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On the formation of vortices in frictionless fluids, with application to the analogy between hydrodynamic phenomena and electrostatic ones.

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(With three figures)

Translated by D. H. Delphenich

I. – Introduction.

1. – Apparent action at a distance can take place between pulsating spheres that are found in a homogeneous, incompressible, and frictionless fluids. All of that suggests that those actions are based in only the pulsating motions, but not on the spherical form that assumed in order to simplify the calculation, and the rigidity of the bodies under deformation that would follow from it. Everything would then make it likely that the same actions at a distance would also appear between pulsating bodies of arbitrary form *in the fluid itself.*

That remark is important because it allows one to glimpse the possibility of developing the theory of hydrodynamical action at a distance by using a simpler and more-general method. Instead of a system that is composed of a fluid and foreign bodies, it must probably be possible to consider a simple fluid system and to deduce the entire theory directly from the equations of motion of the fluid.

2. – A plan for carrying out an investigation in that direction is given by the following argument: When a pulsating fluid mass takes on a motion as a consequence of its attraction to the surrounding fluid, a sliding along the surface that separates the pulsating fluid mass from the surrounding non-pulsating fluid will necessarily take place. However, that sliding can be regarded as the limiting case of a vortex. A *formation of vorticity* will then take place, and it must be possible to draw conclusions of a general nature about the apparent action at a distance that is based in the fluid pressure from the laws of formation of vortices in frictionless fluids.

I have found that the suggested path is completely accessible once I first overcame my cherished prejudice that vorticity could not arise or die off at all in a fluid ([1]). In fact, it proves to be very easy to derive a series of simple theorems on the formation of vorticity. I have already published those theorems before, and after that I especially developed the application of one of them to a discussion of motions in the air and ocean ([2]). Here, I shall return to the application that led me to investigate the formation of vortices, namely, the application to hydrodynamical action at a distance that was suggested above. Therefore, I shall first give a new and independent derivation of the theorem of vortex formation that will be applied.

II. – The hydrodynamical equations of motion.

3. – In what follows, I shall let u denote the velocity, p, the pressure, and q, the density of the fluid. Along with the density, I shall also introduce the specific volume:

(a) $$k = \frac{1}{q},$$

or *mobility*, of the fluid. Moreover, along with the velocity, I shall also introduce the product of velocity and density:

(b) $$\bar{u} = q\,u$$

([1]) The usual categorical emphasis on the conservation of vorticity in frictionless fluids, the parallelism between that result and the principle of the conservation of energy and the comparison between perpetual vortex rings and perpetual atoms, all without expressly underlining the specializing assumptions that were used in the derivation, has certainly broadened the scope of that prejudice considerably, and in that way had an unfavorable effect on research in hydrodynamics, in general. Extended realms in which results must nonetheless be gleaned that are distinguished just as much by their theoretical appeal as by their practical utility are not considered to be approachable in that way as they are in theoretical hydrodynamics. It is especially regrettable that if not for that fact, hydrodynamics would participate more actively in meteorological research. It is probably despite one's instinctual sense of the otherwise striking contradiction with the continual formation of atmospheric vortices that takes place that one often formulates the theorem that fluid vortices cannot be generated "by conservative processes." However, that is entirely incorrect because the formation of vortices by fluid pressure is, in itself, a process of a completely conservative nature, and when the formation of vortices drops off then that is based, in and of itself, upon certain intrinsic properties of the fluid, namely, that it is either homogeneous and incompressible or that its density is a function of only pressure. Since those assumptions are never precisely accurate for natural fluids, the theorems on the conservation of vorticity are pure abstractions, although they are very useful abstractions since the cited conditions can be fulfilled to a very high degree of approximation under some circumstances. By contrast, it is naturally essentially meaningful for the continued formation of vortices to know whether energy can be continuously exhausted by an energy source. For a conservative system, the formation of vortices cannot advance continually in the same direction, but that would be possible with the aid of an energy source, as we see in atmospheric motions.

([2]) **V. Bjerknes**, "Über die Bildung von Cirkulationsbewegungen und Wirbeln in reibunglason Flüssigkeiten," Videnskansselkabers Skrifter, Christiania (1898). "Über einen hydrodynamischen Fundamentalsatz und seine Anwendung auf die Mechanik der Atmosphäre und des Weltmeeres," K. Svenske Vetensksakademiens Handligar, Stockholm (1898). "Das dynamische Prinzip der Circkulationsbewegungen in der Atmosphäre," Meteorologische Zeitschrift (1900). "Cirkulation relative zu der Erde," Öfversigt af K. Vetenskaps-Akad. Föorhandl. (1901), Met. Zeit. (1902).

which is the specific momentum, or *hydrodynamical field intensity*, as the vector quantity for the description of the fluid motion. Finally, I will let f denote the force per unit mass and let $\bar{f}$ denote the force per unit volume that acts upon the individual fluid particles. A relationship exists between $\bar{f}$ and f that is similar to the one that exists between field intensity and velocity, namely:

(c) $$\bar{f} = g f .$$

4. – When I denote the components of the aforementioned vector quantities along the three rectangular axes x, y, z by adding the subscripts x, y, z, resp., the hydrodynamical equations of motion can then be written in the form:

(a) $$\begin{cases} \dfrac{du_x}{dt} = f_x - \dfrac{1}{q}\dfrac{\partial p}{\partial x}, \\ \dfrac{du_y}{dt} = f_y - \dfrac{1}{q}\dfrac{\partial p}{\partial y}, \\ \dfrac{du_z}{dt} = f_z - \dfrac{1}{q}\dfrac{\partial p}{\partial z}. \end{cases}$$

The continuity equation should be added to them:

(b) $$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \frac{1}{q}\frac{dq}{dt} = 0 .$$

The d / dt in those equations has the well-known meaning:

(c) $$\frac{d}{dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} .$$

Therefore, d / dt refers to the changes that one observes in one and the same moving fluid particle, while $\partial / \partial t$ refers to the changes that one observes at one and the same location in space.

