“Sur les transformations canoniques des équations du movement d’un systèmes non holonomes,” C. R. Acad. Sci. Paris **158** (1914), 1064-1068.

On the canonical transformations of the equations of motion of a non-holonomic system

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1. – We shall determine the position of a material system by means of $n + k$ coordinates: q_i, q_{n+q} ($i = 1, 2, \ldots, n$; $\nu = 1, 2, \ldots, k$). Suppose that the motion of a system is subject to non-integrable differential constraints:

$$q'_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\, q'_i + \alpha_\nu \qquad (\nu = 1, 2, \ldots, k), \tag{1}$$

in which $a_{\nu i}$, α_ν are functions of the coordinates q_i , $q_{n+\nu}$, and time t. Let T denote the *vis viva* of the system (in the general case, it is a function of the velocities q'_i , $q'_{n+\nu}$, the coordinates q_i, $q_{n+\nu}$, and time t), and let U denote the force function, which depends upon the same coordinates and time. One can write the equations of motion of such a system (C. R. Acad. Sci. Paris, t. 156, pp. 381) in the following form:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial q'_i} - \frac{\partial\Theta}{\partial q_i} - \sum_{\nu=1}^{k}\frac{\partial\Theta}{\partial q_{n+\nu}}\,a_{\nu i} = \frac{\partial U}{\partial q_i} + \sum_{\nu=1}^{k}\frac{\partial U}{\partial q_{n+\nu}}\,a_{\nu i} + \sum_{\mu=1}^{k}\theta_\mu\left[\frac{da_{\mu i}}{dt} - \frac{\partial q'_{n+\mu}}{\partial q_i} - \sum_{\nu=1}^{k}\frac{\partial q'_{n+\mu}}{\partial q_{n+\nu}}\,a_{\nu i}\right] \tag{2}$$

$$(i = 1, 2, \ldots, n).$$

Here, Θ and θ_μ are the results of eliminating the dependent velocities $q'_{n+\nu}$ from T and $\dfrac{\partial T}{\partial q'_{n+\nu}}$ with the aid of (1). One can take the derivatives of the $q'_{n+\mu}$ upon replacing them with the right-hand side of the equality (1).

Upon introducing the new variables $p_i = \partial\Theta / \partial q'_i$ and the function:

$$H = \sum_{i=1}^{n} p_i\, q'_i - \Theta - U ,$$

in which one expresses them as things that depend upon p_i, q_i, $q_{n+\nu}$, t, we can replace equations (1), (2) with the first-order system of equations:

$$dp_i : dq_i : dq_{n+\nu} = -\frac{\partial H}{\partial q_i} - \sum_{\nu=1}^{k} \frac{\partial H}{\partial q_{n+\nu}} a_{\nu i} + S_i : \frac{\partial H}{\partial q_i} : \sum_{i=1}^{n} b_{\nu i}\, p_i + b_\nu \tag{3}$$

$$(i = 1, 2, \ldots, n\,;\; \nu = 1, 2, \ldots, k),$$

in which

$$S_i\,(p_i, q_i, q_{n+\nu}, t) = \sum_{\mu=1}^{k} \theta_\mu \left[\frac{da_{\mu i}}{dt} - \frac{\partial q'_{n+\mu}}{\partial q_i} - \sum_{\nu=1}^{k} \frac{\partial q'_{n+\mu}}{\partial q_{n+\nu}} a_{\nu i} \right].$$

There will be a change of variables in the right-hand side, as well. We obviously obtain the functions $b_{\nu i}$, b_ν , from the transformations of the constraints (1) to the new variables p_i .

2. – We say that the system (3) is transformed into canonical form if we obtain a system like the following one:

$$dP_s : dQ_s : dT = -\frac{\partial K}{\partial Q_s} : \frac{\partial K}{\partial P_s} : 1 \qquad (s = 1, 2, \ldots, n + k)$$

after introducing the new variables Q_i, $Q_{n+\nu}$, P_i, $P_{n+\nu}$, T, in which K = funct. (Q_s, P_s, T), and the new variables are coupled by k relations:

$$\varphi_\nu\,(Q_s, P_s, T) = 0 \qquad (\nu = 1, 2, \ldots, k), \tag{4}$$

as well.

It is obvious that one can reduce the solution of the problem of the motion of a non-holonomic system to the integration of the Jacobi-Hamilton equation with that transformation.

3. – The system of equations:

$$dx_1 : dx_2 : \ldots : dx_{2m+1} = X_1 : X_2 : \ldots : X_{2m+1}\,,$$

in which $X_1, X_2, \ldots, X_{2m+1}$ are functions of the $x_1, x_2, \ldots, x_{2m+1}$, can be considered to be the first system:

$$\sum_{i=1}^{2m+1} \left(\frac{\partial \Xi_i}{\partial x_j} - \frac{\partial \Xi_j}{\partial x_i} \right) dx_i = 0 \qquad (j = 1, 2, \ldots, 2m + 1)$$

for the Pfaff expression:

$$\sum_{i=1}^{2m+1} \Xi_i \, dx_i \, . \tag{5}$$

The functions Ξ_i (i = 1, 2, …, $2m$ + 1) of the variables $x_1, x_2, \ldots, x_{2m+1}$ always exist and satisfy some conditions that we can write in the symmetric form:

$$\sum_{i=1}^{2m+1} \left(\frac{\partial \Xi_i}{\partial x_j} - \frac{\partial \Xi_j}{\partial x_i} \right) X_i = 0 \qquad (j = 1, 2, \ldots, 2m + 1).$$

