MATHEMATICAL MONOGRAPHS

CONTRIBUTED BY WILHELM BLASCHKE

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KINEMATICS AND QUATERNIONS

BY

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FOREWORD

In the present treatise, a classical theme shall be taken up, namely, the "geometry of motion" or "kinematics," and indeed with the use of the quaternions that L. EULER introduced to that end in 1748. Thus, spherical kinematics will be considered first, in which only the rotations of a rigid body around a fixed point are allowed. They are most intrinsically linked with the geometry of the elliptic three-dimensional space E_3 . The formulas of spherical kinematics may then be extended to spatial kinematics by means of the "dual numbers," according to the pattern of W. K. CLIFFORD, J. HJELMSLEV, A. P. KOTJELNIKOW, and E. STUDY.

I have communicated parts of the following have in 1957 in Helsinki [1]) and in 1958 in Barcelona [2]). J. NINOT encouraged me to prepare it, and further developments by him shall appear in the Collectanea Mathematica in Barcelona. [3]) I especially have H. KUNLE (Freiburg i. B.), H. R. MÜLLER (Berlin), and J. NINOT (Barcelona) to thank for improvements.

Hamburg, 13-9-1960 W. BLASCHKE

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[1]) W. BLASCHKE, "Anwendung dualer Quaternionen auf Kinematik," Ann. Acad. Scient. Fennicae, Helsinki, A I 250/3 (1958).

[2]) W. BLASCHKE, Cursillo de conferencias sobre Cinemática, Sem. Mat. Univ. Barcelona, 1958 (lithographed).

[3]) J. NINOT, "Las congruencias de las rectas y la cinemática biparamétrica." A book on spherical kinematics by H. R. MÜLLER shall appear next.

CONTENTS

CHAPTER ONE

SPHERICAL KINEMATICS

§ 1. Quaternions

A *quaternion* is a higher complex number:

$$\mathfrak{D} = q_0\, e_0 + q_1\, e_1 + q_2\, e_2 + q_3\, e_3\,, \tag{1.1}$$

for which the sum and product are defined thus:

$$\begin{aligned} \mathfrak{D} + \mathfrak{D}' &= \sum (q_j + q'_j)\, e_j\,, \\ \mathfrak{D}\mathfrak{D}' &= \sum q_j q'_k\, e_j e_k\,, \end{aligned} \tag{1.2}$$

with the following *product rules* for the units e:

$$\begin{aligned} e_0\, e_j = e_j\, e_0 &= e_j\,, \\ e_k\, e_k &= -\, e_0 \qquad (k = 1, 2, 3), \\ e_j\, e_k = -\, e_k\, e_j &= e_l \qquad (j, k, l = 1, 2, 3;\ \ 2, 3, 1;\ \ 3, 1, 2). \end{aligned} \tag{1.3}$$

From (1.3), we can set:

$$e_0 = 1, \tag{1.4}$$

and summarize the remaining product rules in the following table:

(1.5)

e_0	$\boxed{e_1}$	e_2	e_3
e_1	$-\,e_0$	$+\,e_3$	$+\,e_2$
e_2	$+\,e_3$	$-\,e_0$	$+\,e_1$
e_3	$+\,e_2$	$+\,e_1$	$-\,e_0$

The q_j shall first mean real numbers. From (1.3), the commutative law is not true for the product, but the associative law is indeed true:

$$\mathfrak{Q}(\mathfrak{Q}'\mathfrak{Q}'') = (\mathfrak{Q}\mathfrak{Q}')\mathfrak{Q}'', \tag{1.6}$$

as one easily verifies with the e_j .

The *conjugate quaternion* to (1.1) will be defined by:

$$\tilde{\mathfrak{Q}} = q_0\, e_0 - q_1\, e_1 - q_2\, e_2 - q_3\, e_3\,, \tag{1.7}$$

and one confirms by calculation that:

$$\mathfrak{Q}\,\tilde{\mathfrak{Q}} = \langle \mathfrak{Q}\ \mathfrak{Q} \rangle = q_0^2 + q_1^2 + q_2^2 + q_3^2 \tag{1.8}$$

and

(1.9) $$\widetilde{\mathfrak{Q}\mathfrak{Q}'} = \tilde{\mathfrak{Q}}'\tilde{\mathfrak{Q}}.$$

One also calls (1.8) the *norm* of $\mathfrak{Q}$:

(1.10) $$\mathfrak{Q}\,\tilde{\mathfrak{Q}} = N(\mathfrak{Q}).$$

From (1.6), (1.8), (1.9) it then follows that:

(1.11) $$N(\mathfrak{Q}\mathfrak{Q}') = N(\mathfrak{Q})\, N(\mathfrak{Q}').$$

A quaternion with:

(1.12) $$\mathfrak{Q} + \tilde{\mathfrak{Q}} = 2q_0 = 0$$

is called a *vector:*

(1.13) $$\mathfrak{q} = q_1\, e_1 + q_2\, e_2 + q_3\, e_3.$$

If one defines the *scalar product* of vectors by:

(1.14) $$<\mathfrak{q}\ \mathfrak{q}'> = q_1 q_1' + q_2 q_2' + q_3 q_3'$$

and their *vector product* by:

(1.15) $$\mathfrak{q} \times \mathfrak{q}' = (q_2 q_3' - q_3 q_2')\, e_1 + (q_3 q_1' - q_1 q_3')\, e_2 + (q_1 q_2' - q_2 q_1')\, e_3$$

then one has for $\mathfrak{Q} = q_0 + \mathfrak{q}$, $\mathfrak{Q}' = q_0' + \mathfrak{q}'$:

(1.16) $$\mathfrak{Q}\mathfrak{Q}' = q_0 q_0' + q_0\, \mathfrak{q}' + q_0'\, \mathfrak{q} - <\mathfrak{q}\ \mathfrak{q}'> + (\mathfrak{q} \times \mathfrak{q}').$$

We recall the well-known *rules of calculation* for the vector product:

(1.17) $$<(\mathfrak{q} \times \mathfrak{q}')\ \mathfrak{q}''> = [\mathfrak{q}\ \mathfrak{q}'\mathfrak{q}''],$$

in which the expression on the right means the determinant of the three vectors, and furthermore:

(1.18) $$<\mathfrak{q}_1 \times \mathfrak{q}_2,\ \mathfrak{q}_3 \times \mathfrak{q}_4> = <\mathfrak{q}_1\ \mathfrak{q}_3><\mathfrak{q}_2\ \mathfrak{q}_4> - <\mathfrak{q}_1\ \mathfrak{q}_4><\mathfrak{q}_2\ \mathfrak{q}_3>,$$

(1.19) $$(\mathfrak{q}_1 \times \mathfrak{q}_2) \times \mathfrak{q}_3 = <\mathfrak{q}_1\ \mathfrak{q}_3>\ \mathfrak{q}_2 - <\mathfrak{q}_2\ \mathfrak{q}_3>\ \mathfrak{q}_1\,,$$

(1.20) $$(\mathfrak{q}_1 \times \mathfrak{q}_2) \times (\mathfrak{q}_3 \times \mathfrak{q}_4) = [\mathfrak{q}_1\, \mathfrak{q}_3\, \mathfrak{q}_4]\ \mathfrak{q}_2 - [\mathfrak{q}_2\, \mathfrak{q}_3\, \mathfrak{q}_4]\ \mathfrak{q}_1\,,$$

and the multiplication law for determinants:

(1.21) $$[\mathfrak{q}_1\, \mathfrak{q}_3\, \mathfrak{q}_4][\mathfrak{q}_1'\, \mathfrak{q}_2'\, \mathfrak{q}_3'] = \begin{vmatrix} <\mathfrak{q}_1\, \mathfrak{q}_1'> & <\mathfrak{q}_1\, \mathfrak{q}_2'> & <\mathfrak{q}_1\, \mathfrak{q}_3'> \\ <\mathfrak{q}_2\, \mathfrak{q}_1'> & <\mathfrak{q}_2\, \mathfrak{q}_2'> & <\mathfrak{q}_2\, \mathfrak{q}_3'> \\ <\mathfrak{q}_3\, \mathfrak{q}_1'> & <\mathfrak{q}_3\, \mathfrak{q}_2'> & <\mathfrak{q}_3\, \mathfrak{q}_3'> \end{vmatrix}.$$

As one easily confirms, the following connection exists between quaternions and determinants:

(1.22) $$\tfrac{1}{2}(\mathfrak{z}\mathfrak{y}\mathfrak{x} - \mathfrak{x}\mathfrak{y}\mathfrak{z}) = [\mathfrak{x}\ \mathfrak{y}\ \mathfrak{z}],$$

and, more generally:

(1.23) $$\tfrac{1}{4}(\mathfrak{Q}_2\tilde{\mathfrak{Q}}_1\mathfrak{Q}_0\tilde{\mathfrak{Q}}_3 + \mathfrak{Q}_3\tilde{\mathfrak{Q}}_0\mathfrak{Q}_1\tilde{\mathfrak{Q}}_2 - \mathfrak{Q}_0\tilde{\mathfrak{Q}}_1\mathfrak{Q}_2\tilde{\mathfrak{Q}}_3 - \mathfrak{Q}_3\tilde{\mathfrak{Q}}_2\mathfrak{Q}_1\tilde{\mathfrak{Q}}_0) = [\mathfrak{Q}_0\ \mathfrak{Q}_1\ \mathfrak{Q}_2\ \mathfrak{Q}_3].$$

Remark: The quaternion product is linked quite simply with the *matrix product.* Namely, if one sets:

(1.24) $$\mathfrak{Q} = \begin{pmatrix} (+q_0 + iq_1) & (q_2 + iq_3) \\ (-q_2 + iq_3) & (q_0 - iq_1) \end{pmatrix}, \qquad i^2 = -1$$

then the one goes over to the other one. [1])

§ 2. EULER's representation of rotations

The quaternions serve to represent the rotations around a fixed point O. Namely, if one introduces the vectors:

(2.1) $$\begin{aligned} \mathfrak{x} &= x_1 e_1 + x_2 e_2 + x_3 e_3, \\ \mathfrak{x}' &= x_1' e_1 + x_2' e_2 + x_3' e_3, \end{aligned}$$

and the *normalized quaternion:*

(2.2) $$\mathfrak{Q}\tilde{\mathfrak{Q}} = N(\mathfrak{Q}) = 1$$

then the transformation $\mathfrak{x}' \to \mathfrak{x}$:

(2.3) $$\mathfrak{x} = \tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}$$

or

(2.4) $$\mathfrak{Q}\mathfrak{x} = \mathfrak{x}'\mathfrak{Q}$$

represents a real-orthogonal substitution of the x_j' with the x_j. In order to see this, we might first verify the following: From the assumption that $\mathfrak{x}'$ is a vector:

(2.5) $$\mathfrak{x}' + \tilde{\mathfrak{x}}' = 0,$$

it follows that the same is true for $\mathfrak{x}$, since:

(2.6) $$\mathfrak{x} + \tilde{\mathfrak{x}} = \tilde{\mathfrak{Q}}(\mathfrak{x}' + \tilde{\mathfrak{x}}')\mathfrak{Q} = 0.$$

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[1]) On quaternions, cf., H. ROTHE, "Systeme geometrischer Analyse," Enc. Math. Wiss., Art III, sec. 11, Leipzig 1921; in particular, pp. 1300-1423.

Furthermore: The substitution (2.3) of the x'_j with the x_j is linear and orthogonal, so from (1.11), (2.2), (2.3), one has:

(2.7) $$N(\mathfrak{x}) = N(\tilde{\mathfrak{Q}})\, N(\mathfrak{x}')\, N(\mathfrak{Q}) = N(\mathfrak{x}'),$$

or

(2.8) $$x_1^2 + x_2^2 + x_3^2 = x_1'^2 + x_2'^2 + x_3'^2 .$$

Finally: The determinant of (2.3) is + 1, since $\mathfrak{Q}$ can always be taken to $\mathfrak{Q} = 1$ while preserving the condition (2.2). One verifies this perhaps by calculation from formula (1.23).

The substitutions (2.3) will define a group G_3 if:

(2.7) $$\mathfrak{x} = \tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}, \qquad \mathfrak{y} = \tilde{\mathfrak{R}}\,\mathfrak{x}\,\mathfrak{R}, \qquad N(\mathfrak{Q}) = N(\mathfrak{R}) = 1$$

implies that:

(2.10) $$\mathfrak{y} = \tilde{\mathfrak{T}}\,\mathfrak{x}'\,\mathfrak{T}, \qquad \mathfrak{T} = \mathfrak{Q}\mathfrak{R}, \qquad N(\mathfrak{T}) = 1.$$

We then show: Any rotation around the origin O in the space R_3 of rectilinear coordinates x_j can be represented using (2.3) by a suitable choice of $\mathfrak{Q}$. Namely, if:

(2.11) $$\mathfrak{Q} + \tilde{\mathfrak{Q}} = 0, \quad \mathfrak{Q} = \mathfrak{q}, \qquad \mathfrak{q}\,\mathfrak{q} = -1$$

then the vector $\mathfrak{x}' = \mathfrak{q}$ goes to itself ($\mathfrak{x} = \mathfrak{q}$) under the rotation (2.2). Moreover, since $\mathfrak{q}\,\mathfrak{q} = -1$, the rotation:

(2.12) $$\mathfrak{x} = -\,\mathfrak{q}\,\mathfrak{x}\,\mathfrak{q}$$

is involutory; i.e., it is different from the identity and gives the identity by two-fold application (it has "period" 2). Thus, (2.12) represents an *inversion* around the axis $\mathfrak{q}$, and thus the rotation around $\mathfrak{q}$ through the angle π. However, this clarifies the fact that any rotation can be arrived at by the composition of such inversions [cf., (2.18) – (2.20) below].

Due to the normalization (2.2), one can set uniquely:

(2.13) $$\mathfrak{Q} = \cos\varphi - \mathfrak{q}\sin\varphi, \quad \mathfrak{q}\,\mathfrak{q} = -1,$$

up to a common change of sign. Then, from (2.3), one will have:

(2.14) $$\mathfrak{x} = \mathfrak{x}'\cos^2\varphi - \mathfrak{q}\,\mathfrak{x}'\mathfrak{q}\sin^2\varphi + (\mathfrak{q}\,\mathfrak{x}' - \mathfrak{x}'\mathfrak{q})\cos\varphi\sin\varphi.$$

For $\mathfrak{x}' = \mathfrak{q}$ one also has $\mathfrak{x} = \mathfrak{q}$. The formula (2.14) thus represents a rotation around the axis $\mathfrak{q}$. In particular, if we take $\mathfrak{x}'$ to be perpendicular to $\mathfrak{q}$:

(2.15) $$\langle \mathfrak{x}' \, \mathfrak{q} \rangle = 0$$

then, from (2.14), one gets:

(2.16) $$\mathfrak{x} = \mathfrak{x}' \cos 2\varphi - (\mathfrak{x}' \times \mathfrak{q}) \sin 2\varphi.$$

Thus, (2.3), (2.13) represent the rotation around the axis $\mathfrak{q}$ through the angle 2φ. Therefore, the signs of $\mathfrak{q}$ and φ are liked with each other, since we have:

(2.17) $$\tilde{\mathfrak{Q}} - \mathfrak{Q} = 2\mathfrak{q} \sin \varphi.$$

The normalized quaternions + $\mathfrak{Q}$ and – $\mathfrak{Q}$ represent the same rotation. $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ belong to inverse rotations. The representation (2.3) of rotations by quaternions goes back to L. EULER [1]).

Any rotation (2.3), (2.13) can be represented by the composition of two inversions in sequence:

(2.18) $$\mathfrak{Q} = \mathfrak{a}_1 \, \mathfrak{a}_2 \, ; \qquad \mathfrak{a}_1 \, \mathfrak{a}_1 = -1, \qquad \mathfrak{a}_2 \, \mathfrak{a}_2 = -1.$$

I. e.: $\mathfrak{a}_1$, $\mathfrak{a}_2$ are assumed to be perpendicular to the rotation axis $\mathfrak{q}$, and include the angle $\varphi + \pi$.

If $\mathfrak{q}$, φ and $\mathfrak{q}'$, φ' are two rotations then we can choose $\mathfrak{a}_2$ to be the common normal of $\mathfrak{q}$, $\mathfrak{q}'$. One then has:

(2.21) $$\mathfrak{Q} = \mathfrak{a}_1 \, \mathfrak{a}_2 \, , \qquad \mathfrak{Q}' = \mathfrak{a}_2 \, \mathfrak{a}_3 \, ,$$

and therefore:

(2.22) $$\mathfrak{Q} \, \mathfrak{Q}' = - \mathfrak{a}_1 \, \mathfrak{a}_3 \, .$$

This construction of the composition of rotations goes back to GAUSS [2]).

§ 3. The elliptic space E_3

We take the coordinates q_j of a normalized quaternion to be homogeneous coordinates in a three-dimensional projective space P_3 . We again refer to the point that the q_j determine in P_3 as $\mathfrak{Q}$. There exists a one-to-one relationship between the rotations (2.3) of R_3 around the fixed point O and the (real) points of P_3 : The projective space P_3 is the *group space* of the group G_3 of rotations of R_3 around O.

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[1]) L. EULER. "Formulae generales pro translatione quacunque corporum rigidorum," Novi Commentarii Acad. Petropolitanae **20** (1776), 189-207.

Due to (2.6), (2.13), the significance of (2.3) or (2.4) also follows by separating the scalar and vector part of (2.4):

$$\langle \mathfrak{q} \, \mathfrak{x} \rangle = \langle \mathfrak{q} \, \mathfrak{x}' \rangle, \quad \mathfrak{x} \cos \varphi - (\mathfrak{q} \times \mathfrak{x}) \sin \varphi = \mathfrak{x}' \cos \varphi + (\mathfrak{q} \times \mathfrak{x}') \sin \varphi.$$

[2]) C. F. GAUSS, Werke, Bd. 8, Göttingen 1900, pp. 256.

This association may be expressed in a somewhat different way. Under the rotation $\mathfrak{Q}$, the axis-cross $\mathfrak{A}_0$ with the origin O and the axes e_1, e_2, e_3 (the *initial cross*) will go to the axis cross $\mathfrak{A}$ with the axes:

(3.1) $$\mathfrak{a}_j = \tilde{\mathfrak{Q}} e_j \mathfrak{Q} \qquad (j = 1, 2, 3)$$

the same origin O, and the determinant:

(3.2) $$[\mathfrak{a}_1 \ \mathfrak{a}_2 \ \mathfrak{a}_3] = +1.$$

Conversely, there is precisely one rotation that takes the initial cross $\mathfrak{A}_0$ to a given axis cross $\mathfrak{A}$ of that kind. It might suffice to set:

(3.3) $$\begin{cases} \mathfrak{v} = (e_1 \times \mathfrak{a}_1) + (e_2 \times \mathfrak{a}_2) + (e_3 \times \mathfrak{a}_3), \\ < \mathfrak{v}\,\mathfrak{v} > = 4\sin^2 2\varphi, \ \ \mathfrak{v} = 2\mathfrak{q}\sin 2\varphi, \\ \qquad \mathfrak{Q} = \cos\varphi - \mathfrak{q}\sin\varphi. \end{cases}$$

Therefore, one can regard the point $\mathfrak{Q}$ of P_3 as the carrier of the axes $\mathfrak{A}$. One then has that:

(3.4) $$\mathfrak{Q} = \tilde{\mathfrak{Q}}_1 \mathfrak{Q}_2$$

is the rotation that takes the axis cross $\mathfrak{Q}_1$ to the cross $\mathfrak{Q}_2$. This rotation $\mathfrak{Q}$ is independent of the choice of initial cross. Namely, if one sets:

(3.5) $$e_j^* = \tilde{\mathfrak{R}}\, e_j \mathfrak{R}, \quad \mathfrak{R}\tilde{\mathfrak{R}} = 1 \qquad (j = 1, 2, 3)$$

then one gets:

(3.6) $$\mathfrak{Q}^* = \tilde{\mathfrak{Q}}_1^* \mathfrak{Q}_2^* = \tilde{\mathfrak{Q}}_1 \mathfrak{R}\tilde{\mathfrak{R}} \mathfrak{Q}_2 = \tilde{\mathfrak{Q}}_1 \mathfrak{Q}_2 = \mathfrak{Q}.$$

For the rotation angle $\mathfrak{Q}$, we have:

(3.7) $$\cos\varphi = \tfrac{1}{2}(\mathfrak{Q} + \tilde{\mathfrak{Q}}) = \tfrac{1}{2}(\tilde{\mathfrak{Q}}_1 \mathfrak{Q}_2 + \tilde{\mathfrak{Q}}_2 \mathfrak{Q}_1) = \sum_0^3 q_j q_j' = <\mathfrak{Q}_1 \ \mathfrak{Q}_2>.$$

We have thus introduced a *metric* into our P_3, in which the "distance" φ between two points $\mathfrak{Q}$, $\mathfrak{Q}'$ is defined by:

(3.8) $$\cos\varphi = \sum_0^3 q_j q_j' = <\mathfrak{Q}\ \mathfrak{Q}'>, \quad <\mathfrak{Q}\ \mathfrak{Q}> = <\mathfrak{Q}'\ \mathfrak{Q}'> = 1.$$

The space P_3 – thus "metrized" – is the *elliptic space* E_3 . One calls the "null" quadric:

(3.9) $$<\mathfrak{Q}\,\mathfrak{Q}> = 0$$

the *absolute quadric* of E_3 .

If the distance between two points $\mathfrak{Q}$, $\mathfrak{Q}'$ is equal to $\pi : 2$, so:

(3.10) $$<\mathfrak{Q}\,\mathfrak{Q}'> = 0,$$

then the points are said to be *(absolutely) conjugate* or *orthogonal*.

If one introduces new coordinates in the space of $\mathfrak{x}$, as well as in the space of $\mathfrak{x}'$, by the orthogonal substitutions:

(3.11) $$\mathfrak{x}'^* = \tilde{\mathfrak{N}}'\mathfrak{x}'\mathfrak{N}', \qquad \mathfrak{x}^* = \tilde{\mathfrak{N}}\mathfrak{x}\mathfrak{N}, \qquad \tilde{\mathfrak{N}}'\mathfrak{N}' = \tilde{\mathfrak{N}}\mathfrak{N} = 1$$

then, in place of the substitution (2.3), one finds the new one:

(3.12) $$\mathfrak{x}^* = \tilde{\mathfrak{Q}}^*\,\mathfrak{x}'^*\,\mathfrak{Q}^*$$

with

(3.13) $$\mathfrak{x}^* = \tilde{\mathfrak{N}}'\mathfrak{Q}\mathfrak{N}.$$

This transition from the q_j to the q'_j is a quaternary "real orthogonal" substitution. One can verify the fact that its determinant equals +1 perhaps by means of the formula (1.23).

In order to recognize that one can represent *any* such substitution by a suitable choice of $\mathfrak{N}$, $\mathfrak{N}'$ in (3.13), we first remark that the transformations (3.13) define a continuous group G_3 . We then remark that for $\mathfrak{N} = \mathfrak{N}'$ one obtains all real, orthogonal, quaternary substitutions from § 2.

Since the orthogonal transformations (3.13) preserve the distances (3.8) in E_3, we can regard them as *elliptic motions*, and thus, as "motions" in E_3 .

§ 4. Sphere map of the lines in E_3

Let $\mathfrak{Q}$, $\mathfrak{Q}'$ be two normalized absolute polar points of E_3 :

(4.1) $$<\mathfrak{Q}\,\mathfrak{Q}> = 1, \qquad <\mathfrak{Q}\,\mathfrak{Q}'> = 0, \qquad <\mathfrak{Q}'\,\mathfrak{Q}'> = 1.$$

We form the products:

(4.2) $$\mathfrak{r} = \tilde{\mathfrak{Q}}\mathfrak{Q}', \qquad \mathfrak{r}' = \mathfrak{Q}'\tilde{\mathfrak{Q}}$$

and establish that $\mathfrak{r}$, $\mathfrak{r}'$ are unit vectors:

(4.3)
$$\mathfrak{r} + \tilde{\mathfrak{r}} = \tilde{\mathfrak{Q}}\mathfrak{Q}' + \mathfrak{Q}'\tilde{\mathfrak{Q}} = 2 < \mathfrak{Q}\mathfrak{Q}' > = 0,$$
$$\mathfrak{r}' + \tilde{\mathfrak{r}}' = \mathfrak{Q}'\tilde{\mathfrak{Q}} + \mathfrak{Q}\tilde{\mathfrak{Q}}' = 2 < \mathfrak{Q}\mathfrak{Q}' > = 0;$$
$$\mathfrak{r}\,\tilde{\mathfrak{r}} = \mathfrak{r}'\,\tilde{\mathfrak{r}}' = 1.$$

Furthermore: Under the motion (3.13) of E_3, $\mathfrak{r}$, $\mathfrak{r}'$ behave as follows:

(4.4)
$$\mathfrak{r}^* = \tilde{\mathfrak{R}}\,\mathfrak{r}\,\mathfrak{R}, \qquad \mathfrak{r}'^* = \tilde{\mathfrak{R}}'\,\mathfrak{r}'\,\mathfrak{R}'.$$

Our orthogonal and normalized points $\mathfrak{Q}$, $\mathfrak{Q}'$ determine their "directed" (oriented) connecting line (*axis*) $\mathfrak{q}$ in E_3 . "The same" axis arises from the point pair:

(4.5)
$$\mathfrak{Q}^* = \mathfrak{Q} \cos \omega - \mathfrak{Q} \sin \omega, \qquad \mathfrak{Q}'^* = \mathfrak{Q} \sin \omega + \mathfrak{Q} \cos \omega.$$

However, the same unit vectors $\mathfrak{x}$ and $\mathfrak{x}'$ belong to this pair, since:

(4.6)
$$\mathfrak{r}^* = \tilde{\mathfrak{Q}}^*\,\mathfrak{Q}'^* = \mathfrak{r}, \qquad \mathfrak{r}'^* = \mathfrak{Q}'^*\,\tilde{\mathfrak{Q}}^* = \mathfrak{r}'.$$

We can express this formula as follows:

Any axis $\mathfrak{g}$ in E_3 has an image point $\mathfrak{r}$ on the left image sphere (direction sphere) K:

(4.7)
$$<\mathfrak{r}\ \mathfrak{r}> = r_1^2 + r_2^2 + r_3^2 = 1,$$

and an image point $\mathfrak{r}'$ on the right image sphere (direction sphere) K′:

(4.8)
$$<\mathfrak{r}'\ \mathfrak{r}'> = r_1'^2 + r_2'^2 + r_3'^2 = 1.$$

The elliptic motion:

$$\mathfrak{Q}^* = \tilde{\mathfrak{R}}'\,\mathfrak{Q}\,\mathfrak{R}, \qquad \mathfrak{R}\tilde{\mathfrak{R}} = \mathfrak{R}'\tilde{\mathfrak{R}}' = 1,$$

when applied to the axis $\mathfrak{g}$, corresponds to the mutually independent unit rotations (4.4) *on the two image spheres K, K′.*

The basic idea of this map goes back to W. K. Clifford in 1873. It was developed further by J. HJELMSLEV (= PETERSEN) and E. STUDY around 1900 [1]).

The *inversion* of the axis:

(4.9)
$$\mathfrak{g} = \{\mathfrak{Q}, \mathfrak{Q}'\} = (\mathfrak{r}, \mathfrak{r}') \to \mathfrak{g}^* = \{\mathfrak{Q}, \mathfrak{Q}'\} = (\mathfrak{r}^*, \mathfrak{r}'^*)$$

[1]) One finds extensive references for § 4 in the encyclopedia article of H. ROTHE that was cited in § 1.

corresponds to the reflections of the image spheres K, K' through their centers:

$$(4.10) \qquad \mathfrak{r}^* = -\,\mathfrak{r}, \qquad \mathfrak{r}'^* = -\,\mathfrak{r}'.$$

Our result essentially contains the fact: *The group of motions* G_6 *of* E_3 *is the "free product" of the rotation groups* G_3, G_3' *of the image spheres* K, K'.

§ 5. Polar tetrahedra in E_3

Four points $\mathfrak{P}_j$ in E_3 define a *polar tetrahedron* relative to the absolute quadric when the following equations are true:

$$(5.1) \qquad \langle\mathfrak{P}_j\, \mathfrak{P}_k\rangle = \delta_{jk}\,, \qquad \{\mathfrak{P}_0\, \mathfrak{P}_1\, \mathfrak{P}_2\, \mathfrak{P}_3\} = +1,$$

where the right-hand formula gives the determinant of the $\mathfrak{P}_j$. We consider the six edges of the tetrahedon:

$$(5.2) \qquad \mathfrak{g}_{jk} = \{\mathfrak{P}_j\,,\, \mathfrak{P}_k\} = (\mathfrak{r}_{jk},\ \mathfrak{r}'_{jk})$$

and assert: The following relations then exist between their spherical images:

$$(5.3) \qquad \begin{aligned} &\mathfrak{r}_{01} + \mathfrak{r}_{23} = 0, \quad &&\mathfrak{r}_{02} + \mathfrak{r}_{31} = 0, \quad &&\mathfrak{r}_{03} + \mathfrak{r}_{12} = 0, \\ &\mathfrak{r}'_{01} - \mathfrak{r}'_{23} = 0, \quad &&\mathfrak{r}'_{02} - \mathfrak{r}'_{31} = 0, \quad &&\mathfrak{r}'_{03} - \mathfrak{r}'_{12} = 0. \end{aligned}$$

In order to prove this, it suffices to recognize: One can take the points $\mathfrak{P}_j$ to the points e_j, respectively, by an elliptic motion (3.13). In this special case, one confirms the relations (5.3) immediately in this way. However, due to (4.4), the relations (5.3) are invariant under the motions (3.13).