5. – In his classic treatise on the conservation of vorticity, **Helmholtz** assumed that the density q of the fluid was constant and defined a new set of equations from (4.a) by differentiating the last equation with respect to y, and the second-to-last equation with respect to z and taking the difference of the two. In that way, an equation arose that no longer included the pressure, and which expressed the principle of the conservation of vorticity when it was combined with two similarly-formed equations.

By contrast, when one drops any specializing assumption on the density, the pressure will not drop out from the new equations that are defined in that way. Instead of **Helmholtz**'s theorem on the conservation of vorticity, one will then come to a more-general one that gives the dependency between the law of vortex formation and the distributions of density and pressure. The law is especially meaningful for the discussion of atmospheric vorticity, and the preservation of the pressure in the formula is advantageous because the pressure can be regarded as known from barometric observations.

However, for the applications that I have in mind here, no *a priori* knowledge of the pressure distribution will be assumed. The elimination of the pressure is essential, and it will be achieved when one multiplies equations (4.a) by the density q before the differentiation. However, the theorems to which one will arrive in that way will first take on their simplest form when one describes the fluid motion in terms of the hydrodynamical field intensity, instead of the velocity.

We next introduce the hydrodynamical field intensity into the general equations of motion. As a result of (3.b), we have:

$$q\frac{du_x}{dt} = \frac{d\bar{u}_x}{dt} - u_x\frac{dq}{dt} = \frac{d\bar{u}_x}{dt} - \bar{u}_x\frac{1}{q}\frac{dq}{dt}.$$

The factor of $\bar{u}_x$ in the last term on the right occurs in the equation of continuity. From (3.a) and (4.b), it is:

$$-\frac{1}{q}\frac{dq}{dt} = \frac{1}{k}\frac{dk}{dt} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}.$$

Each of those three equivalent expressions has the meaning of the rate of cubic expansion per unit volume of the moving fluid particle. To abbreviate, we will denote that quantity by e and then write the continuity equation in the form:

(a) $$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = e$$

from now on, or with a known notation in modern vector analysis, in the form:

(a′) $$\operatorname{div} u = e\,.$$

One will then have:

$$q\frac{du_x}{dt} = \frac{d\bar{u}_x}{dt} + e\bar{u}_x,$$

and the first equation in (4.a) can be written as:

(b) $$\frac{d\bar{u}_x}{dt} + e\bar{u}_x = \bar{f}_x - \frac{\partial p}{\partial x}.$$

6. – The equations that relate to the vorticity $\bar{\xi}$ of the vector $\bar{u}$ shall now be derived from that equations the two corresponding ones that relate to $\bar{u}_y$ and $\bar{u}_z$. That vorticity has the components:

(a)
$$\begin{cases} \bar{\xi}_x = \dfrac{\partial \bar{u}_z}{\partial y} - \dfrac{\partial \bar{u}_y}{\partial z}, \\ \bar{\xi}_y = \dfrac{\partial \bar{u}_x}{\partial z} - \dfrac{\partial \bar{u}_z}{\partial x}, \\ \bar{\xi}_z = \dfrac{\partial \bar{u}_y}{\partial x} - \dfrac{\partial \bar{u}_x}{\partial y}, \end{cases}$$

and will be denoted in modern vector analysis by:

(a′)
$$\bar{\xi} = \text{curl } \bar{u} .$$

This vorticity $\bar{\xi}$ of the field intensity is to be distinguished from the vorticity $\xi = \text{curl } u$, If the density q of the fluid is spatially constant then $\bar{\xi}$ and ξ will point in the same direction everywhere and be proportional to the numerical value:

(a″)
$$\bar{\xi} = q\,\xi ,$$

and there will no longer exist any essential difference between the velocity and field intensity fields. However, we have the general case in mind, where the density can vary arbitrarily in space and time, and the vorticity of the field intensity can deviate arbitrarily much from the vorticity of the velocity in direction, as well as quantity.

Differentiating the first equation in (a) with respect to x, the second, with respect to y, and the third, with respect to z, and adding them will given the known relation:

(b)
$$\frac{\partial \bar{\xi}_x}{\partial x} + \frac{\partial \bar{\xi}_y}{\partial y} + \frac{\partial \bar{\xi}_z}{\partial z} = 0 ,$$

which is generally true for any vector that can be represented as the vorticity of a different vector.

7. – In order to ease the ultimate derivation of the equations that are valid for the formation of vortices, we can convert the general dynamical equation (5.b) somewhat.

If we employ the Euler development (4.c) for the time derivative then the first term on the left in equation (5.b) will assume the form:

$$\frac{\partial \bar{u}_x}{\partial t} + u_x \frac{\partial \bar{u}_x}{\partial x} + u_y \frac{\partial \bar{u}_x}{\partial y} + u_z \frac{\partial \bar{u}_x}{\partial z} .$$

As one easily sees, that can also be written in the form:

$$\frac{\partial \bar{u}_x}{\partial t} + \left(u_x \frac{\partial \bar{u}_x}{\partial x} + u_y \frac{\partial \bar{u}_y}{\partial y} + u_z \frac{\partial \bar{u}_z}{\partial z} \right) + u_z \bar{\xi}_y - u_y \bar{\xi}_z .$$

If we introduce $u_x = k\,\bar{u}_x$, etc., in the trinomial terms in parentheses using (3.a and b) then the expression will assume the form:

$$\frac{\partial \bar{u}_x}{\partial t} + \frac{1}{2} k \frac{\partial}{\partial x} \left(\bar{u}_x^2 + \bar{u}_y^2 + \bar{u}_z^2 \right) + u_z \bar{\xi}_y - u_y \bar{\xi}_z ,$$

or more simply:

$$\frac{\partial \bar{u}_x}{\partial t} + \frac{1}{2} k \frac{\partial \bar{u}^2}{\partial x} + u_z \bar{\xi}_y - u_y \bar{\xi}_z .$$

Equation (5.b) will then become:

$$\frac{\partial \bar{u}_x}{\partial t} + \frac{1}{2} k \frac{\partial \bar{u}^2}{\partial x} + u_x \bar{\xi}_y - u_y \bar{\xi}_x + e \bar{u}_x = \bar{f}_x - \frac{\partial p}{\partial x} .$$

