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On the formation of circulatory motions and vortices in frictionless fluids

by

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I – Introduction.

1. Incompleteness of the usual theories of vortices. – The fundamental laws of the conservation of fluid vortices are known to be valid only when one ascribes certain ideal properties to the fluid, in addition to its lack of viscosity, that are not befitting of natural fluids. Either one assumes complete incompressibility and homogeneity, like the famous originator of the theory ([1]), or one assumes a certain ideal compressibility and homogeneity, like later researchers, that makes the density a function of only the pressure, and indeed in such a way that the fluid will be homogeneous for a constant distribution of pressure ([2]).

Other than the lack of viscosity, the ideal properties then consist of the fact that the fluid can assume at most transient inhomogeneities, and indeed ones that are created by pressure exclusively. In what follows, we will refer to those fluids that were already examined in relation to the motion of vortices as the *homogeneous incompressible* or *ideal compressible* fluids, in contrast to the general ideal fluids, in which any possible transient or persistent inhomogeneity is allowed, and the lack of viscosity remains the single ideal property. We will investigate those ideal fluids of the most-general type in relation to the motion of vortices in what follows.

Due to the heterogeneity in the fluid that we assume to be possible, two different paths can be followed for describing the phenomena of motion since we can employ either the velocity or the product of the velocity and density as the vector quantity. The latter vector quantity, which we would like to call the *momentum*, differs from the velocity by only a constant factor in homogeneous media such that the two ways of describing things will differ from each other only trivially. However, in the case of heterogeneous media, the two ways of describing things deserve a certain independent interest of their own, and that is why we will implement our theory in two

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([1]) **Helmholtz**, *Gesammelte Abhandlungen*, I, pp. 101.

([2]) **Lord Kelvin**, "Vortex motion," Proc. Roy. Soc. Edinburgh (1869); **Kirchhoff**, *Mechanik*, Chap. 15; **Poincaré**, *Théorie des Tourbillions*.

different forms of representation throughout in what follows: Once with the use of velocity as the vector quantity and then with the use of the momentum.

2. Deriving the more-general theory of vortices. – The derivation of more-general laws of vortex motion is extremely easy.

If one employs the velocity as the vector quantity then one can go down any one of the known paths that lead to Helmholtz's laws, except that in the course of calculation, one *does not* perform those reductions that are obtained by specializing the fluid properties, but simply discusses the general expression to which the calculation will lead.

One arrives at the analogous laws for when one employs momentum as the vector quantity in an analogous way once one has replaced velocity with momentum in the general equations of motion.

However, whichever path one goes down for the derivation, one will be repeatedly led to a discussion of vector quantities of a special class that possesses the property that it can be expressed by two scalar quantities. I have dedicated a special investigation to the theory of the vector quantities in the immediately-preceding treatise ([1]), but generally in a much-broader context that is necessary for our immediate purposes on this occasion.

I have communicated the main theorems in the following theory of the formation of circulatory motions and vortices in frictionless fluids for the first time in my lectures on hydrodynamics at the Stockholm Institute in the Spring semester of 1897. As I have now seen for myself, some of those theorems were found before by **Silberstein** in the Fortschritte der Physik, 1896 ([2]). If I communicate them in what follows in connection with my own theorems then I will do that with the unconditional recognition of Silberstein's priority.

3. Velocity and momentum. Velocity vorticity and momentum vorticity. – We will denote the density of the fluid at the geometric point x, y, z by q. Along with the density, we will also consider the reciprocal density or specific volume:

(a) $$k = \frac{1}{q} \;.$$

With regard to the dynamical meaning of those quantities, we can call q the *coefficient of inertia*, and k, the *coefficient of mobility*, of the fluid at the point in question.

We will denote the velocity of the fluid at the geometric point x, y, z by the vector U, which has components U_x, U_y, U_z along the axes. Let the momentum or "quantity of motion" at that point be given by $\bar{U}$, with an analogous notation for its components. Momentum and velocity are coupled by the relation:

([1]) "Zur Theorie gewisser Vektorgrössen," Videnskabsselskabets Skrifter, Kristiania, 1898. – That treatise will be cited simply by the symbol V in what follows.

([2]) **L. Silberstein**, Bulletin International de l'Academie des Sciences de Cracovie, June 1896.

(b) $$\bar{U} = \rho U \qquad U = k\bar{U}\,.$$

Let u and $\bar{u}$ be the vorticities of the respective vector quantities U and $\bar{U}$. u and $\bar{u}$ are therefore two vector quantities that have the rectangular components:

(c) $$u_x = \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}, \qquad u_y = \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}, \qquad u_z = \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y},$$

(d) $$\bar{u}_x = \frac{\partial \bar{U}_z}{\partial y} - \frac{\partial \bar{U}_y}{\partial z}, \qquad \bar{u}_y = \frac{\partial \bar{U}_x}{\partial z} - \frac{\partial \bar{U}_z}{\partial x}, \qquad \bar{u}_z = \frac{\partial \bar{U}_y}{\partial x} - \frac{\partial \bar{U}_x}{\partial y}$$

at the geometric point x, y, z. u shall be called the *vorticity of the velocity*, and $\bar{u}$ shall be called the *vorticity of the momentum* of the fluid at that point. The vorticity of the velocity is known to be equal to twice the angular velocity of the fluid particles that have the coordinates x, y, z at the moment in question.

4. Circulation and rotation. – If the curve s is the curve of a moving material point, U is the velocity of a point on the curve, and U_t is the projection of that velocity onto the tangent to the curve then we will call the integral:

(a) $$\int U_t \,\delta s$$

at a certain time-point t the *tangential velocity* of the curve at that time t. If the tangential component $\dot{U}_t$ of the acceleration of the curve at that point appears under the integral sign instead of U_t then the corresponding integral shall be called the *tangential acceleration* of the curve. By contrast, the total time derivative of the integral (a) shall be called the *acceleration of the tangential motion* of the curve, which is a quantity that is essentially different from the tangential acceleration. If the curve is closed then the three terms above can be replaced with the terms *circulation of velocity*, *circulation of acceleration*, and *acceleration in the circulatory motion*, respectively.