We would like to call the axes:

$$(5.4) \qquad \mathfrak{g} = \{\mathfrak{P}_0\,,\, \mathfrak{P}_1\} = (\mathfrak{r}_{01}\,,\ \mathfrak{r}'_{01}), \qquad \mathfrak{g}^* = \{\mathfrak{P}_2\,,\, \mathfrak{P}_3\} = (\mathfrak{r}_{23}\,,\ \mathfrak{r}'_{23})$$

(absolute) *polars*. It then follows from (5.3) that: The *absolute polarity* $\mathfrak{g} \to \mathfrak{g}^*$ will be represented by:

$$(5.5) \qquad \mathfrak{r}^* = -\,\mathfrak{r}, \qquad \mathfrak{r}'^* = +\,\mathfrak{r}'.$$

From (4.4), the *incidence* of a point $\mathfrak{Q}$ with the axis $(\mathfrak{r}, \mathfrak{r}')$ yields the condition:

$$(5.6) \qquad \mathfrak{Q}\mathfrak{r} - \mathfrak{r}'\mathfrak{Q} = 0.$$

From (5.6), it follows that:

$$(5.7) \qquad \mathfrak{r} = \tilde{\mathfrak{Q}}\ \mathfrak{r}'\, \mathfrak{Q},$$

and that means: The axes of the pencil in E_3 with the vertex $\mathfrak{Q}$ corresponds to a congruence map (5.7) of the two image spheres K, K' onto each other that preserves the direction.

If one composes this map (5.7) with the polarity (5.5) then one sees: The axes of a plane in E_3 correspond to a congruence map that inverts the direction (*transfer*):

$$\mathfrak{r} = -\,\tilde{\mathfrak{Q}}\,\mathfrak{r}'\,\mathfrak{Q} \tag{5.8}$$

of the image spheres.

One calls axes $\mathfrak{g}_1$, $\mathfrak{g}_2$ with $\mathfrak{r}_1 = -\,\mathfrak{r}_2$ *left-parallel* and axes with $\mathfrak{r}_1' = \mathfrak{r}_2'$ *right-parallel*. One can define the angle φ between two intersecting axes $\mathfrak{g}_1$, $\mathfrak{g}_2$ by way of:

$$\cos\varphi = \langle \mathfrak{r}_1\ \mathfrak{r}_2 \rangle = \langle \mathfrak{r}_1' \mathfrak{r}_2' \rangle . \tag{5.9}$$

In particular, for axes that intersect perpendicularly, one has:

$$\langle \mathfrak{r}_1\ \mathfrak{r}_2 \rangle = \langle \mathfrak{r}_1' \mathfrak{r}_2' \rangle = 0. \tag{5.10}$$

§ 6. Motions in E_3

Let $(\mathfrak{a}, \mathfrak{a}')$ be axes in E_3 . From (4.4), (3.43), the rotations of the image spheres K, K' around the rotational axes $\mathfrak{a}$, $\mathfrak{a}'$ through the angles 2φ, $2\varphi'$, namely:

$$\begin{aligned} \mathfrak{r}^* &= (\cos\varphi - \mathfrak{a}\sin\varphi)\,\mathfrak{r}\,(\cos\varphi + \mathfrak{a}\sin\varphi), \\ \mathfrak{r}'^* &= (\cos\varphi' - \mathfrak{a}'\sin\varphi')\,\mathfrak{r}'\,(\cos\varphi' + \mathfrak{a}\sin\varphi'), \end{aligned} \tag{6.1}$$

correspond in E_3 to the elliptic motions:

$$\begin{aligned} \mathfrak{Q}^* &= \tilde{\mathfrak{R}}'\,\mathfrak{Q}\,\mathfrak{R}, \\ \mathfrak{R} &= \cos\varphi + \mathfrak{a}\sin\varphi, \qquad \mathfrak{R}' = \cos\varphi' + \mathfrak{a}'\sin\varphi'. \end{aligned} \tag{6.2}$$

Therefore, since:

$$\mathfrak{Q}\mathfrak{a} - \mathfrak{a}'\mathfrak{Q} = 0 \tag{6.3}$$

any point $\mathfrak{Q}$ of $(\mathfrak{a}, \mathfrak{a}')$ again goes to a point $\mathfrak{Q}^*$ of $(\mathfrak{a}, \mathfrak{a}')$:

$$\mathfrak{Q}^* = (\cos\varphi' - \mathfrak{a}'\sin\varphi')\,\mathfrak{Q}\,(\cos\varphi + \mathfrak{a}\sin\varphi) = \mathfrak{Q}(\varphi' - \varphi) - \mathfrak{Q}\,\mathfrak{a}\sin(\varphi' - \varphi). \tag{6.4}$$

Thus, under the motion (6.2), the points of $(\mathfrak{a}, \mathfrak{a}')$ will be rotated through $\varphi' - \varphi$. It likewise follows that: The points of the polars $(-\,\mathfrak{a}, \mathfrak{a}')$ will be rotated through $\varphi' + \varphi$:

$$\mathfrak{Q}^* = \mathfrak{Q}\cos(\varphi' + \varphi) + \mathfrak{Q}\,\mathfrak{a}\sin(\varphi' + \varphi). \tag{6.5}$$

In particular, for $\varphi' - \varphi = 0$ we get a rotation of E_3 around $(\mathfrak{a}, \mathfrak{a}')$, and for:

(6.6) $$\varphi = \varphi' = \frac{\pi}{2},$$

we obtain the *transfer* around the axis $(\mathfrak{a}, \mathfrak{a}')$ in E_3 :

(6.7) $$\begin{gathered} \mathfrak{Q}^* = -\mathfrak{a}' \mathfrak{Q} \mathfrak{a}, \\ \mathfrak{r}^* = -\mathfrak{a} \mathfrak{Q} \mathfrak{a}, \qquad \mathfrak{r}'^* = -\mathfrak{a}' \mathfrak{r}' \mathfrak{a}'. \end{gathered}$$

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two lines; under the assumption:

$$\mathfrak{r}_1 \times \mathfrak{r}_2 \neq 0, \qquad \mathfrak{r}_1' \times \mathfrak{r}_2' \neq 0,$$

there are then two common normals $(\pm\, \mathfrak{a}, \mathfrak{a}')$ to them:

(6.8) $$\mathfrak{a} = \frac{\mathfrak{r}_1 \times \mathfrak{r}_2}{\sin\varphi}, \qquad \mathfrak{a}' = \frac{\mathfrak{r}_1' \times \mathfrak{r}_2'}{\sin\varphi'}, \qquad \langle \mathfrak{r}_1\, \mathfrak{r}_2 \rangle = \cos\varphi, \qquad \langle \mathfrak{r}_1'\, \mathfrak{r}_2' \rangle = \cos\varphi'.$$

These *common perpendiculars* are polar to each other.

The sections that our lines $\mathfrak{g}_1$, $\mathfrak{g}_2$ cut out on the common perpendiculars $(\pm\, \mathfrak{a}, \mathfrak{a}')$ yield the minimal distances ϑ, ϑ'. Their values can be found by composing the transfers on $\mathfrak{g}_1$, $\mathfrak{g}_2$ for a suitable choice of the values:

(6.9) $$\vartheta = \varphi' + \varphi, \qquad \vartheta' = \varphi' - \varphi.$$

By performing a transfer on the lines:

(6.10) $$\mathfrak{Q}^* = -\,\mathfrak{a}'\, \mathfrak{Q}\, \mathfrak{a},$$

one gets the following formula for the distance ϑ from a point $\mathfrak{Q}$ to a line $(\mathfrak{a}, \mathfrak{a}')$ in E_3 :

(6.11) $$2 \cos 2\vartheta = 2\langle \mathfrak{D}\, \mathfrak{D}^* \rangle = -\,\tilde{\mathfrak{Q}}\, \mathfrak{a}' \mathfrak{Q}\, \mathfrak{a} - \mathfrak{a}\, \tilde{\mathfrak{Q}}\, \mathfrak{a}' \mathfrak{Q}.$$

§ 7. Effect of the line map

We describe this (on pp. 12) in a comparison figure that shows, on the one hand, the effect on the two image spheres K, K', and on the other, in the elliptic space E_3 .

One also observes what the two families of generators of a quadric in E_3 correspond to on the two direction spheres K, K'.

From our formulas, it emerges that, for example:

Reflection through a point and inversion on a line in E_3 are commutable precisely when the point is incident on the lines or their polars.

Gauss's composition of rotations (2.21) carries over, without difficulty, to the composition of motions in E_3 when one regards them as pairs of reflections in lines.

Spherical images K, K'	Elliptic space E_3
Point pair $\mathfrak{r}$, $\mathfrak{r}'$	Axis $(\mathfrak{r}, \mathfrak{r}')$
Reflections $\mathfrak{r}^* = -\mathfrak{r}$, $\mathfrak{r}'^* = -\mathfrak{r}'$	Inversions of axes
Map $\mathfrak{r}^* = -\mathfrak{r}$, $\mathfrak{r}'^* = +\mathfrak{r}'$	Absolute polarity
Congruent point-pair: $\cos\varphi = <\mathfrak{r}_1\, \mathfrak{r}_2> = <\mathfrak{r}'_1 \mathfrak{r}'_2>$	Lines intersecting at an angle φ
Rotation pair: $\mathfrak{r}^* = \tilde{\mathfrak{R}}\, \mathfrak{r}\, \mathfrak{R}, \quad \mathfrak{r}'^* = \tilde{\mathfrak{R}}'\, \mathfrak{r}'\, \mathfrak{R}'$, $\mathfrak{R} = \cos\varphi + \mathfrak{a} \sin\varphi$, $\mathfrak{R}' = \cos\varphi' + \mathfrak{a}' \sin\varphi'$	Screw: $\mathfrak{Q}^* = \tilde{\mathfrak{R}}'\, \mathfrak{r}\, \mathfrak{R}$, Rotational angle around $(\mathfrak{a}, \mathfrak{a}')$: $\varphi' + \varphi$ Rotational angle along $(\mathfrak{a}, \mathfrak{a}')$: $\varphi' - \varphi$
Two point-pairs $\mathfrak{r}_j$, $\mathfrak{r}'_j$ $<\mathfrak{r}_1\, \mathfrak{r}_2> = \cos\varphi, \quad <\mathfrak{r}'_1 \mathfrak{r}'_2> = \cos\varphi'$	Line pair $(\mathfrak{r}_j, \mathfrak{r}'_j)$ with shortest distances $\varphi' \pm \varphi$
Rotation $\mathfrak{r} = \tilde{\mathfrak{Q}}\, \mathfrak{r}' \mathfrak{Q}$	Pencil of lines, vertex $\mathfrak{Q}$
Transfer $\mathfrak{r} = -\tilde{\mathfrak{Q}}\, \mathfrak{r}' \mathfrak{Q}$	Polar plane of $\mathfrak{Q}$
Pair of turns: $\mathfrak{r}^* = -\mathfrak{a}\, \mathfrak{r}\, \mathfrak{a}$, $\mathfrak{r}'^* = -\mathfrak{a}'\, \mathfrak{r}'\, \mathfrak{a}'$	Turn along the line pair $(\pm \mathfrak{a}, \mathfrak{a}')$
Pair of transfers $\mathfrak{r}^* = -\tilde{\mathfrak{P}}\, \mathfrak{r}\, \mathfrak{P}$, $\mathfrak{r}'^* = -\mathfrak{P}'\, \mathfrak{r}\, \tilde{\mathfrak{P}}$	Reflection through a point $\mathfrak{P}$ and its polar plane
Reflection in the planes orthogonal to $\mathfrak{a}$, $\mathfrak{a}'$ $\mathfrak{r}^* = \mathfrak{a}\, \mathfrak{r}\, \mathfrak{a}$, $\mathfrak{r}'^* = \mathfrak{a}'\, \mathfrak{r}'\, \mathfrak{a}'$	$\mathfrak{Q}^* = \mathfrak{a}' \mathfrak{r}\, \mathfrak{a}$, times an axis inversion
$<\mathfrak{r}\, \mathfrak{a}> = c <\mathfrak{r}'\, \mathfrak{a}'>$	Thread (= linear complex) with the axes $(\pm \mathfrak{a}, \mathfrak{a}')$; cf., § 8.

Bibliography. The fundamental paper of CAYLEY on the elliptic metric is: A. CAYLEY, "A sixth memoir upon quantics," Phil. Trans. **149** (1859), 61-90; see also A. CAYLEY, *Collected Mathematical Papers*, v. 2, Cambridge, 1889, pp. 561 to 592.

Klein's interpretation of CAYLEY's metric: F. KLEIN, "Über die sogennante Nicht-Euklidische Geometrie," Gött. Nachr. **17** (1871); cf., also F. Klein, *Gesammelte mathematische Abhandlungen*, Bd. 1, 1921, pp. 244-253.

On the sphere map of lines in E_3, cf., W. K. CLIFFORD, *Preliminary sketch on biquternions*," Proc. London Math. Soc. **4** (1873), 381-395, or also W. K. CLIFFORD, *Mathematical Papers*, London, 1882, pp. 181; J. HJELMSLEV, "Géométrie des droites dans l'espace non euclidien," Kopenhagen Verhandl. Akad., 1900, pp. 308-330; the disseration of G. FUBINI, "Il parallelismo di Clifford negli spazi ellittici," Annali della Scuola Normale, Pisa **9** (1904), 74 pages; E. STUDY, "Beiträge zur Nicht-Euklidischen Geometrie, II: Die Begriffe Links und Rechts in der elliptischen Geometrie," Amer. J. Math. **29** (1907), 116-159.

§ 8. PLÜCKER's line coordinates

In this section, we shall give the relations between our vectors $\mathfrak{r}$, $\mathfrak{r}'$ with the "PLÜCKER" line coordinates g_{jk} that were introduced by G. MONGE in 1771, H. GRASSMANN in 1844, J. PLÜCKER in 1846, and A. CAYLEY in 1857.

If $\mathfrak{Q}$, $\mathfrak{Q}'$ are two orthogonal normalized points on a line then (§ 4) the unit vectors (direction vectors) $\mathfrak{r}$, $\mathfrak{r}'$ will be defined by:

$$\mathfrak{r} = \tilde{\mathfrak{Q}}\,\mathfrak{Q}', \qquad \mathfrak{r}' = \mathfrak{Q}'\,\tilde{\mathfrak{Q}}. \tag{8.1}$$

One thus has:

$$\begin{aligned} \mathfrak{r} &= (q_0 - \mathfrak{q})(q_0' + \mathfrak{q}') = q_0\mathfrak{q}' - q_0'\mathfrak{q} - (\mathfrak{q}\times\mathfrak{q}'), \\ \mathfrak{r}' &= (q_0' + \mathfrak{q}')(q_0 - \mathfrak{q}') = q_0\mathfrak{q}' - q_0'\mathfrak{q} + (\mathfrak{q}\times\mathfrak{q}'). \end{aligned} \tag{8.2}$$

In more detail, this is:

$$\begin{aligned} r_1 &= g_{01} - g_{23}, & r_1' &= g_{01} + g_{23}, \\ r_2 &= g_{02} - g_{31}, & r_2' &= g_{02} + g_{31}, \\ r_3 &= g_{03} - g_{12}, & r_3' &= g_{03} + g_{12}, \end{aligned} \tag{8.3}$$

or

$$\begin{aligned} 2g_{01} &= r_1 + r_1', & 2g_{23} &= r_1' - r_1, \\ 2g_{02} &= r_2 + r_2', & 2g_{31} &= r_2' - r_2, \\ 2g_{03} &= r_3 + r_3', & 2g_{12} &= r_3' - r_3. \end{aligned} \tag{8.4}$$

From this, it follows that:

$$4(g_{01}\,g_{23} + g_{02}\,g_{31} + g_{03}\,g_{12}) = (r_1'^2 + r_2'^2 + r_3'^2) - (r_1^2 + r_2^2 + r_3^2), \tag{8.5}$$

and by constructing the polars for two lines:

(8.6) $$2(g_{01}g'_{23} + g_{23}g'_{01} + \cdots) = (r'_1 s'_1 + \cdots) - (r_1 s_1 + \cdots),$$

where the dots mean cyclic permutations of 1, 2, 3.

From the fact that $\mathfrak{r}$, $\mathfrak{r}'$ are unit vectors, it follows that:

(8.7) $$g_{01}\, g_{23} + g_{02}\, g_{31} + g_{03}\, g_{12} = 0$$

and

(8.8) $$g_{01}^2 + g_{23}^2 + \ldots = 1.$$

From (5.6), one has the dependent equations for a point incident on a line:

(8.9) $$\begin{aligned} q_0 g_{23} &= q_2 g_{03} - q_3 g_{02}, \\ q_0 g_{31} &= q_3 g_{01} - q_1 g_{03}, \\ q_0 g_{12} &= q_1 g_{02} - q_2 g_{01}. \end{aligned}$$

For the shortest distances ϑ, ϑ' between two lines $\mathfrak{g}$, $\mathfrak{g}'$ in E_3, we find:

(8.10) $$\begin{aligned} g_{01}g'_{01} + g_{23}g'_{23} + \cdots &= \cos\frac{\theta}{2}\cos\frac{\theta'}{2}, \\ g_{01}g'_{23} + g_{23}g'_{01} + \cdots &= \sin\frac{\theta}{2}\sin\frac{\theta'}{2}. \end{aligned}$$

A *thread* – or *linear complex* – of lines $\mathfrak{g}$ in E_3 will be defined by a linear equation:

(8.11) $$\sum h^{jk}\, g_{jk} = 0, \quad h^{jk} + h^{kj} = 0.$$

Thus, due to (8.4), it follows for the spherical images that:

(8.12) $$\langle \mathfrak{v}\; \mathfrak{r} \rangle + \langle \mathfrak{v}'\; \mathfrak{r}' \rangle = 0.$$

One then has:

(8.13) $$\begin{aligned} \mathfrak{v} &= e_1(h^{01} - h^{23}) + \cdots \\ \mathfrak{v}' &= e_1(h^{01} + h^{23}) + \cdots \end{aligned}$$

The polar axes ($\pm\, \mathfrak{a}$, $\mathfrak{a}'$) with:

(8.14) $$\mathfrak{a} = c\mathfrak{v}, \qquad \mathfrak{a}' = c'\, \mathfrak{v}'$$

give the "axes" of the thread.

For the algebra of rotations, one should confer § 19.

CHAPTER TWO

COMPULSIVE SPHERICAL ROTATION PROCESSES

§ 9. Map to curves in E_3 [1])

We consider a continuous, one-parameter (= *compulsive*) *rotation process* around the fixed point *O*, for which we set:

$$\mathfrak{r}(t) = \tilde{\mathfrak{Q}}(t)\,\mathfrak{r}'\,\mathfrak{Q}(t), \qquad \mathfrak{Q}\tilde{\mathfrak{Q}} = 1, \tag{9.1}$$

in which $\mathfrak{Q}$, and therefore also $\mathfrak{r}$, depends upon the "time" *t*. We think of the unit vector $\mathfrak{r}'$ as being at rest and referred to an axis-cross at rest (*rest cross*); in (9.1), it corresponds to the moving axis-cross (*moving cross*). The points that are fixed in the moving cross define the *moving system* and the ones that are fixed in the rest cross define the *rest system*. In elliptic space, the point $\mathfrak{Q}(t) = \mathfrak{Q}_0(t)$ describes a curve *L*, which we link with a moving tetrahedron ("associated tetrahedron"), whose corners $\mathfrak{Q}_j(t)$ (j = 0, 1, 2, 3) define an absolute polar tetrahedron:

$$\langle \mathfrak{Q}_j\, \mathfrak{Q}_k \rangle = \delta_{jk}, \qquad [\mathfrak{Q}_0\ \mathfrak{Q}_1\ \mathfrak{Q}_2\ \mathfrak{Q}_3] = +1. \tag{9.2}$$

Thus, $\mathfrak{Q}_1$ shall lie on the tangent to *L* at $\mathfrak{Q}_0$, and $\mathfrak{Q}_2$ shall lie on the osculating plane to *L* in $\mathfrak{Q}_0$. The *differential equations* for *L* then take the form:

$$\begin{array}{rcccccc} d\mathfrak{Q}_0 &=& * & +\mathfrak{Q}_1\rho & * & *, \\ d\mathfrak{Q}_1 &=& -\mathfrak{Q}_0\rho & * & +\mathfrak{Q}_2\sigma & *, \\ d\mathfrak{Q}_2 &=& * & -\mathfrak{Q}_1\rho & * & +\mathfrak{Q}_3\tau, \\ d\mathfrak{Q}_3 &=& * & * & -\mathfrak{Q}_2\tau & *. \end{array} \tag{9.3}$$

In them, ρ means the distance between "neighboring" points of *L* (= arc length) and σ, the angle between neighboring tangents, while τ means the angle between neighboring osculating planes of *L*. The following quantities are the *curvature* and *torsion* of *L*:

$$\frac{\sigma}{\rho} = k, \qquad \frac{\tau}{\rho} = w. \tag{9.4}$$

The spherical image of the edges $\{\mathfrak{Q}_j\ \mathfrak{Q}_k\}$ will be given, according to (5.3), by the products of the $\mathfrak{Q}$ in the following tables:

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[1]) Cf., H. R. MÜLLER, "Die Bewegungsgeometrie auf der Kugel," Monatshefte Wien **55** (1950), 28-42.

(9.5)
$$\begin{array}{c|cccc} & \mathfrak{Q}_0 & \mathfrak{Q}_1 & \mathfrak{Q}_2 & \mathfrak{Q}_3 \\ \hline \tilde{\mathfrak{Q}}_0 & 1 & +\mathfrak{p}_1 & +\mathfrak{p}_2 & +\mathfrak{p}_3 \\ \tilde{\mathfrak{Q}}_1 & -\mathfrak{p}_1 & 1 & -\mathfrak{p}_3 & +\mathfrak{p}_2 \\ \tilde{\mathfrak{Q}}_2 & -\mathfrak{p}_2 & +\mathfrak{p}_3 & 1 & -\mathfrak{p}_1 \\ \tilde{\mathfrak{Q}}_3 & -\mathfrak{p}_3 & -\mathfrak{p}_1 & +\mathfrak{p}_1 & 1 \end{array} \qquad \begin{array}{c|cccc} & \tilde{\mathfrak{Q}}_0 & \tilde{\mathfrak{Q}}_1 & \tilde{\mathfrak{Q}}_2 & \tilde{\mathfrak{Q}}_3 \\ \hline \mathfrak{Q}_0 & 1 & -\mathfrak{p}'_1 & -\mathfrak{p}'_2 & -\mathfrak{p}'_3 \\ \mathfrak{Q}_1 & +\mathfrak{p}'_1 & 1 & -\mathfrak{p}'_3 & +\mathfrak{p}'_2 \\ \mathfrak{Q}_2 & +\mathfrak{p}'_2 & +\mathfrak{p}'_3 & 1 & -\mathfrak{p}'_1 \\ \mathfrak{Q}_3 & +\mathfrak{p}'_3 & -\mathfrak{p}'_2 & +\mathfrak{p}'_1 & 1 \end{array}$$

In them, the first factor of the product is on the left, while the second factor is above; thus, e.g.:

$$\mathfrak{Q}_1\tilde{\mathfrak{Q}}_2 = -\mathfrak{p}'_3 .$$

The *canonical axis-crosses* of the $\mathfrak{p}_j$ on K and the $\mathfrak{p}'_j$ on K' satisfy the conditions:

(9.6)
$$<\mathfrak{p}_j\,\mathfrak{p}_k> = <\mathfrak{p}'_j\,\mathfrak{p}'_k> = \delta_{jk} , \qquad [\mathfrak{p}_1\ \mathfrak{p}_2\ \mathfrak{p}_3] = [\mathfrak{p}'_1\,\mathfrak{p}'_2\,\mathfrak{p}'_3] = +1.$$

Differential equations of the type:

(9.7)
$$d\mathfrak{p}_j = \sum \mathfrak{p}_k \alpha_{jk} , \qquad d\mathfrak{p}'_j = \sum \mathfrak{p}'_k \alpha'_{jk} , \qquad \alpha_{jk} + \alpha_{kj} = \alpha'_{jk} + \alpha'_{kj} = 0$$

are true for them. By derivation of, say:

(9.8)
$$\mathfrak{p}_1 = \tilde{\mathfrak{Q}}_0 \mathfrak{Q}_1$$

it then follows from (9.3), (9.7) that:

(9.9)
$$\mathfrak{p}_2\alpha_{12} + \mathfrak{p}_3\alpha_{13} = \tilde{\mathfrak{Q}}_1\mathfrak{Q}_1\rho + \tilde{\mathfrak{Q}}_0(-\mathfrak{Q}_0\rho + \mathfrak{Q}_2\sigma) ,$$

and therefore, from (9.5):

(9.10)
$$\mathfrak{p}_2\alpha_{12} + \mathfrak{p}_3\alpha_{13} = \mathfrak{p}_1\,\sigma, \qquad \alpha_{12} = \sigma, \qquad \alpha_{13} = 0.$$

Correspondingly, we find:

(9.11)
$$\begin{array}{llll} d\mathfrak{p}_1 = \ * & +\mathfrak{p}_2\lambda & * , & \qquad d\mathfrak{p}'_1 = \ * \quad +\mathfrak{p}'_2\lambda' \quad * , \\ d\mathfrak{p}_2 = -\mathfrak{p}_1\lambda & * & +\mathfrak{p}_3\mu , & \qquad d\mathfrak{p}'_2 = -\mathfrak{p}'_1\lambda' \quad * \quad +\mathfrak{p}'_3\mu' , \\ d\mathfrak{p}_3 = \ * & -\mathfrak{p}_2\mu & * ; & \qquad d\mathfrak{p}'_3 = \ * \quad -\mathfrak{p}'_2\mu' \quad * , \end{array}$$

with the relations:

(9.12)
$$\begin{array}{lcl} \lambda = \sigma,\ \ \mu = \tau - \rho, & & 2\rho = \mu' - \mu, \\ & \sigma = \lambda = \lambda', & \\ \lambda' = \sigma,\ \mu' = \tau + \rho, & & 2\tau = \mu' + \mu. \end{array}$$

In them, λ, λ' mean the arc lengths of the lines $(\mathfrak{p}_1)$, $(\mathfrak{p}'_1)$ on the spheres K, K', and:

(9.13) $$g = \frac{\mu}{\lambda}, \qquad g' = \frac{\mu'}{\lambda}$$

mean their *geodetic curvatures*. Therefore, the spherical curves $(\mathfrak{p}_1), (\mathfrak{p}_1')$ are *isometric* when they are compared to each other for each value of t. Their geodetic curvatures are:

(9.14) $$g = \frac{\tau - \rho}{\sigma}, \qquad g' = \frac{\tau + \rho}{\sigma}$$

or

(9.15) $$g = \frac{w-1}{k}, \qquad g' = \frac{w+1}{k},$$

resp. Conversely, one thus has:

(9.16) $$k = \frac{2}{g' - g}, \qquad w = \frac{g' + g}{g' - g}.$$

Now if, for example, L is planar, so:

(9.17) $$d\mathfrak{Q}_3 = 0, \qquad \tau = 0, \quad \mu = -\rho, \qquad \mu' = +\rho,$$

then the lines $(\mathfrak{p}_1), (\mathfrak{p}_1')$ correspond to each other under a transfer. If $(\mathfrak{p}_1)$ is a great circle $g = 0$ then, from (9.16), L has the fixed torsion $w = + 1$. Correspondingly, $w = - 1$ for $g' = 0$.

We now summarize some formulas for a line $(\mathfrak{p}(t))$ on the unit sphere. If dots mean differentiation with respect to t then we find that:

(9.18) $$\lambda = <\dot{\mathfrak{p}}\dot{\mathfrak{p}}>^{1/2} dt, \qquad g = \frac{[\mathfrak{p}\dot{\mathfrak{p}}\ddot{\mathfrak{p}}]}{<\dot{\mathfrak{p}}\dot{\mathfrak{p}}>^{3/2}},$$

and furthermore, for the line L in E_3, one has:

(9.19) $$\rho = <\dot{\mathfrak{Q}}\dot{\mathfrak{Q}}>^{1/2} dt, \quad 1 + k^2 = \frac{<\dot{\mathfrak{Q}}\dot{\mathfrak{Q}}><\ddot{\mathfrak{Q}}\ddot{\mathfrak{Q}}> - <\dot{\mathfrak{Q}}\ddot{\mathfrak{Q}}>^2}{<\dot{\mathfrak{Q}}\dot{\mathfrak{Q}}>^3}, \quad k^2 w = \frac{[\mathfrak{Q}\dot{\mathfrak{Q}}\ddot{\mathfrak{Q}}]}{<\dot{\mathfrak{Q}}\dot{\mathfrak{Q}}>^3}.$$

§ 10. Velocity

Let $\mathfrak{x}$ be a point in the moving system:

(10.1) $$\mathfrak{x}(t) = \tilde{\mathfrak{Q}}(t)\,\mathfrak{x}'\,\mathfrak{Q}(t).$$

By derivation, it then follows from (10.1), due to the fact that $d\mathfrak{x}' = 0$, that:

(10.2) $$d\mathfrak{x} = d\tilde{\mathfrak{Q}} \cdot \mathfrak{x}' \, \mathfrak{Q} + \tilde{\mathfrak{Q}} \, \mathfrak{x}' \, d\mathfrak{Q},$$

or, when we introduce:

(10.3) $$\mathfrak{x}' = \mathfrak{Q} \, \mathfrak{x} \, \tilde{\mathfrak{Q}}$$

into (10.1), and employ (9.3), (9.5):

(10.4) $$d\mathfrak{x} = (\mathfrak{x} \, \mathfrak{p}_1 - \mathfrak{p}_1 \, \mathfrak{x}) \, \rho = 2(\mathfrak{x} \times \mathfrak{p}_1) \, \rho.$$

In this:

(10.5) $$\frac{d\mathfrak{x}}{dt} = (\mathfrak{x} \, \mathfrak{p}_1 - \mathfrak{p}_1 \, \mathfrak{x}) \, \frac{\rho}{dt} = 2(\mathfrak{x} \times \mathfrak{p}_1) \frac{\rho}{dt}$$

means the *absolute velocity* of point $\mathfrak{x}$ that is fixed in the moving system under our continuous rotation process.