The other two equations are obtained by cyclic permutation of the *x*, *y*, *z*. If we simultaneously move the terms somewhat then that will give the following form for the general hydrodynamical equations of motion:

$$\text{(a)} \qquad \left\{ \begin{aligned} \frac{\partial \bar{u}_x}{\partial t} + u_z \bar{\xi}_y - u_y \bar{\xi}_z &= \bar{f}_x - \frac{\partial p}{\partial x} - e \bar{u}_x - \frac{1}{2} k \frac{\partial \bar{u}^2}{\partial x} , \\ \frac{\partial \bar{u}_y}{\partial t} + u_x \bar{\xi}_z - u_z \bar{\xi}_x &= \bar{f}_y - \frac{\partial p}{\partial y} - e \bar{u}_y - \frac{1}{2} k \frac{\partial \bar{u}^2}{\partial y} , \\ \frac{\partial \bar{u}_z}{\partial t} + u_y \bar{\xi}_x - u_x \bar{\xi}_y &= \bar{f}_z - \frac{\partial p}{\partial z} - e \bar{u}_z - \frac{1}{2} k \frac{\partial \bar{u}^2}{\partial z} , \end{aligned} \right.$$

in which one must recall the relations (3.a and b) and (6.a).

Those three equations can be represented by a single vector equation, namely:

$$\text{(a}'\text{)} \qquad \frac{\partial \bar{u}}{\partial t} + (\text{curl}\,\bar{u}) \times u = f - \nabla p - e \bar{u} - \tfrac{1}{2} k \nabla \bar{u}^2 .$$

∇ is then the symbol for the **Hamiltonian** differential operation with the three differentiation components $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, and following **Gibbs**, the symbol × stands between the two factors in a vector product ([1]).

III. – The general law of vortex formation.

8. – We shall now define a new process that follows **Helmholtz**'s by differentiating the third of equations (7.a) with respect to *y* and the second one with respect to *z* and subtracting them. Initially, the left-hand side of new equation will assume the form:

$$\frac{\partial\bar{\xi}_x}{\partial t}+u_y\frac{\partial\bar{\xi}_x}{\partial y}+u_z\frac{\partial\bar{\xi}_x}{\partial z}-u_x\left(\frac{\partial\bar{\xi}_y}{\partial y}+\frac{\partial\bar{\xi}_z}{\partial z}\right)+\bar{\xi}_x\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)-\bar{\xi}_y\frac{\partial u_x}{\partial y}-\bar{\xi}_z\frac{\partial u_x}{\partial z}$$

when one recalls (6.a). The desired equation will then be the following one when we keep only the term $d\bar{\xi}_x/dt$ on the left-hand side:

(a)
$$\left\{\begin{aligned}\frac{d\bar{\xi}}{dt}= &-\bar{\xi}_x\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\bar{\xi}_x\frac{\partial u_x}{\partial y}+\bar{\xi}_y\frac{\partial u_x}{\partial z}\\ &+\frac{\partial\bar{f}_x}{\partial y}-\frac{\partial\bar{f}_y}{\partial z}-\left(\frac{\partial(e\bar{u}_z)}{\partial y}-\frac{\partial(e\bar{u}_y)}{\partial z}\right)-\frac{1}{2}\left(\frac{\partial k}{\partial y}\frac{\partial\bar{u}^2}{\partial z}-\frac{\partial k}{\partial z}\frac{\partial\bar{u}^2}{\partial y}\right).\end{aligned}\right.$$

The other two equations are obtained by cyclic permutation of *x*, *y*, *z*. Those equations include the most-general law of vortex motion in frictionless fluids, assuming that by "vorticity" one means the vorticity of the hydrodynamical field intensity $\bar{\xi}$, not the vorticity ξ of the velocity.

In order to ease the discussion, we now write:

(b)
$$\frac{d\bar{\xi}_x}{dt}=\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{I}}+\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{II}}+\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{III}}+\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{IV}}$$

and examine the following four equations individually:

($\mathrm{b_I}$)
$$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{I}}=-\bar{\xi}_x\left(\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\bar{\xi}_y\frac{\partial u_x}{\partial y}+\bar{\xi}_y\frac{\partial u_x}{\partial z},$$

([1]) **Grassmann**'s parenthesis notation has the disadvantage that parentheses already have so many other applications. a notation that is useful for other purposes, namely, V, is abandoned by the use of the **Hamiltonian** notations. That is why, of all the notations that have been proposed up to now, it seems to be the most convenient to me. (Cf., **Gibbs-Wilson:** *Vector Analysis*, New York, 1902) (bII) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{II}} = \frac{\partial \bar{f}_x}{\partial y} - \frac{\partial \bar{f}_y}{\partial z},$$

(bIII) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{III}} = -\left(\frac{\partial(e\bar{u}_z)}{\partial y} - \frac{\partial(e\bar{u}_y)}{\partial z}\right),$$

(bIV) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{IV}} = -\frac{1}{2}\left(\frac{\partial k}{\partial y}\frac{\partial \bar{u}^2}{\partial z} - \frac{\partial k}{\partial z}\frac{\partial \bar{u}^2}{\partial y}\right),$$

which give the various factors that influence the motion of vortices.

IV. – Conservation of vorticity.

9. – If we next write out equation (8.bI) and the two that follow from it by cyclic permutation of x, y, z then that will give the system of equations:

(a) $$\begin{cases} \left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{I}} = -\bar{\xi}_x\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \bar{\xi}_y\frac{\partial u_x}{\partial y} + \bar{\xi}_y\frac{\partial u_x}{\partial z}, \\ \left(\frac{d\bar{\xi}_y}{dt}\right)_{\text{I}} = \bar{\xi}_x\frac{\partial u_y}{\partial x} - \bar{\xi}_y\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_x}{\partial x}\right) + \bar{\xi}_z\frac{\partial u_y}{\partial z}, \\ \left(\frac{d\bar{\xi}_z}{dt}\right)_{\text{I}} = \bar{\xi}_x\frac{\partial u_z}{\partial x} - \bar{\xi}_y\frac{\partial u_z}{\partial y} - \bar{\xi}_z\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right). \end{cases}$$

When no vorticity exists at the time considered, so $\bar{\xi}_x = \bar{\xi}_y = \bar{\xi}_z = 0$, the right-hand sides, and therefore the left-hand sides, of those equations will vanish. Thus, when vorticity does not exist, those equations will imply that no new formation of vortices will take place either. As a result, they will imply only modifications that preexisting vortex motion will experience under some circumstances.

