"Ueber einen hydrodynamischen Fundamentalsatz und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres," Kungl. Svenska vetenskapsakademiens Handligar **31** (1898), 1-35.

A FUNDAMENTAL THEOREM OF HYDRODYNAMICS AND ITS APPLICATIONS

ESPECIALLY TO

THE MECHANICS OF THE ATMOSPHERE AND OCEANS

By

V. Bjerknes

With fourteen figures

Submitted to the Royal Academy of Science on 11 May 1898

Checked by R. Rubenson and A. Lindstedt

Translated by D. H. Delphenich

STOCKHOLM 1898
KUNGL. BOKTRYCKERIET. P. A. NORSTEDT & SÖNER

I. – INTRODUCTION.

1. – Fluid motions under restricting and general assumptions about the density.

In theoretical investigations of fluid motions, one mostly cares to introduce certain specializing assumptions on the density. Either one assumes complete homogeneity and incompressibility, or one restricts oneself to the consideration of inhomogeneities that are created by pressure exclusively. However, the results that are obtained under such assumptions have a very restricted application to the motion of natural fluids. In particular, one observes that the large motions in the atmosphere and the oceans arise exclusively as a result of variations of density that *are not* caused by pressure, but mainly depend upon temperature.

That completely explains why rational hydrodynamics has had so little influence on the development of meteorology or the study of the oceans. **Helmholtz**'s famous fundamental theorems ([1]) on fluid vortices and **Lord Kelvin**'s equivalent theorems on the motion of circulation ([2]) allow us to predict certain peculiarities of the atmospheric vortices and circulations, but as is known, the fundamental question of the original creation of such motions is left completely untouched.

However, it does not complicate the discussion of the equations of hydrodynamics when one does not introduce restricting assumptions on the density. I have showed that in a foregoing treatise ([3]) and derived a series of theorems of great generality, one of which especially includes all of the theorems of **Helmholtz** and **Lord Kelvin** in question and can be considered to be equivalent to the general formulation of the equations of motion of hydrodynamics under certain conditions.

2. – The derivation and applications of the fundamental theorem.

The fundamental nature of this theorem will also emerge from the fact that one can find the same thing by considerations of an entirely-elementary nature in a way that is similar to how one derives the general equations of motion for hydrodynamics. In the present work, I will give that elementary derivation from which it will already emerge sufficiently clearly that the theorem possesses sufficient generality to be able to replace the general equations of motion to a greater extent, if it must also be extended by a theorem that has yet to be mentioned here in order to make any recourse to the general equations of motion irrelevant in all cases.

Allow me to refer to the treatise that was cited above for the general analytic derivation.

The elementary derivation of the theorem, as well as all of its uses, is closely related to a geometric representation of the dynamical state of a fluid with the help of two families of surfaces. That representation defines the subject of Section II. The following Section III will include the actual derivation of the fundamental theorem, after which, two more general forms for it will be given in Section IV that are more convenient for most purposes.

([1]) **H. Helmholtz**, *Wissenschaftliche Abhandlungen*, Bd. 1, pp. 101.

([2]) Sir **W. Thomson**, "On vortex motion." Trans. Roy. Soc. Edinburgh (1869), pp. 217.

([3]) **V. Bjerknes**, "Ueber die Bildung von Cirkulationsbewegungen und Wirbels in reibungslosen Flüssigkeiten," Videnskabsselskabets skrifter, Kristiania, 1898.

3. – The meteorological and hydrographic applications of the theorem.

In this last section, I shall restrict myself strictly to a purely-qualitative discussion of the simplest and best-known motions of the two media in this world. It should be pointed out here that in this discussion, I shall address only those results that agree with the intuitions that the meteorologists have already arrived at along other paths. The difference lies not in the results, but only in the new and unified methods that will be proposed that will make it possible to derive the most-disparate results from a single principle.

To that extent, it seems to offer an advantage of only a formal nature. However, I will also show that in a series of cases where old and known controversies exist, the proposed new method will, at the same time, point to a definite path along which one can succeed in resolving those controversies with the help of observations.

The reason why the theorem is especially suited to the treatment of meteorological and hydrographic problems lies in the fact that the large air and ocean currents belong to the realm of motions in which our theorem can replace the general equations of motion of hydrodynamics. There are generally motions in air or water that cannot be treated by our theorem. A typical example is defined by the motion of sound. However, the motions of that type exist entirely in the background for problems of meteorology and hydrographics, such that one can assert that our theorem will include the mechanics of the atmosphere and oceans completely, except for some inessential restrictions.

Obviously, the same thing is true, and indeed with no restrictions, of the general equations of motion of hydrodynamics. When one poses the problem of finding the exact motion of the atmosphere or ocean by exact integration, one will gain no essential advantage if one takes one's starting point to be our theorem instead of the equations of motion. That is because one will encounter the same difficulties in that integration that can be overcome by the current mathematical tools in examples of the simplest kind, in addition.

The use of our theorem will offer no essential advantage over the exact mathematical investigations then. Its essential usefulness lies in the fact that it is suited to qualitative discussions and that one can add approximate quantitative investigations to those qualitative discussions. For example, one can perform calculations of average wind directions and wind strengths, and in that way achieve progressive approximations of arbitrary precision. Indeed, one will arrive at those results by simple graphical methods without meeting up with difficulties of a purely-mathematical nature.

Naturally, in such quantitative applications of our theorem, it will be necessary to bring the influence of the rotation of the Earth and friction under consideration. I have not done that in what follows, where I shall carry out only qualitative discussions, and it would seem all the more superfluous to go into the details of that since the principles of how one considers those factors in problems of meteorology and hydrographics are generally known, and the formal implementation of our theory will not meet up with any complications in that regard.

Finally, it is a pleasant duty for me to thank Herrn Dr. **N. Eckholm** for the many explanations and advice that I have gained from him while I was addressing a science in which I myself did not possess sufficient specialized knowledge. I would like to add that it was through the familiarity

that I gained with his synoptic charts over curves that represent the density of air ([1]) that I was led to the idea of making meteorological applications of a theorem that I had developed while pursuing an entirely-different goal.

([1]) Dr. **N. Eckholm**, "Cartes synoptiques représentant la densité de l'air," Bihang till K. Svenska Vetenskaps Akademiens Handligar, 1891.

II.

GEOMETRIC TOOLS FOR THE DESCRIPTION OF THE DYNAMICAL STATE OF A FLUID

4. – Isobaric and isosteric surfaces and layers.

We imagine that the instantaneous pressure distribution in a fluid is described by surfaces of equal pressure or *isobaric* surfaces. The successive surfaces shall be denoted by the pressure p for small, but constant, intervals:

$$\Delta p \, .$$

We will call the fluid layer between two neighboring surfaces an *isobaric layer*. Inside of such a thing, the pressure can deviate from the value p of the pressure that belongs to a certain middle layer by an amount of at most $\frac{1}{2}\Delta p$.

When one knows the pressure distribution and the external forces that might possibly be acting, one will also know the driving force that acts upon each individual fluid particle. However, the motion that this force generates likewise depends upon the inertia with which the particles oppose the action of the force or the mobility with which it submits to the action of the force. The density of the particles is a measure of that inertia, while the reciprocal value of the density, or the specific volume, gives a measure for the mobility.

That is why we will think of the distribution of the mobility or specific volume in the fluid as being represented with the help of surfaces whose points are all associated with equal values of that quantity. The successive surfaces shall be denoted by the specific volume k for small, but constant, intervals:

$$\Delta k \, .$$

The name *isosteric* surfaces has been proposed for those surfaces ([1]). Two neighboring surfaces bound *isosteric layers*, inside of which the specific volume can deviate from a certain middle value k of the specific volume that belongs to the layer by an amount of at most $\frac{1}{2}\Delta k$.

The isosteric surfaces coincide with surfaces of constant density, to which the terms isopycnic, iso-dense, or equi-dense surfaces have been applied ([2]). By contrast, the isopycnic layers that are associated with constant differences in density will not coincide with the isosteric layers. If we prefer the terms "isosteric" surfaces and layers then that is to emphasize the viewpoint that the thickness of the layer should always refer to constant intervals of specific volume, and not density.

In what follows, we will then imagine the dynamical state of the fluid as always being described with the help of two families of surfaces: viz., the family of isobaric surfaces and the family of isosteric surfaces. Every surface in one of those families will have the property that they can never end inside of the fluid. They must continue until they either end on the boundary surfaces

---

([1]) **A. von Miller-Hauenfels**, *Theoretische Meteorologie*, Vienna, 1883.
([2]) **Ekholm**, *loc. cit.*, p.

of the fluid or return to themselves as a closed surface. Furthermore, two isobaric surfaces that correspond to different values of pressure can never intersect, any more than two isosteric surfaces that correspond to two different constant values of the specific volume.

5. – Isobaric-isosteric curves and tubes.

The isobaric and isosteric surfaces will intersect each other along curves that have constant values of pressure, as well as specific density, and that is why they shall be called *isobaric-isosteric* curves.

Moreover, the two families of surfaces will subdivide the entire fluid into a system of tubes inside of which the pressure and specific volume will deviate from certain constant values p and k that belong to the tubes by an amount of at most $\frac{1}{2}\Delta p$ or $\frac{1}{2}\Delta k$, resp.

It follows immediately from the properties of the two families of surfaces that no isobaric-isosteric curve or tube can terminate insider of the fluid. It must either return to itself or end against the boundary surface of the fluid. Moreover, two different curves of tubes can never intersect each other.

One sees immediately that the isobaric-isosteric tubes have cross-sections that take the form of parallelograms. When the isosteric surfaces approach a state of coincidence with the isobaric surfaces, the angle of the parallelogram will become ever more acute, and parallelogram will stretch to infinity. Ultimately, when the two families of surfaces coincide completely, the tubes will take the form of layers.

We will also employ the expression "isobaric-isosteric tube" in a broader sense, namely, for tubes whose walls are formed of isobaric-isosteric curves exclusively, regardless of whether the cross-sections are small and parallelograms, as was the case with the special tubes that were considered above.

6. – Gradient and mobility vector.

Let m mean the normal to an isobaric surface, while n is the normal to an isosteric surface, Δm is the thickness of an isobaric layer, and Δn is that of an isosteric layer.

The quotients:

$$\pm\frac{\Delta p}{\Delta m}, \qquad \pm\frac{\Delta k}{\Delta n}$$

represent quantities that point along the normals to the surfaces in question. We will call the quantities that are derived in that way *vector quantities*, in particular, when the positive sign is employed, and *gradients* when we choose the negative sign.

The quantity:

(*a*) $$G = -\frac{\Delta p}{\Delta m}$$

will then represent the *pressure gradient*, or more simply, *gradient*, while $G' = -G$ can be called the *pressure vector*.

