"Das dynamische Princip der Circulationsbewegungen in der Atmosphäre," Meteor. Zeit. **17** (1900), 97-106.

The dynamical principle for circulatory motions in the atmosphere

By **V. Bjerknes**, Stockholm.

(With 12 figures)

Translated by D. H. Delphenich

The hydrodynamical equations of motion undoubtedly contain the key to understanding all atmospheric motions. However, one encounters the great difficulty that one cannot represent the integrals of those equations for the complicated conditions that exist in the atmosphere. That is why in order to introduce the methods of rational dynamics into meteorology, one must try to go down a path that will make use of the dynamical principles that are included in those equations without it being necessary for one to be able to integrate the equations. In that way, one will hardly be able to go down a better path than the one that **von Helmholtz** and Lord **Kelvin** went down when the former developed the laws of vortex motion in ideal fluids and the latter developed the laws of circulatory motion, which are mathematically equivalent to them.

As is known, one will arrive at the original Helmholtz-Kelvin laws when one starts from the equations of motion for frictionless fluids and extending it by a restricting assumption that consists of either assuming that the fluid is homogeneous and incompressible or also that the density of the fluid is assumed to be a function of only the pressure. As is known, that assumption is just as inappropriate to the air in the atmosphere as that of being frictionless. The laws also teach that the circulation and vorticial motions can neither arise nor decay and that is why the fundamental question of the primary formation of those motions is left untouched such that they would have only a very restricted application in meteorology. However, one needs only to pursue the same path that led to the Helmholtz-Kelvin laws, but while assuming properties that are closer to those of natural fluids in order to get more general ways that include laws of creation and annihilation of atmospheric circulations or vorticial motions.

One would do best to carry out those generalizations stepwise, and indeed one can advance by way of the following model:

1. With **von Helmholtz** and Lord **Kelvin**, one starts from the equations of motion for frictionless fluids but introduces *no restricting assumptions in regard to the density of the fluid* in the course of calculation.

2. One develops the corresponding laws by starting from the *equations of motion of frictionless fluids.*

A conversion of the laws that are found will ultimately be important in order to put them into a form that would be most convenient for the applications.

3. One refers all laws to a *rotating coordinate system* in order to be able to consider the applications to the circulatory or vorticial motions relative to the rotating Earth.

Of all those generalizations, the first one is, beyond compare, the most important. That is because it is the one by which one will achieve an exhaustive treatment of the primary causes of motions in the atmosphere, which are known to be found in the differences in density that are based upon temperature. The other two generalizations show only how the motion that was created will already be modified in its further evolution, in part by the "deflecting force of Earth rotation," which seeks to alter the direction of the motion that was created as long as one considers it relative to the rotating Earth, and in part by the friction that seeks to compensate for all differences in velocity.

In what follows, I would address the first generalization and its meteorological applications exclusively. Therefore, I will derive the theorem in the context of Kelvin's form of representation only as a theorem regarding circulation. That is because that form has significant practical advantages over the otherwise mathematically equivalent form in which the uses Helmholtz's concept of vorticity as a basis.

In what follows, I will give the derivation of the theorem the most elementary form possible by starting from general dynamical principles and not from the hydrodynamical equations of motion. In regard to the other derivations, as well as other forms of the theorem and other applications than the purely-meteorological ones, allow me to refer to my earlier works ([1]). I shall also refer to an article by **L. Silberstein**, who was the first to examine the generalization of Helmholtz's laws that comes into question here ([2]).

The first of the following five sections include the definition of the concept of circulation and the derivation of the mathematical properties of that concept that are necessary for what follows. The second section describes a geometric representation of the dynamical state of a fluid that is just as important for the derivation of the theorem as it is for the applications. Finally, the third section gives the derivation of the fundamental theorem on circulation, and the last two sections are concerned with the applications of the theorem to the atmospheric motions. In particular, I wish to emphasize that the explanations and advice of Hrn. Dr. **N. Eckholm** have been very useful to me in the writing of the last section.

I. – Circulation.

We consider a connected chain of fluid particles that defines a closed curve. Each of those particles has a well-defined velocity U, and we let the component of that velocity that is tangential to the curve be denoted by U_t . Upon summing those quantities along the curve, that will give:

(1) $$C = \int U_t \, ds ,$$

([1]) **V. Bjerknes**, "Ueber die Bildung von Cirkulationsbewegungen und Wirbeln in reibungslosen Flüssigkeiten," Videnskabsselkabets Skrifter, Christiania (1898). – "Ueber einen hydrodynamsichen Fundamentalsatz und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres," Kongl. svenska Vetenschapsakademiens Handlingar, Stockholm **31** (1898).

([2]) **L. Silberstein**, Bulletin International de l'Académie des Sciences de Cracovie (1896).

in which ds is a line element of the curve. With Lord **Kelvin**, we shall call the quantity C that was just found the *circulation* of the curve s ([1]).

In regard to that concept of circulation of the fluid curves, let it be initially remarked that one can find its true value in the context of arbitrary curves in the atmosphere by observing the wind. As an example, consider a curve that runs along the surface of the Earth as a meridian arc from the pole to the equator and returns from the pole to the equator along a similar meridian arc at the height of the highest cirrus clouds. We can employ suitable degrees of the meridian as curve elements and choose the positive direction on the curve to be the one that leads from the pole to the equator on the Earth's surface and from the equator to the pole high up. The East-West components of the wind that are perpendicular to those curves will not come under consideration, but only the North-South components that point along the curve. For each degree along the meridian, one takes the product of the mean North-South component of the wind along the degree of the meridian and then sums over all of them. The vertical velocities will not come under consideration in the summation because first of all the vertical part of the curve is vanishingly short in comparison to the horizontal part, and secondly, because the vertical velocities have very small values in comparison to the horizontal. On finds the velocities in question on the Earth's surface by the usual wind measurements and the ones at higher altitudes by observing the motions of cirrus clouds. Upon dividing by the total length of the curve, one will get the mean velocity in the tangential direction to the curve.

One can consider the value C of the circulation of that curve to be a measure of the circulatory motion in the atmosphere between the pole and the equator. In that way, the instantaneous value of the circulation that one finds by the use of simultaneous observations, as well as mean values over longer time periods, such as months, seasons, or entire years, can come under consideration.

A simple property of integrals of the form (1) is used in the derivation of the fundamental dynamical law, as well as in all of its practical applications. We consider a series of curves 1, 2, 3, …, n (Fig. 1) that follow each other at their endpoints, and let C_1, C_2, C_3, …, C_n, resp., be the corresponding values of the line integral (1). If we take the sum of all of those line integrals while fixing one and the same positive direction of traversal along all curves then, as one sees from a consideration of the figure, the line integrals along all curve segments that have two curves in common will cancel. That is because the corresponding line integrals will appear once in the sum as positive and once as negative. For that reason, the result of the summation will be simply equal to the line integral C along the external contour:

$$C = C_1 + C_2 + C_3 + \ldots + C_n \,, \tag{2}$$

or:

The sums of the line integrals along a series of curves that follow each other at their endpoints is equal to the line integral along the common exterior contour.