One will easily recognize modifications of that type when one recalls the known kinematical meanings of the differential expressions that appear on the right-hand side. As is known:

$$\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}, \qquad \frac{\partial u_z}{\partial z} + \frac{\partial u_x}{\partial x}, \qquad \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}$$

mean the rates of expansion per unit area of surfaces that are perpendicular to the x, y, or z-axis, respectively. The terms in the diagonal, when endowed with the negative sign, will then say how the vortices will die off along each of the axes in question when an expansion of the mass of the vortical fluid takes place in the plane that is perpendicular to the axis, as one would expect as a result of the decrease in the moment of inertia of the rotating masses.

On the other hand, the differential quotient $\partial u_x / \partial y$ has the meaning of the angular velocity with which the line element of the fluid that is parallel to the y-axis rotated around the z-axis. The term $\bar{\xi}_y \frac{\partial u_x}{\partial y}$ and the remaining terms that are similarly constructed will then give the variations of the individual components of the vorticity that result from the changes in direction of the vortex axis relative to the coordinate system.

Equations (a) then say precisely the same thing about the vorticity $\bar{\xi}$ of the hydrodynamical field intensity that the known **Helmholtz** equations say about the vorticity ξ in homogeneous and incompressible fluids. If the general equations of vorticity (8.a) reduce to equations (a) exclusively then the preexisting vorticity will be preserved with no formation of new vortices.

10. – One also easily verifies that equations (9.a) will reduce to the usual **Helmholtz** equations for the conservation of vorticity when the fluid is homogeneous and incompressible.

If the fluid is initially incompressible then the velocity will fulfill the condition:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 .$$

The first of equations (9.a) can be put into the form:

$$\text{(a)} \qquad \left(\frac{d\bar{\xi}}{dt}\right)_{\mathrm{I}} = \bar{\xi}_x \frac{\partial u_x}{\partial x} + \bar{\xi}_y \frac{\partial u_x}{\partial y} + \bar{\xi}_z \frac{\partial u_x}{\partial z}$$

with the help of that relation. That equation, along with the two corresponding ones that relate to the y and z, are valid for incompressible fluids that can still be heterogeneous, such as, for instance, a mixture of salt water and sugar water, or a mixture of oil and water.

If the fluid is, at the same time, homogeneous, so the density q is constant, then the vorticity of the field intensity $\bar{\xi}$ will be simply proportional to the vorticity of the velocity ξ (6.a″), and after removing the constant factor q, one will get:

$$\text{(b)} \qquad \frac{d\xi_x}{dt} = \xi_x \frac{\partial u_x}{\partial x} + \xi_y \frac{\partial u_x}{\partial y} + \xi_z \frac{\partial u_x}{\partial z} .$$

That equation can also be written in the form:

$$\text{(b}'\text{)} \qquad \frac{d\xi_x}{dt} = \xi_x \frac{\partial u_x}{\partial x} + \xi_y \frac{\partial u_y}{\partial x} + \xi_z \frac{\partial u_z}{\partial x} .$$

Equation (b), along with the corresponding ones for the y and z-axis, give one form for **Helmholtz**'s original equations for the conservation of vorticity ([1]), while equation (b′), along with the corresponding ones for the y and z-axis, give the other.

Now, if the external force is conservative, and the fluid is homogeneous and incompressible, as assumed, then all of the terms to be discussed in the general equations of vortex motion will vanish (cf., in nos. **11**, **13**, **15**). The general equation (8.a) reduces to the form (b) or (b′), and we get back to **Helmholtz**'s result.

V. – Formation of vorticity by external forces.

11. – The equation:

(a) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{II}} = \frac{\partial \bar{f}_z}{\partial y} - \frac{\partial \bar{f}_y}{\partial z},$$

along the corresponding equations for the other two axes, describes the formation of vorticity by external forces, which is obviously always possible.

However, is should be emphasized especially that the external forces can also serve to create vorticity when they are conservative. A force is conservative when its magnitude f *per unit mass* can be represented by a potential function Φ:

(b) $$f_x = \frac{\partial \Phi}{\partial x}, \qquad f_y = \frac{\partial \Phi}{\partial y}, \qquad f_z = \frac{\partial \Phi}{\partial z}.$$

The components of the force *per unit volume* will then be:

(b′) $$\bar{f}_x = q\frac{\partial \Phi}{\partial x}, \qquad \bar{f}_y = q\frac{\partial \Phi}{\partial y}, \qquad \bar{f}_z = q\frac{\partial \Phi}{\partial z}.$$

When one introduces that into (a), one will have:

(c) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{II}} = \frac{\partial q}{\partial y}\frac{\partial \Phi}{\partial z} - \frac{\partial q}{\partial z}\frac{\partial \Phi}{\partial y}.$$

The expression on the right is the first component of a vector product, and equation (c), in conjunction with the two that correspond to the other two axes, can be represented by the single vector equation:

([1]) *Wissenschaftliche Abhandlungen*, I, pp. 110-111.

(c′) $$\left(\frac{d\bar{\xi}}{dt}\right)_{\text{II}} = \nabla q \times \nabla \Phi \, .$$

The vector ∇q is perpendicular to the surfaces of equal density, q = const., and points in the direction of increasing values of the density. In precise analogy with that, the vector $\nabla\Phi$ is perpendicular to the surfaces Φ = const., and points in the direction of increasing values of the potential function Φ. As a vector product of those vectors, $\left(\frac{d\bar{\xi}}{dt}\right)_{\text{II}}$ is perpendicular to both of them. As a result, the axes along which vortices form point along the lines of intersection of the families of surfaces q = const. and Φ = const., and the direction of rotation is the one that takes the first vector factor ∇q to $\nabla\Phi$ the second one along the shortest path.

