, \qquad \Delta(\varphi, \psi) = \frac{1}{E}\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial u} + \frac{1}{G}\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial v}$$

then one will find:

$$\Delta_1 x = 1 - X^2, \qquad \Delta_1 y = 1 - Y^2, \qquad \Delta_1 z = 1 - Z^2, \tag{28}$$

$$\nabla(x, y) = -XY, \qquad \nabla(x, z) = -XZ, \qquad \nabla(y, z) = -YZ. \tag{29}$$

One will then have:

$$\Delta_1 X = \frac{1}{r_2^2}\frac{1}{E}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{1}{r_1^2}\frac{1}{G}\left(\frac{\partial x}{\partial v}\right)^2,$$

and analogous expressions for $\Delta_1 Y$, $\Delta_1 Z$, so:

$$\Delta_1 X + \Delta_1 Y + \Delta_1 Z = \frac{1}{r_1^2} + \frac{1}{r_2^2}. \tag{30}$$

In order to calculate $\Delta_1 x$, one can refer to the general formula (§ 32, page 67):

$$\Delta_2 x = \frac{G\,x_{11} + E\,x_{22} - 2F\,x_{12}}{EG - F^2},$$

in which the x_{rs} are the second covariant derivatives of x with respect to the first fundamental form; however, according to the formulas (I), § 55 (page 5), one has:

$$x_{11} = DX, \qquad x_{12} = D'\,X, \qquad x_{22} = D''\,X,$$

so:

$$\Delta_2 x = \frac{G\,D + E\,D'' - 2F\,D'}{EG - F^2}\,X,$$

or (§ 55):

$$\Delta_2 x = -H\,X = -\left(\frac{1}{r_1} + \frac{1}{r_2}\right) X. \tag{A}$$

This important formula (of Beltrami) proves that for a surface with zero mean curvature (viz., a minimal surface), the sections that one makes with a system of parallel planes will belong to an isothermal system.

Another formula, which has great importance for the theory of applicability, is obtained by the constructing the differential parameter (§ 32, page 68):

$$\Delta_{22}\, x = \frac{x_{11}x_{22} - x_{12}^2}{EG - F^2},$$

which gives:

(*B*) $$\Delta_{22}\, x = (1 - \Delta_1\, x)\, K,$$

from the formulas that were just recalled and from (28).

This is a second-order partial differential equation for x (which is also satisfied for y and z), whose coefficients are defined by only those of the first fundamental form.

An equation of this nature will also be satisfied by:

$$\rho = (x^2 + y^2 + z^2),$$

and indeed, one will find, in the first place (when referred to the lines of curvature):

$$\Delta_1\rho = 2\rho - W^2.$$

Thus, if one observes that the second covariant derivatives of ρ are:

$$\rho_{11} = E + D\, W, \qquad \rho_{12} = F + D'\, W, \qquad \rho_{22} = G + D''\, W$$

then one have directly:

$$\Delta_2\rho = 2 - W\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$

$$\Delta_{22}\,\rho = 1 - W\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + W^2 K,$$

and if one eliminates W, W^2 from the expression for $\Delta_1\rho$, $\Delta_2\rho$, $\Delta_{22}\rho$ then one will obtain the desired formula:

(*C*) $$\Delta_2\rho - \Delta_{22}\rho = 1 + K\,(\Delta_1\rho - 2\rho).$$

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