That is why the circulation between the pole and the equator that was considered as an example above can be considered to be the sum of a series of individual circulations, in which one can have, for example, one of the circulations in the trade wind zone, one of the circulations in the middle latitudes, and one circulation in the polar region. Those individual circulations can be separated

([1]) Sir **W. Thomson**, "On vortex motion," Proc. Roy. Soc. of Edinburgh (1869), **§ 60**.

completely, and afterwards, one can simply obtain the total circulation between the pole and the equator by summation.

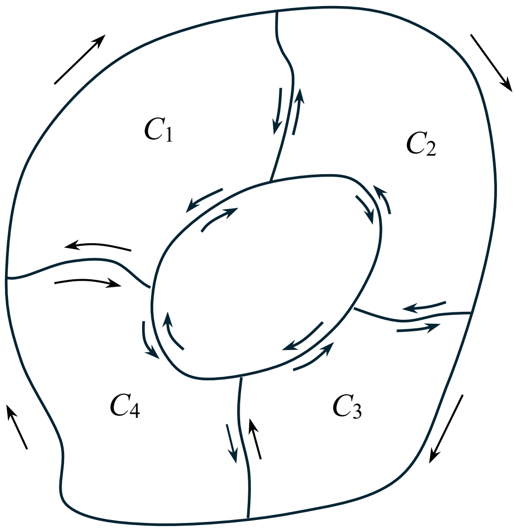

Figure 1.

Our problem is now to find the law by which the circulation in an arbitrary chain of air particles varies over time under the given conditions. In order to prepare to solve that problem, it would be convenient for us to investigate the mathematical expression for the derivative of the circulation with respect to time.

It would be most suitable for us to employ rectangular coordinates x, y, z. If dx, dy, dz mean the projections of the line element ds of the curve along those axes then the expression for the line integral (1) will become:

$$C = \int U_x \, dx + U_y \, dy + U_z \, dz \, .$$

When we differentiate that expression with respect to time, we must recall that the curve is in motion such that not only the velocity components U_x, U_y, U_z, but also the projections dx, dy, dz of the line element ds will vary in time. That is why differentiation will give:

$$\frac{dC}{dt} = \int \frac{dU_x}{dt} dx + \frac{dU_y}{dt} dy + \frac{dU_z}{dt} dz + \int U_x \frac{d}{dt} dx + U_y \frac{d}{dt} dy + U_z \frac{d}{dt} dz \, .$$

In the expression on the right, we initially examine the value of the second integral. The differentiation with respect to time that is denoted by d / dt and the operation by which we divide the curve into line elements for the sake of performing the integration along the curve from a well-defined time-point are mutually-independent operations. We can therefore switch the order of those operations and write the integral in question as:

$$\int U_x \, d \frac{dx}{dt} + U_y \, d \frac{dy}{dt} + U_z \, d \frac{dz}{dt} \, .$$

However, the derivations of x, y, z with respect to t are now nothing but the components of the velocity U_x, U_y, U_z of the point x, y, z on the curve:

$$\frac{dx}{dt} = U_x \, , \qquad \frac{dy}{dt} = U_y \, , \qquad \frac{dz}{dt} = U_z \, ,$$

such the integral in question will become simply:

$$\int U_x \, dU_x + U_y \, dU_y + U_z \, dU_z \, ,$$

or

$$\int \tfrac{1}{2} d\left(U_x^2 + U_y^2 + U_z^2\right).$$

For that reason, only the first integral on the right will remain in the expression above for the time derivative of the circulation C, and that integral has a simple meaning. Namely, the derivatives of the velocity components U_x, U_y, U_z are the components of the acceleration V of point x, y, z on the curve:

$$\frac{dU_x}{dt} = V_x, \qquad \frac{dU_y}{dt} = V_y, \qquad \frac{dU_z}{dt} = V_z.$$

The time derivative of the circulation will then become:

$$\frac{dC}{dt} = \int V_x\,dx + V_y\,dy + V_z\,dz.$$

That means that when we let V_t denote the component of the acceleration in the tangent direction to the curve:

$$\frac{dC}{dt} = \int V_t\,ds \tag{3}$$

or

The increase in the circulation per unit time for a closed curve is equal to the integral of the tangential component of the acceleration along the curve.

In order to find the dynamical law of the variation of circulation in time, we only need to integrate the component of the acceleration, which originates in the forces that are in effect, in the direction that is tangent to the curve. All accelerating forces whose line integral along closed curves is zero will then be meaningless. That leads to a very meaningful simplification of our problem. That is because it is known that all accelerating forces of a conservative nature have that property. Therefore, we never need to include gravity when we are considering the circulation of closed curves in the atmosphere since it is a conservative force.

When we likewise leave aside our assumptions regarding friction and the deflecting force of the Earth rotation, we will only have to address the accelerating force that is due to fluid pressure. It will be easy for us to determine the line integral of that force once we have consider a geometric representation for the dynamical state in the interior of gaseous or fluid media.

II. – Geometric representation of the dynamical state in fluid or gaseous media.

The distribution of the pressure p in any gas or any fluid can be described with the help of surfaces of equal pressure p = const. or *isobars*. The gradient G is perpendicular to the isobaric surfaces, and points in the direction of increasing pressure. If n is the normal to an isobaric surface that points in the direction of increasing values of pressure then the expression for the *gradient* can be written:

(4) $$G = -\frac{dp}{dn}.$$

It would be especially appropriate to indicate the isobaric surfaces for a pressure difference of unity. For a suitable choice of unit, one can always arrange that the isobaric surfaces lie sufficiently close to each other that they will give a sufficiently complete picture of the pressure distribution in the fluid.

The acceleration that the gradient imparts upon a particle in the fluid depends upon its inertia, that is, the density of the particle. It is equal to the gradient divided by the density, or even simpler: It is equal to the gradient multiplied by the specific volume k of the fluid element in question. Therefore, in order to be able to give the distribution of acceleration that is based in the pressure, it is enough to know not only the pressure distribution, but also the distribution of the specific volume in the fluid. That distribution can be described with the help of surfaces of equal specific volume k = const. or *isosteric* surfaces. We also always imagine indicating those surfaces for unit differences in specific volume, and thus choose a unit of appropriate magnitude that the surfaces will lie sufficiently close to each other that they will represent the distribution of specific volume in all regions of the fluid with satisfactory accuracy.

We can define a vector quantity B by the equation:

(5) $$B = \frac{dk}{dn},$$

by analogy with the gradient, in which n is the normal to an isosteric surface that points in the direction of increasing specific volume. B is therefore a vector quantity that points in the direction of increasing specific volume, and since the mobility of the fluid increases with specific volume, we can call B the *mobility vector*. Observe that we have used the positive sign in equation (5), while the negative sign occurs in the defining equation (4) of the gradient. A vector quantity $-B$ that would be defined to be in even-more-complete analogy with (4) would generally have a direction that is opposite to the direction of the gradient with reasonable accuracy since increasing specific volume mostly follows decreasing pressure. On the other hand, the mobility vector B mostly has approximately the same direction as the gradient G, and that is why $-B$ is to be preferred in the applications.

Some general remarks regarding the evolution of isobaric and isosteric surfaces would be important.

First, one should note that an isobaric surface can never vanish in the interior of the fluid: It must either return to itself or end against the boundary surfaces of the fluid. For example, the isobaric surfaces in the atmosphere either surround the entire Earth as a closed surface that mostly coincides very closely with the level surfaces of gravity or it ends on the surface of the Earth, which intersect them along the isobaric curves that we can find by ordinary barometric observations.