If we introduce the *canonical coordinates* x_j for $\mathfrak{x}$ by the Ansatz:

(10.6) $$\mathfrak{x} = x_1\mathfrak{p}_1 + x_2\mathfrak{p}_2 + x_3\mathfrak{p}_3$$

then it follows from (10.4) that:

(10.7) $$d\mathfrak{x} = 2(x_3\mathfrak{p}_2 - x_2\mathfrak{p}_3) \, \rho.$$

On the other hand, it arises from (10.6), due to (9.11), that:

(10.8) $$d\mathfrak{x} = (dx_1 - x_2\lambda) \, \mathfrak{p}_1 + (dx_2 + x_1\lambda - x_3\mu) \, \mathfrak{p}_2 + (dx_3 + x_2\mu) \, \mathfrak{p}_3 \, .$$

A comparison of (10.7), (10.8) yields, due to (9.12), the *guiding conditions:*

(10.9) $$\begin{aligned} dx_1 &= \quad * \quad + x_2\lambda' \quad * \, , \\ dx_2 &= -x_1\lambda' \quad * \quad + x_3\mu', \\ dx_3 &= \quad * \quad - x_2\mu' \quad * \, , \end{aligned}$$

which say that $\mathfrak{x}$ is fixed in the moving system.

Correspondingly, it follows from (10.1) for $d\mathfrak{x} = 0$ that the point:

(10.10) $$\mathfrak{x}' = x_1'\mathfrak{p}_1' + x_2'\mathfrak{p}_2' + x_3'\mathfrak{p}_3'$$

satisfies the *rest conditions:*

(10.11) $$\begin{aligned} dx_1' &= \quad * \quad + x_2'\lambda \quad * \, , \\ dx_2' &= -x_1'\lambda \quad * \quad + x_3'\mu, \\ dx_3' &= \quad * \quad - x_2'\mu \quad * \, . \end{aligned}$$

They say that $\mathfrak{x}'$ is at rest. One also gets the transition from (10.9) to (10.11) when one exchanges the rotation process $\mathfrak{Q}(t)$ with the opposite or *inverse process* $\tilde{\mathfrak{Q}}(t)$, which gives the "rest" system for an observer that is fixed in the "moving" one. One will then switch λ, μ with λ', μ'.

If we introduce the integral:

$$\int_{t_0}^{t} \rho = s, \qquad ds = \rho \tag{10.12}$$

in place of t as the *canonical time*, along with the abbreviations [1]):

$$\frac{\sigma}{\rho} = S, \qquad \frac{\tau}{\rho} = T, \qquad \frac{\lambda}{\rho} = M, \qquad \frac{\mu'}{\rho} = M', \tag{10.13}$$

in which, from (9.12), the relations exist:

$$M' - M = 2, \quad M' + M = 2T, \tag{10.14}$$

then we have for the derivatives of (9.11) with respect to s:

$$\begin{array}{llllll} \dot{\mathfrak{p}}_1 = & * & +L\mathfrak{p}_2 & *, & \dot{\mathfrak{p}}_1' = & * & +L\mathfrak{p}_2' & *, \\ \dot{\mathfrak{p}}_2 = & -L\mathfrak{p}_1 & * & +M\mathfrak{p}_3, & \dot{\mathfrak{p}}_2' = & -L\mathfrak{p}_1' & * & +M'\mathfrak{p}_3', \\ \dot{\mathfrak{p}}_3 = & * & -M\mathfrak{p}_2 & *; & \dot{\mathfrak{p}}_3' = & * & -M'\mathfrak{p}_2' & *, \end{array} \tag{10.15}$$

and from (10.9) and (10.11), we get the guiding and rest conditions:

$$\begin{array}{llllll} \dot{x}_1 = & * & +Lx_2 & *, & \dot{x}_1' = & * & +Lx_2' & *, \\ \dot{x}_2 = & -Lx_1 & * & +M'x_3, & \dot{x}_2' = & -Lx_1' & * & +Mx_3', \\ \dot{x}_3 = & * & -M'x_2 & *, & \dot{x}_3' = & * & -Mx_2' & *. \end{array} \tag{10.16}$$

From the fact that:

$$\mathfrak{p}_1 = \tilde{\mathfrak{Q}}\,\mathfrak{p}_1'\,\mathfrak{Q} \tag{10.17}$$

it follows that:

$$d\mathfrak{p}_1 = \{d\tilde{\mathfrak{Q}}\cdot\mathfrak{p}_1'\,\mathfrak{Q} + \tilde{\mathfrak{Q}}\,\mathfrak{p}_1'\cdot d\mathfrak{Q}\} + \tilde{\mathfrak{Q}}\,d\mathfrak{p}_1'\cdot\mathfrak{Q}\,. \tag{10.18}$$

From (10.2), (10.4), the expression in the brackets vanishes in this, and we find:

$$d\mathfrak{p}_1 = \tilde{\mathfrak{Q}}\,d\mathfrak{p}_1'\cdot\mathfrak{Q}\,. \tag{10.19}$$

We call the line $(\mathfrak{p}_1)$ that is described by the endpoint of the unit vector $\mathfrak{p}_1$ on K the *moving pole path* and the one described by $(\mathfrak{p}_1')$ on K', the *rest pole path*. Then $\lambda = \lambda'$

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[1]) In this, one has $S = k$, $T = w$. The meaning of L then changes.

means that the two pole paths are related to each other isometrically. From (10.19), this says: *The rotation process will be generated in such a way that the moving pole path rolls without slipping on the rest pole path.*

In addition, our geometric formulas illuminate the geometric meaning of the canonical axes: $\mathfrak{p}_1$ is the *instantaneous rotation axis*, whose points possess vanishing guiding velocities, and $\mathfrak{p}_3$ is the common perpendicular of two neighboring $\mathfrak{p}_1(t)$, $\mathfrak{p}_1(t + dt)$, both of which are in the moving system; corresponding statements are true for the $\mathfrak{p}'_j$ in the rest system.

§ 11. Acceleration

If we derive the absolute velocity with respect to canonical time, namely, from (10.7):

$$\dot{\mathfrak{x}} = 2(x_3\,\mathfrak{p}_2 - x_2\,\mathfrak{p}_3) \tag{11.1}$$

with respect to the canonical time *s*, while observing (10.14) to (10.16), then we find the *acceleration vector:*

$$\ddot{\mathfrak{x}} = 2\{- Lx_3\,\mathfrak{p}_1 - 2x_2\,\mathfrak{p}_2 + (Lx_1 - 2x_3)\,\mathfrak{p}_3\}. \tag{11.2}$$

For the determinant of the vectors $\mathfrak{x}$, $\dot{\mathfrak{x}}$, $\ddot{\mathfrak{x}}$, it then follows that:

$$[\mathfrak{x}\,\dot{\mathfrak{x}}\,\ddot{\mathfrak{x}}] = 4\{Lx_3 - 2x_2\,(x_2^2 + x_3^2)\} = 4D. \tag{11.3}$$

The inflection points of the paths on *K* will then be cut out of the third-order surface:

$$Lx_3 = 2x_2\,(x_2^2 + x_3^2). \tag{11.4}$$

For the arc length β of the paths, we find:

$$\beta^2 = <\dot{\mathfrak{x}}\;\dot{\mathfrak{x}}>ds^2 = 4\,(x_2^2 + x_3^2)\;ds^2, \tag{11.5}$$

and for their geodetic curvature:

$$g = \frac{[\mathfrak{x}\,\dot{\mathfrak{x}}\,\ddot{\mathfrak{x}}]}{<\dot{\mathfrak{x}}\,\dot{\mathfrak{x}}>^{3/2}} = \frac{Lx_3 - 2x_1(x_2^2 + x_3^2)}{2(x_2^2 + x_3^2)^{3/2}}. \tag{11.6}$$

Derivation of (11.2) further yields:

$$\tfrac{1}{2}\dddot{\mathfrak{x}} = \{L(M + 4)\,x_2 - \dot{L}x_3\}\mathfrak{p}_1 + \{L(2 - M)\,x_1 - (L_2 + 4)\}\mathfrak{p}_2 + \{\dot{L}x_1 + (L^2 + 4)\,x_2\}\,\mathfrak{p}_3, \tag{11.7}$$

and from this, one gets the determinant:

(11.8) $$\tfrac{1}{16}[\dot{\mathfrak{x}}\,\ddot{\mathfrak{x}}\,\dddot{\mathfrak{x}}] = 3L^2\, x_1\, x_2\, x_3 - \{L(M+4)\, x_2 - \dot{L}x_3\}\,(x_2^2 + x_3^2)\,.$$

Should this vanish for all points on K, then one would have $L = 0$. From (10.15), this would then establish $\mathfrak{p}_1$, $\mathfrak{p}_1'$. The only rotation processes for which all paths become circles are then the rotation processes involving an axis at rest.

The vector product $\dot{\mathfrak{x}}\times\ddot{\mathfrak{x}}$ establishes the location of the osculating plane (the center of curvature) to the path. One finds this vector $\mathfrak{y}$ to be, up to a scalar factor:

(11.9) $$\begin{aligned} y_1 &= Lx_1x_3 - 2(x_2^2 + x_3^2), \\ y_2 &= Lx_2x_3, \\ y_3 &= Lx_3x_3. \end{aligned}$$

Deriving (11.3) yields:

(11.10) $$\dot{D} = \dot{L}x_3 - Lx_2\,(2 + M' - 6x_1^2).$$

Points at which the path lines possess contact of order higher than two with their curvature circle (i.e., a *vertex*) thus satisfy the conditions:

(11.11) $$D = \dot{D} = 0.$$

§ 12. Kinematic image of the polarity

The lines $\mathfrak{Q}_0(t)$, $\mathfrak{Q}_3(t)$ correspond to each other in E_3 under the absolute polarity. This correspondence yields the relations:

(12.1) $$\mathfrak{Q}_0^* = \mathfrak{Q}_3, \qquad \mathfrak{Q}_1^* = \mathfrak{Q}_2, \qquad \mathfrak{Q}_2^* = \mathfrak{Q}_1, \qquad \mathfrak{Q}_3^* = \mathfrak{Q}_0\,,$$

and then, from (9.3):

(12.2) $$\rho^* = -\,\tau, \qquad \sigma^* = -\,\sigma, \qquad \tau^* = -\,\rho,$$

or, from (9.12):

(12.3) $$\lambda^* = -\,\lambda, \qquad \mu^* = +\,\mu, \qquad \lambda'^* = -\,\lambda', \qquad \mu'^* = -\,\mu',$$

and further, due to (9.13), for the geodetic curvature of the pole paths:

(12.4) $$g^* = -\,g, \qquad g'^* = -\,g'.$$

This means:

The rest pole path is preserved for the polar rotation processes, while the moving pole path is reversed.

If the line L that is described by $\mathfrak{Q}_0(t)$ in E_3 is a line $(\mathfrak{a}, \mathfrak{a}')$ then $\mathfrak{p}_1$ is fixed in the moving system and $\mathfrak{p}_1'$ is fixed in the rest system, and our rotation process consists of rotations around a fixed axis.

If the geodetic curvatures g, g' of the pole paths are functions of their common arc length:

(12.5) $$\int \sigma$$

and a linear dependency exists between these functions:

(12.6) $$Ag + Bg' + C = 0,$$

with fixed A, B, C, then it follows for the associated line L in E_3 that:

(12.7) $$Ck + (B + A)\,w + (B - A) = 0$$

is the dependency between the curvature k and the torsion w. Such lines are called BERTRAND curves.

§ 13. Screw lines in E_3

The simplest BERTRAND curves in E_3 are the *screw lines*, for which the curvature k and torsion w remain fixed. They arise as the path of one-parameter groups of motions in E_3 . For example, we take such a group around the axis (e_3, e_3):

(13.1) $$\mathfrak{Q}(t) = (\cos bt - e_3 \sin bt)\,\mathfrak{Q}'\,(\cos at + e_3 \sin at).$$

The point (13.1) with:

(13.2) $$\mathfrak{Q}'(t) = \cos\alpha + e_1 \sin\alpha$$

then describes a screw line S in E_3 . If we set:

(13.3) $$a - b = p, \qquad a + b = q$$

then we get for S:

(13.4) $$\mathfrak{Q}(t) = \cos\alpha\,(\cos pt + e_3 \sin pt) + \sin\alpha\,(e_1 \cos qt - e_2 \sin qt),$$

or, in more detail:

(13.5) $$q_0 = \cos\alpha \cos pt, \quad q_1 = \sin\alpha \cos pt, \quad q_2 = -\sin\alpha \sin pt, \quad q_3 = \cos\alpha \sin pt.$$

Thus, S lies on the quadric:

(13.6) $$\frac{q_0^2 + q_3^2}{\cos^2\alpha} = \frac{q_1^2 + q_2^2}{\sin^2\alpha}.$$

In addition, one has:

(13.7) $$+\frac{q_3}{q_0} = \tan pt, \qquad -\frac{q_2}{q_1} = \tan qt.$$

For $t = 0$, we get:

(13.8) $$\begin{aligned} &\mathfrak{Q} = \cos\alpha + e_1 \sin\alpha, && \dot{\mathfrak{Q}} = e_3 p \cos\alpha - e_1 q \sin\alpha, \\ &\ddot{\mathfrak{Q}} = -p^2 \cos\alpha - e_1 q^2 \sin\alpha, && \dddot{\mathfrak{Q}} = -e_3 p^3 \cos\alpha + e_2 q^3 \sin\alpha. \end{aligned}$$

From this, it follows, from (9.19), for $t = 0$, and thus, for all t, that:

(13.9) $r = c\,dt, \qquad c^2 = p^2\cos^2\alpha + q^2\sin^2\alpha,$

(13.10) $c^2 = a^2 - 2\,ab\cos 2\alpha + b^2,$

$$1 + k^2 = \frac{p^4\cos^2\alpha + q^4\sin^2\alpha}{c^2}, \tag{13.11}$$

$$k^2 w = \frac{pq(q^2 - p^2)^2\cos^2\alpha\sin^2\alpha}{c^6}, \tag{13.12}$$

and furthermore, for $t = 0$:

$$\begin{aligned} \mathfrak{p}_1 &= \tilde{\mathfrak{Q}}\mathfrak{Q}_1 = \frac{\tilde{\mathfrak{Q}}\dot{\mathfrak{Q}}}{c} = \frac{+e_2(p-q)\sin\alpha\cos\alpha + e_3(p\cos^2\alpha + q\sin^2\alpha)}{c}, \\ \mathfrak{p}_1' &= \mathfrak{Q}_1\tilde{\mathfrak{Q}} = \frac{\dot{\mathfrak{Q}}\tilde{\mathfrak{Q}}}{c} = \frac{-e_2(p+q)\sin\alpha\cos\alpha + e_3(p\cos^2\alpha - q\sin^2\alpha)}{c}. \end{aligned} \tag{13.13}$$

The spherical images of S are circles on K, K' that arise from rotation of the vectors (13.13) around e_3. In particular, one has:

$$[e_1\,\mathfrak{p}_1\,\mathfrak{p}_1'] = \sin 2\alpha, \tag{13.14}$$

$$[e_1\,\mathfrak{p}_1\,e_3] = -\frac{b}{a}\sin 2\alpha, \qquad [e_1\,\mathfrak{p}_1'\,e_3] = -\frac{a}{c}\sin 2\alpha. \tag{13.15}$$

One obtains the radii of the circles $(\mathfrak{p}_1)$, $(\mathfrak{p}_1')$ from these relations. In particular, from (13.5), for $q = 0$, one also gets $q_2 = 0$, and therefore S becomes a plane section of the quadric (13.6). More generally: If p/q is rational then S is closed and algebraic.

§ 14. Link quadrangle

We assume that the point $\mathfrak{x}$ of the moving system describes a circle on the unit sphere K. We seek the condition that must be satisfied by the two-parameter rotation process $\mathfrak{Q}$ that is thus determined. To that end, we take, say:

$$\mathfrak{x} = \tilde{\mathfrak{Q}}\,e_3\,\mathfrak{Q}, \tag{14.1}$$

and for the path of $\mathfrak{x}$, the circle in the plane:

$$e_3\,\mathfrak{x} + \mathfrak{x}\,e_3 = 2C. \tag{14.2}$$

This makes:

(14.3) $e_3\,\mathfrak{x} = e_3\,\tilde{\mathfrak{Q}}\,e_3\,\mathfrak{Q} = e_3\,(q_0 - \mathfrak{q})\,e_3\,(q_0 + \mathfrak{q}) = -q_0^2 + q_0\,(e_3\,e_3\,\mathfrak{q} - e_3\,\mathfrak{q}\,e_3) - e_3\,\mathfrak{q}\,e_3\,\mathfrak{q}.$

Since:

(14.4) $$e_3\,\mathfrak{q}\,e_3 = e_1\,q_1 + e_2\,q_2 + e_3\,q_3\,,$$

it follows from (14.2), (14.3) that:

(14.5) $$-\,q_0^2 + q_1^2 + q_2^2 - q_0^2 = C\,(q_0^2 + q_1^2 + q_2^2 + q_0^2)\,,$$

or

(14.6) $$(1 + C)(\,q_0^2 + q_3^2\,) - (1 - C)\,(q_1^2 + q_2^2) = 0.$$

Thus, our two-parameter rotation process corresponds to the quadric (14.6) in E_3, which meets the absolute quadric A in four generators at the planes:

(14.7) $$q_0 \pm i\,q_3 = 0, \quad q_1 \pm i\,q_2 = 0; \quad i^2 = -\,1.$$

If we now examine a one-parameter rotation process, under which the two points of the moving system describe circles then we must bring two quadrics of the type (14.6) together so they intersect; this gives a curve of fourth-order and "first type" that generally possesses the "genus" one, and can be represented by means of elliptic functions. The eight intersection points of C_4 with the absolute quadric A lie in pairs on four generators of the one family, and likewise in pairs on the generators of the other family. For $C = 0$, the circle (14.2) becomes a great circle of the unit sphere, and the quadric (14.5) becomes "apolar" to A.

The rotation process of the spherical link quadrangle thus described is based on the study of the common projective invariants of C_4 and the absolute quadric A. In particular, the analogous process in planar kinematics has been examined in detail many times, due to its engineering interpretation. G. DARBOUX (1879) [1]) has remarked on the connection between the plane link rectangle and elliptic functions.

§ 15. Determination of the canonical axes

Let $\mathfrak{a}_j$ (j = 1, 2, 3) be a rectangular axis-cross with a fixed origin O that is independent of time t. We seek the canonical axes $\mathfrak{p}_j$ – which, for the moment, we would like to denote by $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$ – for the rotation process that is thus determined:

(15.1) $$\begin{aligned} \mathfrak{p}_1 &= \mathfrak{r} = \mathfrak{a}_1 r_1 + \mathfrak{a}_2 r_2 + \mathfrak{a}_3 r_3, \\ \mathfrak{p}_2 &= \mathfrak{s} = \mathfrak{a}_1 s_1 + \mathfrak{a}_2 s_2 + \mathfrak{a}_3 s_3, \\ \mathfrak{p}_3 &= \mathfrak{t} = \mathfrak{a}_1 t_1 + \mathfrak{a}_2 t_2 + \mathfrak{a}_3 t_3. \end{aligned}$$

One has the differential equations for the $\mathfrak{a}_j$:

[1]) One can find many references in A. SCHOENFLIES and M. GRÜBLER, "Kinematik," Enc. math. Wiss., art. IV.3, Leipzig, 1902, as well as W. BLASCHKE and H. R. MÜLLER, *Ebene Kinematik*, Munich, 1956.

$$(15.2)\qquad \begin{aligned} \dot{\mathfrak{a}}_1 &= \quad * \quad -\mathfrak{a}_2 C_3 \ +\mathfrak{a}_3 C_2, \\ \dot{\mathfrak{a}}_2 &= +\mathfrak{a}_1 C_3 \quad * \quad -\mathfrak{a}_3 C_1, \\ \dot{\mathfrak{a}}_3 &= -\mathfrak{a}_1 C_2 \ +\mathfrak{a}_2 C_1 \quad *. \end{aligned}$$

In (10.5), we had:

$$(15.3)\qquad \dot{\mathfrak{a}}_j = 2(\mathfrak{a}_j \times \mathfrak{p}_1)\, R, \qquad R = \frac{\rho}{dt}.$$

From this, it would follow that:

$$(15.4)\qquad \begin{aligned} \tfrac{1}{2}\dot{\mathfrak{a}}_1 &= \quad * \quad -\mathfrak{a}_2 r_3 C \ +\mathfrak{a}_3 r_2 R, \\ \tfrac{1}{2}\dot{\mathfrak{a}}_2 &= +\mathfrak{a}_1 r_3 R \quad * \quad -\mathfrak{a}_3 r_1 R, \\ \tfrac{1}{2}\dot{\mathfrak{a}}_3 &= -\mathfrak{a}_1 r_2 R \ +\mathfrak{a}_2 r_1 R \quad *. \end{aligned}$$

If we introduce the "rotation vector":

$$(15.5)\qquad \mathfrak{c} = \mathfrak{a}_1 C_1 + \mathfrak{a}_2 C_2 + \mathfrak{a}_3 C_3$$

then a comparison of (15.2) and (15.4) gives:

$$(15.6)\qquad \mathfrak{r} = \frac{c}{2R}.$$

Because $\mathfrak{r}$ is a unit vector, if follows from (15.6) that:

$$(15.7)\qquad 4R^2 = \langle \mathfrak{c}\ \mathfrak{c} \rangle.$$

If we set:

$$(15.8)\qquad \frac{\lambda}{dt} = L, \qquad \frac{\mu}{dt} = M$$

then equations (10.15) yield:

$$(15.9)\qquad \begin{aligned} \dot{\mathfrak{r}} &= \quad * \quad +\mathfrak{s}\, L \quad *, \\ \dot{\mathfrak{s}} &= -\mathfrak{r}\, L \quad * \quad +\mathfrak{t}\, M, \\ \dot{\mathfrak{t}} &= \quad * \quad -\mathfrak{s}\, M \quad *. \end{aligned}$$

From this, it follows that:

$$(15.10)\qquad \mathfrak{s} = \frac{\dot{\mathfrak{r}}}{L} = \frac{1}{2}\frac{\dot{\mathfrak{c}}\, R - \mathfrak{c}\, \dot{R}}{LR^2},$$

$$(15.11)\qquad L^2 = \langle \dot{\mathfrak{r}}\, \dot{\mathfrak{r}} \rangle = \frac{1}{4}\frac{\langle \dot{\mathfrak{c}}\, \dot{\mathfrak{c}} \rangle R^2 - 2 \langle \mathfrak{c}\, \dot{\mathfrak{c}} \rangle R\dot{R} + \langle \mathfrak{c}\, \mathfrak{c} \rangle \dot{R}^2}{R^4}.$$

Furthermore, one has:

$$(15.12)\qquad \mathfrak{t} = \mathfrak{r} \times \mathfrak{s} = \frac{1}{4}\frac{\mathfrak{c} \times \dot{\mathfrak{c}}}{LR^2}$$

and

$$M = [\mathfrak{r}\ \mathfrak{s}\ \dot{\mathfrak{s}}] = \frac{1}{8L^2R^3}[\mathfrak{c}\ \dot{\mathfrak{c}}\ \ddot{\mathfrak{c}}]. \tag{15.13}$$

Finally, we have, from (9.12):

$$\begin{aligned} &\sigma = \lambda = \lambda' \qquad = Ldt, \\ &\rho = Rdt, \quad \mu = Mdt, \\ &\tau = \mu + \rho, \ \ \mu' = \mu + 2\rho. \end{aligned} \tag{15.14}$$

§ 16. Kinematics in E_3

If we take a polar tetrahedron in E_3:

$$<\mathfrak{Q}_j\ \mathfrak{Q}_k> = \delta_{jk}, \qquad [\mathfrak{Q}_0\ \mathfrak{Q}_1\ \mathfrak{Q}_2\ \mathfrak{Q}_3] = +1 \tag{16.1}$$

that is independent of time t then a *compulsive process of motion* arises in E_3 with the differential equations:

$$d\mathfrak{Q}_j = \sum_0^3 \mathfrak{Q}_k \omega_{jk}, \qquad \omega_{jk} + \omega_{kj} = 0. \tag{16.2}$$

We map the tetrahedron of $\mathfrak{Q}_k$ onto two rectangular axis crosses (j = 1, 2, 3):

$$\begin{gathered} \mathfrak{p}_j = \tilde{\mathfrak{Q}}_0 \mathfrak{Q}_j, \ \mathfrak{p}'_j = \mathfrak{Q}_j \tilde{\mathfrak{Q}}_0, \ < \mathfrak{p}'_j \mathfrak{p}'_k > = \delta_{jk}, \\ [\mathfrak{p}_1\ \mathfrak{p}_2\ \mathfrak{p}_3] = [\mathfrak{p}'_1\ \mathfrak{p}'_2\ \mathfrak{p}'_3] = +1, \end{gathered} \tag{16.3}$$

such that the formulas of tables (9.5) are again valid. If we then set:

$$d\mathfrak{p}_j = \sum \mathfrak{p}_k\ \alpha_{jk}, \qquad d\mathfrak{p}'_j = \sum \mathfrak{p}'_k \alpha'_{jk}, \qquad \alpha_{jk} + \alpha_{kj} = \alpha'_{jk} + \alpha'_{kj} = 0 \tag{16.4}$$

then it follows from (16.3) by derivation that:

$$\begin{aligned} \alpha_{23} &= \omega_{23} - \omega_{01}, \quad & \alpha'_{23} &= \omega_{23} + \omega_{01}, \\ \alpha_{31} &= \omega_{31} - \omega_{02}, \quad & \alpha_{30} &= \omega_{31} + \omega_{02}, \\ \alpha_{12} &= \omega_{12} - \omega_{03}, \quad & \alpha'_{12} &= \omega_{12} + \omega_{03}. \end{aligned} \tag{16.5}$$

In this way, the motion process in E_3 is mapped onto two rotation processes, one of which is on the unit sphere K, and the other of which is on K'. If, for example, $\mathfrak{a}$ and $\mathfrak{a}'$ are the instantaneous rotational axes of the two rotation processes, and therefore scalar multiplicities of the vectors (α_{23}, α_{31}, α_{12}), $(\alpha'_{23}, \alpha'_{31}, \alpha'_{12})$ – then in E_3 the two polars ($\pm\ \mathfrak{a}$,

$\mathfrak{a}'$) become the *instantaneous screw axes* of the motion process. We would not like to go into this situation further, here.

§ 17. Integral theorems

Let $\mathfrak{Q}(t)$ be a *closed rotation process*, so perhaps, more precisely:

$$(17.1) \qquad \mathfrak{Q}(t+1) = \mathfrak{Q}(t), \qquad \mathfrak{p}_j(t+1) = \mathfrak{p}_j(t).$$

It would then follow from equations (9.11) that:

$$(17.2) \qquad \oint \mathfrak{p}_3 \lambda = 0, \qquad \oint \mathfrak{p}_1 \lambda = \oint \mathfrak{p}_3 \mu, \qquad \oint \mathfrak{p}_3 \mu = 0.$$

We further calculate the integral along a (closed) path curve on the unit sphere:

$$(17.3) \qquad \oint \mathfrak{x} \rho = \oint \tilde{\mathfrak{Q}} \mathfrak{x}' \mathfrak{Q} \rho .$$

If we assume that for $t = 0$:

$$(17.4) \qquad \mathfrak{p}'_j(0) = e_j \qquad (j = 1, 2, 3)$$

then:

$$(17.5) \qquad \oint \mathfrak{x} \rho = \mathfrak{p}_1 x_1 + \mathfrak{p}_2 x_2 + \mathfrak{p}_3 x_3, \qquad \mathfrak{x}'(0) = e_1 x_1 + e_2 x_2 + e_3 x_3,$$

with the vectors:

$$(17.6) \qquad \mathfrak{v}_j = \oint \tilde{\mathfrak{Q}} e_j \mathfrak{Q} \rho .$$

If we further consider the "surface vector: for our paths:

$$(17.7) \qquad \mathfrak{f} = \tfrac{1}{2} \oint \mathfrak{x} \times d\mathfrak{x} = \oint \mathfrak{x} \times (\mathfrak{x} \times \mathfrak{p}_1) \rho = \oint < \mathfrak{x}\, \mathfrak{p}_1 > \mathfrak{x} \rho - \oint \mathfrak{p}_1 \rho$$

then the last vector integral:

$$(17.8) \qquad \mathfrak{v} = \oint \mathfrak{p}_1 \rho$$

does not depend upon the choice of point $\mathfrak{x}$ on the unit sphere. It then remains for us to examine:

$$(17.9) \qquad \mathfrak{w} = \oint < \mathfrak{x}\, \mathfrak{p}_1 > \mathfrak{x} \rho .$$

We have, due to $\mathfrak{x} = \tilde{\mathfrak{Q}} \mathfrak{x}' \mathfrak{Q}$, $\mathfrak{p}_1 = \tilde{\mathfrak{Q}} \mathfrak{Q}_1 = - \tilde{\mathfrak{Q}}_1 \mathfrak{Q}$:

(17.10)
$$\begin{aligned}\mathfrak{w} &= \tfrac{1}{2}\oint(-\tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}_1 + \tilde{\mathfrak{Q}}_1\mathfrak{x}'\mathfrak{Q})\tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}\rho \\ &= \tfrac{1}{2}\oint(-\tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}_1\tilde{\mathfrak{Q}}\mathfrak{x}'\mathfrak{Q}\rho) + \tfrac{1}{2}\mathfrak{v}.\end{aligned}$$

Thus, one gets:

(17.11)
$$\begin{aligned}\mathfrak{f} &= \sum_1^3 \mathfrak{w}_{jk}x_j x_k - \tfrac{1}{2}\mathfrak{v}, \\ \mathfrak{w}_{jk} &= -\tfrac{1}{2}\oint\tilde{\mathfrak{Q}}e_j\mathfrak{Q}_1\tilde{\mathfrak{Q}}e_j\mathfrak{Q}\rho.\end{aligned}$$

For the geodetic curvature C of the moving pole path and the curvature C' of the rest pole path one has, from (9.13):

(17.12)
$$C = \oint\mu, \qquad C' = \oint\mu'.$$

For the surfaces that are traversed by them on K, K' one then has (from GAUSS-BONNET):

(17.13)
$$F = \pi - C, \qquad F' = \pi - C'.$$

§ 18. Rectilinear surfaces in E_3

The line $\{\mathfrak{Q}_0(t), \mathfrak{Q}_1(t)\}$ describes a rectilinear surface (viz., a *ruled surface*) F in E_3. We can choose the points $\mathfrak{Q}_0$, $\mathfrak{Q}_1$ to be the generators of F such that $\{\mathfrak{Q}_0, \mathfrak{Q}_3\}$ and $\{\mathfrak{Q}_1, \mathfrak{Q}_2\}$ become the common perpendiculars to neighboring generators of F; the $\mathfrak{Q}_j$ then define a polar tetrahedron. We thus calculate the vectors $\mathfrak{p}_j$, $\mathfrak{p}'_j$ from the table (9.5). One then has the differential equations:

(18.1)
$$\begin{aligned}
d\mathfrak{Q}_0 &= \quad * \quad + \mathfrak{Q}_1\frac{\mu-\mu'}{2} \quad * \quad + \mathfrak{Q}_3\frac{\lambda-\lambda'}{2}, \\
d\mathfrak{Q}_1 &= +\mathfrak{Q}_0\frac{\mu-\mu'}{2} \quad * \quad + \mathfrak{Q}_2\frac{\lambda'+\lambda}{2} \quad *, \\
d\mathfrak{Q}_2 &= \quad * \quad - \mathfrak{Q}_1\frac{\lambda'+\lambda}{2} \quad * \quad \mathfrak{Q}_3\frac{\mu'+\mu}{2}, \\
d\mathfrak{Q}_3 &= -\mathfrak{Q}_0\frac{\lambda'-\lambda}{2} \quad * \quad - \mathfrak{Q}_2\frac{\mu'+\mu}{2} \quad *.
\end{aligned}$$

In this, one generally has $\lambda \neq \lambda'$. Furthermore, we find:

(18.2)
$$\begin{aligned}
d\mathfrak{p}_1 &= \quad * \quad + \mathfrak{p}_2\lambda \quad *, & d\mathfrak{p}'_1 &= \quad * \quad + \mathfrak{p}'_2\lambda' \quad *, \\
d\mathfrak{p}_2 &= -\mathfrak{p}_1\lambda \quad * \quad + \mathfrak{p}_3\mu, & d\mathfrak{p}'_2 &= -\mathfrak{p}'_1\lambda' \quad * \quad + \mathfrak{p}'_3\mu', \\
d\mathfrak{p}_3 &= \quad * \quad - \mathfrak{p}_2\mu \quad *, & d\mathfrak{p}'_3 &= \quad * \quad - \mathfrak{p}'_2\mu' \quad *.
\end{aligned}$$

For developable surfaces, in particular, one has $\lambda = \lambda'$, and the formulas (18.1), (18.2) agree with (9.3), (9.11).