The quantity:

(*b*) $$B = \frac{\Delta k}{\Delta n}$$

represents the *mobility vector*, while $B' = -B$ is the *mobility gradient.*

For fluid motions under ordinary circumstances, the angle between the pressure gradient and the mobility vector will be acute, and as a result the angle between the mobility gradient and the pressure gradient will be obtuse. On those grounds, in what follows, it would be most convenient to employ the mobility *vector* B, in conjunction with the pressure *gradient* G.

We consider an arbitrary point of the fluid and the surrounding volume element. The gradient G and the mobility vector B will determine a plane (B, G) inside of that volume element that is the common plane of those two quantities. Three surface elements pass through the point then that can be considered to be planar in the infinitesimal limit: The isobaric plane, the isosteric plane, and the gradient plane (B, G). The normal to the isobaric plane coincides with the direction of the gradient, the normal to the isosteric plane coincides with the direction of the mobility vector, and the normal to the gradient plane (B, G) coincides with the isobaric-isosteric line along which the isobaric and isosteric planes intersect.

7. – Isobaric and isosteric lamellae and solenoids.

We assumed above that the two families of surfaces should refer to small, but constant, differences Δp in the pressure and Δk in the specific volume. It is often convenient to assume that those differences have the value 1. However, in that case, we must have the freedom to be able to choose units whose orders of magnitude are appropriate to each individual problem that we address. We can always succeed in doing that in such a way that we choose the units in specialized problems to be decimal fractions of the usual units.

When the differences in the families of surfaces are:

(*a*) $$\Delta p = 1\,, \qquad \Delta k = 1$$

for one such choice of units, the corresponding layers shall be called *isobaric* or *isosteric lamellae*, resp., while the tubes into which the two families of surfaces subdivide space will be called *isobaric-isosteric solenoids*.

If one introduces (*a*) into the defining equations (6.*a*) and (6.*b*) of the gradient and the mobility vector then one will get:

(b) $$G = -\frac{1}{\Delta m}, \qquad B = \frac{1}{\Delta n},$$

or:

The gradient points along the normal to the isobaric lamella in the direction of decreasing pressure, and its numerical value is equal to the reciprocal thickness of the lamella.

The mobility vector points along the normal to the isosteric lamella in the direction of increasing mobility, and its numerical value is equal to the thickness of the lamella.

III.

THE VORTICITY CREATED BY THE GRADIENT

8. – The angular acceleration of different line elements in the fluid.

If one chooses the differences Δp and Δk to be small enough then each volume element of the fluid will contain a large number of isobaric and isosteric surfaces that will represent two families of parallel and equidistant planes. The gradient surfaces (G, B) will be perpendicular to the lines of intersection of those planes and represent a third family of parallel planes.

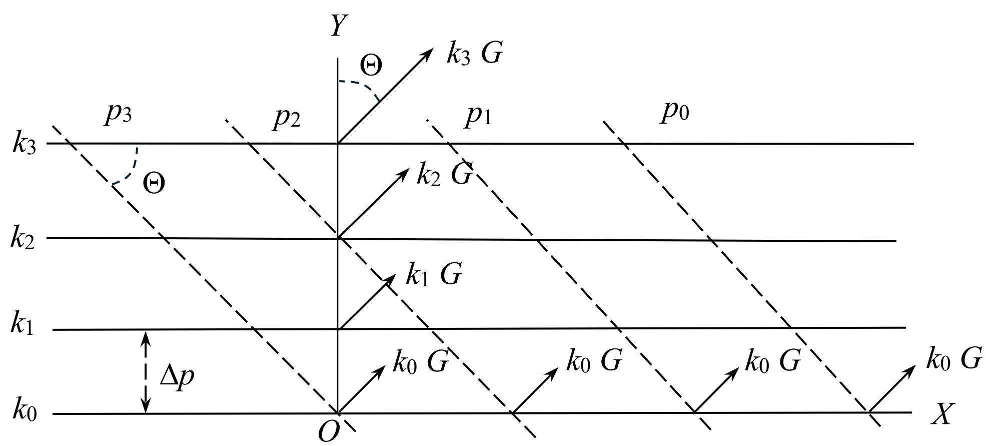

Figure 1.

The isobaric-isosteric tubes will project into a gradient plane (Fig. 1) as a system of mutually-identical parallelograms. The gradient G and the mobility vector B both lie in the plane of the figure. The former is perpendicular to the dashed isobaric side of the parallelogram, while the latter is perpendicular to the solid isosteric side of the parallelogram. They define an angle between them that is equal to the angle Θ of the parallelogram.

The gradient G has the same direction and magnitude at all points of the volume element. By contrast, the acceleration that the gradient imparts upon a fluid particle is equal to the gradient divided by the density, or multiplied by the specific volume k, of the particle:

$$k\,G.$$

The quantity has the direction of the gradient everywhere. However, its numerical values will vary from point to point in the volume element in proportion to the value of k. It follows from this that an arbitrary straight line in the fluid inside of the volume element will generally take on an angular acceleration, in addition to a translatory acceleration. We shall examine that angular acceleration for lines in different directions.

We first note that the acceleration $k\,G$ always lies in the gradient plane (B, G). It then follows from this that the angular acceleration must take place along a normal to that plane. Therefore, straight lines in the fluid will appear as axes of the angular acceleration, so only the isobaric-isosteric lines.

The quantity k does not vary along an isosteric plane. By contrast, it has its fastest variation along the normal to such a plane. All points in the fluid that belong to an isosteric fluid at the moment considered and inside of the volume element in question will therefore take on the same acceleration, while the plane itself and all of the material lines that belong to it will take on only a translatory acceleration, but not an angular one. Those will lines include, among other things, the horizontal solid sides of the parallelograms in Fig.1 and all of the isobaric-isosteric lines that project onto Fig. 1 as points and appear as the axes of each angular acceleration of the fluid lines.

The fluid line *OX* and the fluid line *OZ*, which projects to the point *O*, then take on the simple translatory acceleration $k_0 G$. By contrast, the points of the line *OY*, which is perpendicular to the previous two, and thus has the direction of the mobility vector, will take on accelerations that increase with increasing values of k, such that the lighter end of that line must rush ahead of the heavier one, Therefore, in addition to a translatory acceleration, it must likewise take on an angular acceleration whose direction is illustrated in Fig. 2. It points from the mobility vector B to the gradient G along the shortest path.

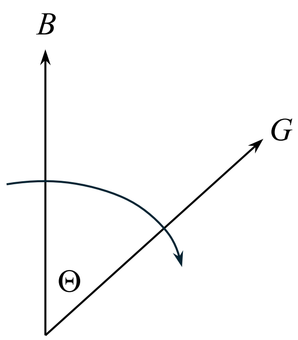

Figure 2.

One ultimately finds the absolute value of that angular acceleration in the following way: From the value $k_1 G$ and $k_0 G$ of the acceleration at the endpoints of an element Δn, one defines the components that are normal to the line:

$$k_1 G \sin\Theta, \qquad k_0 G \sin\Theta,$$

resp. Those are subtracted from each other and the difference is divided by the distance Δn between the points. If we note that the value of the difference is:

$$k_1 - k_0 = \Delta k,$$

and recall the definition (6.*b*) of the mobility vector B then we will get:

$$(b) \qquad B G \sin\Theta$$

as the value of the desired angular acceleration.

One can find the angular acceleration of a straight fluid line of arbitrary direction inside of the volume element. However, since we now know the acceleration of three mutually-perpendicular lines, from the known laws of kinematics, we can derive everything that relates to the rotational motion of the element. We then note the simple result:

The angular acceleration that the gradient imparts upon line elements in the fluid takes place along isobaric-isosteric lines as axes.

All line elements that belong to an isosteric surface at the moment considered will take on no angular acceleration.

All line elements that have the direction of the mobility vector will take on angular accelerations that are equal to the product of the mobility vector and the gradient, multiplied by the sine of the included angle, and will point from the mobility vector towards the gradient.

9. – The vorticial acceleration of surface elements and volume elements in the fluid.

The foregoing theorem already gives a complete description of the vortex-forming power of the gradient. However, one can give it other forms that are more convenient for the applications. That is a purely-kinematical problem that does not touch upon the dynamical content of the theorem.

Instead of relating the vortex-forming power of the gradient to material line elements in various directions, in the applications it would be more convenient to consider the vortex-forming action on material surface elements or column elements, and instead of considering the angular velocity and angular acceleration, it would, at the same time, be most convenient to consider the *vorticial velocity* and the *vorticial acceleration*.

From the principles of the kinematics of continuous systems, we can define the vorticial velocity of a surface element around its normal to be the sum of the angular velocities of two mutually-perpendicular lines in the surface element around the same normal. By comparison, the angular velocity of the surface element is equal to the mean value of the angular velocities of the two perpendicular lines, and therefore it will be only half as big as the vorticial velocity. In a similar way, one defines the vorticial acceleration of the surface element around its normal to be the sum of the angular accelerations of two mutually-perpendicular lines that belong to the surface element around the same normal, and the angular acceleration of the surface element will again be equal to one-half the vorticial acceleration ([1]).

Consistent with those rules, we can derive the vortical acceleration of material surface elements of arbitrary orientation from the theorem above. For example, if we consider a surface that goes through an isobaric-isosteric line then we will find a vorticial acceleration of zero around the normal to the element. On the other hand, any surface element that is perpendicular to an isobaric-isosteric line will take on the vortical acceleration:

$$B\,G \sin \Theta \qquad (a)$$

around its normal, which is the isobaric-isosteric line.

As a result, we can attribute the same vortical acceleration to the entire volume element: The gradient will then impart a vorticial acceleration on each volume element of the fluid whose value

([1]) One should make note of the distinction between angular and vorticial velocity or between angular and vorticial acceleration that is made here. That is because the usage of the words varies, and many authors use those expressions as if they were synonymous. The angular velocity measures the rotational motion of the surface element with a linear measure, while the vortical velocity measures it with an area measure. Therefore, the factor of 2 enters in: The vortical velocity is equal to twice the mean angular velocity in the way that the rate of areal expansion is equal to twice the mean rate of linear expansion.

is (*a*) around an isobaric-isosteric line as its axis according to the sign rule that is represented in Fig. 2.

Finally, if we lay a surface element through the volume element that defines an angle of φ with the isobaric-isosteric line then it will take on the vorticial acceleration:

$$(b) \qquad B\,G \sin \Theta \cos \varphi$$

around its normal.