The isosteric surfaces possess precisely the same property. Those surfaces cannot vanish in the interior of the fluid, any more than the isobars, and otherwise they must continue until they either run back into themselves or end on the boundary surfaces of the fluid. In the atmosphere, largely speaking, they have the same behavior as the isobars. The upper ones surround the Earth, while the lower ones intersect the surface of the Earth along isosteric surfaces.

A second property of the isobaric surfaces is that two successive surfaces that represent different values of the pressure p can never intersect. They must be separated by an isobaric layer over their entire extent that has the same property as the surfaces in its own right, namely, it either returns to itself or ends on the boundary surfaces of the fluid. The successive isosteric will be separated by corresponding isosteric layers.

The two families of surfaces collectively divide all of space into tubular structures that we can refer to as *isobaric-isosteric tubes*. One concludes from the properties that belong to the isobaric and isosteric layers that those tubes will likewise have the property that they either return to themselves or end against the boundary surfaces of the fluid. When the surfaces are indicated for unit differences in the pressure and specific volume, we can call the corresponding tubes *unit tubes*. Assuming that we employ the aforementioned units of suitable magnitude, we will call the corresponding unit tubes *solenoids*, which are regarded as infinitesimal. The cross-section of the larger isobaric-isosteric tubes have the form of curvilinear rectangles, while the cross-sections of the solenoids are rectilinear parallelograms.

Since the solenoids have the property that they either return to themselves or end against the boundary surfaces, every closed curve in the fluid will enclose a well-defined bundle of solenoids: The number A of solenoids in that bundle will be a uniquely-determined number as soon as one has chosen the units for specific volume and pressure.

III. – Derivation of the fundamental dynamical law for circulation.

In order to examine the dynamical conditions for the appearance of circulatory motions as a consequence of fluid pressure, we shall consider a region of the fluid that is so small that we can regard the specific volume and pressure are linearly-varying quantities inside of it. In that region of the fluid, the isobaric surfaces will behave like a family of parallel, equidistant planes, and the isosteric surfaces, like a different family of parallel, equidistant planes. The solenoids are tubes whose cross-sections define a system of mutually-congruent parallelograms. The gradient will have unvarying magnitude and direction everywhere in that region, and the same thing will be true for the mobility vector.

When all particles of the region in question have equal specific volume, the gradient of all points would communicate equally-large accelerations, and the result of the action of the gradient during a time interval would remain a purely-translatory velocity that is superimposed with the previous velocity of that region in the fluid. However, due to the variation of the specific volume from one point to another, different points will assume accelerations of different strengths, in such a way that the lighter regions will rush past the heavier ones. Along with that translation, the gradient will likewise create a rotational motion then under which the fluid masses will be rotated around the lines of intersection of the isobaric and isosteric surfaces as their axes, and indeed in the direction of the mobility vector B along the shortest path to the gradient G (Fig. 2).

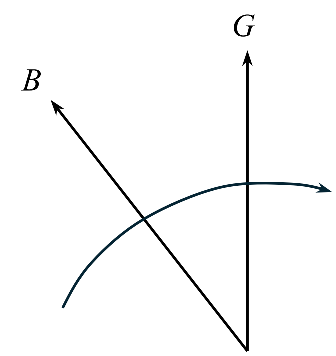

Figure 2.

That rotation of the fluid masses will imply a circulation of all closed curves that consist of fluid particles. Inside of the small region of the fluid being examined, we need to consider only plane curves. One immediately finds the following rule for the sense of the acceleration of the circulation that one of those curves takes on:

Project the gradient and mobility vector onto the plane of the curve: The acceleration of the circulation points in the direction of the shortest path from the projection B of the mobility vector to the projection G of the gradient (Fig. 2).

In order to now find the quantitative law for the acceleration of the circulation that occurs, we recall that from formula (3), the increase in circulation per unit time is the line integral of the component of the acceleration that is tangent to the curve. We first seek to determine the value of that line integral of the acceleration for a curve that is the line of intersection of an isobaric-isosteric tube with an arbitrary plane. That curve will have the form of a parallelogram (Fig. 3) in which two sides of the parallelogram lie on isobaric planes p_0 and p_1 and two of them lie on isosteric planes k_0 and k_1. When h is the distance between two isobaric planes, the gradient will have the numerical values:

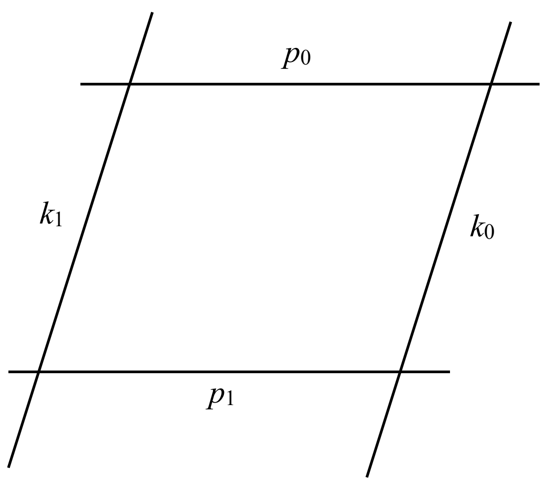

Figure 3.

$$G = \frac{p_1 - p_0}{h} .$$

Since the gradient is perpendicular to the two sides of the isobaric parallelogram, it will not generate any acceleration in the directions that are tangential to those lines. However, it will define an angle of Θ with the sides of the isosteric parallelogram, and as a result it will generate acceleration components $k_1\, G \cos \Theta$ and $k_0\, G \cos \Theta$ in the directions that are parallel to those lines. When we refer both of them to the same direction of traversal along the curve $p_0\, k_0\, p_1\, k_1$, they will become:

$$k_1\, G \cos \Theta , \quad - k_0\, G \cos \Theta ,$$

respectively.

In order to find the value of the line integral, we must multiply those quantities by the lengths of the corresponding line elements and add the products thus-defined. However, both sides of the parallelogram have length:

$$\frac{h}{\cos \Theta} ,$$

such that we will find:

$$(k_1 - k_0)\, G\, h$$

for the value of the line integral. When we introduce the value of the gradient G above, that will become:

$$(k_1 - k_0)\, (p_1 - p_0) .$$

Finally, we can specialize that by the assumption that the isobaric-isosteric tube is a solenoid. By the definition of a solenoid, we will have $k_1 - k_0 = 1$ and $p_1 - p_0 = 1$, and the line integral will take on simply the value 1. We then find the simple result:

The increase per unit time in the circulation of a curve that is the curve of intersection of a solenoid with an arbitrary plane has the numerical value of 1. We have already determined the sense of that increase in circulation, and in order to distinguish the two opposite directions from each other, we can denote the increase in circulation by + 1 when its direction coincides with the chosen positive direction of traversal along the curve, and by – 1, in the opposite case.