§ 19. A theorem of K. STEPHANOS

If, as in § 3, we regard the point $\mathfrak{Q}$ of the elliptic space E_3 as the image of the axis cross with the origin O, where we restrict ourselves to the axis-crosses with the determinant + 1 (“right-hand crosses”), then the condition for the axis-cross $\mathfrak{Q}$ to arise from another one $\mathfrak{Q}'$ by an inversion around an axis $\mathfrak{a}$ reads:

(19.1) $$\mathfrak{Q} = \mathfrak{Q}'\,\mathfrak{a}, \qquad \mathfrak{a} = \tilde{\mathfrak{Q}}'\mathfrak{Q}.$$

From the fact that $\mathfrak{a}$ is a vector, it then follows that:

(19.2) $$\tilde{\mathfrak{Q}}'\mathfrak{Q} + \tilde{\mathfrak{Q}}\mathfrak{Q}' = 2\langle\mathfrak{Q}\,\mathfrak{Q}'\rangle = 0.$$

From this, it follows: All axis-crosses $\mathfrak{Q}$ that emerge from a fixed one $\mathfrak{Q}'$ by inversions have the plane:

(19.3) $$\langle\mathfrak{Q}\,\mathfrak{Q}'\rangle = q_0 q_0' + q_1 q_1' + q_2 q_2' + q_3 q_3' = 0$$

as their image in E_3. If we then take three points $\mathfrak{Q}_j$ (j = 1, 2, 3) in E_3 that do not lie on the same line (i.e., they are not collinear) then they determine a plane uniquely. This gives the following theorem of K. STEPHANOS (1857-1917), which is most simply connected with the construction of GAUSS (§ 2): [1])

For any three right-hand crosses $\mathfrak{Q}_j$ with a common origin O that do not go to each other under rotations around the same axis, there is always a fourth one $\mathfrak{Q}'$ that goes to the $\mathfrak{Q}_j$ by inversions.

Namely, let:

(19.4) $$\mathfrak{R}_1 = \tilde{\mathfrak{Q}}_2\mathfrak{Q}_3, \quad \mathfrak{R}_2 = \tilde{\mathfrak{Q}}_3\mathfrak{Q}_1, \quad \mathfrak{R}_3 = \tilde{\mathfrak{Q}}_1\mathfrak{Q}_2,$$

so:

(19.5) $$\mathfrak{R}_1\,\mathfrak{R}_2\,\mathfrak{R}_3 = 1.$$

We can represent each such rotation by the composition of two inversions in sequence:

(19.6) $$\mathfrak{R}_1 = \mathfrak{b}_2\mathfrak{b}_3, \quad \mathfrak{R}_2 = \mathfrak{b}_3\mathfrak{b}_1, \quad \mathfrak{R}_3 = \mathfrak{b}_1\mathfrak{b}_3,$$

[1]) K. STEPHANOS, “Sur la théorie des quaternions,” Math. Ann. **22** (1883), 589-592.

in which – e.g., $\mathfrak{b}_1$ – is the common perpendicular to the rotational axes of $\mathfrak{R}_2$ and $\mathfrak{R}_3$. One then gets:

(19.7) $$\mathfrak{Q}_1 = \mathfrak{Q}_3 \mathfrak{R}_2 = \mathfrak{Q}_3 \mathfrak{b}_3 \mathfrak{b}_1,$$

and from that:

(19.8) $$\mathfrak{Q}_1 \mathfrak{b}_1 = \mathfrak{Q}_2 \mathfrak{b}_2 = \mathfrak{Q}_3 \mathfrak{b}_3 = \mathfrak{Q}'$$

or

(19.9) $$\mathfrak{Q}_j \mathfrak{b}_j = \mathfrak{Q}',$$

as asserted.

If one replaces $\mathfrak{Q}'$ with the left-hand cross $\mathfrak{Q}^*$ that arise from $\mathfrak{Q}'$ by reflection through the origin O then $\mathfrak{Q}$ goes to $\mathfrak{Q}^*$ by reflections in planes through O, and we obtain a one-to-one correspondence between the left-hand crosses with origin O and the planes in E_3 .

On the basis of (19.3), one can develop a "projective geometry of the axis-crosses around O," although one must admit the "ideal" cross:

(19.10) $$q_1 = q_2 = q_3 = 0.$$

CHAPTER THREE

SURFACE-CONSTRAINED SPHERICAL ROTATION PROCESSES

§ 20. Pfaffian forms [1])

An expression:

(20.1) $$\omega = a(u, v)\, du + b(u, v)\, dv$$

(which is linear in the differentials du, dv of the independent variables u, v) is called a *Pfaffian form*, after J. FR. PFAFF (1765-1825). Following H. GRASSMANN (1809-1877), one introduces the *alternating product* – or *polar product* – of two such forms:

(20.2) $$\omega_j = a_j\, du + b_j\, dv,$$

namely:

(20.3) $$[\omega_1\ \omega_2] = -[\omega_2\ \omega_1] = (a_1 b_2 - a_2 b_1)\, [du, dv],$$

$$[du, dv] = -[dv, du],$$

which can already be obtained naturally from double integrals. The vanishing of the polar product implies the linear dependence of the forms ω_1, ω_2.

One then comes to the *exterior differential* of G. FROBENIUS (1849-1917) and E. CARTAN (1869-1951):

(20.4) $$d\omega = [da, du] + [db, dv] = (b_u - a_v)\, [du, dv].$$

If it vanishes identically then ω is a complete differential:

(20.5) $$\omega = df = f_u\, du + f_v\, dv.$$

Both constructions (20.3) and (20.4) are invariant; i.e., they commute with the introduction of new variables:

(20.6) $$x = x(u, v), \qquad y = y(u, v), \qquad [dx, dy] \neq 0.$$

We add the following two rules of calculation:

(20.7) $$d(f\omega) = [df, \omega] + f\, d\omega,$$

[1]) A throrough presentation of the calculations with alternating differential forms is included in the book by H. REICHARDT: *Vorlesungen über Vektor- und Tensorrechnung*, Berlin, 1957. Cf., also W. BLASCHKE and H. REICHARDT: *Einführung in die Differentialgeometrie*, Berlin-Göttingen-Heidelberg, 1960.

and furthermore, for a domain B in the u, v-plane and its unique surrounding boundary dB:

(20.8)
$$\int_B d\omega = \int_{dB} \omega .$$

This formula (20.8), which converts the double integral on the left into the boundary integral on the right, includes the formulas of GAUSS and STOKES. One also writes:

(20.9)
$$[d\omega]$$
for the exterior differential.

§ 21. Surface-constrained motion processes in E_3

Let $\mathfrak{Q}_j$ (j = 0, 1, 2, 3) be a polar tetrahedron in the absolute quadric in E_3, whose vertices $\mathfrak{Q}_j(u, v)$ depend upon two parameters. A two-parameter (= surface-constrained) process of motion in E_3 is thus defined in this that way. We can bring the formulas of § 16 into play if we interpret the ω in them as Pfaff forms. We then obtain the differential equations:

(21.1)
$$d\mathfrak{Q}_j = \mathfrak{Q}_k\, \omega_{jk} , \qquad \omega_{jk} + \omega_{kj} = 0.$$

On the left, one must sum over the index k that appears in it. The following integrability conditions belong to (21.1):

(21.2)
$$d\omega_{jk} = [\omega_{js}\ \omega_{sk}],$$

in which the sum over s is again implied.

We further set:

(21.3)
$$\mathfrak{p}_j = \tilde{\mathfrak{Q}}_0 \mathfrak{Q}_j , \qquad \mathfrak{p}'_j = \mathfrak{Q}_j \tilde{\mathfrak{Q}}_0 \qquad (j = 1, 2, 3),$$

and find from this, by differentiation, and from (21.1) and (9.5), that:

(21.4)
$$d\mathfrak{p}_j = \mathfrak{p}_k \sigma_{jk} , \quad d\mathfrak{p}'_j = \mathfrak{p}'_k \sigma'_{jk} , \qquad \sigma_{jk} + \sigma_{kj} = \sigma'_{jk} + \sigma'_{kj} = 0,$$

with:

(21.5)
$$\begin{aligned} \sigma_{23} &= \omega_{23} - \omega_{01}, \quad & \sigma_{31} &= \omega_{31} - \omega_{02}, \quad & \sigma_{12} &= \omega_{12} - \omega_{03}, \\ \sigma'_{23} &= \omega'_{23} - \omega'_{01}, \quad & \sigma'_{31} &= \omega'_{31} - \omega'_{02}, \quad & \sigma'_{12} &= \omega'_{12} - \omega'_{03}, \end{aligned}$$

and the integrability conditions:

(21.6)
$$d\sigma_{jk} = [\sigma_{js}\ \sigma_{sk}], \qquad d\sigma'_{jk} = [\sigma'_{js}\ \sigma'_{sj}].$$

For the surface elements:

(21.7)
$$\begin{aligned}
\Omega_1 &= [\sigma_{12}\,\sigma_{13}], \quad &\Omega_2 &= [\sigma_{23}\,\sigma_{21}], \quad &\Omega_3 &= [\sigma_{31}\,\sigma_{32}], \\
\Omega_1' &= [\sigma_{12}'\,\sigma_{13}'], \quad &\Omega_2' &= [\sigma_{23}'\,\sigma_{21}'], \quad &\Omega_3' &= [\sigma_{31}'\,\sigma_{32}'],
\end{aligned}$$

it follows from (21.2) that:

(21.8)
$$\begin{aligned}
\Omega_1 &= +d\omega_{01} - d\omega_{23}, \quad &\Omega_2 &= +d\omega_{02} - d\omega_{31}, \quad &\Omega_3 &= +d\omega_{03} - d\omega_{12}, \\
\Omega_1' &= -d\omega_{01} - d\omega_{23}, \quad &\Omega_2' &= -d\omega_{02} - d\omega_{31}, \quad &\Omega_3' &= -d\omega_{03} - d\omega_{12}.
\end{aligned}$$

We add some relations between our surface elements for the general case that come from (21.5) and (21.7):

(21.9)
$$\begin{aligned}
2[\omega_{02}\omega_{03}] + 2[\omega_{31}\omega_{12}] &= \Omega_1 + \Omega_1', \\
2[\omega_{03}\omega_{01}] + 2[\omega_{12}\omega_{23}] &= \Omega_2 + \Omega_2', \\
2[\omega_{01}\omega_{02}] + 2[\omega_{23}\omega_{31}] &= \Omega_3 + \Omega_3'.
\end{aligned}$$

We would like to apply these formulas. Namely, we would like to find out how the lines $[\mathfrak{Q}_0\ \mathfrak{Q}_3]$ through a point:

(21.10)
$$\mathfrak{P} = \mathfrak{Q}_0 \cos\varphi + \mathfrak{Q}_3 \sin\varphi, \qquad \varphi = \varphi(u, v)$$

can be arranged in order that the surface F that is described by $\mathfrak{P}$ intersects the lines $[\mathfrak{Q}_0\ \mathfrak{Q}_3]$ at right angles. Due to (21.1), one has:

(21.11)
$$\begin{aligned}
d\mathfrak{P} = (-\mathfrak{Q}_0 \sin\varphi + \mathfrak{Q}_3 \cos\varphi)(\omega_{03} + d\varphi) &+ \mathfrak{Q}_1\,(\omega_{01} \cos\varphi + \omega_{31} \sin\varphi) \\
&+ \mathfrak{Q}_2\,(\omega_{02} \cos\varphi - \omega_{23} \sin\varphi).
\end{aligned}$$

Our demand thus implies that:

(21.12)
$$\omega_{03} = d\varphi, \qquad d\omega_{03} = 0, \qquad \varphi = -\int \omega_{03}\,,$$

and φ is determined from this up to an additive constant. There is therefore a family of "parallel surfaces" F with the common normals $[\mathfrak{Q}_0\ \mathfrak{Q}_3]$.

From (21.8) and (21.12), normal congruences are characterized by:

(21.13)
$$\Omega_3 = \Omega_3',$$

i.e., from (21.4) and (21.7), the map of the two image spheres $(\mathfrak{p}_3)$, $(\mathfrak{p}_3')$ is surface-preserving. The fact that any such map:

(21.14)
$$\mathfrak{p}_3(u, v) \to \mathfrak{p}_3'(u, v), \qquad \left[\mathfrak{p}_3\,\frac{\partial\mathfrak{p}_3}{\partial u}\,\frac{\partial\mathfrak{p}_3}{\partial v}\right] = \left[\mathfrak{p}_3'\,\frac{\partial\mathfrak{p}_3'}{\partial u}\,\frac{\partial\mathfrak{p}_3'}{\partial v}\right],$$

belongs to a one-parameter family of "parallel" two-parameter rotation processes can also be seen directly with no detour to E_3, as we would like to briefly discuss.

We thus cover the left-hand sphere $(\mathfrak{p}_3)$ with a family S of curves C that all begin at a point u_0, v_0 and cover a neighborhood of this point simply. Let the family that corresponds under the map (21.14) to the right-hand sphere $(\mathfrak{p}'_3)$ be S', and let its curves be C'. We choose the unit vectors $\mathfrak{p}_1(u_0, v_0)$ and $\mathfrak{p}'_1(u_0, v_0)$ at u_0, v_0 to be perpendicular to $\mathfrak{p}_3(u_0, v_0)$ and $\mathfrak{p}'_3(u_0, v_0)$, but otherwise arbitrary. We then displace $\mathfrak{p}_1$ and $\mathfrak{p}'_1$ parallel to u_0, v_0 along a pair of corresponding points u, v along corresponding lines C, C'. The parallel displacement of Lord KELVIN and LEVI-CIVITA is defined in such a way that along C and C' one must have:

$$\sigma_{12} = \langle\mathfrak{p}_2\,,\, d\mathfrak{p}_1\rangle = 0. \tag{21.15}$$

The association (21.14) is then established by this. This can be proved in two ways. First: The association (21.14) remains unchanged when one varies the families of curves S and S'. We thus see that under such a variation $\mathfrak{p}_1$ and $\mathfrak{p}'_1$ will be rotated through the same angle $\varphi(u, v)$. (This follows from the lemma: The angle between two vectors along the same line remains the same under parallel displacement, and if one displaces a vector along a closed line then the angle between its initial and final positions is equal to the area traversed, up to an additive constant). Secondly: If one rotates the vector $\mathfrak{p}_1(u_0, v_0)$ through the angle α then any axis-cross $\mathfrak{p}'_j(u, v)$ will be rotated around $\mathfrak{p}'_3(u, v)$ through the same angle α.

§ 22. From the theory of surfaces in R_3

To the clarification in the conclusion of § 21, we add some facts from the theory of surfaces in Euclidian R_3 . The axis-cross $\{\mathfrak{x}: \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ depends upon two real parameters u, v. Its origin $\mathfrak{x}$ may therefore describe a surface F, and $\mathfrak{p}_3$ shall mean the unit vector of the surface normal of F at $\mathfrak{x}$. One then has differential equations of the form:

$$\begin{gathered} d\mathfrak{x} = \mathfrak{p}_1\bar{\sigma}_1 + \mathfrak{p}_1\bar{\sigma}_1, \\ d\mathfrak{p}_1 = \mathfrak{p}_2\sigma_3 - \mathfrak{p}_3\sigma_2,\; d\mathfrak{p}_2 = \mathfrak{p}_3\sigma_1 - \mathfrak{p}_1\sigma_3,\; d\mathfrak{p}_3 = \mathfrak{p}_1\sigma_2 - \mathfrak{p}_2\sigma_1, \end{gathered} \tag{22.1}$$

with the integrability conditions:

$$\begin{aligned} d\bar{\sigma}_1 &= +[\sigma_3\bar{\sigma}_1], & d\bar{\sigma}_2 &= +[\bar{\sigma}_1\sigma_3], & 0 &= [\sigma_1\bar{\sigma}_2] + [\bar{\sigma}_1\sigma_2], \\ d\sigma_1 &= -[\sigma_2\sigma_3], & d\sigma_2 &= -[\sigma_3\sigma_1], & d\sigma_3 &= -[\sigma_1\sigma_2]. \end{aligned} \tag{22.2}$$

The ratio of the surface element of the spherical image $(\mathfrak{p}_3)$ of F to the surface element of F gives the Gaussian curvature K at a point of F:

(22.3) $$K = \frac{[\sigma_1\sigma_2]}{[\bar{\sigma}_1\bar{\sigma}_1]} = -\frac{d\sigma_3}{[\bar{\sigma}_1\bar{\sigma}_2]}.$$

Thus, from (22.2), one has:

(22.4) $$\sigma_3 = \frac{d\bar{\sigma}_1}{[\bar{\sigma}_1\bar{\sigma}_2]}\bar{\sigma}_1 + \frac{d\bar{\sigma}_2}{[\bar{\sigma}_1\bar{\sigma}_2]}\bar{\sigma}_2 .$$

Thus, K depends upon the metric on F that is determined by $\bar{\sigma}_1$, $\bar{\sigma}_2$ (GAUSS's *theorema egregium*).

If one continually has $\sigma_3 = 0$ along a line C on F then, according to Lord KELVIN and LEVI-CIVITA, the vectors $\mathfrak{p}_1$ are *parallel* along C on F, and one speaks of the *displacement* of $\mathfrak{p}_1$ along C on F.

Under the rotation:

(22.5) $$\mathfrak{p}_1^* = + \mathfrak{p}_1 \cos\varphi + \mathfrak{p}_2 \sin\varphi, \qquad \mathfrak{p}_2^* = - \mathfrak{p}_1 \sin\varphi + \mathfrak{p}_2 \cos\varphi, \qquad \mathfrak{p}_3^* = \mathfrak{p}_3,$$

our axis cross becomes:

(22.6) $$\sigma_3^* = \langle d\mathfrak{p}_1^*, \mathfrak{p}_2^* \rangle = \sigma_3 + d\varphi.$$

Thus, if the vectors $\mathfrak{p}_1$ are parallel along C on F then the vectors $\mathfrak{p}_1^*$ will also be that way for a fixed φ. Furthermore, for a simply-connected domain B on F and its unique surrounding boundary dB, one has the following integral formal [cf., (20.8)]:

(22.7) $$\int_B [\sigma_1\sigma_2] = \int_B K[\bar{\sigma}_1\bar{\sigma}_2] = -\int_B d\sigma_3 = -\oint_{dB} \sigma_3 .$$

If one takes $\mathfrak{p}_1^*$ to be tangent to dB then one will have:

(22.8) $$\int_B K[\bar{\sigma}_1\bar{\sigma}_2] = -\int_{dB} (\sigma_3^* - d\varphi) = + 2\pi - \oint_{dB} \sigma_3 .$$

The integral:

(22.9) $$\int_{dB} \sigma_3^* = \int g\bar{\sigma}, \qquad g = \frac{\sigma_3^*}{\bar{\sigma}}, \qquad \bar{\sigma}_2 = \bar{\sigma}_1^2 + \bar{\sigma}_2^2$$

of the geodetic curvature yields the total geodetic curvature of the boundary dB (g is the geodetic curvature and $\bar{\sigma}$ is the element of arc length of dB).

We thus obtain the *formula of* GAUSS *and* BONNET (1848):

(22.10) $$\int_B K[\bar{\sigma}_1\bar{\sigma}_2] + \int_{dB} g\bar{\sigma} = 2\pi.$$

§ 23. Surface theory in E_3

We now take the special case of § 21 in which the surface F_0 that is described by $\mathfrak{Q}_0$ intersects the lines $[\mathfrak{Q}_0, \mathfrak{Q}_0]$ at right angles, such that the tangent plane of F_0 at $\mathfrak{Q}_0$ goes through $\mathfrak{Q}_1$, $\mathfrak{Q}_2$. One then has:

(23.1) $$\omega_{03} = 0.$$

We abbreviate:

(23.2) $$\begin{aligned} &\omega_{01} = \alpha_1, \quad \omega_{02} = \alpha_2, \quad \omega_{03} = \alpha_3 = 0; \\ &\omega_{23} = \beta_1, \quad \omega_{31} = \beta_2, \quad \omega_{12} = \beta_3, \end{aligned}$$

and thus find the differential equations:

(23.3) $$\begin{array}{lccccc} d\mathfrak{Q}_0 = & * & +\mathfrak{Q}_1\alpha_1 & +\mathfrak{Q}_2\alpha_2 & * & , \\ d\mathfrak{Q}_1 = & -\mathfrak{Q}_0\alpha_1 & * & +\mathfrak{Q}_2\beta_3 & -\mathfrak{Q}_3\beta_2 & , \\ d\mathfrak{Q}_2 = & -\mathfrak{Q}_0\alpha_2 & -\mathfrak{Q}_1\beta_3 & * & +\mathfrak{Q}_3\beta_1 & , \\ d\mathfrak{Q}_3 = & * & +\mathfrak{Q}_1\beta_2 & -\mathfrak{Q}_2\beta_1 & * & . \end{array}$$

For the two image spheres, we correspondingly set:

(23.4) $$\begin{array}{lccccclcccc} d\mathfrak{p}_1 = & * & +\mathfrak{p}_2\sigma_3 & -\mathfrak{p}_3\sigma_2, & & & d\mathfrak{p}_1' = & * & +\mathfrak{p}_2'\sigma_3' & -\mathfrak{p}_3'\sigma_2', \\ d\mathfrak{p}_2 = & -\mathfrak{p}_1\sigma_3 & * & +\mathfrak{p}_3\sigma_1, & & & d\mathfrak{p}_2' = & -\mathfrak{p}_1'\sigma_3' & * & +\mathfrak{p}_3'\sigma_1', \\ d\mathfrak{p}_3 = & +\mathfrak{p}_1\sigma_2 & -\mathfrak{p}_2\sigma_1 & * \;, & & & d\mathfrak{p}_3' = & +\mathfrak{p}_1'\sigma_2' & -\mathfrak{p}_2'\sigma_1' & * \;. \end{array}$$

The integrability conditions follow from (23.3):

(23.5) $$\begin{array}{lll} d\beta_1 = [\beta_3\,\beta_2], & d\beta_2 = [\beta_1\,\beta_3], & d\beta_3 = [\beta_2\,\beta_1] + [\alpha_2\,\alpha_1], \\ d\alpha_1 = [\beta_3\,\alpha_2], & \omega_{31} = \beta_2, & 0 = [\beta_2\,\beta_1] + [\alpha_2\,\alpha_1]. \end{array}$$

The following relations exist between our Pfaffian forms:

(23.6) $$\begin{aligned} &\beta_j - \alpha_j = \sigma_j, \quad \sigma_j' + \sigma_j = 2\beta_j, \\ &\beta_j + \alpha_j = \sigma_j', \quad \sigma_j' - \sigma_j = 2\alpha_j. \end{aligned}$$

In particular, one has:

(23.7) $$\sigma_3 = \sigma_3' = \beta_3 .$$

We have the integrability conditions for the σ:

(23.8) $$\begin{array}{lll} d\sigma_1 = [\sigma_3\,\sigma_2], & d\sigma_2 = [\sigma_1\,\sigma_3], & d\sigma_3 = [\sigma_2\,\sigma_1], \\ d\sigma_1' = [\sigma_3'\,\sigma_2'], & d\sigma_2' = [\sigma_1'\,\sigma_3'], & d\sigma_3' = [\sigma_2'\,\sigma_1']. \end{array}$$

It follows from (23.7), (23.8) that:

(23.9) $$\Omega = [\sigma_1\ \sigma_2] = [\sigma_1'\sigma_2'],$$

in agreement with (21.8). Conversely, for a congruence ($\mathfrak{p}_3$, $\mathfrak{p}_3'$) with *surface-preserving spherical images:*

(23.10) $$\Omega_3 = \Omega_3' = \Omega$$

there is a family of parallel surfaces that are orthogonal to it.

Between the surface elements Φ_0, Φ_3 of the mutually polar surfaces F_0, F_3 that are described by the points $\mathfrak{Q}_0$, $\mathfrak{Q}_3$, namely:

(23.11) $$\Phi_0 = [\alpha_1\ \alpha_2], \qquad \Phi_3 = [\beta_1\ \beta_2],$$

and the common surface element Ω in the spherical image, there exists the relation:

(23.12) $$\Phi_0 + \Phi_3 = \Omega.$$

For the Gaussian curvature K_0 of the arc length elements of F_0:

(23.13) $$ds_0^2 = \alpha_1^2 + \alpha_2^2,$$

we find, from (22.3), that:

(23.14) $$K_0 = -\frac{1}{\Phi_0} d\left\{\frac{d\alpha_1}{\Phi_0}\alpha_1 + \frac{d\alpha_2}{\Phi_0}\alpha_2\right\} = -\frac{d\beta_3}{\Phi_0} = \frac{\Phi_0 + \Phi_3}{\Phi_0} = \frac{\Omega}{\Phi_0}.$$

Likewise, for the arc length element of F_3:

(23.15) $$ds_3^2 = \beta_1^2 + \beta_2^2$$

one gets the curvature:

(23.16) $$K_3 = -\frac{1}{\Phi_3} d\left\{\frac{d\beta_1}{\Phi_3}\beta_1 + \frac{d\beta_2}{\Phi_3}\beta_2\right\} = -\frac{d\beta_3}{\Phi_3} = \frac{\Phi_0 + \Phi_3}{\Phi_3} = \frac{\Omega}{\Phi_3}.$$

§ 24. Curvature lines. Osculating lines.