Whereas the theorem in the foregoing section referred to line elements of a specially-prescribed direction, that theorem takes surface elements of arbitrary orientation into consideration. That is why we can pursue the motion of one and the same individual surface element with the help of that theorem, or we can apply the theorem to various surface elements that occupy a certain prescribed configuration in succession. In the first case, we link up with the so-called **Lagrangian** picture of hydrodynamical problems, while in the second case, we link up with the **Eulerian** picture. In what follows, we will sometimes employ one of them and sometimes the other.

10. – Other causes of vorticial acceleration.

Previously, we considered only those vorticial accelerations that are generated by the gradient.

If gravity is in effect then one will not have to add any new terms. That force will impart an equally-large acceleration downwards on all points of a volume with no regard for its varying density. As a result, no line will take on angular acceleration, and no vorticity will be formed. Gravity can influence the formation of vorticity only indirectly in such a way that it will act definitively upon the pressure state. Any force of a conservative nature will behave in precisely the same way.

By contrast, previously-existing vortices will always create angular acceleration because in general the rotating mass will experience changes of form during the course of its motion that will vary its moment of inertia. However, that is not the formation of a vortex of a primary nature. Here, we are only dealing with a vorticial acceleration that assumes the prior existence of vortices whose original formation was again reducible to gradients.

One encounters a closely-related source of vorticial acceleration in the "deflecting effect of the Earth rotation" that shows up in terrestrial applications. Once again, the rotation of the Earth cannot act as a primary source of vortex creation, or even the creation of motion, but only serve to modify a motion that was created before.

Things work in a similar manner with friction. It would be capable of generating vorticial acceleration, but only insofar as a velocity already existed before, and it would be, above all, reducible to other causes than friction.

When we direct our attention to only the primary causes of vortex formation, we can then ignore all of those causes. The result that we found above regarding the vortex-forming action of the gradient then solves our problem completely, and we need only to give the theorem the most-intuitive form possible.

11. – General theorem for the vorticial acceleration created by the gradient.

We now make the choice of units that was suggested in no. **7**, and further assume that the surface element to which formula (9.*b*) relates is bounded by just the walls of tube in the form of an isobaric-isosteric solenoid at the moment considered. It must then define a parallelogram. Let α be the area of that parallelogram, so $\alpha \cos \varphi$ will be the area of a normal cross-section of the solenoid. That normal cross-section also defines a parallelogram that has sides of Δm and Δn and a parallelogram angle of Θ (Fig. 1). As a result of a known formula for the area of a parallelogram, we will then get:

$$(a) \qquad \alpha \cos \varphi = \frac{\Delta m \, \Delta n}{\sin \Theta} .$$

With the fixed choice of units, we arrive, at the same time, at the expressions (7.*b*) for the gradient and mobility vector. Since the sign has already been found, we do not need to carry the negative sign in the expression for G, but we can simply perform the calculation numerically.

We will then have:

$$(b) \qquad B \; G \sin \Theta = \frac{\sin \Theta}{\Delta m \, \Delta n} .$$

Multiplying equations (*a*) and (*b*) will give:

$$(c) \qquad \alpha \cdot BG \sin \Theta \cos \varphi = 1$$

or

$$(d) \qquad BG \sin \Theta \cos \varphi = \frac{1}{\alpha} .$$

That formula then says the following: The gradient will impart a vorticial acceleration on the surface element that an arbitrary solenoid cuts out of an arbitrary fluid surface that is equal to the reciprocal area of the element. That acceleration takes place around the normal to the element as its axis, and indeed according to the sign rule that is illustrated in Fig. 2.

If we consider the surface elements in the fluid that follow in succession and define normal cross-sections of a solenoid then all of them will take on a vorticial acceleration around the solenoid axis as its axis. We can attribute that vorticial acceleration to the successive volume elements of the fluid mass that fills up the solenoid at each moment. We then get the following:

(I) *The fluid mass that fills up an isobaric-isosteric solenoid at the moment considered will take on a vorticial acceleration from the gradient at that moment that has the solenoid axis as its axis and an intensity that is equal to the reciprocal of the cross-section of the solenoid.*

The vorticial acceleration points from the mobility vector B towards the gradient G.

When using that theorem, one must always recall that the solenoid defines a purely-geometric system of tubes whose walls allow the fluid to flow freely through them. However, in general, it will always be alternating fluid masses that fill up a solenoid from one moment to the next, and they will receive the vorticial accelerations that were described. The theorem is then connected with the **Eulerian** picture of hydrodynamical problems.

____

IV.

ROTATION AND CIRCULATION IN FLUIDS

12. – The rotational motion of surface elements.

It must always be established that the vorticial acceleration to which the theorem above refers is only the one that is created by the gradient in the moment, and not the total vorticial acceleration. In special cases, it can coincide with the total, for example, at a moment when no vortices have formed completely ([1]). However, although the theorem only refers to a partial vorticial acceleration, it nonetheless allows one to study the actual formation of new vortices. That is why all of the applications that will be made below can be based upon that theorem. However, that does not prevent one from putting it into a form that is more convenient for the applications in which the other sources of angular acceleration (10) will be included. That can be done in full generality with no difficulty. Meanwhile, we will continually ignore the effect of friction or the rotation of the Earth and only extend to those vorticial accelerations that are caused by the variations in the moments of inertia of rotating masses. The vorticial acceleration thus-obtained can be regarded as the total when the motion of the frictionless fluid is assumed to be absolute, and not considered relative to the rotating Earth.

Therefore, let ω_n be the vorticial velocity that a surface element in the fluid possesses around its normal under those assumptions, and let $d\omega_n / dt$ be the angular acceleration around that normal. If the forces that act on it have no moment relative to the normal of the surface element then from the fundamental mechanical principle of surfaces, the product of the vorticial velocity and the moment of inertia will be constant, and since the moment of inertia of the surface element is proportional to its area, one will have the equation:

$$\alpha\, \omega_n = \text{const.}$$

Upon differentiating with respect to time and solving for $d\omega / dt$, one will find that:

$$\text{(a)} \qquad \frac{d\omega_n}{dt} = -\frac{\omega_n}{\alpha} \cdot \frac{d\alpha}{dt}$$

is the expression for the vorticial acceleration that depends upon the changes in the moment of inertia.

If we add the vorticial acceleration (9.*b*) that is due to the gradient then the expression for the vorticial acceleration will become:

$$\text{(b)} \qquad \frac{d\omega_n}{dt} = BG \sin\Theta \cos\varphi - \omega_n \cdot \frac{1}{\alpha} \cdot \frac{d\alpha}{dt}\,.$$

([1]) Compare the result of **L. Silberstein**, Bulletin International de l’Académie des Sciences de Cracovie, June 1896.

If we multiply that equation by α, move the last term to the left-hand side, and reduce everything then that will give:

$$(c) \qquad \frac{d}{dt}(\alpha\,\omega_n) = \alpha \cdot BG \sin\Theta \cos\varphi$$

With Lord Kelvin ([1]), we will call the product $\alpha\,\omega_n$ of the area of the surface element with its vorticial velocity, the *rotation* of the surface element. If we revert to our special choice of units then we will find the following equation for the acceleration in the rotational motion of the fluid element:

$$(d) \qquad \frac{d}{dt}(\alpha\,\omega_n) = 1\ .$$

The rotational motion of every elementary fluid surface that defines a cross-section of a solenoid will then have unit acceleration, and completely independent of what the cross-section is normal or not.

13. – Rotation of fluid surfaces.

An arbitrary fluid surface of finite extent will be subdivided into a certain number A of surface elements α by the solenoids at each point in time. When we apply the notation above to each of those elements, we will find that:

$$(a) \qquad \frac{d}{dt}\sum\alpha\,\omega_n = A\ .$$

With **Lord Kelvin** (*loc. cit.*), we will call the sum $\sum\alpha\,\omega_n$, which extends over all elements of a moving surface, the *rotation* of that surface. The equation expresses the following theorem then:

The rotational motion of a fluid surface has an acceleration that is equal to the number A of solenoids that the surface intersects.

14. – Rotation of fluid tubes.

We can track the motion of a certain fluid surface in the **Lagrangian** way using the theorem above. It will be intersected by an ever-changing number of solenoids during its motion, and the acceleration of the rotational motion will change with that changing number.

However, in many respects, it would be preferable for the theorem to relate to fluid volumes, instead of fluid surfaces. That is why we shall consider a tube whose sheath consists of isobaric-

([1]) "On vortex motion," Trans. Roy. Soc. Edinburgh (1869), **§ 60**.

isosteric curves exclusively, i.e., an isobaric-isosteric tube in the most-general sense of the term (no. **5**). It will include a certain number *A* of solenoids, and the rotational motions of all fluid surfaces that define cross-sections of that tube will have equal acceleration. That is why we can attribute a rotational motion with that acceleration *A* to the entire fluid mass that fills up the tube at the moment in question. We then find the following theorem:

(II) *The fluid mass that fills up an isobaric-isosteric tube has a rotational motion whose acceleration is equal to the number of solenoids that are contained in the tube.*

Whereas we would have thus gained the advantage of referring the theorem to fluid volumes, instead of fluid surfaces, we would then forfeit the advantage of being able to track the motion of one and the same individual fluid particle. The fluid masses that define an isobaric-isosteric tube at one moment will no longer be in such a tube at the next moment. The theorem, just like theorem (I), refers to the ever-changing fluid masses that fill up the successive purely-geometric structure that we call an isobaric-isosteric tube. Obviously, the rotation always refers to axis of the tube, and the sign rule is the same as before (Fig. 2).

15. – Circulation of fluid curves.

We derived Theorem (II) for the motion of a fluid volume from the theorem (no. **13**) on the motion of a fluid surface. We can arrive at a different formulation of the same dynamical fact that is useful in many cases by transforming equation (13.*a*) with the use of the known identity that one calls **Stokes'** theorem. We then go over to the motion of a curve, and we do not need to give up the **Lagrangian** picture during that transition.

The surface whose rotation we consider is bounded by a boundary curve *s*. We will denote an element of that curve by Δs. We will call the velocity of a point on the curve *U* and the projection of that velocity onto the tangent to the curve U_t . With Lord Kelvin (*loc. cit*.), we shall call the sum:

$$\sum U_t \, \Delta s \, , \tag{a}$$

which extends over all elements of the closed curve, the *circulation* of the curve. According to **Stokes'** theorem, the identity:

$$\sum \omega_n \, \alpha = \sum U_t \, \Delta s \tag{b}$$

exists between the rotation of a surface and circulation of its boundary curve.