We can easily go from the result thus-found for the circulation of a curve that is the curve of intersection of a plane with a solenoid to the corresponding general law for an entirely-arbitrary curve. Namely, we can lay a surface through the given arbitrary curve that intersects all of the curves that the solenoid enclosed. The solenoid will determine a system of curves on that surface that each have the form of a parallelogram, and each of which will have an increase in circulation per unit time of either + 1 or – 1. However, according to the law of summation (2) for line integrals, the line integral along the exterior contour will be equal to the sum of the line integrals along all individual contours, and will then be equal to simply the number of solenoids that are enclosed, insofar as all of them rotate in the same direction, and it will otherwise be equal to the excess A of the number of positively-rotating solenoids over the number of negatively-rotating ones. Since that line integral is equal to the increase in circulation C of the figure in question per unit time, we can then represent the result by the formula:

$$\frac{d\varphi}{dt} = A \,. \tag{6}$$

When we regard the enumeration in question algebraically, we can refer to the number A, when provided with a sign, as simply the *number of solenoids that are enclosed by the curve* and reproduce the results as the following theorem:

The increase in circulation per unit time in an arbitrary closed curve is equal to the number of solenoids that are enclosed by the curve.

With the help of that theorem, we can pursue the variation in time of the circulation in a closed chain of fluid particles, assuming that we know the behavior of the isobaric and isosteric surfaces at each time. In that way, the number A will always be alternating, and indeed based upon two facts: On the one hand, since the curve is in motion, and on the other, since the isobaric and isosteric surfaces will change form and position as a result of the varying state of density and pressure, such that the curve will always enclose an alternating bundle of solenoids.

(to be continued)

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The dynamical principle for circulatory motions in the atmosphere

By **V. Bjerknes**, Stockholm.

(continuation)

(With 12 figures)

Translated by D. H. Delphenich

IV. – The most-important circulatory motions in the atmosphere.

We have already made note of the general behavior of the isobaric and isosteric surfaces in the atmosphere. Mostly, the surfaces will follow each other rather precisely since the density increases and decreases with the pressure, in general. They would follow each other absolutely when the density is a function of the pressure. The two systems of surfaces would not intersect each other then, nor would they define any solenoids. The circulation of a curve in the atmosphere would be neither accelerated nor retarded under those conditions but would be a constant that would be characteristic of the curve. That is the known result to which one arrives on the basis of the **Helmholtz-Kelvin** theory.

However, the density or specific volume of the air is never a function of only the pressure, but it can also include the temperature and humidity that vary from point to point. Since the influence of the humidity on the specific volume of the air is insignificant, in the following qualitative arguments, we will focus our attention upon only temperature, for the sake of simplicity. It should be observed that where the temperature is higher, the specific volume of the air will be greater than one would expect from the value of the pressure, and where the temperature is lower, the specific volume will be smaller. That is why in hot regions, one will have the same specific volume for the air on the surface of the Earth that one would again find at higher layers of air in colder regions. The isosteric surfaces must then deviate from the isobaric ones, and indeed always in such a way that in the hotter regions, those surfaces will be lower than the corresponding isobaric surfaces and higher in the colder regions. The two families of surfaces must then necessarily intersect and define solenoids that create circulatory motions in the air. The general nature of the circulatory motions can be easily derived from the known distributions of pressure and temperature with the help of our fundamental theorem.

We can first overlook all seasonal and daily fluctuations of temperature and pressure, and likewise all of the nonuniformities of a local nature in the distribution of land and ocean or the character of the Earth’s surface. The pressure will then be distributed rather uniformly over the entire Earth and will exhibit no essential differences in the polar and equatorial regions. That is why the isobaric surfaces will then lie reasonably parallel to the Earth’s surface. By contrast, the polar regions will have a lower temperature, while the equatorial regions will have a higher one, such that the isosteric surfaces will lie higher in the polar regions and drop down at the equator. The two families of surfaces will intersect each other and define solenoids that surround the entire Earth roughly like parallel circles. Figure 4 illustrates a meridian section through that system of solenoids in which, as in all of the following figures, the isobars are represented by thin lines and the isosteres, by thick ones, and in which the altitude in the atmosphere is greatly exaggerated. The gradient points vertically upwards, while the mobility vector is inclined somewhat downward towards the equatorial side, and the acceleration of the circulation that points from the mobility vector to the gradient will create circulation that makes the air at the Earth's surface flow from the poles to the equator, rise up there, and again flow back to the pole, only to then drop down in the higher latitudes. That is the known *great circulatory motion between the poles and the equator* that appears as a regularly-formed motion in the trade wind zone.

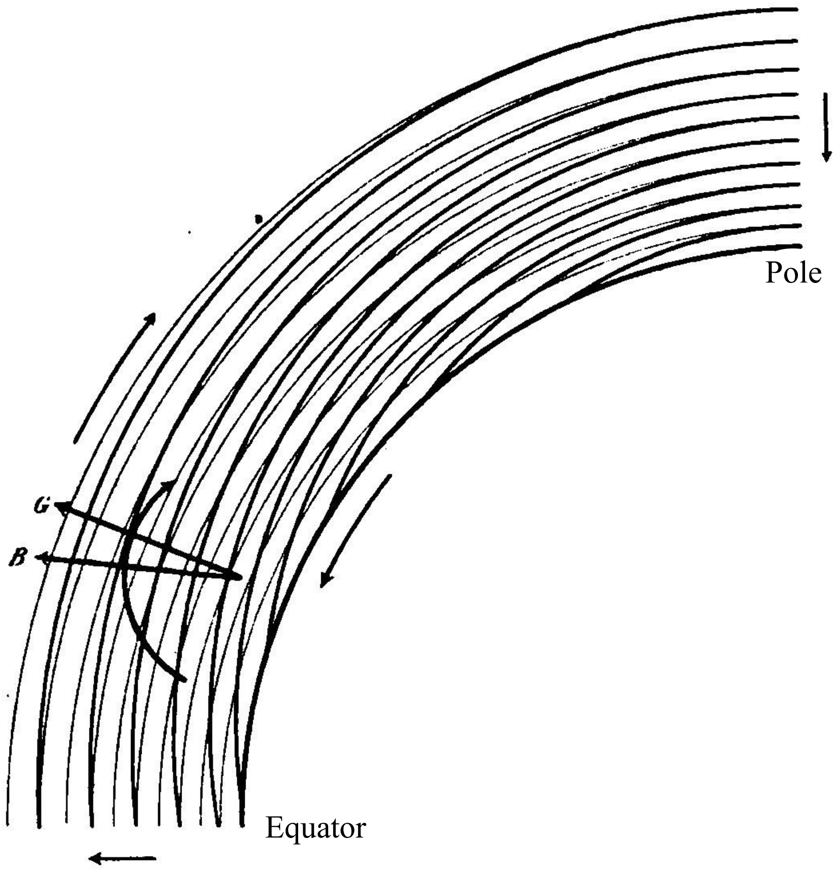

Figure 4. Trade winds.

A series of lesser motions of a periodic nature are connected with those stationary circulatory motions that all originate in the seasonal and daily fluctuations of temperature, in conjunction with the irregular character of the Earth's surface. The significance of the Earth's surface is based upon the fact that the air itself will be heated only very slightly by direct absorption and cooled only very slightly by direct radiation. It is at the surface of the Earth that great temperature fluctuations will occur as a result of the absorption and radiation, and in that way, the closest-lying layer of air will be indirectly heated or cooled. That is why the temperature fluctuations of that layer of air will appear to differ quite sharply according to the character of the Earth's surface.

The most-important thing in that relationship is the difference between the land and the ocean. The land will be heated faster by absorption and cooled faster by radiation than the ocean. That is why the air above the land will be heated more strongly during the daytime and cooled more strongly at night than when it is over the ocean. For that reason, the isosteric surfaces will lie relatively high over the ocean and relatively low over the land during the day. They must intersect the horizontally-lying isobaric surfaces and define a system of solenoids that run along the coastlines.