For precisely the same reasons, the value of the integral:

(b) $$\int \bar{U}_t \,\delta s$$

shall be called the *tangential momentum* of the curve at time t. If the tangential component of the total time derivative of the moment enters under the integral sign then the integral shall be called the *tangential acceleration of the momentum* of the curve, while the total time derivative of the integral (b) shall be called the *acceleration in the tangential momentum* of the curve. In the case of a closed curve, those terms can be replaced with the *circulation of the momentum*, *momentum* *of the circulation of the acceleration*, and *acceleration in the momentum of the circulation*, respectively.

If σ represents a moving material surface, and u_n is the component of the velocity vorticity that is normal to the surface then the value of the surface integral:

$$\text{(c)} \qquad \int u_n \, d\sigma$$

when taken at time t, will be called the *rotation* of the surface σ at time t, and the total derivative will be called the *acceleration in the rotational motion*.

In a precisely-analogous way, if $\bar{u}_n$ is the component of the vorticity of the momentum that is normal to the surface then:

$$\text{(c)} \qquad \int \bar{u}_n \, d\sigma$$

will be called the *rotation of the momentum* of the moving surface at time t, and the total time derivative of this integral the *acceleration of the momentum of the rotation*.

The value of the integral (c) or (d), when calculated over the cross-section of vortex tube, shall be called the *rotation* or *momentum of rotation* of the vortex tube, respectively.

The terminology that is introduced in that way was chosen to be as consistent as possible with the terminology that **Lord Kelvin** used (*loc. cit.*). In order to avoid any misunderstanding, the differing meanings of the words "rotation" and "vorticity" must be emphasized. Vorticity is a vector, while rotation is a composite quantity that is defined by the integral expressions (c) or (d), and they refer to fluid surfaces and vortex tubes, resp.

II. – Circulation of velocity and vorticity of velocity.

5. The equations of motions. – As usual, we will let:

$$\frac{d}{dt}$$

or the Newtonian dot denote the total derivative with respect to time, which refers to the processes that play out at the point of the moving fluid, while the partial derivative $\partial / \partial t$ refers to the time variations that are observed at the fixed geometric points. If the coordinates x, y, z are endowed with moving fluid particles then the components of its velocity will be:

$$\text{(a)} \qquad U_x = \frac{dx}{dt}, \qquad U_y = \frac{dy}{dt}, \qquad U_z = \frac{dz}{dt}.$$

Upon developing the symbol d / dt and considering those relations, we will get:

(b) $$\frac{d}{dt} = \frac{\partial}{\partial t} + U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}.$$

If F denotes the external accelerating force and p denotes the pressure, and we consider x, y, z to be the coordinates of the moving fluid particles then the hydrodynamical equations of motion can be written in the following form:

(c) $$\frac{dU_x}{dt} = F_x - \frac{1}{q}\frac{\partial p}{\partial x}, \quad \frac{dU_y}{dt} = F_y - \frac{1}{q}\frac{\partial p}{\partial y}, \quad \frac{dU_z}{dt} = F_z - \frac{1}{q}\frac{\partial p}{\partial z}.$$

We will use that form exclusively in what follows. If we introduce (b) then we can solve for the coordinates x, y, z that relate to the moving fluid particles and then go over to simply the coordinates of the geometric points. Equations (c) will then become Euler's equations in their usual form.

We especially consider the two vector quantities whose components enter on the right in those equations.

6. The external accelerating force. – The first vector on the right F can take the form of a vector quantity of most-general nature or as a threefold-scalar vector quantity, with the terminology of the foregoing treatise. As such, F will possess a vorticity f that has the components:

(a) $$f_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \quad f_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \quad f_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y},$$

and which can be represented by a system of vortex solenoids of the external acceleration force.

However, it is quite often that one cares to assume that the vorticity components (a) are identically zero everywhere in the field. The external force is then conservative, so F can be represented by a force function φ:

(b) $$F_x = \frac{\partial \Phi}{\partial x}, \quad F_y = \frac{\partial \Phi}{\partial y}, \quad F_z = \frac{\partial \Phi}{\partial z},$$

and thus appears to be a simply-scalar or potential vector quantity, and can be represented geometrically by the equi-scalar family of surfaces:

(c) $$\Phi\,(x, y, z) = \text{const.}$$

and the corresponding equi-scalar lamellae completely (V.12).

Naturally, the intermediate case can occur in which F takes the form of a twofold-scalar vector quantity. However, that case plays no especially-prominent role.

7. Accelerating gradient. – By contrast, the second vector quantity on the right in the equations of motion is a twofold-scalar vector quantity that is reducible to a simply-scalar form in some special cases but can never achieve the generality of a threefold-scalar vector quantity.

That twofold-scalar vector, which we will denote by G, can be written in the normal form (V.6) immediately when we replace the coefficient of inertia q of the fluid with the coefficient of mobility k according to (3.a). The components of G will then be:

$$G_x = -\ k\frac{\partial p}{\partial x},\quad G_y = -\ k\frac{\partial p}{\partial y},\quad G_z = -\ k\frac{\partial p}{\partial z}, \tag{a}$$

or with the vector notation:

$$G = k\,\nabla\,(-\,p)\ . \tag{a$'$}$$

That vector belongs to the conjugate vector (V.7):

$$G' = (-\,p)\ \nabla k\ . \tag{b}$$

We will denote the simple scalar auxiliary vectors $\nabla\,(-\,p)$ and ∇k by $\bar{G}$ and B , resp.

$$\begin{aligned} \bar{G} &= \nabla(-p), \\ B &= \nabla k. \end{aligned} \tag{c}$$

$\bar{G}$ is the vector quantity that one calls the *gradient* in meteorology. We will call G itself the *accelerating gradient* because it appears in the equations of motion (5.c) in precisely the same way as the external accelerating force F. The conjugate auxiliary vector B indicates the direction and magnitude of the greatest increase in the mobility of the fluid, and for that reason, it shall be called the *mobility vector*.