Equation (13.*a*) can also be written:

$$\frac{d}{dt} \sum U_t \, \Delta s = A \tag{c}$$

then. The boundary curve *s* encloses precisely the same bundle of curves that intersects the surface, and we therefore get the following new form for out theorem:

(III) *The circulatory motion of a closed fluid curve has an acceleration that is equal to the number of isobaric-isosteric solenoids that it encloses.*

It is important to note the converse of that theorem (III) and the previous two formulations (I) and (II) of the same dynamical fact. (I) and (II) are linked with the **Eulerian** picture of the hydrodynamical problem: One considers the ever-changing fluid mass that fills up a system of geometric tubes in succession. However, in the formulation (III), we are always free to direct our attention to the same individual fluid curve and the changing number of solenoids that encloses it during its motion.

Even in this case, we can obviously adopt the same sign rule (Fig. 2).

16. – The conservation of circulatory motions and vorticity.

Equation (14.*c*) can be integrated under specialized assumption about the density of the fluid. Namely, if the fluid is homogeneous and incompressible then the isosteric lamellae can be considered to be infinitely dense. There will be no isobaric-isosteric solenoids, the number *A* will reduce to zero, and integrating over time will give:

(*a*) $$\sum U_t \, \Delta s = \text{const.}$$

On the other hand, if the density or the specific volume *k* is a function of pressure *p* then the isosteric surfaces will coincide with the isobaric ones. No lines of intersection between the two families of curves will exist, the number *A* of solenoids will again be zero, and one will again arrive at equation (*a*).

Equation (*a*) represents the known theorem of **Lord Kelvin** (*loc. cit.*) on the conservation of circulatory motion. One can again derive **Helmholtz**'s celebrated theorems on vortex motion from that theorem in the known way.

Those theorems, which are valid only under restricting assumptions on the density, say that any circulatory motion and any vortex are just continuations of previously-existing circulatory motions or vortices. The creation of new motions of that type cannot occur at all under those restricting assumptions about density.

That is why those theorems have only a very limited applicability to the study of the circulatory motions or vortices that evolve in the atmosphere or the oceans. They allow one to study the circulations or vorticities once they have formed completely, and the forces that create circulation or vorticity have ceased to act. The fundamental questions regarding the creation or annihilation of those motions or the conservation of permanent motions of that nature in the presence of frictional resistance are left completely unanswered.

We thus return to the result that was mentioned before in the Introduction that the causes of the great motions in the atmosphere and oceans are always to be sought in the differences in density that *are not* caused by pressure. It again follows from this that the mechanics of those two world media cannot be constructed upon formula (16.*a*), but only upon the more general formula (15.*c*) or upon theorems (I), (II), or (III), which are equivalent to it.

V.

EXAMPLES OF THE APPLICATION OF THE FOREGOING THEOREMS TO THE MOTION OF NATURAL FLUIDS

17. – Formation of vortices by acceleration.

We will now illustrate the applications of the theorems that were developed above by a series of examples that are as simple as possible.

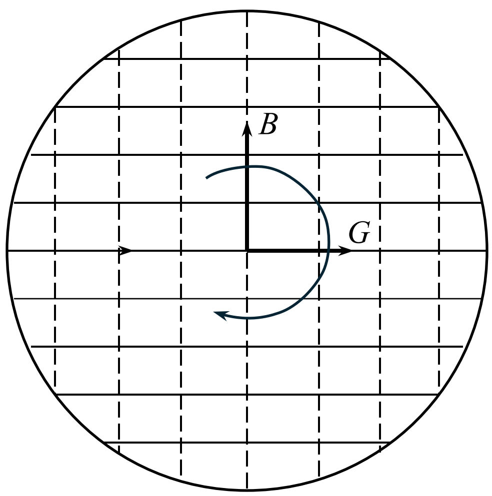

Figure 3.

Let a closed vessel be given that is filled completely with a fluid whose density shall increase above the bottom in such a way that that isosteric surfaces will be equidistant horizontal planes that are represented by the solid lines in Fig. 3. One can arrive at such a density distribution by a suitable mixture of saltwater and sugar water or by heating it in a suitable way.

For the time being, we will imagine that gravity is not in effect and give the vessel an acceleration in the horizontal direction. If the fluid is assumed to be approximately incompressible then a pressure distribution will be created that can be represented by equidistant isobaric planes that are perpendicular to the direction of the acceleration and are represented by the dashed lines in the figure.

The mobility vector B is everywhere-constant and points upwards. The gradient G is likewise constant and has the same direction as the acceleration. The isobaric-isosteric lines are horizontal lines that are perpendicular to B and G, and the solenoids are tubes whose cross-sections define congruent rectangles. From the theorem (I), a vorticial acceleration will be created that will have the same intensity everywhere in the fluid and will be equal to the reciprocal of the area of that rectangle. The vorticity acceleration takes place around the isobaric-isosteric curves as its axis and is directed from the upward-pointing mobility vector B towards the forward-pointing gradient, so it will then admit a rise in the heavier fluid mass along the rear wall of the vessel and a corresponding drop in the lighter mass along the front wall.

Our theorem then leads to the same result that already implied its dynamical sense. The heavier masses must remain behind and displace the lighter ones, which will be driven forwards. The rotational motion of the individual fluid particles must also be accompanied by a deformation, in general, in order for the fluid to always adapt to the form of the vessel. However, if the vessel is spherical then no source of deformation will exist, and the entire fluid mass will take on a rotation like a solid ball around a horizontal diameter.

18. – Periodic formation of vortices created by gravity.

Let the same vessel as above be given, except that the isosteric planes are originally vertical, in such that the mobility vector will be horizontal. If gravity suddenly begins to act then a pressure state will form that is represented by equidistant horizontal planes, and the gradient points upwards against the decreasing pressure.

Under those conditions, from our theorem, a vorticial acceleration must occur that is directed from the mobility vector towards the gradient, and thus induces a sinking of the heavier masses to the bottom and a corresponding rise of the lighter masses.

One can recognize the relationship between that and the foregoing example most easily when one first establishes that the vessel is free to fall. The vessel itself and the fluid masses that it contains will then move downwards with equal acceleration, with no concern for the differences in density. The pressure will be constant everywhere in the fluid and the gradient will be equal to zero everywhere. Should the vessel now be brought to rest, then one would have to impart an acceleration upwards upon it, and from that moment on. the relationships will be precisely the ones in the previous example.

It is also easy for one to predict the further course of the motion. The isosteric surfaces will become ever more horizontal by the rotation of the entire fluid that is created, and they will intersect the isobaric planes at ever-smaller angles, while the solenoids will take on ever-larger cross-sections. One will then have an ever-stronger vorticial velocity and an ever-weaker vorticial acceleration. The result will be a periodic motion under which the isosteric planes will fluctuate around the horizontal position as an equilibrium position.

19. – Creating continuous rotation by heating and gravity.

One will be able to obtain a completely-different result from the one above when one arranges for the isosteric surfaces to always keep their positions despite the motion of the fluid by suitable heating and cooling devices. The solenoids will then suffer no change in cross-section or sign. One will get a vorticial motion with constant acceleration, and as a result, a uniformly-increasing rotation in the same direction whose increase is limited by only friction, instead of the previous periodic motion.

That result is true in full generality, and do not need to have any precise knowledge of the internal mechanism that varies the heat in the volume of the fluid, which is hidden to us. It also remains entirely irrelevant whether we are dealing with weak and continuous changes of volume or strong and discontinuous ones, which appear with the evaporation and condensation of vapors. If one keeps one part of a fluid sufficiently warm, while another is kept sufficiently cool then a circulation will occur that will consist of an over-distillation of the vapor from the warm to the cool part and a corresponding reverse flow of the fluid from the cool to the warm part and will present no essential difference from the circulations that are created by weak changes in volume from our standpoint.

That remark has meteorological interest because it applies to the mechanics of the great circuits of water. However, for applications of that type, it is important to consider the formation of a certain class of vortices, namely, the "sliding vortices (*Gleitungswirbel*)," more precisely.

20. – Sliding vortices on the boundary surfaces of fluid layers of varying density.

In the first example, we assume that the density of the fluid decreases uniformly from the bottom upwards. If we now imagine that the lower half of the vessel is filled with saltwater, for example, while the upper half is filled with sugar water and that there is only a thin boundary layer in which the salt content varies rapidly, but continuously. In that boundary layer, a large number of isosteric planes will be densely-close to each other, and they are represented by the solid horizontal lines in Fig. 4. The isosteric lamellae are very thin discs in that transition region, while the homogeneous saltwater forms only one lamella in the lower half, just as the homogeneous sugar water forms only one lamella in the upper half.

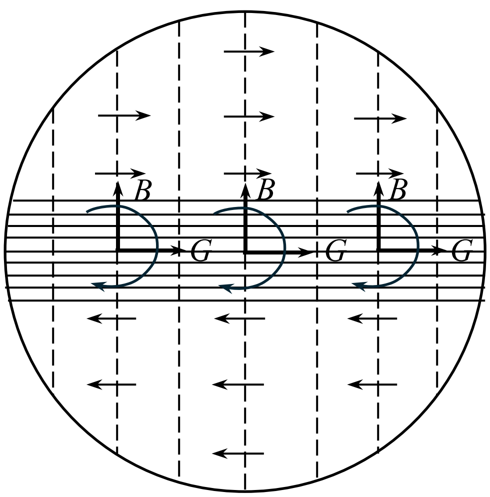

Figure 4.

If the vessel is accelerated in the horizontal direction then the same density distribution will be created as in the first example, which was represented by equidistant vertical planes. By contrast, the solenoids that were previously distributed uniformly in the fluid will be squashed together into very narrow tubes in the transition layer. Vortices are formed in that layer, and only in that layer, but all the more intensely since the intensity is inversely proportional to the solenoid.

One sees immediately that the result of those vortices in the transition layer will be a significant difference between the velocities in the sugar water and the saltwater. The sugar water flows past the saltwater without even forming vortices and seeks to fill up the front half of the vessel, while the saltwater remains behind and fills up the back half. The boundary surface seeks to occupy the vertical location. So far, everything has happened as it did in the first example. However, there is the following difference: Even when the vessel is spherical, deformations must appear in the two homogeneous layers that are superimposed with the purely-translatory motion and seek to create the gradient in the initial moment.

Everything above is valid in general, no matter how fast the transition from saltwater to sugar water might be. We can pass to the limit and assume that the transition is discontinuous: For example, the upper fluid might be oil and the lower one, water. In the boundary layer, vortices with infinitely-small cross-sections, but infinitely-large intensities, will form, or a *sliding* will occur that makes the lighter fluid shoot past the heavier one.

Nothing prevents the lower fluid from being water and the upper one from being air, either. The waves that climb up the rear wall of a vessel that is suddenly put into motion can then be regarded as the result of the formation of a sliding vortex on the boundary surface between the water and the air.