Let $\mathfrak{P}$ be a point of the surface normals $\{\mathfrak{Q}_0, \mathfrak{Q}_3\}$ of our surface F_0 that is described by $\mathfrak{Q}_0$:

(24.1) $$\mathfrak{P} = \mathfrak{Q}_0 \cos\varphi + \mathfrak{Q}_3 \sin\varphi.$$

From (23.3), it then follows that:

(24.2)
$$d\mathfrak{P} = (-\mathfrak{Q}_0 \sin\varphi + \mathfrak{Q}_3 \cos\varphi) d\varphi + \mathfrak{Q}_1(\alpha_1 \cos\varphi + \beta_2 \sin\varphi) + \mathfrak{Q}_2(\alpha_2 \cos\varphi - \beta_1 \sin\varphi).$$

Should $d\mathfrak{P}$ again belong to the surface normals then one would have:

(24.3)
$$\alpha_1 \cos\varphi + \beta_2 \sin\varphi = 0, \qquad \alpha_2 \cos\varphi - \beta_1 \sin\varphi = 0.$$

The defining equation of the *curvature lines* follows from this:

(24.4)
$$\alpha_1\beta_1 + \alpha_2\beta_2 = 0,$$

or, from (23.6):

(24.5)
$$\sigma_1^2 + \sigma_2^2 = \sigma_1'^2 + \sigma_2'^2.$$

Thus, the curvature lines of F_0 have the characteristic property that the spherical images are related to each other in a distance-preserving way. The alternating product of the formulas (24.3) gives:

(24.6)
$$[\alpha_1\alpha_2] \cos^2\varphi - ([\alpha_1\beta_1] + [\alpha_2\beta_2]) \cos\varphi \sin\varphi + [\beta_1\beta_2] \sin^2\varphi = 0.$$

For the roots of these equations, one immediately finds:

(24.7)
$$\tan\varphi_1 \cdot \tan\varphi_2 = \frac{[\alpha_1\,\alpha_2]}{[\beta_1\,\beta_2]}, \qquad \tan\varphi_1 + \tan\varphi_2 = \frac{[\alpha_1\,\beta_1] + [\alpha_2\,\beta_2]}{[\beta_1\,\beta_2]},$$

or, when we introduce the surface elements:

(24.8)
$$\Phi_0 = [\alpha_1\alpha_2], \qquad 2\Psi = [\alpha_1\beta_1] + [\alpha_2\beta_2], \qquad \Phi_3 = [\beta_1\beta_2],$$

we find:

(24.9)
$$\tan\varphi_1 \cdot \tan\varphi_2 = \frac{\Phi_0}{\Phi_3}, \qquad \tan\varphi_1 + \tan\varphi_2 = 2\,\frac{\Psi}{\Phi_3}.$$

We now go on to the determination of the *osculating lines* (asymptotic lines) on our surface F_0. By a differentiation δ, it then follows from (23.3) that:

(24.10)
$$\delta d\mathfrak{Q}_0 = \alpha_1\,\delta\mathfrak{Q}_1 + \alpha_2\,\delta\mathfrak{Q}_2 + \ldots = -\,\mathfrak{Q}_3\,(\alpha_1\beta_2 - \alpha_2\beta_1) + \ldots,$$

in which the dots mean only terms in $\mathfrak{Q}_0$, $\mathfrak{Q}_1$, $\mathfrak{Q}_2$. Should $\delta d\mathfrak{Q}_0$ then lie in the tangent plane then one would have:

(24.11)
$$\alpha_1\beta_2 - \alpha_2\beta_1 = 0.$$

The osculating lines on F_0 are characterized by either this or by:

(24.12) $$\sigma_1\sigma_3' - \sigma_2\sigma_1' = 0.$$

The curvature lines and osculating lines are preserved under the transition from F_0 to the polar surface F_3 .

If we choose the tangents $\{\mathfrak{Q}_0\ \mathfrak{Q}_1\}$ and $\{\mathfrak{Q}_0\ \mathfrak{Q}_2\}$ of F_0 to be tangents to the curvature lines then we get:

(24.13) $$[\alpha_1\,\beta_2] = [\alpha_2\,\beta_1] = 0.$$

§ 25. Surfaces of zero curvature

If the curvature K_0 vanishes identically on F_0 then, from (23.14), one would also have:

(25.1) $$\Omega = \Phi_0 + \Phi_3 = 0,$$

and furthermore, from (23.14), (23.7):

(25.2) $$d\beta_3 = d\sigma_3 = d\sigma_3' = 0.$$

If we fix $\mathfrak{Q}_0$, $\mathfrak{Q}_3$ and rotate $\mathfrak{Q}_1$, $\mathfrak{Q}_2$:

(25.3) $$\mathfrak{Q}_1^* = \mathfrak{Q}_1 \cos \psi + \mathfrak{Q}_2 \sin \psi, \quad \mathfrak{Q}_2^* = -\,\mathfrak{Q}_1 \sin + \mathfrak{Q}_2 \cos \psi$$

then we get:

(25.4) $$\beta_3^* = \beta_3 + d\psi.$$

Due to (25.2), (25.4), we can choose ψ in such a way that:

(25.5) $$\beta_3 = \sigma_3 = \sigma_3' = 0.$$

The differential equations (23.3), (23.4) then simplify to:

(25.6) $$\begin{array}{rcccccc} d\mathfrak{Q}_0 &=& * & +\mathfrak{Q}_1\alpha_1 & +\mathfrak{Q}_2\alpha_2 & * & , \\ d\mathfrak{Q}_1 &=& -\mathfrak{Q}_0\alpha_1 & * & * & -\mathfrak{Q}_3\beta_2 & , \\ d\mathfrak{Q}_2 &=& -\mathfrak{Q}_0\alpha_2 & * & * & +\mathfrak{Q}_3\beta_1 & , \\ d\mathfrak{Q}_3 &=& * & +\mathfrak{Q}_1\beta_2 & -\mathfrak{Q}_2\beta_1 & * & , \end{array}$$

and

(25.7) $$\begin{array}{rccccc} d\mathfrak{p}_1 &=& * & * & -\mathfrak{p}_2\sigma_2, \\ d\mathfrak{p}_2 &=& * & * & +\mathfrak{p}_3\sigma_1, \\ d\mathfrak{p}_3 &=& \mathfrak{p}_1\sigma_2 & -\mathfrak{p}_2\sigma_1 & * & , \end{array} \qquad \begin{array}{rccccc} d\mathfrak{p}_1' &=& * & * & -\mathfrak{p}_2'\sigma_2', \\ d\mathfrak{p}_2' &=& * & * & +\mathfrak{p}_3'\sigma_1', \\ d\mathfrak{p}_3' &=& \mathfrak{p}_1'\sigma_2' & -\mathfrak{p}_2'\sigma_1 & * & . \end{array}$$

These lead to the integrability conditions:

$$d\beta_1 = d\beta_2 = 0, \quad [\beta_2\ \beta_1] + [\alpha_1\ \alpha_1] = 0, \qquad (25.8)$$
$$d\alpha_1 = d\alpha_2 = 0, \quad [\beta_2\ \alpha_1] + [\alpha_2\ \beta_1] = 0,$$

with the relations:

$$\beta_1 - \alpha_1 = \sigma_1, \quad \beta_1 + \alpha_1 = \sigma_1', \qquad (25.9)$$
$$\beta_2 - \alpha_2 = \sigma_2, \quad \beta_1 + \alpha_1 = \sigma_1'.$$

In this case, not only are the lines $\{\mathfrak{Q}_0\ \mathfrak{Q}_3\}$ surface normals to the mutually polar surfaces F_0, F_3, but also, from (25.6), the surfaces F_1, F_2 that are described by $\mathfrak{Q}_1$, $\mathfrak{Q}_2$ are mutually polar with the normals $\{\mathfrak{Q}_1\ \mathfrak{Q}_2\}$.

From (25.8), (25.9), it follows that:

$$d\sigma_1 = d\sigma_2 = 0, \qquad [\sigma_1\ \sigma_2] = 0, \qquad (25.10)$$

and thus the $\mathfrak{p}_j$ depend upon just *one* parameter u, and likewise, the $\mathfrak{p}_j'$ depend upon only one parameter v.

One can see from (25.8) that the curves $(\mathfrak{p}_1)$, $(\mathfrak{p}_2)$ are *carrier lines* (= tractrices) of the curve $(\mathfrak{p}_3)$; i.e., the great circles that contact $(\mathfrak{p}_1)$, $(\mathfrak{p}_2)$, and always run through the corresponding point $\mathfrak{p}_3$; corresponding statements are true for the sphere K'.

If we set:

$$\mathfrak{p}_j(u) = \tilde{\mathfrak{R}}(u)\,\mathfrak{p}_j(0)\,\mathfrak{R}(u), \qquad \mathfrak{p}_j'(v) = \tilde{\mathfrak{R}}'(v)\mathfrak{p}_j'(0)\mathfrak{R}'(v) \qquad (25.11)$$

then we get:

$$\mathfrak{Q}_j(u, v) = \tilde{\mathfrak{R}}(u)\,\mathfrak{Q}_j(0, 0)\,\mathfrak{R}'(v). \qquad (25.12)$$

One then sees that for $j = 0$ the u-curves, and likewise the v-curves are congruent to each other on the surface F_0. The surface F_0 is the is the elliptic analogue in E_3 of a translation surface in Euclidian R_3 .

Such surfaces F_0 in E_3 with $K_0 = 0$ in were first considered by L. BIANCHI (1856-1928) (cf., L. BIANCHI, Opera VIII, Roma, 1958, pp. 256-301).

§ 26. Surface elements in the path surface

We consider a surface-constrained rotation process:

$$\mathfrak{x}(u, v) = \tilde{\mathfrak{Q}}(u,v)\,\mathfrak{x}\,\mathfrak{Q}(u, v). \qquad (26.1)$$

From (26.1), it follows by differentiating (23.3), as in (10.4), that:

$$d\mathfrak{x} = (\mathfrak{x}\ \mathfrak{p}_1 - \mathfrak{p}_1\ \mathfrak{x})\ \alpha_1 + (\mathfrak{x}\ \mathfrak{p}_2 - \mathfrak{p}_2\ \mathfrak{x})\ \alpha_2\ , \qquad (26.2)$$

or:

(26.3) $$d\mathfrak{x} = 2\mathfrak{x} \times (\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2).$$

One arrives at the vectorial surface element of the curve surface from this by constructing the "alternating product" of $2(\mathfrak{x} \times \mathfrak{p}_1)\alpha_1$ with $2(\mathfrak{x} \times \mathfrak{p}_2)\alpha_2$:

(26.4) $$\mathfrak{f} = 4\,[\alpha_1\ \alpha_2]\,(\mathfrak{x} \times \mathfrak{p}_1) \times (\mathfrak{x} \times \mathfrak{p}_2) = 4\,[\alpha_1\ \alpha_2]\,[\mathfrak{x}\ \mathfrak{p}_1\ \mathfrak{p}_2]\,\mathfrak{x},$$

or:

(26.5) $$\mathfrak{f} = 4[\alpha_1\ \alpha_2] <\mathfrak{x}\ \mathfrak{p}_3> \mathfrak{x} = 4\,\Phi_0 <\mathfrak{x}\ \mathfrak{p}_3> \mathfrak{x}\ .$$

The geometric meaning of the vectors $\mathfrak{p}_3$, $\mathfrak{p}_3'$, and also that of Φ_0 is included in this.

To each given surface-preserving map $\mathfrak{p}_3 \to \mathfrak{p}_3'$ of our image spheres $K \to K'$ there exists a one-parameter family of "parallel" associated surface-constrained rotation processes that correspond to the associated parallel surfaces in E_3 . If such a process is known then one obtains the other one from the fact that one rotates each axis-cross of the first one around the associated $\mathfrak{p}_3$ by a fixed angle.

One constructs the rotation process that corresponds to the surfaces F_0 in E_3 with $K_0 = 0$ that were studied in § 25 as follows: A cone of the moving system is constrained to contact a cone of the rest system (both of their vertices are at O) continually along a generator. More generally, one examines the rotation processes:

(26.6) $$\mathfrak{x}(u, v) = \tilde{\mathfrak{R}}(u)\,\mathfrak{x}'\,\mathfrak{R}(v).$$

With this, one confronts the difficult problem of the rotation processes that can be represented in two essentially different ways in the form (26.6), and thus, the translation surfaces in E_3 with multiple generators. In the context of quaternions, this question comes down to the general solution of the equation:

$$\mathfrak{A}_1(u_1)\ \mathfrak{A}_2(u_2)\ \mathfrak{A}_3(u_3)\ \mathfrak{A}_4(u_4) = 1; \qquad \mathfrak{A}_j\tilde{\mathfrak{A}}_j = 1.$$

The corresponding question in Euclidian R_3 (or better, in affine R_3) was solved by S. LIE (1842-1809), 1882, H. POINCARÉ (1854-1912), 1901, W. WIRTINGER (1865-1845), 1938 by resorting to ABEL's theorem [1]).

We would like derive the guiding conditions for a point $\mathfrak{x}$ with the canonical coordinates x_j :

(26.7) $$\mathfrak{x} = x_1\,\mathfrak{p}_1 + x_2\,\mathfrak{p}_2 + x_3\,\mathfrak{p}_3$$

that guarantee that $\mathfrak{x}$ is fixed in the moving system. Differentiation of (26.8) by means of (23.4) gives:

[1]) Cf., also W. BLASCHKE and G. BOL, *Geometrie der Gewebe*, Berlin, 1938, pp. 240.

(26.8)
$$d\mathfrak{x} = \mathfrak{p}_1 dx_1 + \mathfrak{p}_2 dx_2 + \mathfrak{p}_3 dx_3 + x_1(\mathfrak{p}_2\sigma_3 - \mathfrak{p}_3\sigma_2) + x_2(\mathfrak{p}_3\sigma_1 - \mathfrak{p}_1\sigma_3) + x_3(\mathfrak{p}_1\sigma_2 - \mathfrak{p}_2\sigma_1),$$

and from (26.3), (26.7), one gets:

(26.9)
$$d\mathfrak{x} = -2\mathfrak{p}_1\, x_3\, \alpha_2 + 2\mathfrak{p}_3\, x_3\, \alpha_1 + 2\mathfrak{p}_3\,(x_1\alpha_2 - x_2\alpha_1).$$

The comparison of (26.8), (26.9) yields the desired *guiding condition:*

(26.10)
$$\begin{aligned} dx_1 &= \quad * \quad + x_2\sigma_3' - x_3\sigma_2', \\ dx_2 &= -x_1\sigma_3' \quad * \quad + x_3\sigma_1', \\ dx_3 &= +x_1\sigma_2' - x_2\sigma_1' \quad * \quad . \end{aligned}$$

Likewise, for the point:

(26.11)
$$\mathfrak{x}' = x_1'\,\mathfrak{p}_1' + x_2'\,\mathfrak{p}_2' + x_3'\,\mathfrak{p}_3'$$

one gets the *rest conditions:*

(26.12)
$$\begin{aligned} dx_1' &= \quad * \quad + x_2'\sigma_3 - x_3'\sigma_2, \\ dx_2' &= -x_1'\sigma_3 \quad * \quad + x_3'\sigma_1, \\ dx_3' &= +x_1'\sigma_2 - x_2'\sigma_1 \quad * \quad . \end{aligned}$$

Perhaps from the guiding conditions (26.10) or from:

(26.13)
$$d\mathfrak{x} = (x_2\sigma_3' - x_3\sigma_2')\,\mathfrak{p}_1 + (x_3\sigma_1' - x_1\sigma_3')\,\mathfrak{p}_2 + (x_1\sigma_2' - x_2\sigma_1')\,\mathfrak{p}_3\,,$$

it follows that the vectorial surface element of the path surface that is described by $\mathfrak{x}$ is:

(26.14)
$$\tfrac{1}{2}[d\mathfrak{x} \times d\mathfrak{x}] = \{x_1[\sigma_2'\sigma_3'] + x_2[\sigma_3'\sigma_1'] + x_3[\sigma_1'\sigma_2']\}\,\mathfrak{x}\,,$$

and thus, its scalar surface element is:

(26.15)
$$x_1[\sigma_2'\sigma_3'] + x_2[\sigma_3'\sigma_1'] + x_3[\sigma_1'\sigma_2'].$$

§ 27. On line congruences in E_3

We consider the *congruence* of lines $\{\mathfrak{Q}_0, \mathfrak{Q}_3\}$ that intersect a surface-constrained motion process in E_3 (§ 21). In order to ascertain the “focal point” on $\{\mathfrak{Q}_0, \mathfrak{Q}_3\}$, we take the point:

(27.1)
$$\mathfrak{P} = \mathfrak{Q}_0 \cos\varphi + \mathfrak{Q}_3 \sin\varphi\,,$$

and demand that, from (21.1):

(27.2)
$$d\mathfrak{P} = \mathfrak{Q}_1(\omega_{01}\cos\varphi + \omega_{21}\sin\varphi) + \mathfrak{Q}_2(\omega_{02}\cos\varphi - \omega_{23}\sin\varphi)$$
$$+(-\mathfrak{Q}_0\sin\varphi + \mathfrak{Q}_3\cos\varphi)(d\varphi + \omega_{03})$$

lies on our line. This yields the equations:

(27.3)
$$\omega_{01}\cos\varphi + \omega_{31}\sin\varphi = 0, \qquad \omega_{02}\cos\varphi - \omega_{23}\sin\varphi = 0.$$

Increasing φ yields the defining equation of the developable surfaces (*Torsen*) in our congruence:

(27.4)
$$\omega_{01}\,\omega_{23} + \omega_{02}\,\omega_{31} = 0.$$

The alternating product of the equations (27.3) yields the *focal point:*

(27.5)
$$[\omega_{01}\,\omega_{02}]\cos^2\varphi - \{[\omega_{01}\,\omega_{23}] + [\omega_{02}\,\omega_{31}]\}\cos\varphi\sin\varphi + [\omega_{23}\,\omega_{31}]\sin^2\varphi = 0.$$

The corresponding calculation for the congruence of the lines $\{\mathfrak{Q}_1, \mathfrak{Q}_2\}$ that are polar to $\{\mathfrak{Q}_0, \mathfrak{Q}_3\}$ yields:

(27.6)
$$\omega_{01}\cos\psi + \omega_{03}\sin\psi = 0, \qquad \omega_{31}\cos\psi - \omega_{23}\sin\psi = 0,$$

and from this, by preserving (27.4) for the focal point:

(27.7)
$$[\omega_{01}\,\omega_{31}]\cos^2\psi - \{[\omega_{01}\,\omega_{23}] + [\omega_{02}\,\omega_{31}]\}\cos\psi\sin\psi + [\omega_{23}\,\omega_{31}]\sin^2\psi = 0.$$

If one chooses the points $\mathfrak{Q}_0$, $\mathfrak{Q}_3$ in such a way that the focal points (27.5) are harmonically separated, and correspondingly for the points $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, then from (27.5), (27.7), one gets:

(27.8)
$$[\omega_{01}\,\omega_{31}] = 0, \qquad [\omega_{02}\,\omega_{31}] = 0,$$

and we have, in our case:

(27.9)
$$[\sigma_{23}\;\sigma'_{23}] = 0, \qquad [\sigma_{31}\;\sigma'_{31}] = 0.$$

One calls congruences whose spherical images $(\mathfrak{p}_3)$, $(\mathfrak{p}'_3)$ are conformally (= angle-preserving) related to each other *isotropic congruences.* Due to (21.4):

(27.10)
$$d\mathfrak{p}_3 = \sigma_{31}\,\mathfrak{p}_1 - \sigma_{23}\,\mathfrak{p}_2, \qquad d\mathfrak{p}'_3 = \sigma'_{31}\,\mathfrak{p}'_1 - \sigma'_{23}\,\mathfrak{p}'_2,$$

we have, in our case:

(27.11)
$$\sigma_{23}\sigma'_{31} - \sigma_{31}\sigma'_{23} = 0, \qquad \sigma'_{23} = c\,\sigma_{23}, \qquad \sigma'_{31} = c\,\sigma_{31},$$

and from this:

(27.12)
$$[\sigma_{23}\;\sigma'_{23}] = 0, \quad [\sigma_{23}\;\sigma'_{31}] = [\sigma'_{23}\,\sigma_{31}], \quad [\sigma_{31}\;\sigma'_{31}] = 0.$$

For the common perpendicular $(\mathfrak{p}_3, \mathfrak{p}'_3)$ of neighboring lines of the congruence, one has:

$$
\begin{aligned}
\mathfrak{p} &= f \cdot \mathfrak{p}_3 \times d\mathfrak{p}_3 = f \cdot (\mathfrak{p}_1 \sigma_{33} + \mathfrak{p}_2 \sigma_{31}), \\
\mathfrak{p}' &= f' \cdot \mathfrak{p}'_3 \times d\mathfrak{p}'_3 = f' \cdot (\mathfrak{p}'_1 \sigma'_{33} + \mathfrak{p}'_2 \sigma'_{31}).
\end{aligned} \tag{27.13}
$$

In our case, one then has:

$$
\mathfrak{p}' = \pm f \cdot (\mathfrak{p}'_1 \sigma_{23} + \mathfrak{p}'_2 \sigma_{31}) . \tag{27.14}
$$

The pencils of rays of the $\mathfrak{p}$ and $\mathfrak{p}'$ are thus congruences. From this, it follows: For isotropic congruences, the common perpendicular of a line of the congruence and its neighbor define two mutually polar pencils of rays. As in the Euclidian case, these isotropic congruences are closely clinked with the isotropic curves (imaginary curves with null length).

Another noteworthy case of congruences in elliptic space is the one in which the spherical images are curves:

$$
[\sigma_{23}\, \sigma_{31}] = [\sigma'_{23}\, \sigma'_{31}] = 0, \qquad \mathfrak{p}_3 = \mathfrak{p}_3(u), \qquad \mathfrak{p}'_3 = \mathfrak{p}'_3(v) . \tag{27.15}
$$

They are the normal congruences of translation surfaces.

Finally, one must observe the TSCHEBYSCHEFF *congruences*, for which the arc length elements of their spherical images may be brought into the following form:

$$
\begin{aligned}
\sigma_{23}^2 + \sigma_{31}^2 &= du^2 + 2du\, dv \cdot \cos\lambda + dv^2, \\
\sigma_{23}'^2 + \sigma_{31}'^2 &= du^2 + 2du\, dv \cdot \cos\lambda' + dv^2.
\end{aligned} \tag{27.16}
$$

The curve nets u, v = fixed on K, K' are then TSCHEBYSCHEFF nets.

On the situation that was treated in § 21-27, cf., W. BLASCHKE, "Sulle congruenze rettilinee nello spazio ellittico," Annali di Mat. (4) **48** (1959), 209-221.

Similar to what we did in § 17, one can also derive integral theorems for surface-constrained motion processes. Thus, the GAUSS-BONNET formula for the surface F_0 in E_3 can also be regarded kinematically.

CHAPTER FOUR

ALGEBRA OF SPATIAL KINEMATICS

§ 28. Dual line coordinates

Let $\underline{\mathfrak{a}}$ be an "axis" in Euclidian R_3, $\mathfrak{a}$, a unit vector on $\underline{\mathfrak{a}}$ (its *direction vector*), and let $\overline{\mathfrak{a}}$ be its moment vector about the origin O, namely:

(28.1) $$\overline{\mathfrak{a}} = \mathfrak{x} \times \mathfrak{a},$$

if $\mathfrak{r}$ means any point on $\underline{\mathfrak{a}}$ (more precisely, this means the vector from O to $\mathfrak{x}$). $\overline{\mathfrak{a}}$ is then independent of the choice of point $\mathfrak{x}$ on $\underline{\mathfrak{a}}$ since:

(28.2) $$(\mathfrak{x} + f\mathfrak{a}) \times \mathfrak{a} = \mathfrak{x} \times \mathfrak{a}.$$

Conversely: If an axis $\underline{\mathfrak{a}}$ is given by the two vectors $\mathfrak{a}$, $\overline{\mathfrak{a}}$ then (28.1) characterizes the point $\mathfrak{x}$ on $\underline{\mathfrak{a}}$.

The dependencies:

(28.3) $$\langle\mathfrak{a}\ \mathfrak{a}\rangle = 1, \qquad \langle\mathfrak{a}\ \overline{\mathfrak{a}}\rangle = 0$$

exist between the two vectors $\mathfrak{a}$, $\overline{\mathfrak{a}}$. One can combine them into a single one by the introduction of ε with:

(28.4) $$\varepsilon^2 = 0,$$

when one sets:

(28.5) $$\underline{\mathfrak{a}} = \mathfrak{a} + \varepsilon\,\overline{\mathfrak{a}}.$$

One then has, in fact:

(28.6) $$\langle\underline{\mathfrak{a}}\ \underline{\mathfrak{a}}\rangle = \langle\mathfrak{a}\ \mathfrak{a}\rangle + 2\varepsilon\langle\mathfrak{a}\ \overline{\mathfrak{a}}\rangle = 1.$$

One calls numbers of the form (a, $\overline{a}$; b, $\overline{b}$ real):

(28.7) $$\underline{a} = a + \varepsilon\,\overline{a}, \quad \underline{b} = b + \varepsilon\,\overline{b}$$

with

(28.8) $$\underline{a}\,\underline{b} = ab + \varepsilon(a\overline{b} + b\overline{a})$$

dual numbers. For calculations with them, one must observe that the division by "null parts" $\varepsilon\,\overline{a}$ is not allowed.

From (28.6), it emerges that: The axes $\underline{\mathfrak{a}}$ in R_3 may be mapped to the "dual points" on the unit sphere (28.6) in a one-to-one way.

For two axes $\underline{\mathfrak{a}}$, $\underline{\mathfrak{a}}'$, we have:

(28.9) $$<\underline{\mathfrak{a}}\ \underline{\mathfrak{a}}'> = <\mathfrak{a}\ \mathfrak{a}'> + \varepsilon\{<\bar{\mathfrak{a}}\ \mathfrak{a}'> + <\mathfrak{a}\ \bar{\mathfrak{a}}'>\}.$$

Let $\mathfrak{r}$, $\mathfrak{r}'$ be the base points of the common perpendicular to $\underline{\mathfrak{a}}$, $\underline{\mathfrak{a}}'$ on these axes, and let φ be the angle between them, while $\bar{\varphi}$ is their shortest distance. One then has:

(28.10) $$\begin{aligned} &<\mathfrak{a}\,\mathfrak{a}'> = \cos\varphi, \\ &<\bar{\mathfrak{a}}\,\mathfrak{a}'> + <\mathfrak{a}\,\bar{\mathfrak{a}}'> = -[\mathfrak{r}' - \mathfrak{r}, \mathfrak{a}, \mathfrak{a}'] = -\bar{\varphi}\sin\varphi. \end{aligned}$$

If we set:

(28.11) $$f(\varphi + \varepsilon\,\bar{\varphi}) = f(\varphi) + \varepsilon\bar{\varphi}\,f'(\varphi)$$

then we can combine equations (28.10) into one:

(28.12) $$<\underline{\mathfrak{a}}\ \underline{\mathfrak{a}}'> = \cos\underline{\varphi}, \qquad \underline{\varphi} = \varphi + \varepsilon\,\bar{\varphi}.$$

In particular:

(28.13) $$<\underline{\mathfrak{a}}\ \underline{\mathfrak{a}}'> = 0$$

means the perpendicular intersection of $\underline{\mathfrak{a}}$, $\underline{\mathfrak{a}}'$.

§ 29. Motions in line space

Let:

(29.1) $$\underline{\mathfrak{Q}} = \sum_0^3 e_j(q_j + \varepsilon\bar{q}_j) = \mathfrak{Q} + \varepsilon\bar{\mathfrak{Q}}$$

be a "dual quaternion." In it, one shall have $e_j\,\varepsilon = \varepsilon\,e_j$. We take $\underline{\mathfrak{Q}}$ to be normalized; i.e., we let:

(29.2) $$<\underline{\mathfrak{Q}}\ \underline{\mathfrak{Q}}> = <\mathfrak{Q}\ \mathfrak{Q}> + 2\varepsilon<\mathfrak{Q}\ \bar{\mathfrak{Q}}> = 1,$$

so:

(29.3) $$<\mathfrak{Q}\ \mathfrak{Q}> = 1, \quad <\mathfrak{Q}\ \bar{\mathfrak{Q}}> = 0.$$

Then:

(29.4) $$\underline{\mathfrak{a}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{a}}'\underline{\mathfrak{Q}}, \quad \underline{\mathfrak{Q}}\tilde{\underline{\mathfrak{Q}}} = 1,$$

in which the tilde means the sign change in e_1, e_2, e_3, represents a dual-orthogonal substitution of the dual unit vectors $\underline{\mathfrak{a}}$, and thus a dual rotation of the unit sphere. If the axes $\underline{\mathfrak{a}}$ are interpreted as being in line space R_3 then, due to the invariance of (28.9) this gives a motion of R_3 that is applied to its axes.

One might see that one obtains all of the motions of the continuous, six-parameter group G_6 of R_3 from (29.4) as follows: First, the transformations define a group, and then by composition of the motions $\underline{\mathfrak{Q}}_1$, $\underline{\mathfrak{Q}}_2$ one produces the motion $\underline{\mathfrak{Q}}_1\,\underline{\mathfrak{Q}}_2$. Furthermore, from § 2, G_6 includes the group G_3 of all rotations $\mathfrak{Q}$, $\tilde{\mathfrak{Q}}\mathfrak{Q} = 1$ around the origin. On the other hand, G_6 includes the group G_3' of all translations:

$$(29.5) \qquad \underline{\mathfrak{Q}} = 1 + \varepsilon\bar{\mathfrak{Q}}, \qquad \bar{\mathfrak{Q}} + \tilde{\bar{\mathfrak{Q}}} = 0, \qquad \bar{\mathfrak{Q}} = \mathfrak{q}.$$

From (29.4), (29.5), it follows that, in fact:

$$(29.6) \qquad \begin{aligned} &\underline{\mathfrak{a}} = (1-\varepsilon\mathfrak{q})\underline{\mathfrak{a}}'(1+\varepsilon\mathfrak{q}), \\ &\mathfrak{a} = \mathfrak{a}', \quad \bar{\mathfrak{a}} = \bar{\mathfrak{a}}' - \mathfrak{q}\mathfrak{a}' + \mathfrak{a}'\mathfrak{q} = \bar{\mathfrak{a}}' + 2(\mathfrak{a}' \times \mathfrak{q}). \end{aligned}$$

A translation is produced, in fact, for the points:

$$(29.7) \qquad \mathfrak{x} = \mathfrak{x}' + \mathfrak{v}, \qquad \mathfrak{x} \times \mathfrak{a} = (\mathfrak{x}' \times \mathfrak{a}) + (\mathfrak{v} \times \mathfrak{a}),$$

or for the axes:

$$(29.8) \qquad \mathfrak{a} = \mathfrak{a}', \qquad \bar{\mathfrak{a}} = \bar{\mathfrak{a}}' + (\mathfrak{v} \times \mathfrak{a}').$$

It then suffices to set $\mathfrak{v} = -2\mathfrak{q}$ in (29.6) in order to obtain the translation (29.7). The fact that the determinant of (29.4) is equal to + 1 might follow from continuity or the fact that it is true for the two groups G_3, G_3'.