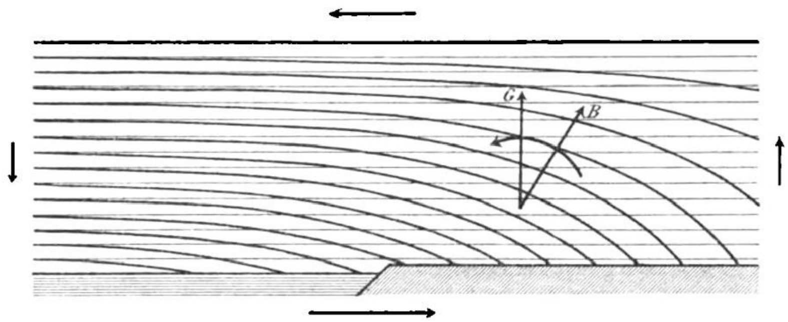

Figure 5. – Wind over land and sea.

Fig. 5 illustrates a section of that system of solenoids. The acceleration of circulation that points from the mobility vector to the gradient will create a circulation under which the air flows across the surface of the Earth from the ocean to the land, rises up there, and flows back over the ocean at a high altitude, only to drop down again. Everything will be reversed at night. The isosteric surfaces will then lie higher over land than over the ocean, the solenoids will have their signs changed, and they will generate the opposite circulation: One gets the well-known phenomenon of *winds over land and sea.*

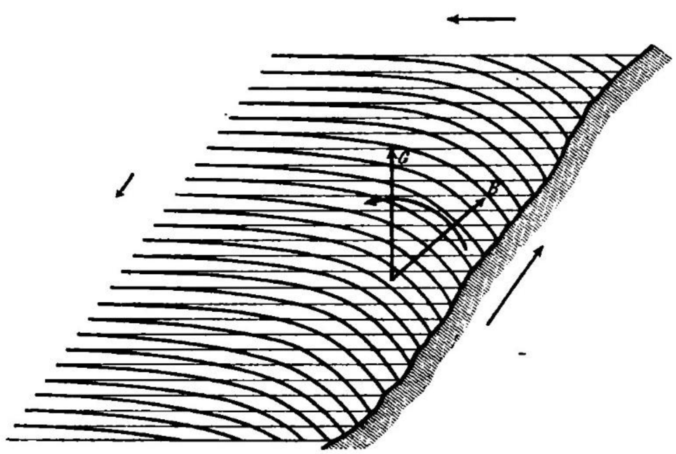

Figure 6. Wind over mountains and valleys.

The seasonal change in temperature takes place in a manner that is similar to the daily change: In the Summer, the isosteric surfaces will have lower altitudes, on average, than the corresponding isobaric surfaces over the continents and higher altitudes over the oceans. The solenoids that are created in that way and run along the coastlines will generate a circulation under which the wind direction at the surface of the Earth will point from the sea to the land more than the opposite direction, on average. In Winter, the isosteric surfaces will lie, on average, higher than the corresponding isobaric surfaces over the continents and lower, on average, over the oceans. The solenoids that run along the coastlines will have the opposite sign and create a circulation under which the wind at the Earth's surface will predominately point from the land to the sea. One then comes to the known phenomenon of *monsoon winds*.

Along with the distribution of land and ocean, the form that the surface of the Earth takes will also come under consideration. The layer of air that is heated by absorption or cooled by radiation will have the same form as the surface of the Earth before that relationship is modified by the motions in the air. The air layer over a plane will have the form of a horizontal disc, and the isosteric surfaces will keep the form of horizontal planes, despite their motions up and down as a result of changes in temperature, such that they cannot intersect the isobaric surfaces, which likewise take the form of horizontal planes. By comparison, at the slope of a mountain, the air layer in question will have an inclined configuration. During the daytime, when that layer is heated more strongly than the rest of the air, the isosteric surfaces that lie as horizontal planes at a great distance will drop when they intersect that layer and intersect the isobaric surfaces that lie as horizontal planes. Along the slope, a system of solenoids will be defined, and a section of them is illustrated in Figure 6. The acceleration of circulation that points from the mobility vector to the gradient will create a circulation under which the air will move upwards along the slope, rise to the summit of the mountain, and then move down the other side to a great distance. At night, that layer of air will be colder than the remaining air, the isosteric surfaces, which lie horizontal and flat at a great distance, will lie higher in that layer, the solenoids will have the opposite sign, and they will generate the opposite circulation, under which the cold air flows downwards in order to sink as deeply as possible into the valley, while the air that was driven out of it will rise upwards in order to gradually replace the air that flows higher up. That explains the *regular day and night winds that appear in the mountainous areas*, in which the daily wind points from the valley to the mountain, and the night wind points from the mountain to the valley.

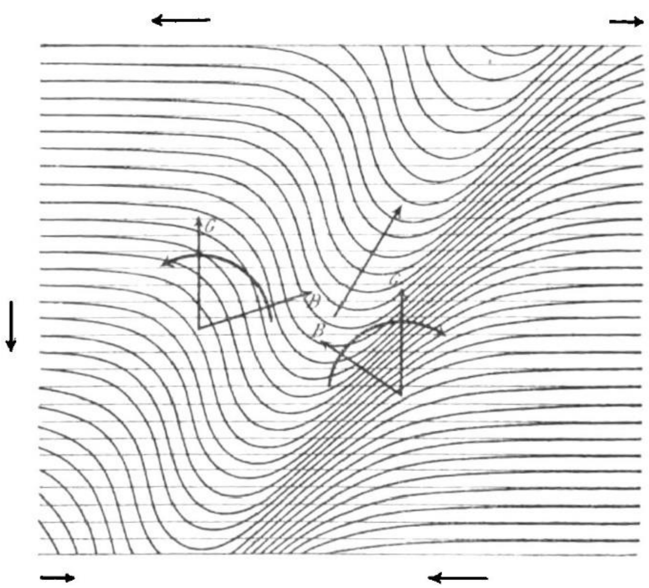

Figure 7. Local rising air currents.

That phenomenon will appear all the more forcefully the higher the mountain is. By contrast, the smaller the unevenness in the terrain, the weaker that the intensity of the solenoid will be, and the more irregular that it will become. Without creating noticeable winds of a well-defined directions in the surface of the Earth, those solenoids will be distributed irregularly and create local rising air currents that are the causes of the cumulus cloud formation on beautiful days. As long as the rising air masses are warmer than the surrounding ones, the isosteric surfaces will have depressions in them where the intersect those air masses. Figure 7 shows a section of one such column of rising warmer air, and figure 8 illustrates the extreme case in which a separate air mass is heated so strongly that its specific volume is itself will be greater than that of the air that lies vertically above it. That air mass is then surrounded by closed isosteric