(I.A) *One then finds the formation of vorticity around the lines of intersection of the equi-dense surfaces q = const. and the equipotential surfaces Φ = const., and indeed in the direction that points from the density vector ∇q to the potential vector $\nabla\Phi$.*

According to that law, a vortex will form in the atmosphere, for example, around the lines of intersection of the equi-dense surfaces and the level surfaces of the force of gravity. When f is the acceleration of gravity, and Φ is the gravitational potential, according to the definition (b), the vector $\nabla\Phi$ will point vertically upwards. Now, if the density vector points sideways then the rotation of the density vector to the potential vector will be a motion in which the denser masses sink on the one side, while the heavier ones rise on the other, as one would expect. That theorem can be exploited for the derivation and quantitative study of the formation of atmospheric vortices in precisely the same way as the theorem that this formation of vorticity will lead back to the distribution of density, in conjunction with the distribution of pressure. However, we shall not go into the details of the applications of that kind on this occasion.

If the fluid is homogeneous and incompressible then the vector ∇q will be zero, and therefore the vector product (c′), as well. Hence, the formation of vortices by conservative external forces cannot take place in homogeneous and incompressible (cf. no. **10**).

12. – The consideration of the following special case will be especially informative for the discussion that follows.

In a homogeneous fluid of density $q = q_0$, one finds a bounded fluid mass that has the greater density $q = Q$, and which we will briefly refer to as the *body*, in contrast to the surrounding fluid. Fig. 1 shall represent a vertical section through the body and the fluid. In the transition layer, one sees a number of densely-neighboring surfaces of equal density that are cut from the horizontal level surfaces of the force of gravity. It is only in that transition layer that the density vector ∇q will have a non-zero value, and that is the only place where vorticity can form. However, the vorticity that emerges in the direction from the potential vector to the density vector will be possible, as one sees from a consideration of the figure, only when the body takes on a motion that is downwards with respect to the surrounding fluid. On the other hand, if the fluid body has a density that is less than that of the surrounding fluid then the vector ∇q will have the opposite direction in the transition layer, so the vortices will form in the opposite direction, and the body must take on a motion that is upwards relative to the surrounding fluids.

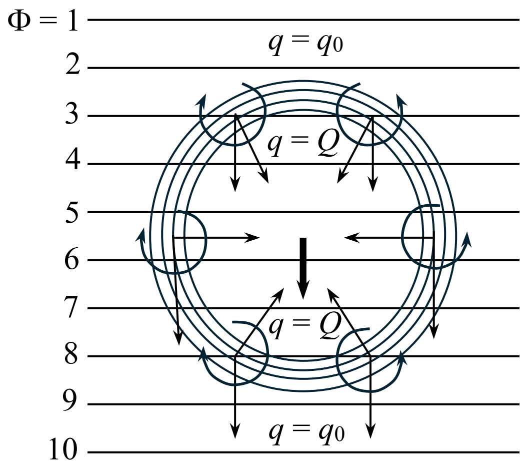

Figure 1.

Nothing prevents us now from introducing forces that keep the individual points of the homogeneous interior of the fluid body in positions that are fixed with respect to each other. That is because those forces would be conservative and would not cause vortices to form in the homogeneous interior of the fluid body and would therefore not cause any modification of our result. In other words, we can also replace the interior nucleus of the fluid body with a solid body. Nothing prevents us from making the transition layer ever thinner until we reach the limiting case in which the density suddenly changes from the value Q to the value q_0 and the vorticity begins to slide on that discontinuity surface. As a consequence of the theory of vorticity, we then find the result that:

(I.B) *Relative to the surrounding massive fluid, a denser body will take on a downward motion while a less-dense body will take on an upward motion.*

That is nothing but the known phenomenon that one would otherwise explain with the help of Archimedes' principle. Here, by way of comparison, we will place it with the theorems that will be derived below. However, it is important to emphasize that the result (I.B) is obtained by dynamical considerations, and that is why it does, in fact, have a much broader significance than the usual one that one derived from Archimedes' principle by static considerations. The result (I.b) is valid regardless of the motions that the body and fluid possess with respect to each other from the outset.

VI. – Formation of vorticity by changes of volume in a flow field.

13. – The equation:

(a) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{III}} = -\left(\frac{\partial(e\bar{u}_z)}{\partial y} - \frac{\partial(e\bar{u}_y)}{\partial z}\right),$$

in conjunction with the two corresponding ones that relate to the *y* and *z* axes, can be represented by the single vector equation:

(a′) $$\left(\frac{d\bar{\xi}}{dt}\right)_{\text{III}} = -\operatorname{curl} e\bar{u}.$$

When the vector $e\bar{u}$ has a curl, in the terminology of vector analysis, the dynamical formation of vorticity will occur.

The vector $e\bar{u}$ vanishes identically when the rate of expansion *e* is equal to zero. Thus, that cause of vortex formation will be absent in a homogeneous and incompressible fluid (cf., no. **10**).

We now go on to the consideration of the especially-important special case of the formation *primary* vortices. When $\bar{\xi} = 0$ at the initial time, the hydrodynamical field intensity $\bar{u}$ can be represented by a potential:

(b) $$\bar{u}_x = \frac{\partial\bar{\varphi}}{\partial x}, \qquad \bar{u}_y = \frac{\partial\bar{\varphi}}{\partial y}, \qquad \bar{u}_z = \frac{\partial\bar{\varphi}}{\partial z}.$$

Substituting that in (a) will give:

(c) $$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\text{III}} = -\left(\frac{\partial e}{\partial y}\frac{\partial\bar{\varphi}}{\partial z} - \frac{\partial e}{\partial z}\frac{\partial\bar{\varphi}}{\partial y}\right).$$

Two other equations are given by cyclic permutation of *x*, *y*, *z*. The three equations are the equations for the components of the vector equation:

(c′) $$\left(\frac{d\bar{\xi}}{dt}\right)_{\text{III}} = -\nabla e \times \nabla\bar{\varphi}.$$

That law of formation of vorticity will then be exactly similar to the foregoing one (11.c′). One only needs to replace density *q* in (11.c′) with the rate of cubic expansion *e*, the potential Φ of the force of gravity with the potential *φ* of the hydrodynamical field intensity and remember to invert the signs in order to derive the following theorem from (I.A):

(II.A). *Formation of vorticity will take place along the lines of intersection of the equi-expansion surfaces e = const. and the equipotential surfaces* $\bar{\varphi}$ *= const., and indeed in the direction from the potential vector* $\nabla\bar{\varphi}$ *to the expansion vector* ∇e.