All of those vector quantities can be described by two systems of equi-scalar surfaces (V.13):

$$\begin{aligned} p\,(x, y, z) &= \text{const.}, \\ k\,(x, y, z) &= \text{const.} \end{aligned}$$

The first ones are then the surfaces of equal pressure or *isobaric* surfaces. The second ones are the surfaces of equal mobility or equal specific volume. The name *isosteric* surfaces was proposed for such surfaces in meteorology. Any isosteric surface k = const. overlaps with an *equi-dense* surface q = const. However, if we prefer the term "isosteric surfaces" then that is because the *representative family of surfaces* (V.8) must refer to the specific volume k and not to the density q.

The two representative families of surfaces split the flow field into *isobaric* and *isosteric lamellae*, respectively. Furthermore, they intersect along *isobaric-isosteric curves*, and they collectively split the entire space into *isobaric-isosteric solenoids*.

The gradient $\bar{G}$ points along the normals to the isobaric lamellae and is numerically equal to the reciprocal density of the lamella. The mobility vector B will be represented by isosteric lamellae in that way. Finally, the accelerating gradient G has the direction of the gradient $\bar{G}$ everywhere and is numerically equal to the reciprocal value of the distance between the two isobaric boundary surfaces of the solenoid times its mobility coefficient. Finally, G' the conjugate vector points in the same direction as the mobility coefficient, and one will find its numerical value when one multiplies the reciprocal of the distance between the isosteric boundary surfaces of a solenoid with the constant value of the pressure inside of the solenoid.

Finally, *the vorticity g of the accelerating gradient* has the expression (V.13):

(d) $$g = \mathrm{V}\, \nabla k\, \nabla\, (-p)$$

or

(d′) $$g = \mathrm{V}\, B \bar{G}\,,$$

and is then equal to *the vector product of the mobility vector and the gradient. The vortex lines are the isobaric-isosteric curves, and the vorticity solenoids are the isobaric-isosteric solenoids*.

8. Degeneration of the accelerating gradient to a simply-scalar form. – There are three cases in which the accelerating gradient degenerates into a simply-scalar or potential vector quantity, so the vorticity g will then vanish (V.11 and 14): When the pressure is constant, when density is constant, and finally, when a relation exists between density and pressure.

Whereas an arbitrary motion can naturally appear to fulfill one of those conditions instantaneously, we can immediately overlook the first case, viz., the constancy of the pressure, along with the vanishing of the gradient and accelerating gradient that will follow from it. By contrast, the other two degeneracy conditions can be fulfilled by certain natural fluids to good approximation, such that we would be justified by representing fluids with those ideal properties in such a way that those conditions are fulfilled exactly. Those fluids are the *homogeneous incompressible* ones, in which the density is independent of coordinates and time, and the *homogeneous ideal-compressible* fluids, for which one does not consider the temperature in the relation that exists between the density and pressure. The accelerating gradient then degenerates to a simply-scalar or potential vector quantity with a single-valued potential exactly for the fluids for which one has confirmed the laws of conservation of circulation and vortex motion, but otherwise not. Our investigation will then essentially refer to the general case in which the accelerating gradient preserves its twofold-scalar nature.

Another case that is especially interesting in the applications is the improper degeneration into multiply-connected spaces (V.11): The density can appear to be a function of pressure everywhere, but that function can (for example, as a result of a forced temperature state) have different forms in different channels of space. The accelerating gradient will also be vortex-free in the entire field then; however, the potential itself will be multi-valued.

9. The tangential acceleration of a fluid curve. – We can replace the three equations of motion (5.c) with the single vector equation:

(a) $$\dot{U} = F + G\,,$$

while at the same time:

(b) $$U = \frac{dr}{dt}$$

appears in place of equations (5.a), where r is the radius vector with the projection x, y, z. At a given time-point t, we project the three vectors in equation (a) onto a curve s, and integrate from the starting point 0 to the endpoint 1 of the curve:

(c) $$\int_0^1 \dot{U}_t\,\delta s = \int_0^1 F_t\,\delta s + \int_0^1 G_t\,\delta s\,.$$

That equation says that the tangential acceleration of a fluid curve s is equal to the sum of the line integral of the external accelerating force and the accelerating gradient along the curve. That theorem, which is dynamically self-evident, can be considered to be simple a conversion of the general equations of motion that can be useful for some special purposes. All of the following theorems on the formations of velocity circulation or velocity vorticity will again be simple conversions of the theorem (c) or adaptations of them to special situations.

10. Converting the expression for the tangential acceleration. – The first integral on the right in (c) can be converted in a known way. One needs only to observe that since the integration is performed along the curve to the well-defined time t is an operation that is independent of time, the sequence of the operations d / dt and integration can be reversed. The sequence of operations δ and d / dt can be inverted for the same reason. Finally, one must use the vector equation (9.b), which is equivalent to the relations (5.a), when one observes the identity of the differentials dr and ds when r is the radius vector of the curve. After a simple calculation in Cartesian form or with vector symbols, one will then find the known formula ([1]):

([1]) Lord Kelvin, *loc. cit.*

(a) $$\int_0^1 \dot{U}_t\,\delta s = \frac{d}{dt}\int_0^1 U_t\,\delta s - \tfrac{1}{2}(U_1^2 - U_0^2)\,.$$

If we consider a closed curve, in particular, then the last term will drop out, and we will have:

(b) $$\int_0^1 U_t\,\delta s = \frac{d}{dt}\int_0^1 U_t\,\delta s\,.$$

With the use of the definitions that we established in no. **4**, we will then see that for moving curves, the tangential acceleration will be different from the acceleration in the tangential motion, in general, while the circulatory acceleration will be equal to the acceleration in the circulatory motion.