Here, we have always confined ourselves to only qualitative discussions. However, we can make everything rigorously quantitative with no difficulty. For example, we can very easily derive the simple law from our theorem that at the initial moment of the motion, the component of the velocity that is tangent to the boundary surface will behave inversely to the densities of the two fluids. At a later stage of the motion, that same law will be true for the tangential velocity that was newly-formed at the last moment, while the total tangential velocity will obey a more complicated law.

21. – Periodic sliding vortices created by gravity.

We now imagine that gravity is in effect, and we imagine that an initial state is given in which the boundary surface of the two fluids is not horizontal in our vessel. The isobaric surfaces, which lie as exactly or approximately horizontal planes, must then intersect the isosteric surfaces and define solenoids. Sliding vortices will be created on the boundary surface, and as a result of that, the heavy fluid will try to reach the bottom and drive the lighter fluid upwards. The boundary surface seeks to occupy a horizontal position. When that horizontal position has been attained, the acceleration of the sliding vortex will be zero, while its velocity will be largest. The motion will continue, and periodic fluctuations will occur that make the boundary surface oscillate about its horizontal position. During the entire motion, the two homogeneous fluids will be vortex-free, and the entire progress of the vorticity will reduce to sliding on the boundary surface.

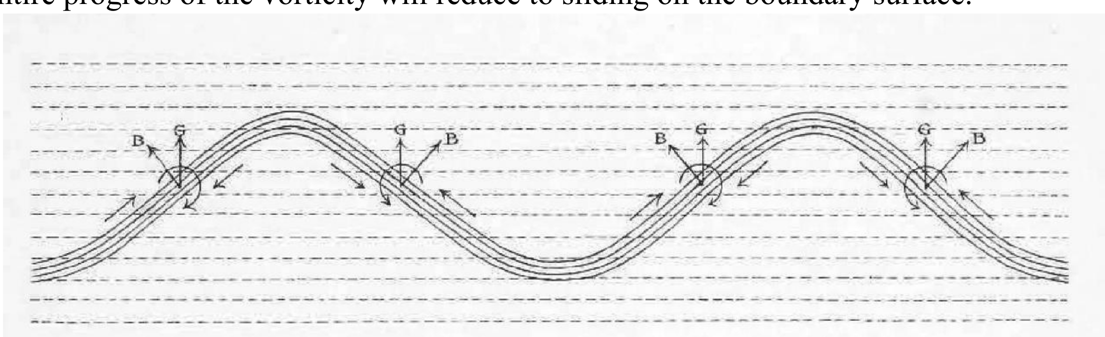

Figure 5.

If the boundary surface of the two fluids is curved in a wave-like manner then the horizontal or almost-horizontal isobaric surfaces will intersect the wave-like isosteric surfaces repeatedly (Fig. 5). They will define solenoids, and a vorticial acceleration will exist that always tries to product a downward flow of the upper heavier fluid from the wave crest to the wave trough and an upward flow of the lighter fluid from the wave trough to the wave crest. Those vorticial accelerations will have their greatest value at the moment when the waves have attained their greatest heights. By contrast, the vorticial accelerations will be zero and the vorticial velocities will attain their maximal value at the moment when the boundary surface is horizontal.

Therefore, we can use our theory in order to study the possible wave motions at the boundary of water and air. In particular, our theory will immediately imply that no wave motion will be possible when the two fluids in contact with each other have the same density under the pressure that prevails.

By contrast, our theory does not explain why waves are created as a result of differences in velocity between the two fluids in contact with each other, for example, the formation of water waves by wind. That is based upon **Helmholtz**'s investigation of an instability in the state of motion in the case where the boundary surface is planar. It follows from that instability that the original simple sliding vortex that we had on a planar boundary surface will resolve into a system of partial sliding vortices with the result that the boundary surface will take on a wavelike form. It is important to emphasize that from our standpoint, we are not dealing with a formation of primary vortices with that formation of waves (which cannot be derived directly from our theory, but only with the help of stability considerations), but only with the decomposition of a pre-existing vortex into partial vortices.

One can easily move on from the periodic sliding vortices that were just discussed to the permanent ones that one encounters in the process of distillation. As an example, it should be mentioned that the inclined isosteric surfaces and the horizontal isobaric surfaces in the boundary layer between the air and the water define the flux of boundary-surface solenoids, from which one derives the downward-pointing motion of the water in flux. Those boundary-surface solenoids then belong to the large and complicated system of solenoids that is completed by the global cycles of water.

22. – Archimedes' principle.

One encounters an interesting example of the sliding vortices that are created by gravity in the form of buoyancy according to Archimedes' principle.

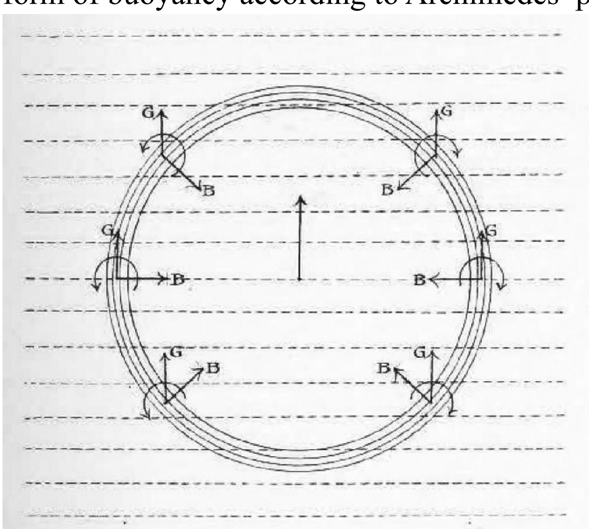

Figure 6.

We consider a bounded fluid mass that is placed in a medium with a different specific gravity, for example, a spherical accumulation of hydrogen in air. In the transition layer between hydrogen and air, where a partial mixing occurs, a large number of isosteric surfaces will surround the hydrogen bubble concentrically (Fig. 6). The mobility vector B points along the radius towards the center of the sphere. By contrast, the isobaric surfaces will intersect the surrounding air and the hydrogen bubble as approximately-horizontal planes. The gradient G points upwards.

As a result, the isobaric-isosteric curves will surround the ball as parallel horizontal circles, and the solenoids will be tubes with a similar behavior. From our theorem, the mixture of gases in those tubes admit a vorticial motion around the axes of those tubes that goes from the mobility vector towards the gradient, and those vortices will admit a rise in the inner, lighter mass with respect to the outer, heavier mass.

No intersections of isobaric or isosteric surfaces are found in the homogeneous interior of the ball, such that vortices will be strictly confined to the transition layer, while the ball will take on a translatory motion upwards.

If the sphere contains carbon dioxide, instead of hydrogen, then the mobility vector will point outwards, and the gas mixture in the solenoid will take on the opposite vorticial motion. The resulting motion of the ball would then point downwards.

If we consider a drop of oil or an air bubble in water then the transition layer will be discontinuous, and the actual vortices will go over to a sliding motion, as was explained above.

We can add the remark that the result above will not be modified at any point when an arbitrary system of forces of a conservative nature acts inside of the ball. In particular, we can assume that this system of forces fixes all fluid particles in the same mutual configuration. In other words: We can go over to the fluid masses in a solid body and thus come to the law of buoyancy for the case of solid bodies, as well.

23. – The motion of air in a chimney.

The discussion of the motion of hot air in a chimney is particularly interesting due to its analogy with the great circulatory motions in the atmosphere. Even in a purely-mathematical context, the application of our theorems to this case deserves some attention.

To begin with, we can easily survey the general evolution of isosteric surfaces. For the sake of simplicity, we can assume that they define horizontal planes in the interior space, as well as in the exterior space. However, the matching parts of one and the same surface do not lie on the same level. That is because where higher temperatures prevail, the isosteric surfaces will be displaced downwards. Theoretically, the difference between the levels can reach infinitely-large values. Namely, with sufficient heating, the air inside of the chimney will take on a specific volume that one would otherwise find in the highest layers of the atmosphere.

Therefore, in order to orient ourselves regarding the simultaneously evolution of the isobaric surfaces, we can first consider two equilibrium cases: On the one hand, the chimney can be closed at the upper end, and on the other, at the lower end, by means of a cover.

If the chimney is closed at the top then the hot air that it encloses will be found to be in stable equilibrium. At the lower mouth, the cold air outside and the warm air inside will have the same pressure. However, the isobaric surfaces inside of the chimney will otherwise all be displaced upwards, and the pressure difference will be greatest at the cover at its top, which corresponds to the difference in weight between the hot air in the chimney and an equal column of cold air. The level differences between the isobaric surfaces are then included between a narrow range of limits and cannot exceed the height of the chimney, in any case. If one suddenly opens the cover then a compensating pressure difference must come about. A pressure drop must occur everywhere inside of the chimney, and as a result, all of the isobaric surfaces must be displaced downwards. By contrast, in the outside air above the mouth of the chimney, a pressure increase must occur such that the isobaric surfaces, which were previously precisely planar, will contain rises.

If the chimney is closed below then the hot air will be found to be in labile equilibrium. The pressure will now have the same value at the upper mouth that is has inside and outside the chimney. However, all of the isobaric surfaces, exactly like the isosteric ones, would otherwise be displaced downwards in the interior space, except that once again, the possible level differences in the isobaric surfaces can be contained between narrow limits, as in the previous case, and therefore they can never exceed the height of the chimney, whereas there are no limits to the possible level differences between the isosteric surfaces. If one opens the cover then a compensating pressure difference would occur at the opening. It would displace the isobaric surfaces in the interior somewhat upwards, while the isobaric surfaces below the mouth would contain depressions.

When a stationary state of motion takes place, a pressure distribution must exist that must lie between the two static pressure distributions in question. At the bottom of the chimney and in its upper part, isobaric surfaces must be displaced somewhat downwards. In the upper part and above the mouth, they are displaced upwards. However, those displacements are regarded as small in comparison to the displacements that the isosteric surfaces experience, which always point downwards.

In general, the air at the upper mouth will be hot when it exists. However, in order to obtain simple results, we first assume that cooling takes place in the upper part of the chimney, such that the air flowing out of it will already have the temperature of the exterior space. The isosteric surfaces will then follow the isobaric surfaces at the upper mouth, exactly as they did at the lower one. Fig. 7 refers to that idealized case.