14. – One further concludes from this directly that an expanding or contracting fluid body will behave similarly in a potential flow field to the way that heavier or lighter body will behave in a heavy fluid according to Archimedes' principle. The expanding body has a constant rate of expansion in its interior. However, in a transition layer, the rate of expansion will rapidly drop down to the value zero. A number of equi-expansion surfaces will be packed close to each other in that transition layer, and it is only in that layer that the vector ∇e will have a non-zero value. Thus, the formation of vorticity is bounded by that transition layer. Since the vector ∇e points inward, the vortices will have a relative motion of the expanding body in the direction from the potential vector to the expansion vector and opposite to the direction of the potential vector as a consequence (Fig. 2). When the body contracts, the expansion vector will have the opposite direction in the transition layer, so the formation of vorticity will occur in the opposite direction, and the contracting body will seek to take on a motion relative to the surrounding fluid that is in the direction of the potential vector.

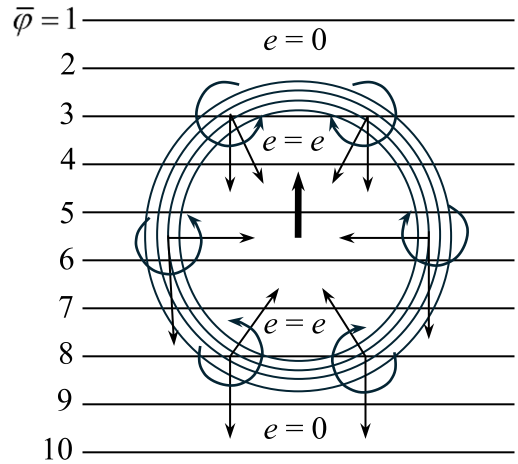

Figure 2.

Since the direction of the potential vector is the same as the direction of the current, the result can also be expressed in the following way:

(II.B). *When considered relative to the surrounding moving fluid, a contracting body will take on a motion that follows the current, while an expanding body will take on a motion that is opposite to it.*

We have derived that result from formula (13.c′), which assumed a vortex-free distribution in the hydrodynamical field intensity. That would say that the body whose volume is changing had a well-defined motion relative to the fluid beforehand. Theorem (II.B) then implies that a supplementary motion will get added to that motion. However, one easily sees from the discussion of the general formula (13.a′) that the result is true in full generality in the form (II.B). It is only the relationship between the formation of vorticity and the lines of intersection between the two families of surfaces (II.A) that will go away at the same moment that the equipotential surfaces cease to exist.

The result (II.B) includes the dynamical principle that is at the basis for the attraction or repulsion of pulsating balls ([1]), except that its derivation is now completely general with no restriction in regard to the form of the pulsating bodies.

VI. [*sic*] – Formation of vorticity by mobility differences in a flow field.

15. – Equation ($8.\mathrm{b}_{\mathrm{IV}}$):

(a)
$$\left(\frac{d\bar{\xi}_x}{dt}\right)_{\mathrm{IV}} = -\frac{1}{2}\left(\frac{\partial k}{\partial y}\frac{\partial \bar{u}^2}{\partial z} - \frac{\partial k}{\partial z}\frac{\partial \bar{u}^2}{\partial y}\right),$$

in conjunction with the corresponding equations for the other two axes, are the component equations for the vector equation:

(a′)
$$\left(\frac{d\bar{\xi}}{dt}\right)_{\mathrm{IV}} = -\tfrac{1}{2}\nabla k \times \nabla \bar{u}^2 .$$

That equation again has the same form as (11.c′) and (13.c′), and we will then get a second dynamical analogue for the simple formation of vorticity by conservative external forces.

When we make the comparison of that to formula (11.c′), we will find differences at the following three points: First of all, a negative sign appears on the right. Secondly, the specific volume k, or reciprocal density, enters in place of the density q. Thirdly, the square of the hydrodynamical field intensity $\bar{u}$ will enter in place of the potential Φ of the force of gravity. The theorem will then change into the following one:

(III.A). *Formation of vorticity will take place along the lines of intersection of the surfaces k = const. of equal specific volume and the surfaces* $\bar{u}^2$ *= const. of equal values of the square of the hydrodynamical field intensity, and indeed in the direction from the vector* $\nabla\bar{u}^2$ *to the vector* ∇k.

When $\nabla k = 0$, and therefore the fluid is homogeneous, the formation of vorticity will cease.

16. – When we apply that result to a moving fluid in which a fluid body of different specific volume is found, we will get a second dynamical analogue for the motion of a heavy body in a heavy fluid according to Archimedes' principle.

Let the fluid initially have a greater specific volume than the surrounding fluid (Fig. 3). The specific volume shall drop continuously, but rapidly, in a transition layer from the value $k = K$ to the value $k = k_0$ that is has in the surrounding fluid. A vector ∇k will exist only in the transition layer then. Since that vector points inward, the formation of vorticity that occurs in the direction from the vector $\nabla\bar{u}^2$ to the vector ∇k will have a motion of the body relative to the surrounding

([1]) **V. Bjerknes**, *Vorlesungen über hydrodynamische Fernkräfte nach C. A. BJERKNES' Theorie*, Leipzig, 1900-1902. Bd. I, pp. 198, Bd. II, pp. 8. – In what follows, that work will be referred to by the symbol H. F.

fluid as a consequence that has the opposite direction to the vector $\nabla \overline{u}^2$. If the body has a lower specific volume than the surrounding fluid then the vector ∇k will point in the opposite direction in the transition layer, and the oppositely-directed vortices will have a relation motion of the body in the direction of the vector $\nabla \overline{u}^2$ as a consequence. Now, the square of the hydrodynamical field intensity will be proportional to the kinetic energy of the current in the surrounding homogeneous fluid, and we can say that the vector $\nabla \overline{u}^2$ will point in the direction of increasing energy of the flow field. The result will then be the following one:

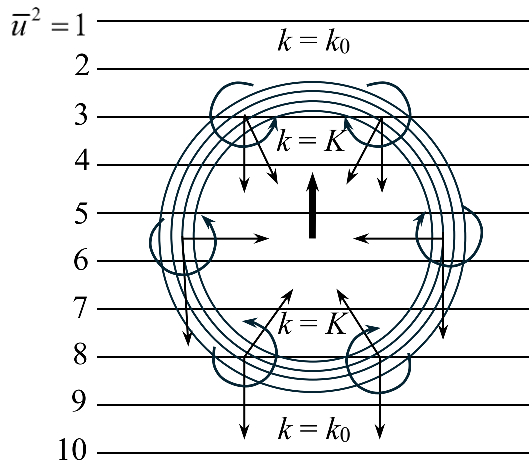

Figure 3.