11. The circulatory motion of fluid curves and the rotational motion of fluid surfaces. – After introducing (10.a) into (9.c), we can discuss the tangential motion of an arbitrary fluid curve completely with the assistance of the general theorems on line integrals that were developed in the foregoing treatise, and especially the twofold-scalar vector quantities.

However, we shall immediately restrict ourselves to the case in which the curve is closed. Our integral theorem can then be exhibited in two equivalent forms:

(a_1) $$\frac{d}{dt}\int U_t\,\delta s = \int F_t\,\delta s + \int G_t\,\delta s\,,$$

(a_2) $$\frac{d}{dt}\int u_n\,d\sigma = \int f_n\,d\sigma + \int g_n\,d\sigma\,.$$

The first form (a_1) follows immediately from (9.c) by substitution of (10.b). The second form (a_2) follows from (a_1) by transforming it with Stokes's theorem. $d\sigma$ is then the surface element of an arbitrary fluid surface σ that has the fluid curve s as the boundary curve. u_n is the component of the velocity vorticity (3.c) that is normal to the surface, f_n is the corresponding normal component of the vorticity f (6.a) of the external accelerating force, and g_n is that of the accelerating gradient (7.d).

With the use of the terminology that was introduced in no. **4**, those equations then say the following:

The acceleration in the circulatory motion of a fluid curve is, at any time, equal to the sum of the line integral of the external accelerating force and the accelerating gradient along the curve.

The acceleration in the rotational motion of a fluid surface at each point in time is equal to the sum of the surface integral of the vorticity of the external accelerating force and the accelerating gradient over a surface.

Both theorems can be combined into a single theorem on vortex solenoids (V.25). The integral under the sign *d* / *dt* on the left in both equations represents the number of vortex solenoids in the velocity, and the two integrals on the right represent the number of vortex solenoids in the external accelerating force and the accelerating gradient that is enclosed by the curve. During its motion, the curve will surround an always-varying number of those three classes of solenoids. However, the following relationship will exist between those three numbers at each time-point during the motion:

The time derivative of the number of vortex solenoids in the velocity is equal to the number of vortex solenoids in the external accelerating force and the accelerating gradient.

In that way, one must note that the vortex solenoids in the accelerating gradient are identical to the isobaric-isosteric solenoids (7). In the case of a multiply-connected field, we also include the virtual solenoids outside of the field.

12. Conservative external force. – If the external force F is conservative then the corresponding vorticity will be zero, and equations (11.a_1 and a_2) reduce to:

$$\frac{d}{dt}\int U_t\,\delta s = \int G_t\,\delta s\,, \tag{a_1}$$

$$\frac{d}{dt}\int u_n\,d\sigma = \int g_n\,d\sigma\,. \tag{a_2}$$

We then have only two systems of vortex lines to consider in this case: those of the velocity, which are vortex lines in the original sense of the term, and the vortex lines of the accelerating gradient, which coincide with the isobaric-isosteric curves.

In order to better understand the content of the formulas (a), we consider the state of motion at a well-defined time-point *t* and let the curves of the two classes experience variations, in one case, along the actual vortex lines, and in the other, along the isobaric-isosteric lines, and we then employ the theorem that the variations of the vortex lines leaves the line integral of the primary vector along a closed curve unchanged, and indeed that is the only variation that has that property (V.22).

For the isobaric-isosteric variation, that will create a tube that consists of isobaric-isosteric solenoids. The integral on the right (a) is kept constant during that variation, and therefore the time derivative of the integral that occurs on the left, but not the integral itself. The belt-shaped curves (V.15) that run along that tube will then have different circulation velocities, but equal acceleration of the circulation velocity, in general, and all cross-sectional surfaces in that tube generally have different rotations, but equal accelerations in their rotational motion. That is why we can ascribe that rotational acceleration to the entire tube. If we apply that result to the isobaric-isosteric solenoid then we will find that:

At each point in time, the isobaric-isosteric solenoid bound a system of material tubes in the fluid that have rotational motion with unit acceleration.

In that way, one must recall the meaning that is used for the word "rotation." Moreover, one must always include the virtual solenoids in multiply-connected spaces when one would like to discuss the motion of the entire fluid with the help of that theorem.

The curve will generate a vortex tube in the proper sense of the term under the other variation. That variation can leave the integral that enters under the differentiation sign on the left in (a) constant, while the integral that enters on the right must generally vary. All belt-shaped curves of that tube will then have equal circulation velocity, but different acceleration in their circulatory motion at the moment in question, and all cross-sectional surfaces of the tube will have equal rotation, but different accelerations in their rotational motion. A moment later, the same material curves will then have differing circulatory motions, and the same surfaces will have differing rotational motions, and the material tube considered, which was a vortex tube at time t, will no longer be a vortex tube at the next moment. It is only in one case that the tube remains a vortex tube during its motion, namely, when the vortex lines coincide with the isobaric-isosteric curves. However, the simplest examples already show that such a coincidence is not necessary. We are then justified in reaching the following conclusion:

Vortex lines, vortex surfaces, and vortex tubes appear to be purely-geometric structures, in general, which have entirely-different motions than the material structures, which temporarily coincide with them at one moment.

If we mention a result of such a negative nature then that is only due to the fact that it contradicts the famous positive result of von Helmholtz, which is valid under special circumstances, and it will be all-the-more important for one to strongly emphasize that negative result, since it is not rare for one to tacitly ascribe greater generality to Helmholtz's laws than they actually possess.