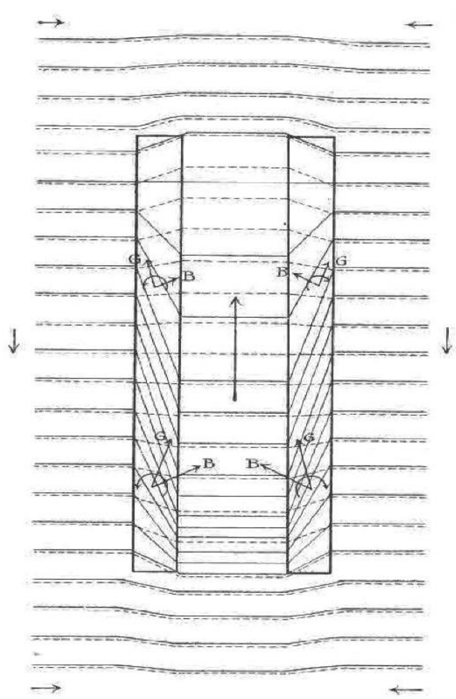

Figure 7.

The remarkable situation will then come about that the isobaric and the isosteric surfaces will coincide in the air, while one will find lines of intersection and solenoids *only in masonry of the chimney*, assuming that one has, however, coupled the separate parts of the surface that belong together.

We then conclude from our theorem (I) that no actual vorticity can arise, except for the "virtual vortices" in the masonry.

However, anyone who is familiar with the general conception of **Stokes'** theorem ([1]) will see immediately that theorem (III) will still be valid, assuming that one has included the virtual solenoids in the calculation. The motion will be "vortex-free circulatory motion." According to theorem (III), each curve that consists of air particles that goes up the chimney and returns to itself in the exterior space will take on a circulatory motion with an acceleration that is equal to the number of solenoids that it encloses. That number will be the same for all curves in the air of that type, namely, it will be equal to the total number of virtual solenoids. The mobility vectors that correspond to those

([1]) Confer circulatory motions, etc., in **§ 11**.

solenoids point towards the interior of the chimney, while the deviation of the gradients from the vertical will always be less than that of the mobility vector. The circulation that points from mobility vector *B* towards the gradient *G* will then admit a *rise* of the air inside of the chimney and a corresponding drop of the air in all of the outside space.

24. – Vortices in escaping smoke or steam.

If we drop the assumption that there is cooling at the upper end of the chimney that was made above then the air that leaves it will be hot. The isosteric surface at the upper mouth will not coincide with the isobaric surfaces, as in Fig. 7, but will possess depressions where the isobaric surfaces have rises. The two families of surfaces will also intersect each other in the air now and define real solenoids, and instead of vortex-free circulation, a mixed circulatory and vorticial motion will be created.

One always observes those vortices in escaping hot smoke. The vorticity of the steam that escapes from the chimney of a locomotive is even more violent than that of smoke. *One* cause of that might be sought in the condensations that occurs on the contact surfaces between the hot steam and the cold air, which will make the behavior of the isosteric surfaces even more complicated.

The vorticity in escaping smoke or steam are generally distinguished by the fact that the cooled or condensed masses outside will remain behind the warm masses inside, which agrees completely with our sign rule for the formation of vorticity. However, it is still undecided whether we have been given a complete explanation for those complicated phenomena in that way. Namely, we will probably once more encounter the case in which a simpler motion would be unstable, such that the large vortices will decompose into smaller ones in a complicated way.

VI.

THE MECHANICS OF THE ATMOSPHERE AND THE OCEANS

25. – Generalities regarding the dynamical discussion of meteorological-hydrographic phenomena.

As was noted before (no. **15**), all of the large circulatory or vorticial motions in the atmosphere or ocean arise exclusively from density differences that *do not* originate in the pressure, but in other causes. On those grounds, the dynamical phenomena of those two global media lie beyond the scope of any discussion of the foundations of the **Helmholtz-Kelvin** theorems on vorticity and circulatory motions. If one attempts to generalize the theorems in order to obtain a useful foundation for the mechanics of meteorology and hydrographics then the first step must necessarily lead to our theorem in one of the forms (I), (II), (III), or perhaps another equivalent form. Further generalizations in which one considers the rotation of the Earth and friction are also easy to follow through with and will become unavoidable for any quantitative investigations. However, they would be unnecessary for the qualitative discussion that follows.

Close interactions exist between the motions in the atmosphere and the ones in the ocean such that when discussing the mechanics of one medium, one must always keep the other medium in mind. It is important to note, in that regard, that our theorems possess enough generality that we can treat the atmosphere and the ocean as *a single* fluid while employing them. That would become immediately clear upon conferring several of the examples that were treated before.

26. – Isobaric and isosteric surfaces in the atmosphere and oceans.

The isobaric surfaces in the atmosphere mostly coincide very closely with the level surfaces of gravity. The gradient points upwards and will generally deviate only slightly from the vertical. The successive surfaces are separated from each other by isobaric lamellae that are layers of air that lie on top of each other, and whose densities will increase with altitude in a roughly geometric sequence. The surfaces and lamellae in the upper layers of air surround the entire Earth, while the lower ones will meet the Earth's surface from time to time or continue into the ocean. The isosteric surfaces and lamellae have an entirely-similar behavior, except that they can never pass through the boundary surface.

The isobaric and isosteric surfaces and lamellae behave in roughly the same way in the ocean, except that they are more precisely coupled with the level surfaces of gravity, and all lamellae will have much smaller thicknesses. None of those surfaces surround the entire Earth since they will all be bounded by the continents and islands in a complicated way. We will always be justified in replacing the discontinuous transition between air and water in our representation with a continuous transition layer in which a large number of isosteric surfaces lie densely close to each other.

In the case of complete equilibrium, the isobaric and isosteric surfaces will coincide completely with the level surfaces of gravity. They would define no lines of intersection or solenoids, and no source of vorticity or circulation would exist. However, that equilibrium would be perturbed by the nonuniform supply of heat from the Sun. That heating acts primarily on the behavior of the isosteric surfaces, and in that way, it will act secondarily on the behavior of the isobaric surfaces. A consequence of that will be that, as was described in the example of the chimney, the isosteric surfaces will suffer relatively large changes in level, while the isobaric surfaces will suffer relatively small ones.

The greatest difference in pressure that is observed on a barometer at sea level (which amounts to around 100 millimeters of mercury) can serve as an example. That would correspond to level changes of around 1000 meters for the isobaric surfaces. At the same time, the density in the air at sea level can suffer changes that can reach a ratio of 2 : 3 in any event, even when one does not go to the most extreme cases, and that ratio would correspond to level changes of more than 2000 meters. The fact that a lively and changing intersection of the families of surfaces with each other must occur during such level changes is understandable. In the ocean, the possible changes must obviously lie within very narrow limits. However, the results of observations also agree those of the theoretical argument there, according to which the greatest fluctuations will occur on the isosteric surfaces.

Nonetheless, no matter how lively the motions of the two families of surface might be, they will always consist of only relatively-small fluctuations of each surface about certain mean positions. The periodic or also completely-irregularly-varying system of winds will depend upon those fluctuations, while the mean positions of the surface must be well-defined for the stationary system of winds.

That is why we first consider the mean positions of the two families of surfaces and after that, the most important fluctuations that the surfaces can experience.

27. – Mean positions of the isobaric and isosteric surfaces. Trade winds.

If we overlook all periodic and random barometric fluctuations then that would imply that the pressure (when reduced to sea level) is distributed rather uniformly over the surface of the Earth such that the isobaric surfaces, in their mean positions, will behave like level surfaces in the first approximation. That is why we would make no great mistake by representing the mean positions of the isobaric surfaces as concentric spheres about the center of the Earth, as are represented by the dashed circles in Fig. 8.

By comparison, when we overlook all annual and daily periods, temperature will be distributed very nonuniformly over the Earth with higher mean temperatures in the tropics and lower mean temperatures in the polar regions. The influence of those temperature differences on the specific volume of air will be increased by the abundant supply of humidity in the equatorial regions and the corresponding dryness of the air in the polar regions. Therefore, the isosteric surfaces must lie high in the polar regions in order for them to approximate the surface of the Earth in the equatorial ones, as is shown by the solid curves in Fig. 8.

The two families of curves must then define lines of intersection and solenoids that will surround the entire Earth somewhat like parallel circles when we ignore small deviations of a local nature. The mobility vector will deviate from the gradient towards the equatorial side such that a rotation of the mobility vector *B* towards the gradient *G* will give a circulation that goes from the equator to the pole in the upper layers of air, while it would point from the pole to the equator on the surface of the Earth and would be observed as the trade winds.

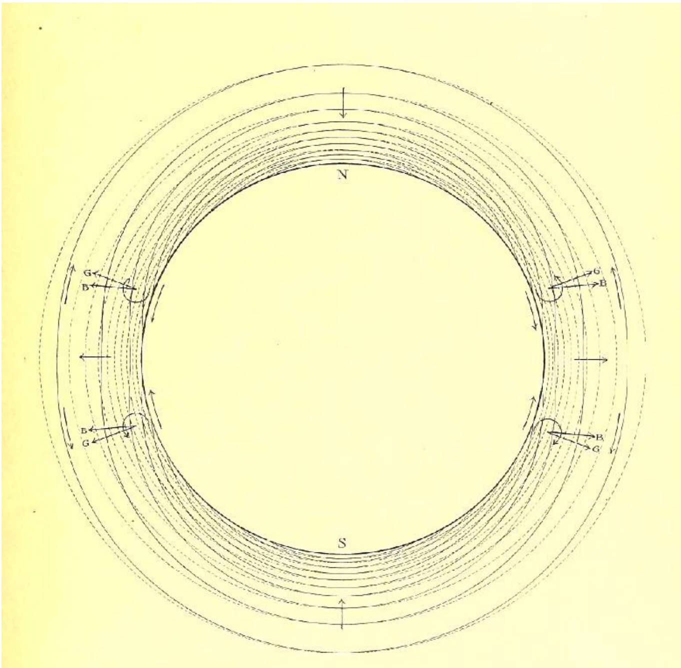

Figure 8.

All of the data that is required for a mechanical discussion of the trade winds will first be present when one has established the mean configurations of the two families of surfaces from the pole to the equator by observation. According to theorem (III), a simple enumeration of the solenoids will give the force that maintains that circulations, and by considering the influence of the rotation of the Earth and friction, one will be able to calculate the mean wind direction and wind strength that those circulations define at each individual location on the Earth.

One last question that involves the greatest difficulty from the mechanical standpoint is that of deciding whether the circulatory motion that defined according to the rules above is stable at all locations on the Earth or whether there are regions where the large, regular circulations split into smaller, partial circulations and vortices. That conjecture is closely related to the fact that the regions of the Earth where one observes no regular trade winds are such zones of instability. However, the data required for resolving that question still does not exist.

28. – Daily motions of the isosteric surfaces.