(III.B). *Relative to the surrounding current, a body of greater specific volume will be driven in the direction of decreasing energy of the flow, while a body of lower specific volume will be driven in the direction of increasing energy.*

A rigorous derivative of that result was previously carried out only for spherical bodies ([1]). The derivation that was given now is free of any restriction in regard to the form of the body.

VII. – Analogy between vortex-forming forces in fluids and ponderomotive forces in electrostatic fields.

17. – We return to the general equation of vortex formation (8.a). We can imagine that the external forces are arranged such that they will cancel the formation of vortices that takes place in the fluid on dynamical grounds. Under those conditions, we will always have:

(a) $$\overline{\xi}_x = \overline{\xi}_y = \overline{\xi}_z = 0$$

then. The left-hand side of equation (8.a) will then vanish identically. The terms on the right-hand side that refer to the conservation of vorticity will likewise drop out. Finally, as a result of (a), the

([1]) H. F., I, pp. 208.

hydrodynamical field intensity will continually depend upon a potential $\bar{\varphi}$. That is why the third row on the right of equation (8.a) will then assume the form (13.c). It can then be written in the form:

$$\frac{\partial \bar{f}_z}{\partial y}-\frac{\partial \bar{f}_y}{\partial z}=\frac{\partial e}{\partial y}\frac{\partial \bar{\varphi}}{\partial z}-\frac{\partial e}{\partial z}\frac{\partial \bar{\varphi}}{\partial y}+\frac{1}{2}\left(\frac{\partial k}{\partial y}\frac{\partial \bar{u}^2}{\partial z}-\frac{\partial k}{\partial z}\frac{\partial \bar{u}^2}{\partial y}\right).$$

One will obtain two other equations by cyclic permutation of the symbols *x*, *y*, *z*, and all three of them can be represented by the single vector equation:

(b) $$\text{curl } \bar{f} = \nabla e \times \nabla \bar{\varphi} + \tfrac{1}{2} \nabla k \times \nabla \bar{u}^2 .$$

The external forces that prevent the formation of vorticity fluid must also fulfill that equation. As a result, we can define the equal and opposite force:

$$\bar{f}_e = -\bar{f}$$

to be the force that acts to create vorticity in the fluid and must then be canceled by the external forces. The foregoing discussion has shown how that force will have, as a consequence, progressive motions that are partially pulsating fluid masses and partially light or heavy ones moving through the surrounding flow field. In other words, $\bar{f}_e$ is the force that **C. A. Bjerknes** called the "force of hydrodynamical energy" for the case of spherical bodies ([1]). The vector equation:

(c) $$\text{curl } \bar{f}_e = -\nabla e \times \nabla \bar{\varphi} - \tfrac{1}{2} \nabla k \times \nabla \bar{u}^2$$

will be true for the force of hydrodynamical energy then, or when we make no further use of the auxiliary quantity $\bar{\varphi}$, and therefore replace $\nabla \bar{\varphi}$ with $\bar{u}$:

(c′) $$\text{curl } \bar{f}_e = -\nabla e \times \bar{u} - \tfrac{1}{2} \nabla k \times \nabla \bar{u}^2 .$$

18. – On the other hand, we do know the ponderomotive forces that act upon the individual material unit volumes in the electrostatic fields. For ease of comparison with the corresponding hydrodynamical formulas, I shall refer to the corresponding electric and hydrodynamical quantities by the same symbols. If $\bar{u}$ means the electrostatic field intensity, *k* means the dielectric constant (i.e., the polarizability), and *e* means the true electric charge density then the ponderomotive force $\bar{f}$ per unit volume will have the components:

([1]) **H. F.**, I, pp. 133, II, pp. 20.

$$\bar{f}_x = e\bar{u}_x - \tfrac{1}{2}\bar{u}^2 \frac{\partial k}{\partial x},$$
$$\bar{f}_y = e\bar{u}_y - \tfrac{1}{2}\bar{u}^2 \frac{\partial k}{\partial y},$$
$$\bar{f}_z = e\bar{u}_z - \tfrac{1}{2}\bar{u}^2 \frac{\partial k}{\partial z},$$

which would now seem to be generally assumed ([1]).

If we now define the vorticity of that force and recall the condition that the electrostatic field intensity $\bar{u}$ is a vortex-free vector then we will arrive at equation (17.c′), but with the opposite sign for the terms on the right-hand side. It will then be:

$$\text{curl } \bar{f} = -\text{ curl } \bar{f}_e .$$

Thus:

The ponderomotive forces in the electrostatic field and the force of energy in the hydrodynamical flow field have equal and opposite vorticity.

One cannot conclude the relationship between those forces from this relationship between the vorticities of the two forces being compared with no other assumptions. The two forces can possibly differ from each other by a non-vortical part, and therefore a potential part. Therefore, how that can be the case cannot be concluded from the theory of formation of vortices in fluids that was just developed. However, the practical value of the result on the equal and opposite vorticity of the two forces will become clear from the detailed discussions that were presented above. The motion of a body through a surrounding medium is necessarily coupled with the formation of vorticity in a boundary layer between the body and the medium. Conversely, the appearance of vorticity in such a boundary layer is necessarily coupled with a motion of the body through the surrounding medium. The opposite equality of the two vorticities will then lead to the following result:

The forces of energy in the hydrodynamical flow field and the ponderomotive forces in the electrostatic field create equal and opposite motions of the bodies in question through the fields.

By contrast, it might still be possible that the two forces can behave differently in regard to deformations of a body that one can no longer refer to as motions of the body as a whole through the surrounding medium.