A further proof employs formula (1.23) for the determinant:

$$(29.9) \qquad -4[\mathfrak{Q}_0\, \mathfrak{Q}_1\, \mathfrak{Q}_2\, \mathfrak{Q}_3] = \mathfrak{Q}_0\tilde{\mathfrak{Q}}_1\mathfrak{Q}_2\tilde{\mathfrak{Q}}_3 + \mathfrak{Q}_3\tilde{\mathfrak{Q}}_2\mathfrak{Q}_1\tilde{\mathfrak{Q}}_0 - \mathfrak{Q}_2\tilde{\mathfrak{Q}}_1\mathfrak{Q}_0\tilde{\mathfrak{Q}}_3 - \mathfrak{Q}_3\tilde{\mathfrak{Q}}_0\mathfrak{Q}_1\tilde{\mathfrak{Q}}_2 .$$

Bibliography. The use of dual numbers in geometry goes back to W. K. CLIFFORD (1845-1879). One can learn about this significant geometer in the book: W. K. CLIFFORD, *The common sense of the exact sciences*, New York, 1955. Further, see A. P. KOTJELNIKOW (1865-1944), *Die Schraubenrechnung und ihre Anwendungen auf Geometrie und Mechanik*, Kazan, 1895; J. PETERSEN (= HJELMSLEV) (1873-1950), *Géométrie des droites dans l'espace non euclidien*, Kopenhagen Verhandl. Akad., 1900, pp. 308-330; G. FUBINI, "Il parallelismo di Clifford negli spazi ellittici," Annali della Scuola Normale, Pisa **9** (1904), 74 pages (dissertation); E. STUDY, *Geometrie der Dynamen*, Leipzig, 1903; E. Study, "Ziele der analytischen Kinematik," Sitzungsber. Berlin. Math. Ges. **12** (1913), 36-60.

One finds a thorough discussion of quaternions, their history, generalizations, and geometric applications, along with the associated references, in H. ROTHE, "Systeme geometrischer Analyse," Enc. math. Wiss., art. III, sec. 11, Leipzig, 1921. Cf., also W. BLASCHKE, *Differentialgeometrie*, Bd. I, Berlin 1921 and later editions; R. v. MISES (1883-1953), "Motorrechnung, ein neues Hilfsmittel der Mechanik," Z. angew. Math. Mech. **4** (1924), 155-181; R. v. MISES, "Anwendungen der Motorrechnung," Z. angew. Math. Mech. **4** (1924), 193-213; E. A. WEISS (1900-1942), *Einführung in die Liniengeometrie und Kinematik*, Leipzig und Berlin, 1935; W. BLASCHKE, *Nicht-Euclidische Geometrie und Mechanik*, Leipzig und Berlin, 1942; W. BLASCHKE, *Analytische Geometrie*, 2nd ed., Basel/Stuttgart, 1954.

§ 30. An invariant for three axes

If we take three pair-wise skew axes:

(30.1)
$$\underline{\mathfrak{a}}_j = \mathfrak{a}_j + \varepsilon \overline{\mathfrak{a}}_j , \qquad < \underline{\mathfrak{a}}_j \, \underline{\mathfrak{a}}_j > = 1,$$
and define the determinant:

(30.2)
$$[\underline{\mathfrak{a}}_1 \, \underline{\mathfrak{a}}_2 \, \underline{\mathfrak{a}}_3] = [\mathfrak{a}_1 \; \mathfrak{a}_2 \; \mathfrak{a}_3] + \varepsilon\{[\overline{\mathfrak{a}}_1 \, \mathfrak{a}_2 \; \mathfrak{a}_3] + [\mathfrak{a}_1 \, \overline{\mathfrak{a}}_2 \, \mathfrak{a}_3] + [\mathfrak{a}_1 \; \mathfrak{a}_2 \; \overline{\mathfrak{a}}_3]\}$$

then this yields the following invariant under the motions of R_3:

(30.3)
$$J = \frac{[\overline{\mathfrak{a}}_1 \mathfrak{a}_2 \mathfrak{a}_3] + [\mathfrak{a}_1 \overline{\mathfrak{a}}_2 \mathfrak{a}_3] + [\mathfrak{a}_1 \mathfrak{a}_2 \overline{\mathfrak{a}}_3]}{[\mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{a}_3]}.$$

We would like to interpret this geometrically. If one substitutes, perhaps, $\overline{\mathfrak{a}}_1$ in J by way of:

(30.4)
$$\underline{\mathfrak{a}}_1^* = c_1 \underline{\mathfrak{a}}_1 + c_2 \underline{\mathfrak{a}}_2 + c_3 \underline{\mathfrak{a}}_3 ,$$

with real c_j , that satisfies the condition:

(30.5)
$$< \underline{\mathfrak{a}}_1^* \, \underline{\mathfrak{a}}_1^* > = 1$$

then J remains unchanged; however, $\underline{\mathfrak{a}}_1^*$ is any line of the ruled family of the quadric through $\underline{\mathfrak{a}}_1$, $\underline{\mathfrak{a}}_2$, $\underline{\mathfrak{a}}_3$. Thus, J depends upon only this ruled family. If we take the quadric in the form:

(30.6)
$$\sum_0^3 a_{jk} x_j x_k = 0 \qquad (x_0 = 1,\ a_{jk} = a_{kj})$$

in rectangular coordinates x_1, x_2, x_3 then we find that:

(30.7)
$$J = \frac{A}{A_0} (a_{11} + a_{22} + a_{33})$$
with

(30.8)
$$A = \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad A_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

If $\mathfrak{x}_j$ means a point on $\underline{\mathfrak{a}}_j$ then the formula also yields:

(30.9) $$J = \frac{<\mathfrak{a}_2\,\mathfrak{a}_3><\mathfrak{x}_2-\mathfrak{x}_3,\mathfrak{a}_1>+<\mathfrak{a}_3\,\mathfrak{a}_1><\mathfrak{x}_3-\mathfrak{x}_1,\mathfrak{a}_2>+<\mathfrak{a}_1\,\mathfrak{a}_2><\mathfrak{x}_1-\mathfrak{x}_2,\mathfrak{a}_3>}{[\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3]}.$$

Thus, if:

(30.10) $$J = 0,$$

in particular, then it must follow that:

(30.11) $$a_{11} + a_{22} + a_{33} = 0.$$

E. A. WEISS also concerned himself with the invariant J in the reference cited in § 29 with the use of the "complex symbolism" of line geometry that was introduced by R. WEITZENBOECK (1885-1955).

§ 31. The spatial hexangle with only right angles

Let $\underline{\mathfrak{a}}_j$ (j = 1, 2, 3) be pair-wise skew lines and let $\underline{\mathfrak{b}}_j$ be the common perpendicular from $\underline{\mathfrak{a}}_{j-1}$ to $\underline{\mathfrak{a}}_{j+1}$ (j mod 3). The six lines $\underline{\mathfrak{a}}_2$, $\underline{\mathfrak{b}}_1$, $\underline{\mathfrak{a}}_3$, $\underline{\mathfrak{b}}_2$, $\underline{\mathfrak{a}}_1$, $\underline{\mathfrak{b}}_3$ define a spatial hexangle with nothing but right angles. Up to a dual scalar factor, one then has:

(31.1) $$\underline{\mathfrak{b}}_1 = \underline{\mathfrak{a}}_2 \times \underline{\mathfrak{a}}_3, \qquad \underline{\mathfrak{b}}_2 = \underline{\mathfrak{a}}_3 \times \underline{\mathfrak{a}}_1, \qquad \underline{\mathfrak{b}}_3 = \underline{\mathfrak{a}}_1 \times \underline{\mathfrak{a}}_2 .$$

Let $\underline{\mathfrak{c}}_j$ be the common perpendicular of the opposite sides $\underline{\mathfrak{a}}_j, \underline{\mathfrak{b}}_j$. One again has, up to a dual-scalar factor:

(31.2) $$\underline{\mathfrak{c}}_1 = \underline{\mathfrak{a}}_1 \times \underline{\mathfrak{b}}_1 = \underline{\mathfrak{a}}_1 \times (\underline{\mathfrak{a}}_2 \times \underline{\mathfrak{a}}_3) = <\underline{\mathfrak{a}}_3\,\underline{\mathfrak{a}}_1>\underline{\mathfrak{a}}_2 - <\underline{\mathfrak{a}}_1\,\underline{\mathfrak{a}}_2>\underline{\mathfrak{a}}_3,$$

and cyclic permutations of 1, 2, 3. It then follows that:

(31.3) $$\underline{\mathfrak{c}}_1 + \underline{\mathfrak{c}}_2 + \underline{\mathfrak{c}}_3 = 0,$$

and that means: *The three common perpendiculars* $\underline{\mathfrak{c}}_j$ *of the opposite sides* $\underline{\mathfrak{a}}_j, \underline{\mathfrak{b}}_j$ *again have a common perpendicular* $\underline{\mathfrak{d}}$.

This FIGURE of PETERSEN (= HJELMSLEV) and MORLEY (1898) is connected, on the one hand, with the fact that the altitudes in a spherical triangle intersect, and on the other hand, with the FIGURE of DESARGUES in projective geometry. Our figure that is composed of the ten lines $\underline{\mathfrak{a}}_j, \underline{\mathfrak{b}}_j$, $\underline{\mathfrak{c}}_j$, $\underline{\mathfrak{d}}$ has, in fact, the symmetry property that each of its lines will be met at right angles by three other ones.

If, for a line:

(31.4) $$\underline{\mathfrak{a}} = \underline{a}_1 e_1 + \underline{a}_2 e_2 + \underline{a}_3 e_3$$

one introduces the ratios of the duals $\underline{\mathfrak{a}}_j$ as its homogeneous coordinates, and one considers the group G_{16} of line transformations:

(31.5) $$\underline{a}_j^* = \sum_1^3 \underline{c}_{jk}\underline{a}_k$$

with dual $\underline{c}_{jk}$ and determinant 1 then this defines a counterpart to plane projective geometry that E. STUDY called "dual-projective." One is thus compelled to introduce "ideal lines." for which the coordinates:

(31.6) $$\underline{a}_j = a_j + \varepsilon\overline{a}_j$$

have all a_j equal to zero.

If one carries over the figure of DESARGUES to dual-projective geometry then one obtains the aforementioned figure of the hexangle. In a similar way, one may carry over the figure of PAPPOS to line space. It consists of two nonets of lines on different "sheets" in such a way that every line of the one sheet meets precisely three other ones at right angles.

One obtains a hexangle with only right angles, for example, from the twelve edges of a cube when one omits the six edges that meet at two opposite corners.

§ 32. The cylindroid

In this and the following section, we give a brief overview of the simplest figures in dual-projective geometry, which one calls *chains* (Ketten). K. G. CH. v. STAUDT (1798-1867), in his investigations into complex projective geometry, considered chains of such points in the points of a line; i.e., the totality of all of them that determine a real double ratio with three given ones.

Correspondingly, among the lines that cut a given one at right angles, one can determine chains of such lines when one measures the double ratio by means of the double ratio of the values of the tangent of the half "dual angle" with a fixed line of that type $\underline{\mathfrak{g}}'$. Thus, this dual angle will be defined by:

(32.1) $$\underline{\varphi} = \varphi + \varepsilon\overline{\varphi},$$

in which φ means the angle and $\overline{\varphi}$ means the shortest distance from $\underline{\mathfrak{g}}'$ to $\underline{\mathfrak{g}}$. We thus have:

(32.2) $$\tan\frac{\varphi+\varepsilon\overline{\varphi}}{2} = \tan\frac{\varphi}{2} + \varepsilon\left(1+\tan^2\frac{\varphi}{2}\right)\frac{\overline{\varphi}}{2}.$$

Obviously, such a chain of lines is determined uniquely by three of its elements, no two of which run parallel.

In projective geometry, four points of a line have the same double ratio as their connecting line with a point that does not lie on the initial line. In our dual-projective geometry, this corresponds to the statement: Let $\mathfrak{g}_j$ be four lines that meet a given one $\mathfrak{a}$ at right angles. The common perpendiculars $\underline{\mathfrak{g}}'_j$ of the $\mathfrak{g}_j$ with a further line $\underline{\mathfrak{a}}'$ then have the same double ratio. If the lines $\underline{\mathfrak{g}}$ that meet $\underline{\mathfrak{a}}$ at right angles define a chain then the common perpendicular $\underline{\mathfrak{g}}'$ of $\underline{\mathfrak{g}}$ with $\underline{\mathfrak{a}}'$ also define a chain (W. R. Ball). In particular, the lines of a pencil define a chain, since all of their double ratios turn out to be real. Thus, the common perpendiculars $\underline{\mathfrak{g}}'$ of the lines $\underline{\mathfrak{g}}$ of a pencil define a chain. One easily shows that one can generate all chains in this way. Following A. CAYLEY (1821-1895), one calls the ruled surface that is swept out by the lines of a chain a *cylindroid*. With a suitable choice of rectangular coordinates x, y, z, its equation reads:

$$(x^2 + y^2)\, z = 2axy, \tag{32.3}$$

with the single invariant of the motion a. The z-axis is a double line of the surface. Its generators lie in the space $z^2 \leq a^2$. The cylindroid was introduced by W. R. HAMILTON (1805-1865) in 1830 and was investigated in 1868 by J. PLÜCKER (1801-1868), and was treated thoroughly by W. R. BALL (1840-1913); cf., W. R. Ball, *Theory of Screws*, London, 1902 and E. STUDY (1862-1930), *Geometrie der Dynamen*, Leipzig, 1903.

If a circular cylinder Z rolls without slipping inside of another one Z' of double radius then every plane section of Z describes a cylindroid, as long as its interior does not lie on Z'. For $a = 0$, the cylindroid degenerates into a pencil of lines.

§ 33. Two-parameter chains

We now turn our attention to the two-dimensional case. We take three dual vectors (lines):

$$\underline{\mathfrak{g}}_j = \mathfrak{g}_j + \varepsilon\bar{\mathfrak{g}}_j \qquad \bar{\mathfrak{g}}_j \neq 0 \;(j = 1, 2, 3), \tag{33.1}$$

and we consider the *two-parameter chain* of lines:

$$\underline{\mathfrak{g}} = c_1\,\underline{\mathfrak{g}}_1 + c_2\,\underline{\mathfrak{g}}_2 + c_3\,\underline{\mathfrak{g}}_3 \tag{33.2}$$

with real c_j . We set:

$$\mathfrak{g}_j = \sum_1^3 g_{jk} e_k\,, \qquad \bar{\mathfrak{g}}_j = \sum_1^3 \bar{g}_{jk} e_k \tag{33.3}$$

and demand that:

$$\text{Det}\; g_{jk} \neq 0. \tag{33.4}$$

We can then eliminate the c_j from the equations:

$$h_k = \sum c_j g_{jk}\,, \qquad \bar{h}_k = \sum c_j \bar{g}_{jk} \tag{33.5}$$

and find that:

$$\bar{h}_j = \sum a_{jk} h_k \tag{33.6}$$

with real c_j for our chain. If:

$$\bar{h}'_j = \sum b_{jk} h'_k\,, \qquad \text{Det}\; b_{jk} \neq 0 \tag{33.7}$$

is a second chain then one has:

(33.8) $$\sum (h_j + \varepsilon \bar{h}_j)(h'_j + \varepsilon \bar{h}'_j) = \sum h_j h'_j + \varepsilon (a_{jk} + b_{jk}) h_j h'_k .$$

If we then take:

(33.9) $$a_{jk} + b_{jk} = c\ \delta_{jk}$$

then we see: Between our two chains, one has the reciprocal relationship that to any line of the one, a one-parameter chain of the second comes about that meets the first line at right angles.

PLÜCKER, BALL, and STUDY, in his "Geometrie der Dynamen," have examined such chains (also three-parameter ones). Their classification under the group G_{18} of dual-projective geometry or under the group G_6 of Euclidian motions raises no difficulties, although it especially misleads one into excessive terminology by the introduction of ideal and imaginary elements (BALL, STUDY).

§ 34. Relationship with projective line geometry

If we replace the inhomogeneous rectangular coordinates x_1, x_2, x_3 with homogeneous ones, where we write $x_j : x_0$ in place of x_j , then the "PLÜCKER line coordinates" (§ 8) for the line connecting two points x_j , y_j are written in the form:

(34.1) $$g_{jk} = x_j\, y_k - x_k\, y_j .$$

For the determinant of four points x, y; x', y', one finds:

(34.2) $$D(g, g') = g_{01} g'_{23} + g_{23} g'_{01} + \cdots ,$$

in which the dots mean cyclic permutations of 1, 2, 3.

(34.3) $$D(g, g') = 0$$

is then the condition for the intersection of the lines g, g', and:

(34.5) $$D(g, g) = 2(g_{01}\, g_{23} + \ldots) = 0$$

gives the dependency of the line coordinates g_{jk} for a line g. By comparison, if one takes the g'_{jk} to be arbitrary in (34.2), but possibly skew-symmetric:

(34.6) $$g'_{jk} + g'_{kj} = 0.$$

then (34.3) represents a three-parameter totality of lines g, which R. STURM (1840-1949) called a *thread* and J. PLÜCKER (1801-1868) called a *linear complex*.

If one introduces the value of g from (34.1) into (34.2) then one obtains the bilinear equation:

(34.6) $$\sum g'_{rs} x_j y_k = 0,$$

where one sums over all even permutations of j, k, r, s that arise from 0, 1, 2, 3. If the determinant of g'_{rs}, namely:

(34.7) $$\mathrm{Det}\ g'_{rs} = \tfrac{1}{2}\mathrm{Det}(g', g'),$$

is not equal to zero then (34.6) represents an involutory correlation that associates any point x with a point y that represents a plane through x, and for which any line of the thread (34.3) corresponds to itself. Following A. F. MÖBIUS (1790 to 1868), one calls this correlation a *null system*. If the determinant (34.7) vanishes then the thread consists of all lines of intersection of g' (viz., a degenerate thread).

We now return to the general case (34.7). If y is a "point at infinity" – i.e., $y_0 = 0$ – then we find for the null point x the plane at infinity:

(34.8) $$g'_{rs}\, x_j = 0 \qquad (k = 1, 2, 3)$$

or

(34.9) $$g'_{03} x_2 - g'_{02} x_3 = 0,$$

and cyclic permutations of 1, 2, 3. The direction of the pole x of the plane at infinity thus agrees with the direction g'_{01}, g'_{02}, g'_{03}. A line h that is perpendicular to it satisfies the equation:

(34.10) $$h_{01} g'_{01} + h_{02} g'_{02} + h_{03} g'_{03} = 0,$$

and therefore lies on the degenerate thread g'':

(34.11) $$\begin{aligned} &g''_{01} = 0, \quad g''_{02} = 0, \quad g''_{03} = 0, \\ &g''_{23} = g'_{01}, \ g''_{31} = g'_{02}, \ g''_{12} = g'_{03}. \end{aligned}$$

In the "pencil of threads" $g' + tg''$ there is then a degenerate thread for which:

(34.12) $$D(g' + tg'', g' + tg'') = D(g', g') + 2t\, D(g', g'') = 0,$$

or, more precisely:

(34.13) $$g_{01}\, g_{23} + \ldots + t(g_{01}^2 + \ldots) = 0.$$

For:

(34.14) $$g_{01}^2 + \ldots \neq 0$$

the line $g' + tg''$ is therefore defined uniquely, and one calls it the *axis of the thread* g', where the term "axis" is now given a different meaning than the one that it had in § 28.

After this digression into line geometry, we now return to kinematics.

§ Special motions

. In § 29, we established that the orthogonal dual transformation $\underline{\mathfrak{r}}' \rightarrow \underline{\mathfrak{r}}$:

$$(35.1) \qquad \underline{\mathfrak{r}} = \underline{\tilde{\mathfrak{Q}}}\,\underline{\mathfrak{r}}'\,\underline{\mathfrak{Q}}, \qquad \underline{\mathfrak{Q}}\,\underline{\tilde{\mathfrak{Q}}} = 1, \qquad \underline{\mathfrak{Q}} = \mathfrak{Q} + \varepsilon\bar{\mathfrak{Q}}$$

represents a motion in Euclidian R_3, when applied to the lines $\underline{\mathfrak{r}}$ in R_3, and that any such motion of the group of motions G_6 can be represented in that way, from which the dual, normalized quaternion $\underline{\mathfrak{Q}}$ is defined uniquely, except for its sign. If we set:

$$(35.2) \qquad \underline{\mathfrak{Q}} = \cos\underline{\varphi} + \underline{\mathfrak{a}}\sin\underline{\varphi}, \qquad < \underline{\mathfrak{a}}\;\underline{\mathfrak{a}} > = 1, \qquad \underline{\varphi} = \varphi + \varepsilon\bar{\varphi}$$

then $\underline{\mathfrak{a}}$ means the *screw axis* of the motion $\underline{\mathfrak{Q}}$, 2φ, is its rotation angle, and $2\bar{\varphi}$ is a "shift" along $\underline{\mathfrak{a}}$. The determination of $\underline{\mathfrak{a}}$ breaks down for $\mathfrak{Q} = 1$ – i.e., for the *displacements*.

We now look for the *involutory motions* – i.e., the motions that have period 2:

$$(35.3) \qquad \underline{\mathfrak{Q}}\,\underline{\tilde{\mathfrak{Q}}} = +1, \qquad \underline{\mathfrak{Q}}\,\underline{\mathfrak{Q}} \neq \pm 1.$$

From:

$$(35.4) \qquad \begin{aligned} \underline{\mathfrak{Q}} &= \underline{q}_0 + \underline{\mathfrak{q}}, \\ \underline{\tilde{\mathfrak{Q}}}\,\underline{\mathfrak{Q}} &= \underline{q}_0^2 + < \underline{\mathfrak{q}}\,\underline{\mathfrak{q}} > = +1, \\ \underline{\mathfrak{Q}}\,\underline{\mathfrak{Q}} &= \underline{q}_0^2 - < \underline{\mathfrak{q}}\,\underline{\mathfrak{q}} > + 2\underline{q}_0\underline{\mathfrak{q}} = \pm 1, \end{aligned}$$

it follows that either:

$$(35.5) \qquad \underline{q}_0^2 = 1, \qquad \underline{\mathfrak{q}} = 0$$

or

$$(35.6) \qquad \underline{q}_0 = 0, \qquad < \underline{\mathfrak{q}}\;\underline{\mathfrak{q}} > = 1.$$

The first case leads to the identity. In the second case, one has:

$$(35.7) \qquad \underline{\mathfrak{r}} = -\underline{\mathfrak{q}}\,\underline{\mathfrak{r}}'\,\underline{\mathfrak{q}}, \qquad < \underline{\mathfrak{q}}\;\underline{\mathfrak{q}} > = 1,$$

and we obtain the *reversal* (i.e., the rotation through the angle π) of the line $\underline{\mathfrak{q}}$.

§ 36. Incidence

For a quaternion:

$$(36.1) \qquad \underline{\mathfrak{Q}} = (q_0 + \varepsilon\bar{q}_0) + (\mathfrak{q} + \varepsilon\bar{\mathfrak{q}}),$$

we have defined the "conjugate: by way of:

(36.2) $$\tilde{\underline{\mathfrak{Q}}} = (q_0 + \varepsilon\overline{q}_0) - (\mathfrak{q} + \varepsilon\overline{\mathfrak{q}}).$$

In addition, we would like to consider the quaternion $\underline{\mathfrak{Q}}_\varepsilon$ that arises from $\underline{\mathfrak{Q}}$ by changing the sign of ε:

(36.3) $$\underline{\mathfrak{Q}}_\varepsilon = (q_0 - \varepsilon\overline{q}_0) + (\mathfrak{q} - \varepsilon\overline{\mathfrak{q}}).$$

One then has, in more detail:

(36.4)
$$\begin{aligned}
\underline{\mathfrak{Q}} &= q_0 + \varepsilon\overline{q}_0 + \mathfrak{q} + \varepsilon\overline{\mathfrak{q}}, & 4q_0 &= \underline{\mathfrak{Q}} + \tilde{\underline{\mathfrak{Q}}} + \underline{\mathfrak{Q}}_\varepsilon + \tilde{\underline{\mathfrak{Q}}}_\varepsilon, \\
\tilde{\underline{\mathfrak{Q}}} &= q_0 + \varepsilon\overline{q}_0 - \mathfrak{q} - \varepsilon\overline{\mathfrak{q}}, & 4\overline{q}_0 &= \underline{\mathfrak{Q}} + \tilde{\underline{\mathfrak{Q}}} - \underline{\mathfrak{Q}}_\varepsilon - \tilde{\underline{\mathfrak{Q}}}_\varepsilon, \\
\underline{\mathfrak{Q}}_\varepsilon &= q_0 + \varepsilon\overline{q}_0 - \mathfrak{q} - \varepsilon\overline{\mathfrak{q}}, & 4\mathfrak{q} &= \underline{\mathfrak{Q}} - \tilde{\underline{\mathfrak{Q}}} + \underline{\mathfrak{Q}}_\varepsilon - \tilde{\underline{\mathfrak{Q}}}_\varepsilon, \\
\tilde{\underline{\mathfrak{Q}}}_\varepsilon &= q_0 - \varepsilon\overline{q}_0 - \mathfrak{q} + \varepsilon\overline{\mathfrak{q}}, & 4\varepsilon\overline{\mathfrak{q}} &= \underline{\mathfrak{Q}} - \tilde{\underline{\mathfrak{Q}}} - \underline{\mathfrak{Q}}_\varepsilon + \tilde{\underline{\mathfrak{Q}}}_\varepsilon.
\end{aligned}$$

We have associated a point with the rectangular coordinates x_j with the vector:

(36.5) $$\mathfrak{x} = x_1\, e_1 + x_2\, e_2 + x_3\, e_3\,.$$

We would now like to assign it to the quaternion:

(36.6) $$\underline{\mathfrak{X}} = 1 + \varepsilon\,\mathfrak{x} = 1 + e(x_1\, e_1 + x_2\, e_2 + x_3\, e_3),$$

moreover. Analogously, a plane whose equation in HESSE normal form reads:

(36.7) $$u_0 + u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0, \qquad u_1^2 + u_2^2 + u_3^2 = 1$$

shall correspond to the direction unit vector:

(36.8) $$\mathfrak{u} = u_1\, x_1 + u_2\, x_2 + u_3\, x_3, \qquad \langle\mathfrak{u}\ \mathfrak{u}\rangle = 1,$$

and the quaternion:

(36.9) $$\underline{\mathfrak{U}} = \mathfrak{u} + \varepsilon\, u_0\,.$$

We then have for points that:

(36.10) $$\tilde{\underline{\mathfrak{X}}} - \underline{\mathfrak{X}}_\varepsilon = 0, \quad \underline{\mathfrak{X}} + \tilde{\underline{\mathfrak{X}}} = 2, \quad \underline{\mathfrak{X}}\tilde{\underline{\mathfrak{X}}} = 1,$$

and for planes that:

(36.11) $$\tilde{\underline{\mathfrak{U}}} + \underline{\mathfrak{U}}_\varepsilon = 0, \quad \underline{\mathfrak{U}}\tilde{\underline{\mathfrak{U}}} = 1.$$

For a point $\underline{\mathfrak{X}}$ and a line $\underline{\mathfrak{g}}$:

(36.12) $$\underline{\mathfrak{g}} = \mathfrak{g} + \varepsilon\overline{\mathfrak{g}}, \quad \langle\underline{\mathfrak{g}}\ \underline{\mathfrak{g}}\rangle = 1,$$

we calculate the expression:

(36.13) $$\underline{\mathfrak{g}}\,\underline{\mathfrak{X}} - \underline{\mathfrak{X}}\,\underline{\mathfrak{g}}_\varepsilon = 2\varepsilon\{\,\overline{\mathfrak{g}} - (\mathfrak{x} \times \mathfrak{g})\}.$$

Thus:

(36.15) $$\underline{\mathfrak{g}}\,\underline{\mathfrak{X}} - \underline{\mathfrak{X}}\,\underline{\mathfrak{g}}_\varepsilon = 0$$

means the *incidence of the point and the plane.*

For a line and a plane, we find:

(36.16) $$\underline{\mathfrak{g}}\,\underline{\mathfrak{U}} + \underline{\mathfrak{U}}\,\underline{\mathfrak{g}}_\varepsilon = 2\{- <\mathfrak{g}\ \mathfrak{u}> + \varepsilon(u_0\,\mathfrak{g} + (\,\overline{\mathfrak{g}} \times \mathfrak{u})\}.$$

Should this expression vanish, we would then have:

(37.17) $$<\mathfrak{g}\ \mathfrak{u}> = 0.$$

Furthermore, if $\mathfrak{x}$ is a point of $\underline{\mathfrak{g}}$ then one will have:

(36.18) $$\overline{\mathfrak{g}} = \mathfrak{x} \times \mathfrak{g},$$

and from the vanishing of (36.16), it would then follow, due to (36.17), (36.18), that:

(36.19) $$u_0\,\mathfrak{g} + (\,\overline{\mathfrak{g}} \times \mathfrak{u})) = u_0\,\mathfrak{g} + (\mathfrak{x} \times \mathfrak{u}) \times \mathfrak{u} = \{u_0 + <\mathfrak{x}\ \mathfrak{u}>\}\mathfrak{g} = 0,$$

and thus:

(36.20) $$u_0 + <\mathfrak{x}\ \mathfrak{u}> = 0.$$

One thus shows that:

(36.21) $$\underline{\mathfrak{g}}\,\underline{\mathfrak{U}} + \underline{\mathfrak{U}}\,\underline{\mathfrak{g}}_\varepsilon = 0$$

is the *condition for the incidence of the plane and the line.*

Finally, we have for a point and a plane:

(36.22) $$\underline{\mathfrak{U}}\,\underline{\mathfrak{X}}_\varepsilon - \underline{\mathfrak{X}}\,\underline{\mathfrak{U}}_\varepsilon = 2e\{u_0 + <\mathfrak{u}\ \mathfrak{x}>\},$$

and this means that:

(36.23) $$\underline{\mathfrak{U}}\,\underline{\mathfrak{X}}_\varepsilon - \underline{\mathfrak{X}}\,\underline{\mathfrak{U}}_\varepsilon = \underline{\mathfrak{U}}\,\underline{\mathfrak{X}} + \underline{\mathfrak{X}}\,\underline{\mathfrak{U}} = 0$$

is the *condition for incidence of the point and the plane.*

§ 37. Motions of points and planes

We consider the motion in R_3 that is applied to its *lines:*

(37.1) $$\underline{\mathfrak{g}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{g}}'\,\underline{\mathfrak{Q}}, \qquad <\underline{\mathfrak{Q}}\,\underline{\mathfrak{Q}}> = 1.$$

We assert: The same motion, when applied to the *points* $\underline{\mathfrak{X}}$ of R_3, gives:

(37.2) $$\underline{\mathfrak{X}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{X}}'\,\underline{\mathfrak{Q}}_\varepsilon .$$

In order prove this, we first observe that under (37.2) points $\underline{\mathfrak{X}}'$ go to points $\underline{\mathfrak{X}}$, since the equations (36.10) are preserved:

(37.3) $$\begin{aligned} \underline{\mathfrak{X}}_\varepsilon - \tilde{\underline{\mathfrak{X}}} &= \tilde{\underline{\mathfrak{Q}}}_\varepsilon(\underline{\mathfrak{X}}'_\varepsilon - \tilde{\underline{\mathfrak{X}}}')\underline{\mathfrak{Q}} = 0, \\ \underline{\mathfrak{X}}\tilde{\underline{\mathfrak{X}}} &= \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{X}}'\underline{\mathfrak{Q}}_\varepsilon\tilde{\underline{\mathfrak{Q}}}_\varepsilon\,\tilde{\underline{\mathfrak{X}}}'\underline{\mathfrak{Q}} = 1. \end{aligned}$$

We then need only to show that the condition (36.15) remains true for the incidence of a line and a point under (37.1), (37.2):

(37.4) $$\underline{\mathfrak{g}}\underline{\mathfrak{X}} - \underline{\mathfrak{X}}\underline{\mathfrak{g}}_\varepsilon = \tilde{\underline{\mathfrak{Q}}}_\varepsilon(\underline{\mathfrak{g}}'\underline{\mathfrak{X}}'_\varepsilon - \tilde{\underline{\mathfrak{X}}}'\underline{\mathfrak{g}}'_\varepsilon)\underline{\mathfrak{Q}} .$$

Correspondingly, we see: The motion (37.1), when applied to *planes*, gives:

(37.5) $$\underline{\mathfrak{U}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{U}}'\,\underline{\mathfrak{Q}}_\varepsilon ,$$

since

(37.6) $$\tilde{\underline{\mathfrak{U}}} + \underline{\mathfrak{U}}_\varepsilon = \tilde{\underline{\mathfrak{Q}}}_\varepsilon\,(\underline{\mathfrak{U}}' + \underline{\mathfrak{U}}'_\varepsilon)\,\underline{\mathfrak{Q}} = 0, \qquad \underline{\mathfrak{U}}\,\tilde{\underline{\mathfrak{U}}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{U}}'\underline{\mathfrak{Q}}_\varepsilon\underline{\mathfrak{Q}}'_\varepsilon\,\tilde{\underline{\mathfrak{U}}}'\underline{\mathfrak{Q}} = 1.$$

§ 38. Screws

We take a motion $\underline{\mathfrak{Q}}$, as in § 29, to be:

(38.1) $$\underline{\mathfrak{Q}}\,\tilde{\underline{\mathfrak{Q}}} = 1, \qquad \underline{\mathfrak{Q}} = \mathfrak{Q} + \varepsilon\bar{\mathfrak{Q}}, \qquad \langle\mathfrak{Q}\ \mathfrak{Q}\rangle = 1, \quad \langle\mathfrak{Q}\ \bar{\mathfrak{Q}}\rangle = 0$$

and

(38.2) $$\underline{\mathfrak{Q}} = \cos\underline{\omega} + \underline{\mathfrak{a}}\sin\underline{\omega}, \qquad \underline{\mathfrak{a}} + \tilde{\underline{\mathfrak{a}}} = 0, \qquad \langle\underline{\mathfrak{a}}\ \underline{\mathfrak{a}}\rangle = 1.$$

For dual arguments of a – perhaps analytic – function *f*, one sets:

(38.3) $$f(x + \varepsilon y) = f(x) + \varepsilon\, y\, f'(x).$$

One then has:

(38.4) $$\begin{aligned} q_0 &= \cos\omega, \qquad \mathfrak{q} = \mathfrak{a}\sin\omega, \\ \bar{q}_0 &= -\bar{\omega}\sin\omega, \ \ \bar{\mathfrak{q}} = \bar{\mathfrak{a}}\sin\omega + \mathfrak{a}\bar{\omega}\cos\omega, \end{aligned}$$

and thus:

(38.5) $$\begin{aligned} \underline{\mathfrak{a}} &= \frac{\underline{\mathfrak{q}}}{\sqrt{1-q_0^2}} + \varepsilon\frac{q_0\bar{q}_0\ \underline{\mathfrak{q}}}{(1-q_0^2)^{3/2}}, \\ \bar{\mathfrak{a}} &= \frac{\bar{\mathfrak{q}}}{\sqrt{1-q_0^2}} + \varepsilon\frac{q_0\bar{q}_0\ \mathfrak{q}}{(1-q_0^2)^{3/2}}. \end{aligned}$$

In the case of a translation, this formula says only that:

(38.6) $q_0 = 1, \qquad q_1 = q_2 = q_3 = 0, \qquad \underline{\mathfrak{Q}} = 1 + \varepsilon \bar{\mathfrak{q}}$.

For $\omega, \bar{\omega}$, we have:

(38.7) $$\cos\omega = q_0, \qquad \sin\omega = \sqrt{1-q_0^2}, \qquad \bar{\omega} = -\frac{\bar{q}_0}{\sqrt{1-q_0^2}}.$$

If we set:

(38.8) $$\underline{\mathfrak{Q}} = \underline{\mathfrak{R}}\,\underline{\mathfrak{S}} = \underline{\mathfrak{S}}\,\underline{\mathfrak{R}},$$

with:

(38.9) $$\underline{\mathfrak{R}} = \cos\omega + \underline{\mathfrak{a}}\sin\omega, \qquad \underline{\mathfrak{S}} = 1 + \varepsilon\,\mathfrak{a}\,\bar{\omega},$$

then $\underline{\mathfrak{R}}$ represents the rotation around the axis $\underline{\mathfrak{a}}$ through the angle 2ω. From (37.2), when $\underline{\mathfrak{S}}$ is applied to points, it gives the displacement:

(38.10) $$\underline{\mathfrak{X}} = (1 - \varepsilon\,\mathfrak{a}\,\bar{\omega})\,\underline{\mathfrak{X}}'(1 - \varepsilon\,\mathfrak{a}\,\bar{\omega}),$$

or

(38.11) $$\mathfrak{x} = \mathfrak{x}' - 2\,\mathfrak{a}\,\bar{\omega}.$$

Thus, $-2\bar{\omega}$ is the shift in the direction $\mathfrak{a}$. One sees that $\underline{\mathfrak{R}}$ actually represents a rotation around $\underline{\mathfrak{a}}$ from the fact that the point $\underline{\mathfrak{X}}$ remains fixed on $\underline{\mathfrak{a}}$:

(38.12) $$\underline{\mathfrak{a}}\,\underline{\mathfrak{X}} - \underline{\mathfrak{X}}\,\underline{\mathfrak{a}}_{\varepsilon} = 0$$

under:

(38.13) $$\underline{\mathfrak{X}} = (\cos\omega - \underline{\mathfrak{a}}\sin\omega)\,\underline{\mathfrak{X}}'(\cos\omega + \underline{\mathfrak{a}}_{\varepsilon}\sin\omega).$$

One recognizes the fact that we are dealing with a rotation through the angle 2ω by considering the "spherical part:"

(38.14) $$\underline{\mathfrak{g}} = \tilde{\underline{\mathfrak{R}}}\,\underline{\mathfrak{g}}'\,\underline{\mathfrak{R}}$$

of our rotation, as in § 2.

In particular, the inversion along the axis $\underline{\mathfrak{a}}$ will be represented by:

(38.15) $$\underline{\mathfrak{g}} = -\,\underline{\mathfrak{a}}\,\underline{\mathfrak{g}}'\,\underline{\mathfrak{a}}.$$

Any motion $\underline{\mathfrak{Q}}$ can be represented as the product of two inversions:

(38.16) $$\underline{\mathfrak{Q}} = \underline{\mathfrak{a}}_1\,\underline{\mathfrak{a}}_2.$$

The axes $\underline{\mathfrak{a}}_1$, $\underline{\mathfrak{a}}_2$ cut the screw axis perpendicularly with an angle and shortest distance between them that equals one-half the rotation angle and shift of the screw.

§ 39. Transfers

If one composes the reflection through the origin O, namely:

(39.1) $$\underline{\mathfrak{g}} = - \underline{\mathfrak{g}}'_\varepsilon , \qquad \underline{\mathfrak{X}} = \underline{\mathfrak{X}}'_\varepsilon , \qquad \underline{\mathfrak{U}} = - \underline{\mathfrak{U}}'_\varepsilon ,$$

with a motion (37.1), (37.2), (37.5):

(39.2) $$\underline{\mathfrak{g}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{g}}'\,\underline{\mathfrak{Q}} , \qquad \underline{\mathfrak{X}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{X}}'\,\underline{\mathfrak{Q}}_\varepsilon , \qquad \underline{\mathfrak{U}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{U}}'\,\underline{\mathfrak{Q}}_\varepsilon$$

then one obtains the formula for a *transfer* in R_3 :

(39.3) $$\underline{\mathfrak{g}} = - \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{g}}'_\varepsilon\,\underline{\mathfrak{Q}} , \qquad \underline{\mathfrak{X}} = \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{X}}'_\varepsilon\,\underline{\mathfrak{Q}}_\varepsilon , \qquad \underline{\mathfrak{U}} = - \tilde{\underline{\mathfrak{Q}}}\,\underline{\mathfrak{U}}'_\varepsilon\,\underline{\mathfrak{Q}}_\varepsilon .$$

In order for a transfer to be involutory – i.e., to be of period 2 – it is necessary that:

(39.4) $$\underline{\mathfrak{Q}}_\varepsilon\,\underline{\mathfrak{Q}} = \pm 1.$$

If one observes that:

(39.5) $$\underline{\mathfrak{Q}}\,\tilde{\underline{\mathfrak{Q}}} = 1, \qquad \langle \mathfrak{Q}\,\mathfrak{Q} \rangle = 1, \qquad \langle \mathfrak{Q}\,\bar{\mathfrak{Q}} \rangle = 0$$

then it follows from (39.4) that:

(39.6) $$(2\,q_0^2 - 1) + 2q_0\,\mathfrak{q} + \varepsilon\,(\mathfrak{q}\,\bar{\mathfrak{q}} - \bar{\mathfrak{q}}\,\mathfrak{q}) = \pm 1,$$

or

(39.7) $$2\,q_0^2 - 1 = \pm 1, \qquad q_0\,\mathfrak{q} = 0, \qquad \mathfrak{q} \times \bar{\mathfrak{q}} = 0.$$

From the second equation, it follows that either:

(39.8) $$\mathfrak{q} = 0$$

or

(39.9) $$q_0 = 0.$$

In the former case, (39.8) gives, from (39.5):

(39.10) $$q_0 = \pm 1.$$

From (39.5), one further has:

(39.11) $$q_0\,\bar{q}_0 + \langle \mathfrak{q}\,\bar{\mathfrak{q}} \rangle = 0,$$

and therefore, due to (39.8), (39.10):

(39.12) $$\bar{q}_0 = 0,$$

so:

(39.13) $$\underline{\mathfrak{Q}} = +1 + \varepsilon\,\bar{\mathfrak{q}} .$$

Since the choice of sign for $\underline{\mathfrak{Q}}$ is inessential, it suffices to take the positive sign in (39.13). The second formula (39.3) then gives:

(39.14) $$1 + \varepsilon\,\mathfrak{x} = (1 - \varepsilon\,\bar{\mathfrak{q}})(1 - \varepsilon\,\mathfrak{x})(1 - \varepsilon\,\bar{\mathfrak{q}}),$$

or

(39.15) $$\mathfrak{x} = -\,\mathfrak{x}' - 2\,\bar{\mathfrak{q}}\,.$$

This is the *reflection* (symmetry) *through the point* $-\bar{\mathfrak{q}}$.

In the second case (39.9), one will have:

(39.16) $$\underline{\mathfrak{Q}} = \mathfrak{q} + \varepsilon\,(\bar{q}_0 + \bar{\mathfrak{q}})\,, \quad \mathfrak{q}\times\bar{\mathfrak{q}} = 0.$$

The second formula (39.3) now gives:

(39.17) $$\mathfrak{x} = 2\,\bar{q}_0\,\mathfrak{q} + \mathfrak{q}\mathfrak{x}'\mathfrak{q}, \qquad \langle\mathfrak{q}\;\mathfrak{q}\rangle = 1,$$

or

(39.18) $$\mathfrak{x}\;\mathfrak{q} + \mathfrak{q}\;\mathfrak{x}' + 2\,\bar{q}_0 = 0.$$

The points of the plane:

(39.19) $$\langle\mathfrak{q}\;\mathfrak{x}\rangle - \bar{q}_0 = 0$$

go to themselves under the map (39.18). Therefore, (39.17) means the *reflection through the plane* (39.19). For the reflection in a plane $\underline{\mathfrak{U}}$, one has the following formulas:

For lines:

(39.20) $$\underline{\mathfrak{g}} = -\,\underline{\mathfrak{U}}\,\underline{\mathfrak{g}}'_{\varepsilon}\,\tilde{\underline{\mathfrak{U}}} = +\,\underline{\mathfrak{U}}\,\underline{\mathfrak{g}}'_{\varepsilon}\,\underline{\mathfrak{U}}_{\varepsilon}\,,$$

For points:

(39.21) $$\underline{\mathfrak{X}} = -\,\underline{\mathfrak{U}}\,\underline{\mathfrak{X}}'_{\varepsilon}\,\underline{\mathfrak{U}} = -\,\underline{\mathfrak{U}}\,\tilde{\underline{\mathfrak{X}}}'\,\underline{\mathfrak{U}}\,,$$

For planes:

(39.22) $$\underline{\mathfrak{B}} = +\,\underline{\mathfrak{U}}\,\underline{\mathfrak{B}}'_{\varepsilon}\,\underline{\mathfrak{U}} = -\,\underline{\mathfrak{U}}\,\tilde{\underline{\mathfrak{B}}}'\,\underline{\mathfrak{U}}\,.$$

We still need to determine the meaning of $\underline{\mathfrak{Q}}$ in the general case of the transfer (39.3). In order to do that, if we compose the reflection through the point $\mathfrak{p}$:

(39.23) $$\underline{\mathfrak{g}} = -\,\tilde{\underline{\mathfrak{P}}}\,\underline{\mathfrak{g}}'_{\varepsilon}\,\underline{\mathfrak{P}}\,, \qquad \underline{\mathfrak{P}} = 1 - \varepsilon\mathfrak{p}$$

with the rotation around the axis $\underline{\mathfrak{a}}$ through $\mathfrak{p}$:

(39.24) $$\underline{\mathfrak{a}} = \mathfrak{a} + \varepsilon(\mathfrak{p}\times\mathfrak{a}) = \mathfrak{a} + \varepsilon\,\frac{\mathfrak{p}\mathfrak{a} - \mathfrak{a}\mathfrak{p}}{2}$$

through the angle 2ω.

(39.25) $$\underline{\mathfrak{g}}^* = \{\cos\omega - (\mathfrak{a} + \varepsilon\,\mathfrak{p}\times\mathfrak{a})\sin\omega\}\,\underline{\mathfrak{g}}\,\{\cos\omega + (\mathfrak{a} + \varepsilon\,\mathfrak{p}\times\mathfrak{a})\sin\omega\}$$

then we get (39.3), with:

(39.26) $$\underline{\mathfrak{Q}} = (1 - \varepsilon\,\mathfrak{p})\,\{\cos\omega + (\mathfrak{a} + \varepsilon\,\mathfrak{p}\times\mathfrak{a})\sin\omega\}$$

and

(39.27) $$q_0 = \cos\omega, \quad \bar{q}_0 = \langle \mathfrak{p}\,\mathfrak{a}\rangle \sin\omega, \qquad \mathfrak{q} = \mathfrak{a}\sin\omega, \quad \bar{\mathfrak{q}} = -\mathfrak{p}\cos\omega.$$

§ 40. Simple manifolds of axis-crosses

In the projective space P_7 with homogeneous coordinates q_j, $\bar{q}_j$ (j = 0, 1, 2, 3), we have the two "absolute quadrics":

(40.1) $$\langle\mathfrak{Q}\,\mathfrak{Q}\rangle = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 0$$

and

(40.2) $$\langle\mathfrak{Q}\,\bar{\mathfrak{Q}}\rangle = q_0\bar{q}_0 + q_1\bar{q}_1 + q_2\bar{q}_2 + q_3\bar{q}_3 = 0.$$

The manifold M_6 of all axis-crosses (right-angled crosses) $\underline{\mathfrak{Q}}$ in P_7 satisfies the homogeneous equation (40.2).

We next consider the manifold M_3 of all $\underline{\mathfrak{Q}}$ that arise from $\underline{\mathfrak{Q}}$ = 1 by rotations around the origin O. We find the linear M_3 for them on the quadric (40.2):

(40.3) $$\bar{q}_0 = \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 0.$$

For the axis-crosses that arise from $\underline{\mathfrak{Q}}$ = 1 by all translations, that yields the linear manifold M_3' on (40.2):

(40.4) $$\bar{q}_0 = q_1 = q_2 = q_3 = 0.$$

Finally, for the axis-crosses that are found by means of all reflections of $\underline{\mathfrak{Q}}$ = 1 through all planes, we find the linear manifold M_3'' on (40.2):

(40.5) $$q_0 = \bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 0.$$

Thus, M_3' and M_3'' have the characteristic property that any two of their axis-crosses go to each other under a rotation. The number 3 is the highest dimension number for a linear manifold on the quadric (40.2). [1])

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[1]) Cf., perhaps, E. BERTINI, *Einführung in die projektive Geometrie mehrdimensionale Raüme*, Vienna, 1924, chap. VI, E. STUDY, "Grundlagen und Ziele der analytischen Kinematik," Sitzungsber. Berlin. Math. Ges. **12** (1913), 36-60, and the next book to appear in this series by W. BURAU.

We now give two examples of quadratic manifolds of axis-crosses, namely, first, the manifold M_5 of axis-crosses that are coupled with a line $\underline{\mathfrak{g}}$ and cut a fixed line $\underline{\mathfrak{h}}$. From § 29, we have:

(40.6) $$\underline{\mathfrak{g}} = \underline{\tilde{\mathfrak{Q}}}\,\underline{\mathfrak{g}}'\underline{\mathfrak{Q}},$$

and in more detail:

(40.7) $$\mathfrak{g} = \tilde{\mathfrak{Q}}\,\mathfrak{g}'\,\mathfrak{Q}, \qquad \bar{\mathfrak{g}} = \bar{\tilde{\mathfrak{Q}}}\mathfrak{g}'\mathfrak{Q} + \tilde{\mathfrak{Q}}\mathfrak{g}'\bar{\mathfrak{Q}} + \tilde{\mathfrak{Q}}\mathfrak{g}'\mathfrak{Q}.$$

From (28.10), the condition for the intersection of $\underline{\mathfrak{g}}$ and $\underline{\mathfrak{h}}$ reads:

(40.8) $$-2\{<\mathfrak{h}\,\bar{\mathfrak{g}}> + <\bar{\mathfrak{h}}\,\mathfrak{g}>\} = \mathfrak{h}\,\bar{\mathfrak{g}} + \bar{\mathfrak{g}}\,\mathfrak{h} + \bar{\mathfrak{h}}\,\mathfrak{g} + \mathfrak{g}\,\bar{\mathfrak{h}} = 0.$$

(40.8), (40.7) yield the vanishing of the real part (i.e., scalar part) of:

(40.9) $$\{\mathfrak{h}\,\bar{\tilde{\mathfrak{Q}}}\,\mathfrak{g}' + \mathfrak{h}\,\tilde{\mathfrak{Q}}\,\bar{\mathfrak{g}}' + \bar{\mathfrak{h}}\,\tilde{\mathfrak{Q}}\,\mathfrak{g}'\}\mathfrak{Q} + \mathfrak{h}\,\tilde{\mathfrak{Q}}\,\mathfrak{g}'\bar{\mathfrak{Q}}.$$

If we take, in particular:

(40.10) $$\mathfrak{q} = \mathfrak{h} = e_3, \qquad \bar{\mathfrak{g}}' = \bar{\mathfrak{h}} = 0$$

then the vanishing of the real part of:

(40.11) $$\mathfrak{h}(\bar{\tilde{\mathfrak{Q}}}\,\mathfrak{g}'\mathfrak{Q} + \tilde{\mathfrak{Q}}\,\mathfrak{g}'\bar{\mathfrak{Q}})$$

would follow from that. From that, it follows, due to (40.5), that the equation of our M_5 is:

(40.12) $$q_0\bar{q}_0 - q_1\bar{q}_1 - q_2\bar{q}_2 + q_3\bar{q}_3 = 0.$$

We further take the M_5' of all axis-crosses $\underline{\mathfrak{Q}}$ whose origin O lies in the plane $x_3 = 0$. If we apply the motion (37.2) to the origin $\underline{\mathfrak{X}}' = 1$ then we find:

(40.13) $$\underline{\mathfrak{X}} = 1 + \varepsilon\,\mathfrak{x} = \underline{\tilde{\mathfrak{Q}}}\,\underline{\mathfrak{Q}} = 1 + 2\varepsilon\{\bar{q}_0\,\mathfrak{q} - q_0\,\bar{\mathfrak{q}} + (\mathfrak{q} \times \bar{\mathfrak{q}})\}.$$

This then gives the equation for our M_5' as:

(40.14) $$q_0\bar{q}_3 - q_0\bar{q}_3 + q_1\bar{q}_2 - q_2\bar{q}_1 = 0.$$

Finally, we consider the manifold M_4 of all axis-crosses $\underline{\mathfrak{Q}}$ that result from a fixed $\underline{\mathfrak{Q}}'$ by the inversions in all lines $\underline{\mathfrak{g}}$:

(40.15) $$\underline{\mathfrak{Q}} = \underline{\mathfrak{Q}}'\,\underline{\mathfrak{g}}.$$

One then has:

(40.16) $$\underline{\mathfrak{g}} = \underline{\tilde{\mathfrak{Q}}}'\,\underline{\mathfrak{Q}},$$

and therefore:

(40.17) $$\underline{\mathfrak{g}} + \tilde{\underline{\mathfrak{g}}} = \tilde{\underline{\mathfrak{Q}}}' \underline{\mathfrak{Q}} + \tilde{\underline{\mathfrak{Q}}} \underline{\mathfrak{Q}}' = 2 < \underline{\mathfrak{Q}} \underline{\mathfrak{Q}}' > = 0.$$

Our M_3' thus satisfies the condition:

(40.18) $$< \underline{\mathfrak{Q}} \underline{\mathfrak{Q}}' > = 0.$$

From this, according to STEPHANOS (cf. § 19), it follows that:

To any three positions $\underline{\mathfrak{Q}}_1$, $\underline{\mathfrak{Q}}_2$, $\underline{\mathfrak{Q}}_3$ of a rigid body, there is, in general, precisely one fourth one $\underline{\mathfrak{Q}}'$ that goes to the $\underline{\mathfrak{Q}}_j$ by inversions through lines. The single exception arises when the $\underline{\mathfrak{Q}}_j$ are permuted by rotations around parallel axes.

One can also investigate the geometry of axis-crosses that are based in the group G_{28} of projective transformation that take the quadric (40.2) to itself. [1])

[1]) On the projective geometry of the quadric (40.2) and the associated "triality principle" (viz., the relationship between its points and its two families of "generators" M_3), cf., also E. A. WEISS, *Punktreihengeometrie*, Leipzig and Berlin, 1939, pp. 154.

CHAPTER FIVE

COMPULSIVE SPATIAL MOTION PROCESSES

§ 41. The canonical axis-cross

We now consider a one-parameter (= compulsive) motion process in R_3 whose effect on lines $\underline{\mathfrak{g}}$ is represented by means of an equation:

$$(41.1) \qquad \underline{\mathfrak{g}}(t) = \tilde{\underline{\mathfrak{Q}}}(t)\,\underline{\mathfrak{g}}'\,\underline{\mathfrak{Q}}(t), \qquad \underline{\mathfrak{Q}}\tilde{\underline{\mathfrak{Q}}} = 1,$$

and its effect on points $\underline{\mathfrak{X}}$ by means of (37.2):

$$(41.2) \qquad \underline{\mathfrak{X}}(t) = \tilde{\underline{\mathfrak{Q}}}(t)\,\underline{\mathfrak{X}}'\,\underline{\mathfrak{Q}}_\varepsilon(t).$$

In these equations, the real variable t means "time."

By extending equations (9.3) to "dual" ones, we obtain the following differential equations ($\underline{\mathfrak{Q}} = \underline{\mathfrak{Q}}_0$):

$$(41.3) \qquad \begin{array}{rcccc}
d\underline{\mathfrak{Q}}_0 = & * & +\underline{\mathfrak{Q}}_1\underline{\rho} & * & *, \\
d\underline{\mathfrak{Q}}_1 = & -\underline{\mathfrak{Q}}_0\underline{\rho} & * & +\underline{\mathfrak{Q}}_2\underline{\sigma} & *, \\
d\underline{\mathfrak{Q}}_2 = & * & -\underline{\mathfrak{Q}}_1\underline{\sigma} & * & +\underline{\mathfrak{Q}}_3\underline{\tau}, \\
d\underline{\mathfrak{Q}}_3 = & * & * & -\underline{\mathfrak{Q}}_2\underline{\tau} & *,
\end{array}$$

with

$$(41.4) \qquad \underline{\rho} = \rho + \varepsilon\bar{\sigma}, \quad \underline{\sigma} = \sigma + \varepsilon\bar{\sigma}, \quad \underline{\tau} = \tau + \varepsilon\bar{\tau}.$$

Separating the real and dual parts in (41.3) then gives the spherical part of our spatial motion process in the form of formulas (9.3), along with:

$$(41.5) \qquad \begin{array}{rcccccccc}
d\bar{\mathfrak{Q}}_0 = & * & +\mathfrak{Q}_1\bar{\rho} & * & * & * & +\mathfrak{Q}_1\rho & * & *, \\
d\bar{\mathfrak{Q}}_1 = & -\mathfrak{Q}_0\bar{\rho} & * & +\mathfrak{Q}_2\bar{\sigma} & * & -\bar{\mathfrak{Q}}_0\rho & * & +\bar{\mathfrak{Q}}_2\sigma & *, \\
d\bar{\mathfrak{Q}}_2 = & * & -\mathfrak{Q}_1\bar{\sigma} & * & +\mathfrak{Q}_3\bar{\tau} & * & -\bar{\mathfrak{Q}}_1\sigma & * & +\bar{\mathfrak{Q}}_3\tau, \\
d\bar{\mathfrak{Q}}_3 = & * & * & -\mathfrak{Q}_2\bar{\tau} & * & * & * & -\bar{\mathfrak{Q}}_2\tau & *.
\end{array}$$

One then comes to the "dualization" of the product table (9.5). It gives, for example:

$$(41.6) \qquad \tilde{\mathfrak{Q}}\bar{\mathfrak{Q}}_1 + \tilde{\bar{\mathfrak{Q}}}\mathfrak{Q}_1 = \bar{\mathfrak{p}}_1, \qquad \bar{\mathfrak{Q}}_1\tilde{\mathfrak{Q}} + \mathfrak{Q}_1\tilde{\bar{\mathfrak{Q}}} = \bar{\mathfrak{p}}_1'.$$

Since:

$$(41.7) \qquad \langle \underline{\mathfrak{p}}_j\, \underline{\mathfrak{p}}_k \rangle = \delta_{jk}, \qquad [\underline{\mathfrak{p}}_1\, \underline{\mathfrak{p}}_2\, \underline{\mathfrak{p}}_3] = +1$$

the lines $\underline{\mathfrak{p}}_j$ define a right-angled axis-cross that we call *canonical* relative to the moving system; likewise, the $\underline{\mathfrak{p}}'_j$ define the canonical axis-cross in the rest system. Along with the differential equations (9.11), one then has:

$$\begin{array}{llllllll} d\overline{\mathfrak{p}}_1 = & * & +\mathfrak{p}_2\overline{\lambda} & * & * & +\overline{\mathfrak{p}}_2\lambda & *, \\ d\overline{\mathfrak{p}}_2 = & -\mathfrak{p}_1\overline{\lambda} & * & +\mathfrak{p}_3\overline{\mu} & -\overline{\mathfrak{p}}_1\lambda & * & +\overline{\mathfrak{p}}_3\mu, \\ d\overline{\mathfrak{p}}_3 = & * & -\mathfrak{p}_3\overline{\mu} & * & * & -\overline{\mathfrak{p}}_2\mu & *, \end{array} \tag{41.8}$$

and

$$\begin{array}{llllllll} d\overline{\mathfrak{p}}_1' = & * & +\mathfrak{p}_2'\overline{\lambda}' & * & * & +\overline{\mathfrak{p}}_2'\lambda' & *, \\ d\overline{\mathfrak{p}}_2' = & -\mathfrak{p}_1'\overline{\lambda}' & * & +\mathfrak{p}_3'\overline{\mu}' & -\overline{\mathfrak{p}}_1'\lambda' & * & +\overline{\mathfrak{p}}_3'\mu', \\ d\overline{\mathfrak{p}}_3' = & * & -\mathfrak{p}_3'\overline{\mu}' & * & * & -\overline{\mathfrak{p}}_2'\mu' & *. \end{array} \tag{41.9}$$

The following connection exists in them:

$$\begin{array}{ll} \sigma = \lambda = \lambda', & \overline{\sigma} = \overline{\lambda} = \overline{\lambda}', \\ 2\rho = \mu' - \mu, & 2\overline{\rho} = \overline{\mu}' - \overline{\mu}, \\ 2\tau = \mu' + \mu, & 2\overline{\tau} = \overline{\mu}' + \overline{\mu}. \end{array} \tag{41.10}$$

Corresponding to (10.4), we have:

$$d\underline{\mathfrak{g}} = 2(\underline{\mathfrak{g}} \times \underline{\mathfrak{p}}_1)\underline{\rho}, \tag{41.11}$$

and therefore:

$$d\underline{\mathfrak{g}} = 0 \qquad \text{for} \qquad \underline{\mathfrak{g}} = \underline{\mathfrak{p}}_1. \tag{41.12}$$

This means: $\underline{\mathfrak{p}}_1$ is the axis of the instantaneous screw in the moving system. From the fact that:

$$< \underline{\mathfrak{p}}_3\ \underline{\mathfrak{p}}_1 > = 0, \qquad < \underline{\mathfrak{p}}_3\ d\underline{\mathfrak{p}}_1 > = 0, \tag{41.13}$$

it follows: $\underline{\mathfrak{p}}_3$ is the common perpendicular to two neighboring screw axes $\underline{\mathfrak{p}}_1$, $\underline{\mathfrak{p}}_1 + d\underline{\mathfrak{p}}_1$. This explains the meaning of the canonical axis-cross.