13. The conservation of circulatory and rotational motion. – When the twofold-scalar density and pressure fields generate to a simply-scalar field, the vorticity g vanish identically. If the degeneration is a proper one, as we will assume, then the line integral of the accelerating gradient will also vanish, and even in multiply-connected spaces, and equations (12.a) reduce to:

$$\frac{d}{dt}\int U_t\,\delta s = 0\,, \tag{a$_1$}$$

$$\frac{d}{dt}\int u_n\,d\sigma = 0\,. \tag{a$_2$}$$

If that degeneration takes place instantaneously then those formulas say that at the moment of degeneration, all closed fluid curves will have maximal or minimal circulation of velocity, and all fluid surfaces will have maximal or minimal rotation.

However, if the degeneration takes place continually, as it must for the homogeneous incompressible or ideal incompressible fluids, in particular, then the equations can be integrated:

(b1) $$\int U_t \, \delta s = \text{const.},$$

(b2) $$\int u_n \, d\sigma = \text{const.},$$

and we find Kelvin's known equivalent for Helmholtz's laws, which says that *any fluid curve has constant circulation, and any fluid surface has constant rotation.*

We now have *only one* system of vortex lines in the fluid to consider, namely, the vortex lines in the proper sense of the term. A vortex tube will be generated by varying the curve *s* along those vortex lines, and its belt-shaped curves will have equal circulation, while its cross-sectional surface will have equal rotation.

We then consider the same material tube at a later time-point. From equation (b1), the circulation will not change for any of the belt-shaped curves. If one performs an arbitrary variational motion of one of those curves on the sheath of the tube then the value of the integral (b1) will remain unchanged. However, since, as long as vortices exist at all, the variation of the vortex line is the only one that leaves the value of the integral (b1) unchanged (V.22), that variation must be a variation of the vortex line, and as a result the tube in question will continue to be a vortex tube.

If we imagine that the cross-section of the tube decreases at infinity then it will go to a vortex line in the limit, and we will find a material curve that is, on the one hand, a vortex line, but must always remain a vortex line. We have then arrived at Helmholtz's celebrated result:

In a homogeneous incompressible or ideal incompressible fluid, we can make the remarkable simplification that *the geometric structures that we call vortex lines, vortex surfaces, and vortex tubes will carry the motion of those material structures that coincide with them once along with them.*

14. The acceleration of vorticity. – We again return to the frictionless fluids of the most-general nature. We perform the differentiation with respect to time in the left-hand side of equation (12.a2). The left-hand side will then split into two terms, one of which we move to the right-hand side. That will then give:

(a) $$\int \dot{u}_n \, d\sigma = - \int u_n \frac{d}{dt} d\sigma + \int g_n \, d\sigma .$$

If we now recall that the vorticity *u* if equal to twice the angular velocity and the vorticity of the acceleration $\dot{u}$ is equal to twice the angular acceleration.

In the degenerate case when *g* is equal to zero, the equation will reduce to:

(b) $$\int \dot{u}_n \, d\sigma = - \int u_n \frac{d}{dt} d\sigma .$$

If the vorticity u is zero then the right-hand side will vanish, and as a result, u_n, as well. Therefore, when no vorticity is present, no acceleration of vorticity will exist either and no formation of vortices can take place. However, if a vortex u exists then an acceleration of vorticity will also exist, in general, namely, as long as the area of the material surface element varies. In other words, there exist accelerations of vorticity that must appear as a result of changes in the moments of inertia of the rotating masses, and that is the only source of acceleration of vorticity that appears at all in the case where the circulatory and rotational motion remains preserved.

However, in the general case, the acceleration of vorticity will depend upon two superimposed effects: Namely, on the one hand, the moment of inertia of the rotating mass will constantly change, and on the other, the creation of new vortices will originate in the direct intervention of the vortex-generating vector g.

15. The formation of vortices. – In order to study this creation of new vortices under the simplest-possible set of relationships, we shall consider the case in which the vorticity is originally zero everywhere in the fluid. Formula (14.a) will then become:

(a) $$\int \dot{u}_n \, d\sigma = \int g_n \, d\sigma \ .$$

Since that equation is valid for any arbitrary fluid surface σ, one can conclude that under the assumed circumstances, a complete geometric identity will exist between the vector quantities $\dot{u}$ and g. We can then immediately deduce the following fact about the acceleration of vorticity at a moment when the fluid is vortex-free from our knowledge of the vector g, namely, the accelerating gradient:

Vortex acceleration will be found around the line of intersection of the isobaric and isosteric surfaces, which will be its axis, and its intensity will be equal to the reciprocal of the area of the cross-section of the isobaric-isosteric solenoid.

When that acceleration of vorticity takes place during a time element, vortices of indefinitely-weakening intensity will be completed, and the state of vorticity in the fluid that thus arises will be described by the following theorem, except for higher-order quantities:

In the first moments, the isobaric-isosteric curves are vortex lines, and the isobaric-isosteric solenoids are vortex tubes of infinitely-small, but equal, rotation.

However, during the continued motion those simple relationships that would emerge from the above will no longer exist, in general.

Those two theorems overlap completely with **Silberstein**'s theorems that were mentioned in the Introduction, except for inessential differences in their formulation.

III. – Circulation of momentum and vorticity of momentum.

16. Introduction of the momenta into the equations of motion. – After multiplying the general equations of motion (15.c) by the density q, the term on the left in the first equation can be converted in the following way:

$$q\frac{dU_\alpha}{dt} = \frac{d\bar{U}_\alpha}{dt} - \dot{U}_\alpha \cdot \frac{1}{q}\frac{dq}{dt},$$

in which $\bar{U}_\alpha$ is the x-component of the momentum, according to (3.b)

The continuity equation, in one of its known forms, then gives a different expression for $\frac{1}{q}\frac{dq}{dt}$. If we write:

(a) $$e = -\frac{1}{q}\frac{dq}{dt} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z},$$

for brevity, then the equations of motion will become:

(b) $$\begin{aligned} \frac{d\bar{U}_x}{dt} &= qF_x - e\bar{U}_x - \frac{\partial p}{\partial x}, \\ \frac{d\bar{U}_y}{dt} &= qF_y - e\bar{U}_y - \frac{\partial p}{\partial y}, \\ \frac{d\bar{U}_z}{dt} &= qF_z - e\bar{U}_z - \frac{\partial p}{\partial z}. \end{aligned}$$