([1]) Cf., on this, e.g., **Helmdoltz**, *Wissenschaftliche Abhandlungen*, I, pp. 811; **Kirchhoff**, *Vorlesungen über Elektrizität und Magnetismus*, pp. 180; **Heaviside**, *Electromagnetic Theory*, I, pp. 107; **E. Cohn**, *Das elektromagnetische Feld*, pp. 37.

VIII. – Analogy between electrostatic and hydrodynamical phenomena.

19. – We can now summarize the hydrodynamical formulas that refer to the case in which suitable external forces keep the hydrodynamical forces of energy in equilibrium. We can likewise summarize the formulas for the electrostatic field, and in that way, make the same assumption that suitable external forces will preserve equilibrium with the moving ponderomotive electric forces.

The conditions that we then impose upon the external forces in the two cases will correspond with each other precisely. That is because the hydrodynamical forces of energy will not be canceled by the external forces, so vorticity in the hydrodynamical field intensity $\bar{u}$. On the other hand, if the ponderomotive forces of the electrical system are not canceled then the system will no longer be strictly static, and according to **Maxwell**'s theory, vorticity must appear in the electrical field intensity $\bar{u}$.

The two formal systems will be identical to each other, except for the difference in signs in the formulas that refer to the ponderomotive forces. When we denote the corresponding quantities with the same symbols, it will then be unnecessary to write out the system of formulas twice, so we can write down a single system that can be interpreted in two different ways.

Initially, under the assumptions that were made, the hydrodynamical field intensity will be a vortex-free vector quantity:

(a) $$\text{curl } \bar{u} = 0 .$$

The same equation will be true for the electric field intensity $\bar{u}$, which is always a vortex-free vector quantity, according to **Maxwell**'s theory.

Upon multiplying the hydrodynamical field intensity by the specific volume or mobility k, one will obtain a new vector, namely, the velocity u. Upon multiplying the electric field intensity by the dielectric constant or the polarizability k of the medium, one will get a new vector quantity, namely, the electric polarization (i.e., induction) u:

(b) $$u = k\bar{u} .$$

The divergence of the velocity gives the rate of cubic expansion per unit volume e (5.a′). The divergence of the polarization gives the true electric density e ([1]):

(c) $$\text{div } u = e .$$

The electrostatic field is known to be determined uniquely when the true density e and the value of the dielectric constant k are given everywhere. Since the system of equations is the same, as a result, the hydrodynamical flow field will also be necessarily determined uniquely when the rate of cubic expansion e and the value of the specific volume k are given everywhere. In that way, one will obtain the following general result:

([1]) I shall employ the Heaviside "rational system of units" for electric quantities. Confer, H. F., II, pp. 288-294.

Let the specific volume in a hydrodynamical system be distributed in precisely the same way as the dielectric constant in an electrostatic field and let the rate of cubic expansion per unit volume in a hydrodynamical system be distributed in precisely the same way that true electric density is distributed in the electrostatic field. The velocity in the hydrodynamical system will be distributed in precisely the same way as the polarization in the electric system, and the hydrodynamical field intensity will be distributed in space in precisely the same way as the electric field intensity.

In those two systems, which are not distinct from each other in a geometric context then, certain forces will appear that must be canceled by external forces in order for the condition that the field intensity should have a vortex-free nature to continue to be fulfilled. We still do not know the expression for those forces in both cases. However, we know their vorticity:

(d) $$\text{curl } \bar{f}_e = \pm \nabla e \times \bar{u} \pm \tfrac{1}{2} \nabla k \times \nabla \bar{u}^2,$$

in which the upper sign refers to the electrostatic case, and the lower one, to the hydrodynamical case.

20. – Since the one system is actually a static one, in any event, as far as our conception of accessible motion is concerned, while the other one is a moving one, the analogy can ordinarily exist for only one moment, namely, for only the moment when the moving system passes through a well-defined configuration. However, when an oscillating motion about that configuration exists, the analogy can continue to exist. If the oscillations evolve with unnoticeably-small amplitudes then the analogy will also be complete on the surface of things insofar as the hydrodynamical system seems to be a strictly-static system insofar as external observations are concerned.

One can apply the same system of equations (18.a to d) in order to describe the intersection state of that oscillating system, which is true for the instantaneous state of motion, except that one interprets the field quantities as the time-independent quadratic mean values of those quantities that they originally represented. In that regard, naturally, everything behaves in precisely the same way as in the special case that was developed for spherical bodies ([1]).

IX. – Concluding remarks.

20 [*sic*]. – The theory of vortex formation in frictionless fluids that was developed has thus permitted us to achieve an extended generalization of the previously-known results about the analogy of hydrodynamical phenomena with electric or magnetic ones. However, not all of the previously-known results will be subject to generalization. In addition to the hydrodynamical action at a distance, which is based upon either the pulsating motions of bodies or the differences in densities between bodies and fluid, there are other ones that are caused by translatory proper

([1]) H. F., I, pp. 30, 64, and 190.

motions of the body. Obviously, oscillating spheres are known to behave like permanent magnets, except for the consistent difference in the sign of the ponderomotive force. However, the theory of vortices that we developed gives us no explanation for effects of that kind.

That does not say that the theory of vorticity cannot accomplish that goal. The case of *autonomous* proper motions of the body is excluded by the assumption that the external forces should prevent the appearance of vorticity in the field intensity (no. **17**). Only well-defined proper motions can occur, but they would not have vorticity in the field intensity as a consequence. However, other methods besides the one that was developed here that are based upon vorticity considerations will become preferable when one wishes to consider those autonomous proper motions, whether those proper motions are generated by foreign, non-hydrodynamical forces or they are based in the effect of hydrodynamical forces of energy, since no external forces were introduced as a reaction to them, and indeed other methods are absolutely necessary in order to answer the question of the analogy between electricity and magnetism completely. That is because the theory of vortices gives only the vorticity of the desired ponderomotive force itself, but not the expression for the force itself.

Nonetheless, the new methods will always pertain to the discussion of fluid motions under circumstances in which the formation of vortices takes place, and the discussion of the formation of vorticity can be considered to be a useful introductory problem.

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