If the expression (5) is represented in the normal form:

$$\sum_{i=1}^{2m+1} \Xi_i \, dx_i = - \Phi \, (y_1, y_2, \ldots, y_m \, ; z_1, z_2, \ldots, z_m, y_0) \, dy_0 + \sum_{i=1}^{m} z_i \, dy_i \, ,$$

in which $y_0, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_m$ are functions of the variables x_i (i = 1, 2, …, $2m$ + 1), then the first system will correspond to the canonical form:

$$dz_i : dy_i : dy_0 = - \frac{\partial \Phi}{\partial y_i} : \frac{\partial \Phi}{\partial z_i} : 1 \qquad (i = 1, 2, \ldots, m).$$

In the case where relations exist between the variables $x_1, x_2, \ldots, x_{2m+1}$, one will have the same number of relations between the new variables y_0, y_i, z_i .

Hence, in order to transform a given system into canonical form, it is necessary and sufficient that one must know a system of particular solutions to the following equations:

$$- \sum_{i=1}^{2m+1} \left(\frac{\partial \Phi^*}{\partial x_j} \frac{\partial y_0}{\partial x_i} - \frac{\partial \Phi^*}{\partial x_i} \frac{\partial y_0}{\partial x_j} \right) X_i + \sum_{i=1}^{2m+1} \sum_{l=1}^{m} \left(\frac{\partial z_l}{\partial x_j} \frac{\partial y_l}{\partial x_i} - \frac{\partial z_l}{\partial x_i} \frac{\partial y_l}{\partial x_j} \right) X_i = 0 \tag{6}$$

$$(j = 1, 2, \ldots, 2m + 1),$$

in which Φ^*, z_i, y_i are unknown functions (Φ^* denotes the result of the expression of the function Φ as a function of the variables $x_1, x_2, \ldots, x_{2m+1}$). The system of solutions must be such that one can express the variables $x_1, x_2, \ldots, x_{2m+1}$ as functions of the variables y_0, y_i, z_i .

4. – The relations (6) give the necessary and sufficient for the canonical transformation of equations (3). We will get the relations (4) upon introducing supplementary variables $p_{n+\nu} = f_{n+\nu} \, (q_1, q_2, \ldots, q_{n+k} \, ; p_1, p_2, \ldots, p_n, t)$, in which $f_{n+\nu}$ are arbitrarily-chosen functions.

We shall confine ourselves to examining the case in which there are conditions:

$$a_\nu = 0, \qquad \frac{\partial T}{\partial q_{n+\nu}} = \frac{\partial U}{\partial q_{n+\nu}} = \frac{\partial a_{\mu i}}{\partial q_{n+\nu}} = 0 \qquad (\nu, \mu = 1, 2, \ldots, k),$$

and the *vis viva* T is a homogeneous quadratic function of the velocities. In that case, the system (3) will reduce to the system:

$$dp_i : dq_i : dt = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} \sigma_{ij} \frac{\partial H}{\partial p_j} : \frac{\partial H}{\partial p_i} : 1 \tag{7}$$

in which $\sigma_{ij} = \sum_{j=1}^{n} \theta_\mu \left(\frac{\partial a_{\mu i}}{\partial q_j} - \frac{\partial a_{\mu j}}{\partial q_i} \right)$ and θ_μ is expressed as a function of the p_i, q_i, t, and the quadratures (1) after integrating the system (7).

For the canonical transformation of (7), it is necessary and sufficient to fulfill the following conditions that must be satisfied by the functions K^*, Q_i, P_i, T :

$$\frac{\partial(K^*, T)}{\partial(\alpha, t)} - [\alpha, t] + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \left\{ \frac{\partial(K^*, T)}{\partial(\alpha, t)} - [\alpha, q_i] \right\} + \sum_{i=1}^{n} \left(-\frac{\partial H}{\partial p_i} + \sum_{j=1}^{n} \sigma_{ij} \frac{\partial H}{\partial p_j} \right) \left\{ \frac{\partial(K^*, T)}{\partial(\alpha, t)} - [\alpha, p_i] \right\} = 0 .$$

Here, α is one of the variables t, q_i, p_i, and we can set:

$$\frac{\partial(K^*, T)}{\partial(\alpha, t)} = \frac{\partial K^*}{\partial \alpha} \frac{\partial T}{\partial \beta} - \frac{\partial K^*}{\partial \beta} \frac{\partial T}{\partial \alpha}, \qquad [\alpha, \beta] = \sum_{l=1}^{n} \left(\frac{\partial P_l}{\partial \alpha} \frac{\partial Q_l}{\partial \beta} - \frac{\partial P_l}{\partial \beta} \frac{\partial Q_l}{\partial \alpha} \right) .$$

The conditions become simpler when one seeks transformations for which the coordinates will remain unchanged. Such a search belongs to Chaplygin's theory of the *reducing multiplier* N (q_1, q_2) of (Moscow Mathematical Society, v. XXVIII, pp. 303), in which the author has examined a class of such transformations: $P_1 = N\, p_1$, P_2 , $N\, p_2$, $T = \int N\, dt$ for the systems with two degrees of freedom that are subject to such conditions.

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