The first thing that appears on the right is the external force:

(c_1) $$\bar{F} = qF.$$

If the force is conservative then the accelerating force F will be a potential vector quantity (6.b), and $\bar{F}$ will be twofold-scalar vector quantity that is represented in normal form:

(c_2) $$\bar{F} = q\,\nabla\Phi.$$

In order to study it, one must indicate the *equi-dense* surfaces q = const. and the *equipotential* surfaces Φ = const. in order to be able to employ our theorems on twofold-scalar vector quantities directly. In particular, the vorticity $\bar{f}$ of the external force is equal to the vector product of the auxiliary vectors ∇q and $\nabla\Phi$, the first of which is the external accelerating force, while we shall call the second one the *density vector:*

(c_3) $$\bar{f} = \mathrm{V}\,\nabla q\,\nabla\Phi.$$

Of the degeneracy conditions, only the constancy of the density will possess a more-general physical meaning. The assumption that a relation exists between potential and density would not carry as much weight from the physical standpoint as the previously-considered relation between density and pressure. We will then consider only the homogeneity of the fluid to be the potential condition for the external force.

The vector quantity:

$$e\bar{U}\,, \tag{d$_1$}$$

whose components enter into the second term on the right of the equations of motion, behaves in an entirely-similar way. It is a vector quantity of unrestricted generality that nonetheless transforms like a twofold-scalar vector quantity when the momentum $\bar{U}$ is a simply-scalar or potential vector quantity. If $\bar{\varphi}$ is the momentum potential in this case then our vector will become:

$$e\nabla\bar{\varphi} \tag{d$_2$}$$

and possesses the normal form directly.

The scalar quantity e represents the rate of cubic expansion of the fluid at the point x, y, z. We can refer to the surfaces e = const. as the *equi-expansion surfaces*. The field will be described completely by way of them and the equipotential surfaces $\bar{\varphi}$ = const. of the momentum. The vorticity of that vector is the vector product of the auxiliary vectors $\nabla\bar{\varphi}$ and ∇e, the first of which is the momentum, while we will call the second one the *expansion vector:*

$$\bar{d} = \mathrm{V}\nabla e\nabla\bar{\varphi}\,. \tag{d$_3$}$$

Finally, the gradient:

$$\bar{G} = -\nabla p$$

will occur as a third vector quantity, which is always a potential vector quantity, in contrast to the accelerating gradient (7).

17. The tangential acceleration of the momentum of a fluid curve. – We can replace the three equations of motion (16.b) with the one vector equation:

$$\dot{\bar{U}} = q\,F - e\bar{U} + \bar{G}\,. \tag{a}$$

We project all vector quantities into the tangent to a curve s at a given time-point t and integrate along the curve:

(b) $$\int_0^1 \dot{\bar{U}}\,\delta s = \int_0^1 q\,F_t\,\delta s - \int_0^1 e\bar{U}_t\,\delta s + \int_0^1 \bar{G}_t\,\delta s\,.$$

That equation says that the tangential acceleration of the momentum of a fluid curve is the sum of three line integrals that are calculated along the curve: the integral of the external force, that of the gradient, and minus the integral of a vector that is the product of the momentum $\bar{U}$ and the rate of expansion *e*. That equation corresponds to formula (9.c), but without being dynamically self-evident to the same degree, which is then due to the fact that the equations of motion, in the form (16.b) or (17.a), differ essentially from the usual form that one gives to the equations of motion in the mechanics of material points.

All of the following theorems on circulation of momentum and vorticity of momentum are simple conversions or specializations of this theorem.

18. Converting the expression for the tangential acceleration of the momentum of a fluid curve. – The expression for the tangential acceleration of the momentum of a curve can converted in way that is similar to the expression for the tangential acceleration above (10). One easily finds that:

(a) $$\int_0^1 \dot{\bar{U}}\,\delta s = \frac{d}{dt}\int_0^1 \bar{U}_t\,\delta s - \int_0^1 \bar{U}_t\,\delta U\,,$$

in which the last integral, in contrast to the case (10.a), is not immediately integrable.

The quantity that appears under the integral sign in the last integral can also be written:

$$q\,\delta(\tfrac{1}{2}U^2)\ .$$

Moreover, $\delta(\tfrac{1}{2}U^2)$ is the product of the line element δs with the component of the vector:

(b) $$C = \nabla(\tfrac{1}{2}U^2)$$

that is tangential to the curve, and the formula (a) can then be written in the form:

(c) $$\int \dot{\bar{U}}_t\,\delta s = \frac{d}{dt}\int U_t\,\delta s - \int q\,C_t\,\delta s\,.$$

Since *C* is, by definition (b), a simply-scalar vector quantity, the vector quantity *q C* that occurs in the last integral on the right will again be twofold-scalar.

The field of this vector quantity will be described with the help of the *equi-dense* surfaces q = const., and the surfaces of constant velocity-squared, or *isokinetic* surfaces $\tfrac{1}{2}U^2$ = const., and the vorticity of that vector is equal to the vector product of the density vector ∇q and the kinetic vector $\nabla(\frac{1}{2}U^2)$, or C. Thus:

(d) $$\bar{c} = \mathrm{V}\nabla q \nabla(\tfrac{1}{2}U^2) \ .$$

From now on, we shall no longer direct our attention to the tangential acceleration of the momentum of our curve, but to the first expression on the right in formula (b), or the acceleration of the tangential momentum of the curve. When we substitute the formula (18.a) in (17.b), we will find that quantity to be:

(e) $$\frac{d}{dt}\int_0^1 \bar{U}_t\,\delta s = \int_0^1 q F_t\,\delta s - \int_0^1 e\bar{U}_t\,\delta s + \int_0^1 q C_t\,\delta s + \int_0^1 \bar{G}_t\,\delta s \, .$$

That expression can now be discussed in general with the help of our theorems regarding line integrals of vector quantities. However, we shall move on to discuss the special case of closed curves.