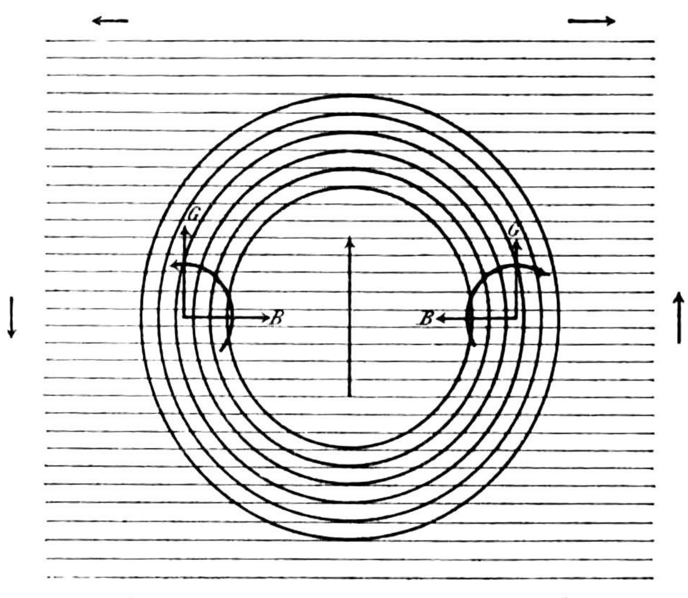

Figure 8. Rising air masses.

surfaces that are indicated by circles. In order to simplify the drawing, the variation of the specific volume in the surrounding air with altitude was ignored. In both cases, the acceleration of circulation that points from the mobility vector to the gradient will create a circulation in which the lighter air mass in the interior must rise up relative to the heavier one outside of it. In the latter case, where the isosteric surfaces are closed, that motion upwards will also follow Archimedes' law of buoyancy. That is why that law can be regarded as a singular special case of our law of circulation.

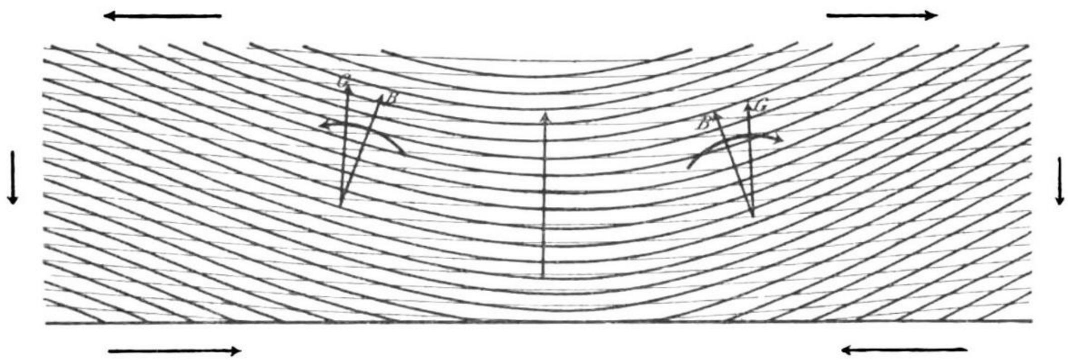

Figure 9. Cyclone.

The influence of the rotation of the Earth on the day-to-night variation of the winds over land and sea or mountains and valleys will only be minor since one has directions of motion that vary rapidly one after the other such that the deflecting forces have no effect when summed over a long period of time. However, one can also think of such relationships as being given in such a way that the air over large areas of the Earth will be heated more strongly over several days than the surrounding air. As a consequence of absorption, that can happen most easily over extended planes where the ventilation is less effective, due to the local rising currents that were just considered. Over the ocean, the warm ocean currents that are surrounded by cold masses of water will admit heating that will continue without interruption during the day and night. Inside of those heated air masses, the isosteric surfaces will have depressions. At the same time, the isobaric surfaces can also contain depressions as a result of the smaller weight of the warm air masses and the smaller pressure that follows from that. The depressions in the isosteric surfaces will then be the strongest ones since those surfaces must drop just as much as the isobars as a result of the reduced pressure, and because a depression that is due to the higher temperature must be added to the latter depression. For that reason, the isosteric surfaces must intersect the isobaric ones and define a system of solenoids that surround the hot air masses in an annular way. Figure 9 illustrates a section through that system of solenoids, and as one sees, the acceleration of circulation that points from the mobility vector to the gradient will generate a circulation under which the air masses along the Earth's surface will flow into a central region from all directions, rise up, and then flow higher up, only to again drop down at a great distance.

All of the air motion is completely analogous to the one that was considered above (Figs. 7 and 8), except that the assumptions differ in two regards: The heated air mass is much larger, and the relationships are no longer such that everything needs to invert during the night, and regardless of whether the heating takes place over land or sea, moreover. The deflecting force of the Earth's rotation will become appreciable during the resulting ongoing motion of the air over great distances, and the originally radially-directed downward flow of air, and corresponding upward flow will change into motions of a spiral nature. The original circulation in the vertical plane will then be superimposed with a rotation in the horizontal plane, and friction will first set a limit on the intensity of the two motions. When that motion has achieved a sufficient intensity that the motion is strong in comparison to the forces that generate the motion, relationships will exist such that one would be justified in employing the **Helmholtz-Kelvin** laws of vorticity in the first approximation: The vortex that is defined will then "seek to preserve its individuality," so-to-speak, and can vary only slowly as a result of the vortex-forming or vortex-destroying forces that act upon it. Hence, if the relationships for that to happen are favorable then the entire air mass that comes under consideration can be led along by the large atmospheric currents while preserving its vorticial motion. An advancing motion of the vortex can also take place in such a way that the center of the vortex-forming forces will displace, and therefore a more or less continuous formation of a new vortex will happen along with the old one. Such a displacement of the center of the vortex-forming forces is possible as long as the motion has advanced to the point that the warm air that exists in the center will no longer be the air that was heated due to the given local conditions *in situ*, but the air that flowed in from the outside. That is because the air that flows in from the various directions will generally have differing temperatures, and due to that asymmetry, the places where the air is hottest, and where the isosteric surfaces will have their greatest depressions for that reason, will not coincide with the instantaneous center of the vortex. The system of solenoids will then displace, and a new vortex will be formed along with the old one and combine with it into a somewhat-displaced vortex. The same game will be repeated with that new vortex, and we will get a vortex that displaces from place to place, as is observed with *cyclones*.

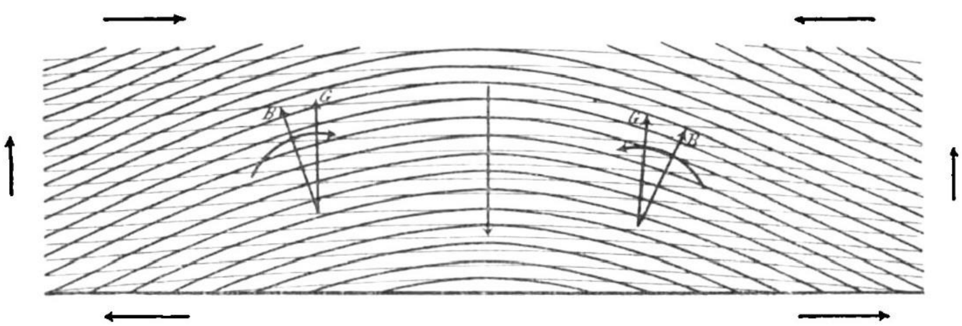

Figure 10. Anticyclone.