Barometric pressure shows daily and yearly fluctuations with relatively-small amplitudes. By contrast, the simultaneously-occurring fluctuations in air temperature and the changes in its humidity and specific volume that follow them can be very significant. That is why it would not be a serious mistake to consider the isobaric surfaces to be stationary, while the isosteric surfaces experience significant daily and yearly changes in level.

On the nighttime side of the Earth, the isosteric surfaces will lie high up, while on the daytime side, they will be lower. Thus, the isosteric surfaces must intersect the isobaric ones in the twilight region and define a system of lines of intersection and solenoids that behave like meridians and make a circuit around the entire Earth from east to west in twenty-four hours. Those solenoids must always strive to create a circulation that must go from night to day on the surface of the Earth and from day to night in the higher air layers.

Since it is always rapidly-changing air masses that are exposed to the accelerating effect of those solenoids, large wind strengths cannot arise in that way, and the corresponding periodic winds might not be confirmed with any certainty. However, the general character of those winds is easy to glimpse.

Let us consider the situation at the times of the equinoxes. At the equator, that wind will appear to be a pure west wind at sunrise and as a pure east-to-west wind at sunset, and it will therefore have a purely-oscillating nature. By comparison, it will always appear at the poles, and indeed as a wind that points towards the Sun that moves along the horizon. A similar, if also less regular, situation will also exist at the other times of year and can be characterized as a tendency of the wind direction to "follow the Sun." The difficulties associated with isolating those wind components are based, first of all, upon their weakness, but then also upon the fact that they are, at the same, combined with a land and sea wind that fluctuates with a daily period. However, if the Earth always turned the same side towards the Sun then that wind would have a much greater intensity than the trade winds now possess.

29. – Nonuniformities in the daily motion of the isosteric surfaces. Winds over land and sea.

If the effect of the daily collective motion of the isosteric surface moves more to the background then certain nonuniformities in that motion would have an even stronger influence. Due to the large daily temperature fluctuations over the land or sea, the fluctuations in level of the isosteric surfaces would take place with greater amplitude over land than over the sea. The local heating that takes place over land during the day will have the consequence that the isosteric surfaces would drop below the isobaric ones (Fig. 9). That is why a system of isobaric-isosteric solenoids would form along the coastlines during the daytime, around which a circulation of air would have to form that would be directed from the sea to the land at the Earth's surface.

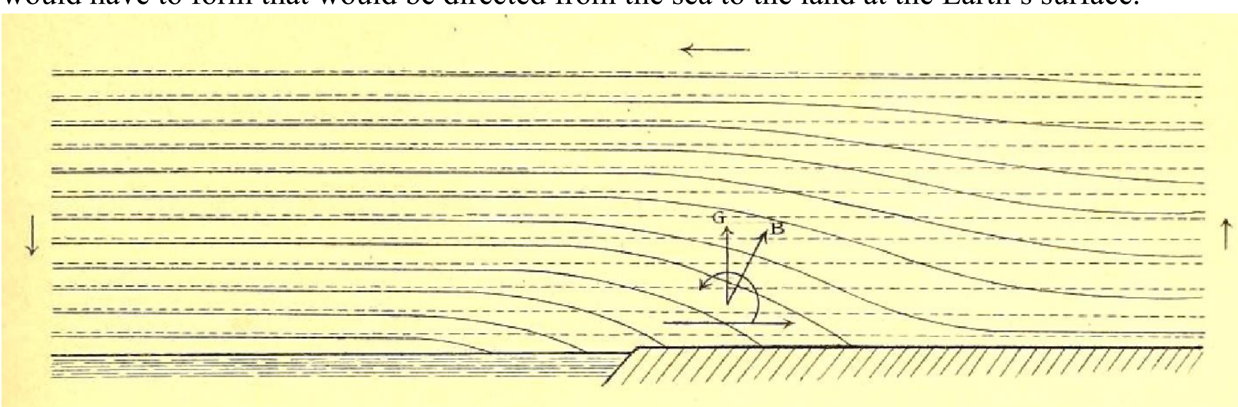

Figure 9.

During the night, the situation will be reversed due to the stronger local cooling over land. The isosteric surfaces will be gradually rising in the evening, so the solenoids that run along the coast will vanish for a moment only to then appear with the opposite sign, and a circulation with the opposite direction will then form.

There is no doubt that the system of solenoids that appears at twilight is much more powerful than the one that forms over coasts. However, whereas the first one distributes its accelerating effect over every-changing air masses, the stationary coastal solenoids will concentrate their effect on the same air masses, and that is why they can create significant wind strengths over the course of a day.

30. – Annual motions of the isosteric surfaces.

We shall now ignore the daily period in the motions of the isosteric surfaces in order to consider only the yearly periods.

In our summer, the isosteric surfaces must lie lower in in the northern hemisphere and higher in the southern hemisphere, while in our winter the opposite situation must be true. Therefore, if we ignore the influence of local conditions, especially the distribution over land and sea, then a system of solenoids must appear that takes the form of something like parallel circles and whose sign will change with the time of year. A circulation must form around that system of solenoids that points from winter towards summer on the surface of the Earth. One would then have to expect large seasonal fluctuations in the direction and strength of the wind over large parts of the Earth, which is why an excess of southerly wind will occur during our summer, and an excess of northerly wind will occur during our winter. However, that wind, whose presence one can assume on theoretical grounds, like the wind that points from night towards day, seems too weak to be able to make itself felt appreciably in the presence of stronger winds. Nonetheless, such an annual period might be discovered by more precise investigations, for example, as a period in the intensity of the trade winds.

31. – Nonuniformities in the annual motion of the isosteric surfaces. Monsoons.

However, various nonuniformities will become more noticeable than the collective motion, exactly like in the case of daily motions.

Thus, summer does not occur simultaneously in all parts of the northern hemisphere. Rather, the summer begins earliest in the southern countries, while winter might still prevail in the northern ones. As a result, the solenoids will develop strongly in the boundary region and create the north winds that often appear in spring.

Nonetheless, the nonuniformities that make themselves felt most strongly are connected with the distribution of land and sea. The annual temperature fluctuation will behave exactly like the daily one, namely, with greater amplitude over land than over sea. It follows from this that the isosteric surfaces will lie lower, on average, over land during summer, while they will lie higher over the sea in the winter, on average. That will imply annual fluctuations in the mean wind directions and wind strengths such that during the summer, at the Earth's surface, the wind over the sea will exceed the wind over land, on average, while the situation will be reversed in winter.

The best-known example of such a wind is the Indian monsoon.

32. – Cyclones.

Even more irregular motions occur that were unforeseen, in addition to the periodic motions of the isosteric surface that were considered above.

A random supply of heat or humidity to the air will always create a local depression in the isosteric surfaces such that they will drop below the isobaric surfaces, and the lowest isobaric surfaces will intersect the Earth along isobaric curves that surround the barometric minimum. The isobaric surfaces will probably have rises in the higher air layers, precisely as in the case of the chimney. However, the detailed behavior of the surfaces has no meaning for our qualitative discussion. It is enough to observe that the depressions in the isosteric surfaces develop more strongly than they do in the isobaric ones (Fig. 10).

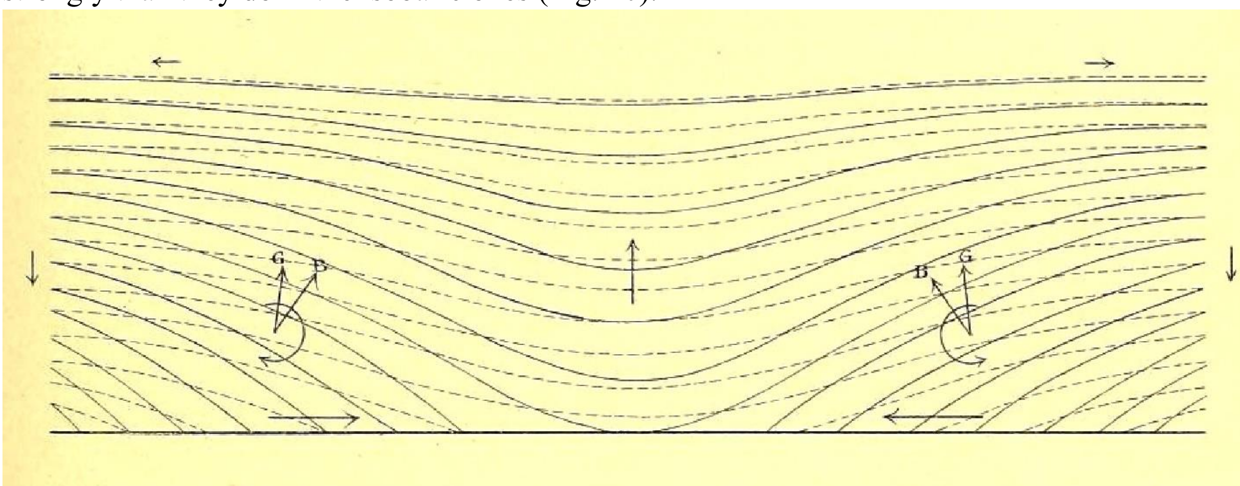

Figure 10.

The two families of surfaces will intersect along closed curves such that every density and pressure minimum will generally be surrounded by a system of annular isobaric-isosteric solenoids. From our sign rule, a circulation will form around those solenoids that behaves as follows: First, it takes place towards the center on the Earth's surface, then upwards in the vicinity of that surface, furthermore, it goes outwards in the upper air layers, and then it slowly sinks towards the center at great distances away.

The mean wind direction and wind strength of the collective circulation, as well as in the individual regions of the cyclone, can be calculated from the number and configuration of the solenoids, while simultaneously considering the rotation of the Earth and friction.

33. – On some points of contention in the theory of cyclones.

The explanations of the causes of the trade winds, the winds over land and sea, and cyclones do not differ at any point, in principle, from the explanations that are ordinarily given by meteorologists. The difference consists of only their special formulation since everything was traced back to the mutual configuration of the isobaric and isosteric surfaces.

We shall now attempt to verify one advantage of the formulation that is proposed here with an example. The causes of the trade winds or the land and sea winds have never been the subject of any fundamental disagreement. By contrast, the way that cyclones form is the subject of debate since, as opposed to the explanation that was reproduced above, one asserts that cyclones form in a purely-mechanical way as a result of conflicts between the large air currents in the atmosphere. In particular, one would then have to expect that cyclones would appear in the possible zones of instability in the trade winds where the possibility would exist for the large circulatory motions to decompose into smaller vortices. If that conception of things is correct then from our point of view that would mean that the solenoids that created the cyclones are not present in the cyclones themselves but belong to the large system of stationary solenoids that cause the trade winds and the large circulatory motions in the atmosphere.