19. The circulation of momentum of fluid curves and the rotation of the momentum of fluid surfaces. – If the curve is closed then the last integral (18.d) will vanish due to the potential nature of the gradient G.

We write out the formula directly in dualistic form by transforming all integrals using Stokes's theorem and introducing the vorticities $\bar{f}$, $\bar{d}$, $\bar{c}$ of the vector quantities $q\,F$, $q\bar{U}$, $q\,C$ that were defined above:

(a₁) $$\frac{d}{dt}\int \bar{U}_t\,\delta s = \int q F_t\,\delta s - \int e\,\bar{U}_t\,\delta s + \int q C_t\,\delta s$$

(a₂) $$\frac{d}{dt}\int \bar{u}_t\,d\sigma = \int \bar{f}_n\,d\sigma - \int \bar{d}_n\,d\sigma + \int \bar{e}_n\,d\sigma \, .$$

The first equation refers to the acceleration in the circulation of the momentum of the fluid curve s, while the last one refers to the acceleration in the rotation of the momentum of the surface σ. Both formulas can be expressed in one and the same theorem about the varying number of vortex solenoids of four different vector quantities that intersect the surface s or are surrounded by the belt-shaped boundary curve s, namely:

The time derivative of the number of vortex solenoids of the momentum is equal to the sum of the numbers of vortex solenoids of the external force, the product of the kinetic vector and density, and minus the product of the rate of cubic expansion and the momentum.

In order to be able to better understand the content of this theorem, we will consider three special cases such that only one of those vector quantities is active in each case.

20. Formation of vorticity in the momentum by external forces exclusively. – When the last two integrals on the right in (19.a) vanish, for whatever reason, those equations will reduce to:

$$\text{(a}_1\text{)} \qquad \frac{d}{dt}\int \bar{U}_t \,\delta s = \int q\,F_t\,\delta s\,,$$

$$\text{(a}_2\text{)} \qquad \frac{d}{dt}\int \bar{u}_t \,d\sigma = \int \bar{f}_n \,d\sigma\,.$$

After a discussion that is completely analogous to the one in no. **12**, one will come to the following theorem:

At each point in time, the vortex solenoid of the external force will bound a system of material tubes in the fluid that have an acceleration of the momentum of rotation equal to unity.

That theorem can be extended by a theorem of a negative nature by analogy with the second theorem in no. **12**. However, the negative result is not especially interesting here or in what follows.

When the external force is conservative, $q\ F$ will degenerate into a twofold-scalar vector quantity, and one will find the vortex solenoid simply by intersecting the equipotential surfaces Φ = const. and the equi-dense surfaces q = const.

We consider the case in which the force is conservative, and no vorticity is present in the momentum at the moment in question. We will then be led to the following theorem by a discussion that is patterned on the one in no. **15**:

The acceleration of the vorticity of the momentum takes place along the lines of intersection of the equipotential surfaces and the equi-dense surfaces, and with an intensity that is equal to the reciprocal area of the cross-section of the equi-dense-equipotential solenoid.

The fact that one can exhibit a further theorem by analogy with the second theorem in no. **15** is immediately clear. However, we do not need to formula that theorem explicitly.

21. Formation of vorticity in the momentum from the rate of dilatation and momentum exclusively. – If the first and last integrals on the right in (19.a) vanish, for whatever reason, then those equations will reduce to:

$$\text{(a}_1\text{)} \qquad \frac{d}{dt}\int \bar{U}_t \,\delta s = -\int e\,\bar{U}_t\,\delta s\,,$$

(a2) $$\frac{d}{dt}\int \bar{u}_t\, d\sigma = -\int \bar{d}_n\, d\sigma .$$

By analogy with the previous section, we find the following two theorems:

At each point in time, the vortex solenoid of the vector $e\bar{U}$ *in the fluid bound a system of material tubes that have an acceleration in the momentum of rotation that is equal to unity.*

If the momentum of the fluid is distributed in a vortex-free way at the moment in question then the equi-expansion surfaces and the equipotential surfaces will determine a system of vortex solenoids of the vector $e\, U$. However, since a formation of vorticity in the momentum will come about under those circumstances, one will only be able to find the vortex solenoid of $e\bar{U}$ at the initial moment of the formation of vorticity in that simple way. However, the following theorem will be true at that initial moment:

The acceleration of the vorticity of the momentum is found to take place along the lines of intersection of the equi-expansion surfaces and the equipotential surfaces of the momentum, and with an intensity that is equal to the reciprocal area of the cross-section of the solenoid that is determined by those families of surfaces.

22. Defining vorticity in the momentum in terms of the square of velocity and density exclusively. – Finally, if the first two integrals on the right in formulas (19.a) vanish for whatever reason then they will reduce to:

(a1) $$\frac{d}{dt}\int \bar{U}_t\, \delta s = \int q\, C_t\, \delta s ,$$

(a2) $$\frac{d}{dt}\int \bar{u}_t\, d\sigma = \int \bar{c}_n\, d\sigma .$$

In this case, in contrast to the previous two, we have to deal with a twofold-scalar vector quantity and its vorticity from the outset.

At each point in time, the representative equi-dense and isokinetic families of surfaces in the fluid determine a system of material tubes that have an acceleration of the rotation of momentum equal to unity,

and *at the moment of the formation of the first vortex,* the following theorem is true:

The acceleration of the vorticity of the momentum is found to take place along the lines of intersection of the equi-dense and isokinetic surfaces with an intensity that is equal to the *reciprocal of the area of the cross-section of the solenoid that is determined by those families of surfaces.*

Moreover, one can find two other equivalent forms for those theorems since one can also replace the surfaces of equal velocity-squared by the surfaces of equal momentum-squared or by the surfaces of equal kinetic energy. All of those three families of surfaces have common lines of intersection with the equi-dense surfaces.