Due to the many rising currents in the atmosphere, overcrowding would take place when no corresponding motions downwards are present. That motion downwards can either be uniformly distributed over large areas, and therefore drop off very little, or it can be localized into better-defined bounded descending air currents. It can be especially easy for the latter to occur when large masses of air have accumulated in the high air layers at a certain location that are colder, and therefore heavier, than the surrounding ones. Due to the higher pressure in the heavier air mass, the isobaric surfaces will have rises there. However, at the same time, the isosteric surfaces will have larger rises, namely, first of all, as a result of the higher pressure in a rise that is equal to that of the isobaric surfaces, and then an extra one that results from the lower temperature. The isobaric and isosteric surfaces must then intersect each other and define solenoids that surround the cold air masses in an annular way. Fig. 10 illustrates a section through one such system of solenoids, once the motion has already advanced for some time such that the heavier air masses have reached the surface of the Earth. The acceleration of the circulation that points from the mobility vector to the gradient will create a circulation under which the air in the higher layers will flow in from all directions into a central region where it drops, flows outward along the surface of the Earth, and rises again at a great distance. If that circulation advances for a long period of time then the radial influx high up, as well as the radial outflux lower down, will exhibit motions of a spiral nature as a result of the influence of the rotation of the Earth. In that way, one will arrive at the phenomenon of the *anticyclone*.

We have based the development above upon the so-called physical theory of cyclones and anticyclones, according to which one regards the primary cause of the formation of cyclones to be a rising current of warmer air and the primary cause of the formation of anticyclones to be descending current of colder air. As is known, one can also present another theorem, namely, the so-called mechanical one, according to which the primary cause is to be sought in the collision of large atmospheric currents in the higher layers of air, or according to which those of the cyclones and anticyclones are regarded as the boundary structures in the large circulations of the atmosphere. A more-precise implementation of the mechanics of cyclones or anticyclones on the basis of that theory has never been carried out, as far as I know, and that is why we cannot go into a positive discussion of that theory. By comparison, we can formulate a criterion by which we can decide the extent to which the physical theorem will give a satisfactory explanation for the mechanics of cyclones or anticyclones from a purely-empirical investigation.

To that end, we imagine that we know the distribution of density, pressure, and wind at each time since the formation of a cyclone or anticyclone. We can then construct the isosteric and isobaric surfaces and find the number of solenoids that are present at each time, and we can likewise calculate the circulation of various closed curves at various time form wind observations. Now, if the physical theory is correct then the number of solenoids that exist or are coming into existence must suffice to explain the circulations or wind velocities that are present. In general, one will first be able to perform precise calculations when one likewise brings friction and the rotation of Earth under consideration. However, if one finds, for example, that only so many solenoids have come into existence in a cyclone that took place over three days, such that even when one ignores friction, they can create only a small percentage of the circulations that were present over the course of the three days, then one must conclude that other forces besides the ones that the solenoids represent have been active during the formation of the cyclone. By contrast, if one finds that the number of solenoids that are present is sufficient to generate the circulations or wind velocities that are present over the course of a few hours when one ignores the rotation of the Earth and friction then one must conclude that a large excess of force was present during those three days in order to overcome the forces of resistance that were not considered explicitly. However, it is only once the inadequacy of the frictional forces that originate in the solenoids that are locally present in the cyclone has been proved that one would have the grounds for seeking out other causes of the formation of cyclones and bringing the more remote solenoids of the large atmospheric circulation under consideration.

That example is especially interesting due to the fact that here one is not dealing with Gedanken experiments that cannot be performed, but with investigations that are practicable and have even been carried out already, at least in part. Generally, one has still never completely researched a cyclone by simultaneous observations in the higher air layers. However, a section through a passing anticyclone and cyclone was obtained at **Blue Hill** by observations over four successive days from 21 to 24 September in 1898. My student **Sandström** has constructed the isobaric and isosteric surfaces of that cyclone from the observations that **Helm-Clayton** published, to the extent that it could be done by combining the observations that were made on different days, and hopefully his work on that can be published soon. All that should be pointed out here is that the number of solenoids that were found to be present in that way was so large that they sufficed to generate the large wind velocities that were present over the course of a few hours. From the lively interest that now exists for performing observations in the higher air layers, it will hopefully not be long before we can also succeed in tracking the complete history of the development of cyclones by means of systematic simultaneous observations at various locations, instead of constructing a hypothetical instantaneous state for it from observations performed at different times, as in the aforementioned. Based upon such simultaneous observations, one would also be able to decide between the mechanical and physical theory of the advancing of the cyclone. If the physical theory is correct, such that a continuous formation of new vortices would take place, along with the old ones, then the system of solenoids would have to rush ahead of the actual vortex somewhat. By contrast, if the cyclone is led along by the large atmospheric currents then the system of solenoids (when such a thing is present at all) would only follow the vortex exactly.

In the foregoing, we have considered trade winds, monsoons, land and sea winds, mountain and valley winds, cyclones, and anticyclones to be isolated phenomena, for the sake of simplicity. However, in reality, a complete isolation of those systems of winds from each other cannot be accomplished since the winds that actually appear always have more or less complex causes, and we have employed this schematic subdivision into isolated phenomena only in order to simplify this overview. However, for the immediate application of the theory that was proposed here to meteorological practice, in which one observes the density and pressure distribution that is actually present, in conjunction with the actual winds, any such subdivision would seem to be artificial. Regardless of how complicated the relationships might be, one always nonetheless works with the actual winds and their actual causes. In that way, this theory differs essentially from the usual dynamical theories that are based upon exhibiting particular integrals of the equations of motion, and in which one must first perform an idealization of the relationships that exist that is mostly taken too far before one can employ the theory.

Therefore, if one would wish to study the motion in the atmosphere with the use of the theory that is developed here then the problem would be to find the actual behavior of the isobaric and isosteric surfaces in the atmosphere and the behavior of the solenoids that those families of surfaces define. In that investigation, one would find the ideal relationships that were assumed in the foregoing only in exceptional cases, if not never. One would never find the solenoids that follow the degrees of latitude precisely, and therefore generate pure trade winds. Just as rarely would one find solenoids that follow the coastlines precisely over long distances, and thus generate pure land and sea winds. Rather, one would find that the actual solenoids would mostly encircle the entire Earth as tubes or curves of rather irregular form, and that they would mostly have variations in direction that are more or less strong under the transition from land to sea and would therefore always be moving with the changing of day and night or Summer and Winter. During the day or the Summer, the solenoids will deflect towards the polar side over the solid land, while during the night or Winter, they will deflect towards the equatorial side. If one were to use those actual solenoids as a basis then one would gain the advantage that one would see the actual winds in conjunction with the actual and complete causes. For example, one would treat the Indian monsoons in that way as neither pure land or sea winds nor as pure trade winds, but as what they actually are, namely, a combination of land and sea winds and trade winds.

For similar reasons, one must not always expect to see the assumed annular systems of solenoids develop completely in cyclones or anticyclones. That is because at the same time as one brings the local temperature increase in the central region of the cyclone under consideration, one will also have to bring the general temperature drop from the equator to the pole. The isosteric surfaces in which the local depression appears do not lie parallel to the Earth's surface then but drop towards the equator. Their lines of intersection with the isobaric surfaces, which lie roughly-parallel to the Earth's surface, and the corresponding solenoids will then have the appearance that is illustrated in Fig. 11 when they are projected onto the surface of the Earth. Most of the solenoids belong to the system of solenoids that encircles the entire Earth, except that in the region of the cyclone they will deflect towards the polar side. It is only in the central region that the annular, closed solenoids will occur in the cyclone itself, and indeed only when the depression in the isosteric surface is sufficiently deep. Otherwise, as is shown in Fig. 12, all of the solenoids would lie where all of them encircle the whole Earth, except that they would have the aforementioned bulges in the region of the cyclone. In the study of that actual behavior of the solenoids in cyclones, one would therefore also see the system of winds in the cyclone in conjunction with the large circulatory motions in the atmosphere. It deserves to be pointed out on this occasion that the usual propagation of the cyclone from W to E that takes place at our latitude can generally be interpreted by saying that the bulges propagate like waves in the system of solenoids that surrounds the entire Earth.