Deciding between those theories will happen immediately when one succeeds in confirming the existence or nonexistence of solenoids in sufficient numbers and with the correct sign in the cyclone itself. That question can be resolved relatively easily by observations that one makes with the meteorological kites that are now employed in North America and France especially. Any kite that is equipped with recording instruments that record at a series of different altitudes will produce a complete cross-section of the passing cyclone that is similar to Fig. 10 and can even indicate a single cross-section regardless of whether the solenoids in question are there or not.

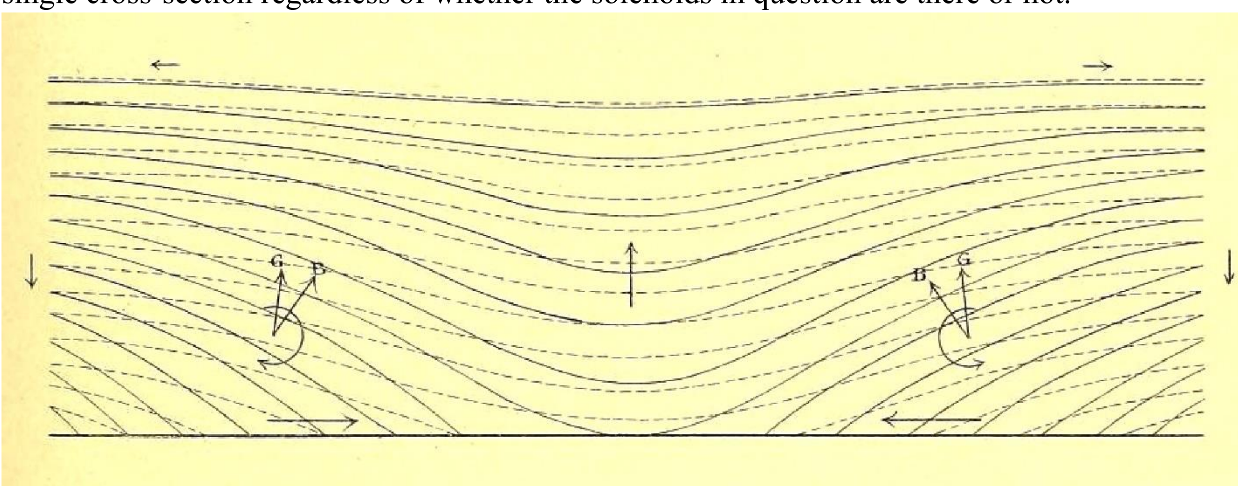

Figure 10.

The source of the controversy regarding the primary cause of the formation of cyclones is in the fact that the question of whether there are regions in which cyclones are created or annihilated, as well as the details regarding them, has still not been resolved definitively. The usual assumption, which is also disputed by the various factions, is that the cyclones that pass over North and Central Europe form in the Atlantic ocean over the region where the Gulf stream is found, while they tend to disappear gradually when they move over Russia and Asia.

We will once more be able to resolve those question by observations made with meteorological kites. Cyclones will form wherever one finds cross-sections as in Fig. 10 with a large number of solenoids. In regions where the isobaric and isosteric surfaces in the cyclone coincide completely (Fig. 11), the motion will continue only as a consequence of inertia and must gradually decrease due to the effect of friction. Finally, in regions where the isosteric surfaces rise above the isobaric surfaces (Fig. 12), the cyclone will be rapidly destroyed and can perhaps be employed even in anticyclones.

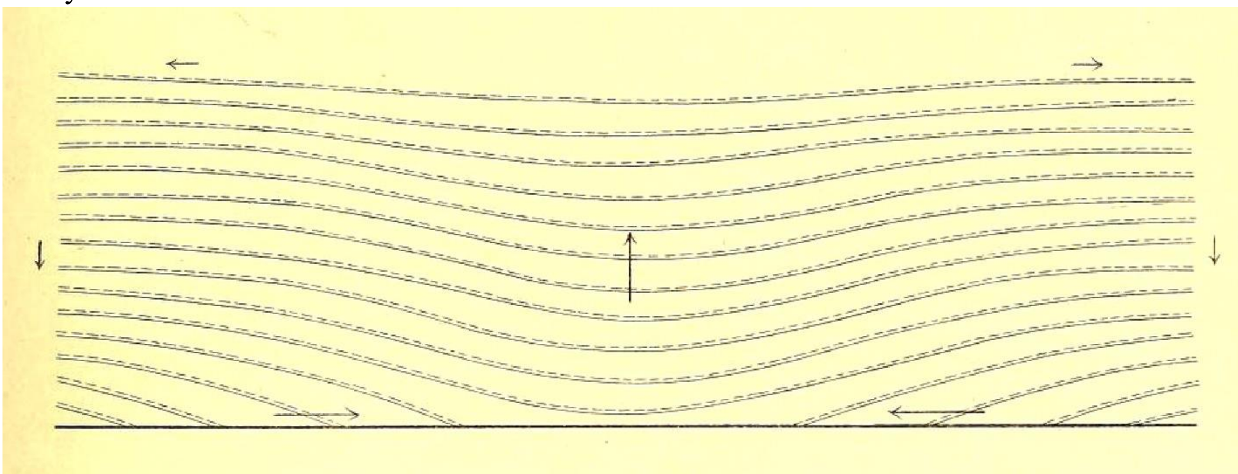

Figure 11.

Most of those observations can be made at fixed stations on the land. The investigation of the Gulf stream region would be carried out by expeditions that are sent out into those regions to perform hydrographic studies, while likewise being equipped with meteorological kites.

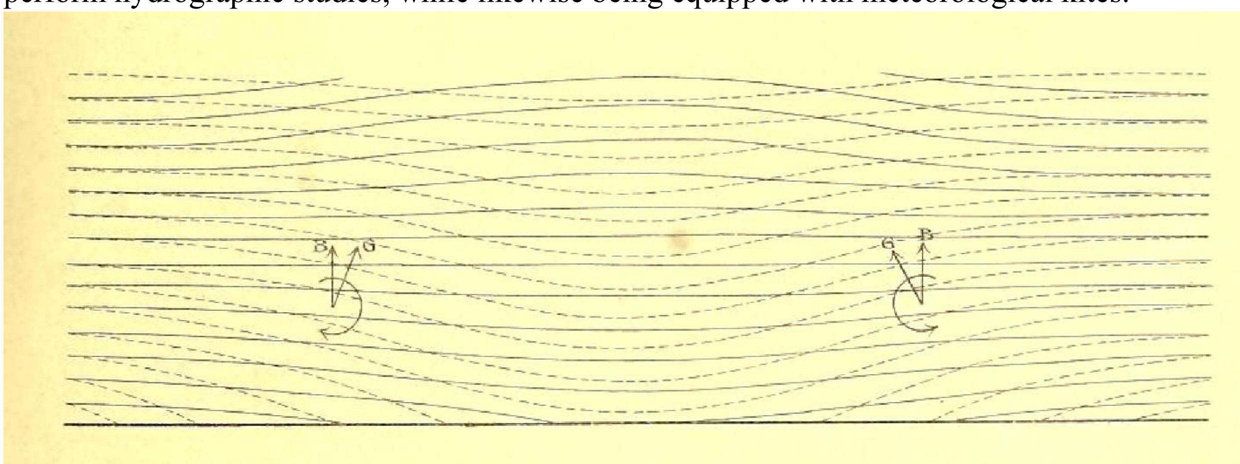

Figure 12.

34. – Anticyclones.

One encounters uncertainties in the theory of anticyclones that are similar to the ones that one encounters with cyclones. They can be created by a local cooling of the air at the surface of the Earth that will be followed by a rise of the isosteric surfaces and then a second rise of the isobaric surfaces (Fig. 13). That will result in the formation of an annular system of solenoids that creates an upward-moving air current in the central region from which an outward-pointing air current forms the Earth's surface and one that rises up and returns to itself at great distances from the center.

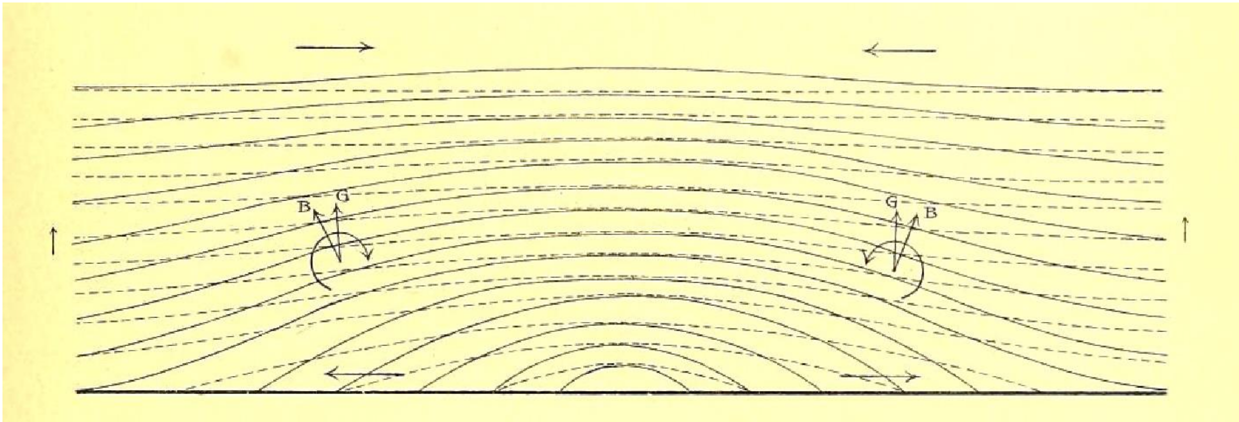

Figure 13.

Another explanation is more mechanical in character. The large number of rising air currents would produce an overflow in the upper layers of air when there are no corresponding downward currents. The upward motion can be distributed uniformly and proceed slowly over large parts of the Earth's surface. However, they can also be localized in the form of intense air currents.

If the first explanation is correct then the solenoids that create the anticyclonic motion must be found in the anticyclone itself. If the second picture is correct then the solenoids that create the motion must be found in distant regions, namely, in the cyclones or among the stationary solenoids that create the large circulatory motions in the atmosphere.

As in the case of cyclones, the question in dispute will be resolved when one investigates the behavior of the isobaric and isosteric surface in anticyclones by direct observations.

35. – The causes of the large ocean currents.

The formation of the large ocean currents can be explained in two different ways, either as the result of the differences in density that arise from changes of temperature and salinity or as friction between the wind and the surface of the ocean. Of course, when both effects are present, the question will be which of them is dominant.

(final pages 34 and 35 might or might not be missing from the original file, along with Fig. 14.)