VI. – Concluding remarks.

23. Comparing the theorems of vorticity of velocity and momentum. – When one compares the theorem thus-obtained on vorticity of velocity and momentum, probably the first impression that one gets is that the latter theorems are much more complicated. That difference is explained immediately by the binomial form of the fundamental theorem (11.a) and the trinomial form of the corresponding fundamental theorem (19.a). That comes down to the fact that in the case of vorticity of momentum, one has to work with representations that are kinematically and dynamically less familiar to us. For example, one notes that one will have momentum vortices at the boundary surface of two bodies of different densities that are coupled together in a fixed way only when they move as a whole along a tangent direction to the boundary surface.

One further notes that one will come to no new theorems on the conservation of vortex motion by the results on vorticity of momentum. In general, one can exhibit the conditions for the conservation of vorticity of momentum mathematically and derive theorems that are analogous to Helmholtz's. However, the relations that appear as equations of conditions in that way (for example, between the force potential and the density or between the square of velocity and the density) remain only assumptions of a mathematical nature that cannot be based in real or ideal fluid properties. One will encounter the single degenerate case of the twofold-scalar field that is reducible to fluid properties when one assumes that q = const. and thus assumed homogeneity and incompressibility of the fluid. However, in that case, momentum and velocity, and the vorticity of momentum and velocity will differ by only a constant factor, and one will come back to simply the Helmholtz laws for homogeneous, incompressible fluids.

Nonetheless, if the momentum theorems might seem more complicated and less fruitful than the velocity theorems on first glance then there are still some cases in which the former are to be preferred absolutely. That is based, above all, on a certain difference in the way that the vorticities of velocity and momentum are defined to which we will now draw attention, but without going into that question in full generality.

24. Impulsive and progressive formation of vortices. – We imagine that no external forces act on the fluid, such that the cause of any formation of vortices that might occur would be purely hydrodynamical, and we consider only the onset of the motion from a moment of rest. The circulation of the velocity or the vorticity of the velocity will then arise from equations (12.a), while the circulation of momentum and vorticity of momentum will be defined by equations (21.a) and (22.a).

In order for a motion from the rest state to take place at all, the accelerating gradient G must be non-zero at the initial moment of the motion. The same thing must be true of the vorticity g of the accelerating gradient when no degenerate case is present. Thus, the right-hand sides of equations (12.a), and as a result, the left-hand sides, as well, will be non-zero at the initial moment, or the motion will already begin from the first moment on with finite values for the acceleration of the circulatory motion or the rotational motion.

However, at the initial moment of the motion, the velocity and the momentum will be zero at each point in the fluid, and as a result, the square of velocity and the kinetic vector C, as well. The left-hand sides of equations (21.a) and (22.a) will then vanish identically, and the acceleration in the circulation of the momentum of a fluid curve or in the rotation of the momentum of a fluid surface will have the value zero at the initial moment. It is only later in the course of motion that this acceleration will achieve a finite value, namely, when the velocity has attained a finite value.

We then have the important difference that the vorticity of the velocity already develops with full force from the first instant of the motion, while the vorticity of the momentum is created only progressively once the motion has achieved a finite intensity.

If we then consider the velocity distribution or the momentum distribution a short time after the onset of the motion then the following theorem will be true:

While the momentum will possess a potential function at first moments of the motion, except for higher-order quantities, the velocity will not, in general.

If one considers the special case of an incompressible, but still heterogeneous, fluid in which a finite motion is created instantaneously by an impulsive motion of the wall of the vessel then that result will reduce to a known theorem that goes back to Lord Kelvin ([1]).

25. Applications of the theory. – I will not go into the details of the use of the theorems that were developed above here. Only the two most important domains of application will be suggested now.

The motions of the atmosphere and the oceans are always motions of a circulating or vorticial nature, and that is why they can be discussed with the help of our theorems. The primary cause of the motions in those two terrestrial media always appears in the form of those differences in density that *are not* created by pressure, but are based in other sources, namely, in the first place, the nonuniform heating, and in the second place, in the material heterogeneity that follows from the varying humidity of air and the varying salinity of seawater. When one neglects differences in density of that nature, as one ordinarily does in hydrodynamical investigations, all of the driving forces in the atmosphere and oceans will drop out, and that is why the results that are obtained after neglecting such things can be utilized by the meteorologists and hydrographers only to a limited extent. We did not neglect those things above, and that is why we are justified in asserting

([1]) *Treatise on Natural Philosophy*, I., § 317.

that all motions of the ocean and the air must take place in agreement with our theorems. Even the force of friction, which have not considered explicitly, can be considered to be something that is included in the most-general forms of our theorems (11.a) or (19.a) since the foreign force *F* is subject to no restrictions and can therefore also include the force of friction.

For applications of that nature, the theorems on circulation of velocity and vorticity of velocity would prove to be most suitable, while the momentum theorems would offer only occasional advantages.

One will find the most-important applications of the momentum theorems in a different domain of problems. It is known that deep-rooted analogies exist between hydrodynamical phenomena, on the one hand, and electric and magnetic phenomena, on the other. The full scope of those analogies is still not known, and that is why its exploitation for the introduction of more-general methods in mathematical physics, as well as for a more rigorous systematization of that science is, still not possible. However, the three theorems (20), (21), and (22) seem to be the best tool for the researching the extent of those analogies completely, but the first theorem will first require a conversion that is closely linked with one of the greatest difficulties that one encounters in the investigation of those analogies.

It was the ambition to find a more-general derivative of the theorems of Professor C. A. Bjerknes on those analogies that led me to the theorems regarding the circulation of momentum and the vorticity of momentum. The simpler theorems on the circulation of velocity and the vorticity of velocity were then easy to exhibit.

I will return to the applications of the theorems in the two suggested directions in later works.