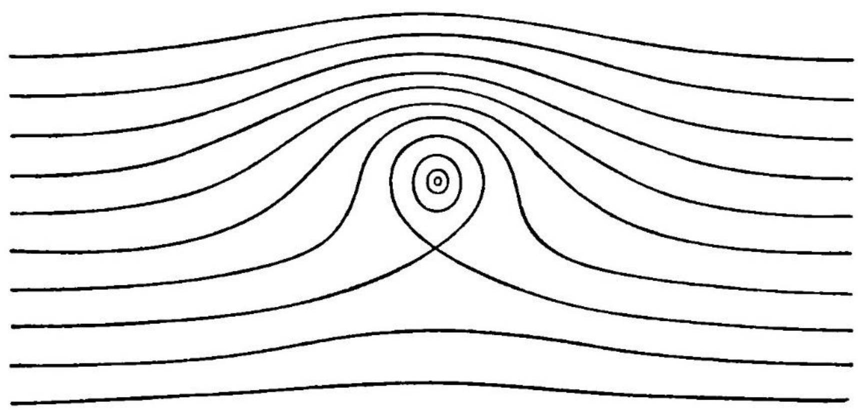

Figure 11.

Entirely-similar considerations can be applied to the anticyclones. If the physical theory of anticyclones is correct such that they contain colder air in their immediate neighborhood then the solenoids that surround the entire Earth must exhibit bulges in the anticyclone region that points towards the equatorial side, in contrast to the bulges in the cyclone regions.

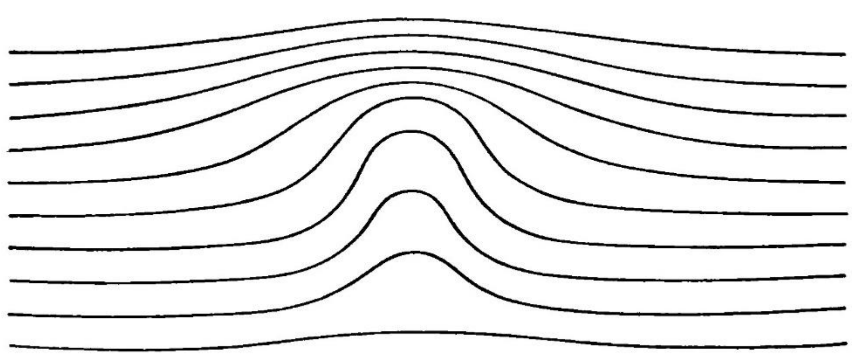

Figure 12.

V. – Concluding remarks.

We have used our fundamental theorem for a purely-qualitative discussion of the most-important atmospheric motion, However, the theorem itself is a quantitative one, and therefore admits a more precise quantitative way of approaching the phenomena. However, it would undoubtedly be hasty to imply the theorem directly as a foundation for more extensive calculations concerned with atmospheric motions. The formal incompleteness that exists in it due to the fact that we have not considered the rotation of the Earth or friction already obstructs those computational applications for the time being. That oversight would not be difficult to eliminate in a formal mathematical context. The two generalizations that were already suggested as necessary in the Introduction, namely, when one brings friction and the rotation of the Earth under consideration, will simply consist of extending the fundamental equation (6) by two terms on the right-hand side, the first of which is the line integral of the frictional force, when taken along the curve, and the second one is the line integral of the deflecting force of the Earth rotation. However, one will first encounter an essential difficulty in the applications because the frictional resistance depends upon the velocities that the neighboring air particles have relative to each other, and that is why a calculation of the frictional resistance that is based upon rational principles would require knowledge of the motion of the air, not from one degree of latitude to the next, but from millimeter to millimeter. That situation would imply that one would necessarily need to go down a somewhat-different path and that the suggested theoretical generalization would not have the great practical significance that one might have expected at first glance.

Therefore, the value of the theory that was developed here is not so much in the formal possibility that it opens up of carrying out calculations of the atmospheric motions. Rather, its great significance is to be sought in the fact that the theory gives a rational dynamical principle by which one can group the facts that have been found by observations. In that way, one will also establish the best foundation for any future quantitative dynamical meteorology. The problem will always be to simply record the number of positions of the solenoids and the corresponding distributions and intensities of the winds. One will then learn how to bring the simultaneous influence of the rotation of the Earth and friction under consideration by experiment, instead of by calculation. For periodic winds of short periods, such as the land and sea winds or mountain and valley winds that follow the alternation of day and night, one will probably find in that way that the circulations that actually occur do not deviate especially much from the values that one calculated from our theorem when one neglects the rotation of the Earth and friction. That is because the work done by the solenoid in that case probably consists of essentially the work done overcoming the inertia of the air masses. On the other hand, the study of solenoid counts and wind strengths for periodic winds of long period, such as the monsoons, or for stationary winds, such as the trade winds, will lead to knowledge of the equilibrium relationships between the moving forces that the solenoids represent and the forces that appear in the stationary state of motion and resist them. In the cyclones, one has an opportunity to study, not only the state of acceleration, where the essential resistance is viscous, but also that of the stationary motion, in which the solenoids just suffice to keep the created motion below the resisting forces, and the decaying motion, in which the resisting forces are exceeded. A more exact knowledge of the cyclones in that context would have great significance for weather forecasting, in particular. One would be able to conclude from the number of solenoids the extent to which the wind would increase or decrease in intensity in the near future. Ultimately, everything would depend upon only whether one could obtain sufficiently-many systematic observations of the higher air layers and the engineering of that type of observations has already been developed to the extent that the possibility of obtaining such observations with sufficient regularity that they could be utilized in daily weather forecasts can no longer be in doubt.

In that regard, I would also like to make note of the fact that the theory that was developed here has the same applicability to the motions in the oceans as it does to motions in the atmosphere. In that way, temperature and salinity play the same role in calculating density in the ocean that temperature and humidity do in calculating the density of air. Ultimately, the theory will also preserve its applicability when one considers the atmosphere and the ocean together as a single fluid medium. That has greater significance due to the extensive interactions between the motions of the air and the ones in the ocean. It will therefore offer an excellent opportunity for the simultaneous solution of great meteorological and hydrographic problems when the plans that were drawn up at the Hydrographic Congress in Stockholm 1899 are realized, such that the hydrographic expeditions that are undertaken several times a year from the participating countries might likewise include meteorological ones that employ instruments for the study of the higher air layers. In particular, the north Atlantic ocean would offer great interest in Fall and Winter in that regard. There one might perhaps succeed in studying the formation of cyclones that frequently take place most likely in the region of the Gulf stream, and at the same time, measure the amount of heat that is consumed by the formation of the cyclone and that comes from the ocean.

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