Now, let $\mathfrak{z}$ be the origin of the canonical axis-cross:

$$\mathfrak{z} \times \mathfrak{p}_j = \overline{\mathfrak{p}}_j. \tag{41.14}$$

If we differentiate (41.14) for $j = 1, 2, 3$ then it follows, with the use of the differential equations:

$$\begin{aligned} (d\mathfrak{z} \times \mathfrak{p}_1) + (\mathfrak{z} \times \mathfrak{p}_2)\lambda &= +\mathfrak{p}_3\overline{\lambda} + \overline{\mathfrak{p}}_2\lambda, \\ (d\mathfrak{z} \times \mathfrak{p}_3) - (\mathfrak{z} \times \mathfrak{p}_2)\mu &= -\mathfrak{p}_2\overline{\mu} - \overline{\mathfrak{p}}_2\mu, \end{aligned} \tag{41.15}$$

and thus, from (41.14), that:

(41.16) $$d\mathfrak{z} \times \mathfrak{p}_1 = +\mathfrak{p}_2\bar{\lambda}, \qquad d\mathfrak{z} \times \mathfrak{p}_3 = -\mathfrak{p}_2\bar{\mu}.$$

From this, it follows that:

(41.17) $$d\mathfrak{z} = +\mathfrak{p}_1\bar{\mu} + \mathfrak{p}_3\bar{\lambda},$$

and correspondingly in the rest system:

(41.18) $$d\mathfrak{z}' = +\mathfrak{p}'_1\bar{\mu}' + \mathfrak{p}'_3\bar{\lambda}'.$$

The point $\mathfrak{z}$ is the intersection point of the screw axis $\underline{\mathfrak{p}}_1$ with the common perpendicular $\underline{\mathfrak{p}}_3$ to $\underline{\mathfrak{p}}_1$, $\underline{\mathfrak{p}}_1 + d\underline{\mathfrak{p}}_1$, and for that reason it is called the *center* of the axis surface $(\underline{\mathfrak{p}}_1)$ on $\underline{\mathfrak{p}}_1$. The meanings of $\underline{\rho}, \underline{\sigma}, \underline{\tau}$ are obtained with no further assumptions by dualizing the meanings of ρ, σ, τ in § 9.

We now consider some simple special cases.

If we have:

(41.19) $$\lambda = \lambda' = 0$$

for all t then we have:

(41.20) $$d\mathfrak{p}_1 = d\mathfrak{p}'_1 = 0,$$

and therefore the axis surfaces $(\underline{\mathfrak{p}}_1)$, $(\underline{\mathfrak{p}}'_1)$ are cylinders. One then also calls the process of motion *cylindrical*. For:

(41.21) $$\bar{\rho} = 0, \qquad \bar{\mu} = \bar{\mu}'$$

the two axis surfaces $(\underline{\mathfrak{p}}_1)$, $(\underline{\mathfrak{p}}'_1)$ will be developable onto each other, and the motion process will be generated by the rolling without slipping of $(\underline{\mathfrak{p}}_1)$ on $(\underline{\mathfrak{p}}'_1)$.

If (41.19) and (41.21) are true simultaneously then we are dealing with a *planar motion process* that arises from the rolling without slipping of the cylinder $(\underline{\mathfrak{p}}_1)$ on the cylinder $(\underline{\mathfrak{p}}'_1)$, The relations:

(41.22) $$\rho = \sigma = \tau = 0$$

characterize the translation processes whose spherical component is the identity.

§ 42. The axis surface

From the formula:

(42.1) $$d\underline{\mathfrak{p}}_1 = \underline{\mathfrak{p}}_2\underline{\lambda}, \qquad \underline{\lambda} = \lambda + \varepsilon\bar{\lambda},$$

it follows that λ is the angle and $\bar{\lambda}$ is the shortest distance between neighboring screw axes $\underline{\mathfrak{p}}_1$, $\underline{\mathfrak{p}}_1 + d\underline{\mathfrak{p}}_1$. Corresponding statements are true for the axis surface $(\underline{\mathfrak{p}}'_1)$ in the rest system.

From:

(42.2) $$\lambda = \lambda', \qquad \bar{\lambda} = \bar{\lambda}'$$

we see that the axis surface in the have the same *twist* (= division parameter) in corresponding axes:

(42.3) $$\frac{\lambda}{\bar{\lambda}} = \frac{\lambda'}{\bar{\lambda}'}.$$

The axis surfaces contact each other at each time point t along the common screw axis $\underline{\mathfrak{p}}_1$, and have their common center there. One speaks of the "grinding" of the moving surface on the rest surface. From (41.17), (41.18):

(42.3) $$\bar{\mu} - \bar{\mu}' = -\,2\bar{\rho}$$

measures the shift along $\underline{\mathfrak{p}}_1$. Since:

(42.5) $$< \underline{\mathfrak{p}}_1\ \underline{\mathfrak{p}}_3 > = 0, \quad < \underline{\mathfrak{p}}_1,\ \underline{\mathfrak{p}}_3 + d\underline{\mathfrak{p}}_3 > = 0,$$

$\underline{\mathfrak{p}}_1$ is the common perpendicular of $\underline{\mathfrak{p}}_1$, $\underline{\mathfrak{p}}_3 + d\underline{\mathfrak{p}}_3$. The relationship between the ruled surfaces $(\underline{\mathfrak{p}}_1)$, $(\underline{\mathfrak{p}}_3)$ is therefore reciprocal.

We then seek the *curvature axis* $\underline{\mathfrak{q}}$ of $(\underline{\mathfrak{p}}_1)$ that has same angle and distance from three "neighboring" generators. This follows by means of the basic equations:

(42.6) $$\begin{gathered} < \underline{\mathfrak{q}}\ \underline{\mathfrak{p}}_1 > = \underline{c}, \qquad < \underline{\mathfrak{q}}\ \underline{\mathfrak{p}}_2 > = 0, \\ < \underline{\mathfrak{q}}\ \underline{\mathfrak{p}}_1 > \underline{\lambda} - < \underline{\mathfrak{q}}\ \underline{\mathfrak{p}}_3 > \underline{\mu} = 0. \end{gathered}$$

From this, one has:

(42.7) $$\underline{\mathfrak{q}} = \frac{\underline{\mathfrak{p}}_1 \underline{\mu} + \underline{\mathfrak{p}}_3 \underline{\lambda}}{\sqrt{\underline{\lambda}^2 + \underline{\mu}^2}},$$

or, in more detail:

(42.8) $$\mathfrak{q} = \frac{\mathfrak{p}_1 \mu + \mathfrak{p}_3 \lambda}{\sqrt{\lambda^2 + \mu^2}}, \qquad \bar{\mathfrak{q}} = \frac{\mathfrak{p}_1 \bar{\mu} + \mathfrak{p}_3 \bar{\lambda} + \bar{\mathfrak{p}}_1 \mu + \bar{\mathfrak{p}}_3 \lambda}{\sqrt{\lambda^2 + \mu^2}} - \frac{\lambda\bar{\lambda} + \mu\bar{\mu}}{\lambda^2 + \mu^2}\mathfrak{q}.$$

Furthermore, one has:

(42.9) $$\underline{c} = \frac{\underline{\mu}}{\sqrt{\underline{\lambda}^2 + \underline{\mu}^2}}, \qquad c = \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}, \qquad \bar{c} = \frac{\bar{\mu}}{\sqrt{\lambda^2 + \mu^2}} - \frac{\lambda\bar{\lambda} + \mu\bar{\mu}}{\lambda^2 + \mu^2}c.$$

Therefore, $\underline{\mathfrak{q}}$ is the common perpendicular to $\underline{\mathfrak{p}}_2$, $\underline{\mathfrak{p}}_2 + d\underline{\mathfrak{p}}_2$. If we set:

(42.10) $$\underline{c} = \cos\, \underline{\varphi} = \cos\varphi - \varepsilon\, \bar{\varphi} \sin\varphi$$

then we have:

(42.11) $$\cos\varphi = \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}\ , \quad \bar{\varphi} = \frac{\lambda\bar{\lambda} + \mu\bar{\mu}}{\lambda^2 + \mu^2}\frac{\mu}{\lambda} - \frac{\bar{\mu}}{\lambda}\,.$$

Corresponding formulas are true for the rest axis surface $(\underline{\mathfrak{p}}_1')$.

§ 43. Velocity

Differentiation of formula (41.2), while observing the basic equations, gives:

(43.1) $$d\underline{\mathfrak{X}} = \tilde{\underline{\mathfrak{Q}}}_1 \underline{\mathfrak{X}}' \tilde{\underline{\mathfrak{Q}}}_\varepsilon \underline{\rho} + \tilde{\underline{\mathfrak{Q}}}\, \underline{\mathfrak{X}}' \tilde{\underline{\mathfrak{Q}}}_{1\varepsilon} \underline{\rho}_\varepsilon = \tilde{\underline{\mathfrak{Q}}}_1 \underline{\mathfrak{Q}}\, \underline{\mathfrak{X}}\, \underline{\rho} + \underline{\mathfrak{X}}\, \tilde{\underline{\mathfrak{Q}}}_\varepsilon \tilde{\underline{\mathfrak{Q}}}_{1\varepsilon} \underline{\rho}_\varepsilon\,,$$

and from that, according to (9.5):

(43.2) $$d\underline{\mathfrak{X}} = -\,\underline{\mathfrak{p}}_1 \underline{\mathfrak{X}}\, \underline{\rho} + \underline{\mathfrak{X}}\, \underline{\mathfrak{p}}_{1\varepsilon} \underline{\rho}_\varepsilon\,.$$

In more detail:

(43.3) $$d(1 + \varepsilon\mathfrak{x}) = -\,(\mathfrak{p}_1 + \varepsilon\bar{\mathfrak{p}}_1)(1 + \varepsilon\mathfrak{x})(\rho + \varepsilon\bar{\rho}) + (1 + \varepsilon\mathfrak{x})(\mathfrak{p}_1 - \varepsilon\bar{\mathfrak{p}}_1)(\rho - \varepsilon\bar{\rho}),$$

and therefore:

(43.4) $$d\mathfrak{x} = 2\{(\mathfrak{x} \times \mathfrak{p}_1) - \bar{\mathfrak{p}}_1\}\rho - 2\mathfrak{p}_1 \bar{\rho}\,.$$

If one introduces the origin $\mathfrak{z}$ of the axis cross, which fulfills the equation:

(43.4) $$\mathfrak{z} \times \mathfrak{p}_1 = \bar{\mathfrak{p}}_1\,,$$

then one gets:

(43.6) $$d\mathfrak{x} = 2\{(\mathfrak{x} - \mathfrak{z}) \times \mathfrak{p}_1\}\rho - 2\mathfrak{p}_1 \bar{\rho}\,.$$

If we set:

(43.7) $$\frac{\rho}{dt} = R, \qquad \frac{\bar{\rho}}{dt} = \bar{R}$$

then we get:

(43.8) $$\mathfrak{v} = \dot{\mathfrak{x}} = 2\{(\mathfrak{x} - \mathfrak{z}) \times \mathfrak{p}_1\}R - 2\mathfrak{p}_1 \bar{R}$$

for the vector $\mathfrak{v}$ of the absolute velocity, or when we introduce *canonical coordinates:*

(43.9) $$\mathfrak{x} = \mathfrak{z} + x_1\mathfrak{p}_1 + x_2\mathfrak{p}_2 + x_3\mathfrak{p}_3,$$

this becomes:

(43.10) $$\tfrac{1}{2}\mathfrak{v} = \tfrac{1}{2}\dot{\mathfrak{x}} = (x_3\mathfrak{p}_2 - x_2\mathfrak{p}_3)\, R - \mathfrak{p}_1 \bar{R}\,.$$

From (41.17), one has:

(43.11) $$\dot{\mathfrak{z}} = \mathfrak{p}_1\overline{M} + \mathfrak{p}_3\overline{L}\,; \qquad \overline{M} = \frac{\overline{\mu}}{dt}, \qquad \overline{L} = \frac{\overline{\lambda}}{dt}.$$

If one differentiates the canonical point coordinates:

(43.12) $$x_j = \langle \mathfrak{x} - \mathfrak{z},\, \mathfrak{p}_j \rangle$$

then one gets the *guiding conditions:*

(43.13) $$\begin{aligned} \dot{x}_1 &= -\overline{M}' + x_2 L, \\ \dot{x}_2 &= -x_1 L + x_3 M', \\ \dot{x}_3 &= -\overline{L} - x_2 M', \end{aligned}$$

which confirm that a point (43.9) is fixed in the moving system. Correspondingly, for:

(43.14) $$\mathfrak{x}' = \mathfrak{z}' + x_1'\mathfrak{p}_1' + x_2'\mathfrak{p}_2' + x_3'\mathfrak{p}_3'$$

one gets the *rest conditions:*

(43.15) $$\begin{aligned} \dot{x}_1' &= -\overline{M} + x_2' L, \\ \dot{x}_2' &= -x_1' L + x_3' M', \\ \dot{x}_3' &= -\overline{L} - x_2' M'. \end{aligned}$$

They are true for a point $\mathfrak{x}'$ at rest.

For a *plane* with the canonical equation:

(43.16) $$u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0,$$

one finds by differentiating (43.16), and by means of (43.13), the guiding conditions:

(43.17) $$\begin{aligned} \dot{u}_0 &= +u_1\overline{M}' + u_3\overline{L}, \\ \dot{u}_1 &= +u_2 L, \\ \dot{u}_2 &= -u_1 L + u_3 M', \\ \dot{u}_3 &= -u_2 M', \end{aligned}$$

and the rest conditions:

(43.18) $$\begin{aligned} \dot{u}_0' &= +u_1'\overline{M} + u_3'\overline{L}, \\ \dot{u}_1' &= +u_2' L, \\ \dot{u}_2' &= -u_1' L + u_3' M, \\ \dot{u}_3' &= -u_2' M. \end{aligned}$$

For a *line* with the coordinates $g_j, \overline{g}_j$ relative to the canonical axis-cross, the guiding conditions read:

(43.19)
$$\begin{aligned}&\dot{g}_1 = g_2 L, \quad \dot{g}_2 = -g_1 L + g_3 M', \quad \dot{g}_3 = -g_2 M',\\ &\dot{\bar{g}}_1 = +g_2\bar{L} + \bar{g}_2 L,\\ &\dot{\bar{g}}_2 = -g_1\bar{L} + g_3\bar{M}' - \bar{g}_1 L + \bar{g}_3 M',\\ &\dot{\bar{g}}_3 = -g_2\bar{M}' - \bar{g}_2 M',\end{aligned}$$

and the rest conditions:

(43.20)
$$\begin{aligned}&\dot{g}'_1 = g'_2 L, \quad \dot{g}'_2 = -g'_1 L + g'_3 M, \quad \dot{g}'_3 = -g'_2 M,\\ &\dot{\bar{g}}'_1 = +g'_2\bar{L} + \bar{g}'_2 L,\\ &\dot{\bar{g}}'_2 = -g'_1\bar{L} + g'_3 M' - \bar{g}'_1 L + \bar{g}'_3 M,\\ &\dot{\bar{g}}'_3 = -g'_2\bar{M} - \bar{g}'_2 M.\end{aligned}$$

§ 44. Normal thread

Should the line:

(44.1) $$\underline{\mathfrak{n}} = \mathfrak{n} + \varepsilon\bar{\mathfrak{n}}$$

be perpendicular to the path tangent at $\mathfrak{x}$ then, from (43.10), one would have:

(44.2) $$n_1\bar{\rho} - n_2 x_3 \rho + n_3 x_2 \rho = 0.$$

On the other hand, since $\underline{\mathfrak{n}}$ runs through $\mathfrak{x}$, one has:

(44.3) $$\mathfrak{x} \times \mathfrak{n} = \bar{\mathfrak{n}}, \qquad x_3 n_2 - x_2 n_3 = \bar{n}_1.$$

It then follows that all of the path normals at an instant t satisfy the linear equation:

(44.4) $$n_1\bar{\rho} + \bar{n}_1\rho = 0$$

or, more generally:

(44.5) $$\langle\mathfrak{n}\,\mathfrak{p}_1\rangle\bar{\rho} + \{\langle\mathfrak{n}\,\bar{\mathfrak{p}}_1\rangle + \langle\bar{\mathfrak{n}}\,\mathfrak{p}_1\rangle\}\rho = 0.$$

One calls the structure that is represented by a linear equation in the line coordinates n_j, $\bar{n}_j$ a "linear complex" or a "thread" (§ 34); (44.4) or (44.5) then represents the desired normal thread. For $\bar{\rho} = 0$, the thread degenerates into the set of lines of intersection of the screw axis $\mathfrak{p}_1$.

One finds the following canonical line coordinates for the *path tangents:*

$$(44.6)\qquad \begin{aligned} g_1 &= -\bar{\rho}, & \bar{g}_1 &= -(x_2^2 + x_3^2)\rho, \\ g_2 &= +x_3\rho, & \bar{g}_2 &= +x_3\bar{\rho} + x_1x_2\rho, \\ g_3 &= -x_2\rho, & \bar{g}_3 &= +x_2\bar{\rho} + x_1x_3\rho. \end{aligned}$$

They define the quadratic complex:

$$(44.7)\qquad (g_2\bar{g}_2 + g_3\bar{g}_3)\rho + (g_2^2 + g_3^2)\bar{\rho} = 0.$$

§ 45. Twist and center of the path ruled surface

Let $\underline{\mathfrak{g}}(t)$ be a ruled surface. The common perpendicular $\underline{\mathfrak{h}}$ of two neighboring generators is then:

$$(45.1)\qquad \begin{aligned} \mathfrak{h} &= \frac{\mathfrak{g}\times d\mathfrak{g}}{\alpha}, \quad \alpha^2 = < d\mathfrak{g}, d\mathfrak{g} >, \\ \bar{\mathfrak{h}} &= \frac{(\bar{\mathfrak{g}}\times d\mathfrak{g}) + (\mathfrak{g}\times d\bar{\mathfrak{g}})}{\alpha} - \frac{\bar{\alpha}}{\alpha}\mathfrak{h}. \end{aligned}$$

The center $\mathfrak{y}$ on $\underline{\mathfrak{g}}$ is defined by the point of intersection of $\underline{\mathfrak{g}}$ and $\underline{\mathfrak{h}}$:

$$(45.2)\qquad \mathfrak{y} \times \mathfrak{g} = \bar{\mathfrak{g}}, \qquad \mathfrak{y} \times \mathfrak{h} = \bar{\mathfrak{h}}.$$

From the latter relation, it follows that:

$$(45.3)\qquad [\mathfrak{y}, \mathfrak{h}, d\mathfrak{g}] = <\bar{\mathfrak{h}}, d\mathfrak{g}>,$$

or, due to (45.1):

$$(45.4)\qquad <\mathfrak{y}\ \mathfrak{g}> = \frac{[\mathfrak{g}, d\mathfrak{g}, d\bar{\mathfrak{g}}]}{< d\mathfrak{g}, d\mathfrak{g} >}.$$

If we apply these formulas to the ruled surfaces that are described by the lines in moving system then, since $\bar{\mathfrak{p}}_1 = 0$, one has:

$$(45.5)\qquad \begin{aligned} d\mathfrak{g} &= 2(\mathfrak{g}\times\mathfrak{p}_1)\rho, \\ d\bar{\mathfrak{g}} &= 2(\bar{\mathfrak{g}}\times\mathfrak{p}_1)\rho + 2(\mathfrak{g}\times\mathfrak{p}_1)\bar{\rho}. \end{aligned}$$

The angle α between two neighboring generators will then come from:

$$(45.6)\qquad \alpha^2 = <d\mathfrak{g}, d\mathfrak{g}> = 4(g_2^2 + g_3^2)\rho^2,$$

and for its shortest distance $\bar{\alpha}$, one has:

(45.7) $$\alpha\bar{\alpha} = \langle d\mathfrak{g},\ d\bar{\mathfrak{g}}\rangle = 4(g_2\bar{g}_2 + g_3\bar{g}_3)\rho^2 + 4(g_2^2 + g_3^2)\rho\bar{\rho}\,.$$

It then follows that the *twist* $\alpha : \bar{\alpha}$ of our ruled surface is:

(45.8) $$\frac{\bar{\alpha}}{\alpha} = \frac{g_2\bar{g}_2 + g_3\bar{g}_3}{g_2^2 + g_3^2} + \frac{\bar{\rho}}{\rho}.$$

Thus, for intersecting lines ($\bar{\alpha} = 0$), in particular, one has:

(45.9) $$(g_2\bar{g}_2 + g_3\bar{g}_3)\rho + (g_2^2 + g_3^2)\bar{\rho} = 0.$$

Due to (45.5), it then follows from (45.4) that:

(45.10) $$\langle\mathfrak{y}\ \mathfrak{g}\rangle = \frac{g_2\bar{g}_3 - g_3\bar{g}_2}{g_2^2 + g_3^2}\, g_1,$$

and thus, due to (45.2):

(45.11) $$\mathfrak{y} = A\mathfrak{g} + (\mathfrak{g}\times\bar{\mathfrak{g}}),\qquad A = \frac{g_2\bar{g}_3 - g_3\bar{g}_2}{g_2^2 + g_3^2}\, g_1.$$

For the common perpendicular of neighboring generators of a path ruled surface, it follows from (45.1), (45.5) that:

(45.12) $$\mathfrak{h} = (\bar{g}_1\mathfrak{g} - \mathfrak{p}_1)\frac{2\rho}{\alpha}, \qquad \bar{\mathfrak{h}} = \mathfrak{y}\times\mathfrak{h}.$$

§ 46. Intersection of neighboring planes and lines

Let:

(46.1) $$u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

be a plane in the moving system at the time t. In order to ascertain its intersection with its neighboring position at time $t + dt$, we differentiate (46.1), when we apply the rest conditions (43.15) to the x_j and the guiding conditions (43.17) to the u_j :

(46.2) $$u_1\bar{\rho} + (u_3x_2 - u_2x_3)\rho = 0.$$

The coordinates of the intersection line $\underline{\mathfrak{r}}$ of (46.1), (36.2) are, up to a common factor:

(46.3) $$\begin{aligned} r_1 &= -(u_2^2 + u_3^2)\rho, & r_2 &= +u_1u_2\rho, & r_3 &= +u_1u_3\rho,\\ \bar{r}_1 &= -u_1^2\bar{\rho}, & \bar{r}_2 &= u_0u_3\rho - u_1u_2\bar{\rho}, & \bar{r}_3 &= -u_0u_2\rho - u_1u_3\bar{\rho}. \end{aligned}$$

We further seek the intersection point of neighboring lines. We have:

(46.4) $$\bar{\mathfrak{g}} = \mathfrak{x} \times \mathfrak{g}, \qquad d\bar{\mathfrak{g}} = \mathfrak{x} \times d\mathfrak{g},$$

and from (45.5):

(46.5) $$\begin{aligned} d\mathfrak{g} &= 2(\mathfrak{g} \times \mathfrak{p}_1)\rho, \\ d\bar{\mathfrak{g}} &= 2(\bar{\mathfrak{g}} \times \mathfrak{p}_1)\rho + 2(\mathfrak{g} \times \mathfrak{p}_1)\bar{\rho}. \end{aligned}$$

It follows from (46.4), (46.5) that:

(46.6) $$g_2 x_2 + g_3 x_3 = 0, \quad g_3\bar{\rho} + \bar{g}_3\rho - g_2 x_1 \rho = 0, \quad g_2\bar{\rho} + \bar{g}_2\rho + g_3 x_1 \rho = 0,$$

and from this, it follows that:

(46.7) $$g_1 \bar{g}_1 \rho = (g_2^2 + g_3^2)\bar{\rho},$$

in agreement with (45.9).

§ 47. Acceleration

For the sake brevity, if we once more employ the "canonical time":

(47.1) $$s = \int_{t_0}^{t} \rho, \qquad ds = \rho$$

then one has that:

(47.2) $$R = 1$$

in (43.7) and thus, in (43.8), the absolute velocity becomes:

(47.3) $$\dot{\mathfrak{x}} = 2\{(\mathfrak{x} - \mathfrak{z}) \times \mathfrak{p}_1\}R - 2\mathfrak{p}_1 \bar{R},$$

or, in more detail:

(47.4) $$\tfrac{1}{2}\dot{\mathfrak{x}} = -\mathfrak{p}_1 \bar{R} + x_3\,\mathfrak{p}_2 - x_2\,\mathfrak{p}_3 .$$

By differentiating with respect to *s*, it then follows, when one applies the guiding conditions (43.13) to *x* and equations (10.15) to $\mathfrak{p}$, that the *acceleration* is:

(47.5) $$\tfrac{1}{2}\ddot{\mathfrak{x}} = -\left(\dot{\bar{R}} + x_3 L\right)\mathfrak{p}_1 - (\bar{L} + \bar{R}L + 2x_2)\mathfrak{p}_2 + (x_1 L - 2x_3)\,\mathfrak{p}_3 .$$

In this, from (41.10), (47.1), one sets:

(47.6) $$M' - M = 2.$$

For the vector product:

(47.7) $$\mathfrak{w} = \tfrac{1}{4}\dot{\mathfrak{x}} \times \ddot{\mathfrak{x}},$$

we find that:

$$
\begin{aligned}
w_1 &= -x_2(L\bar{R}+\bar{L}) - 2(x_2^2 + x_3^2) + x_1 x_3 L, \\
w_2 &= x_1 L\bar{R} + x_2\dot{\bar{R}} - 2x_3\dot{\bar{R}} + x_2 x_3 L, \\
w_3 &= \bar{R}(L\bar{R}+\bar{L}) + 2x_2\bar{R} + x_3\dot{\bar{R}} + x_3^2 L.
\end{aligned}
\tag{47.8}
$$

For the *inflection points* of a path, one has $\mathfrak{w} = 0$, or:

$$\ddot{\mathfrak{x}} + f\,\dot{\mathfrak{x}} = 0. \tag{47.9}$$

From (47.4), (47.5), one then has:

$$
\begin{array}{cccl}
* & * & -x_3 L & = \dot{\bar{R}} + f\bar{R}, \\
* & -2x_2 & +x_3 f & = L\bar{R} + \bar{L}, \\
x_1 L & -x_2 f & -2x_3 & = 0.
\end{array}
\tag{47.10}
$$

For $L \neq 0$, this gives the location of the inflection points as:

$$
\begin{aligned}
-x_1 &= \frac{1}{2L^2}\left\{4\dot{\bar{R}} + f[L(L\bar{R}+\bar{L}) + 4\bar{R}] + f^2\dot{\bar{R}} + f^3\bar{R}\right\}, \\
-x_2 &= \frac{1}{2L}\left\{(L\bar{R}+\bar{L})L + f\,\dot{\bar{R}} + f^2\bar{R}\right\}, \\
-x_3 &= \frac{1}{L}\left\{\dot{\bar{R}} + f\,\bar{R}\right\}.
\end{aligned}
\tag{47.11}
$$

For $\bar{R} \neq 0$, this is a cubic line C_3 that goes through the point at infinity of $\mathfrak{p}_1$ ($f = \infty$), where it has the line at infinity of the plane $x_3 = 0$ for its tangent and the plane at infinity for its osculating plane. It lies on the parabolic cylinder $w_3 = 0$. For:

$$L \neq 0, \quad \bar{R} = 0, \quad \dot{\bar{R}} \neq 0, \tag{47.12}$$

C_3 degenerates into a parabola, and for:

$$L = 0, \quad \bar{R} \neq 0, \tag{47.13}$$

into a line. Finally, for:

$$L = 0, \quad \bar{R} = 0, \quad \dot{\bar{R}} = 0, \quad \bar{L} \neq 0, \tag{47.14}$$

we obtain the locus of the inflection points of the circular cylinder:

$$2(x_2^2 + x_3^2) + x_2\bar{L} = 0. \tag{47.15}$$

For the inverse motion process, we have:

$$\rho^* = -\rho, \qquad \bar{\rho}^* = -\bar{\rho}, \qquad \lambda^* = +\lambda, \qquad \bar{\lambda}^* = +\bar{\lambda}. \tag{47.16}$$

For the locus through which three neighboring lines go, we must, for that reason, apply the map:

$$x_1^* = -x_1, \qquad x_2^* = -x_2, \qquad x_3^* = +x_3 \tag{47.17}$$

to C_3 .

§ 48. Three neighboring positions of lines

If we consider three "neighboring" positions of a line $\underline{\mathfrak{g}}$ that is defined in the moving system then they generally uniquely determine a quadric that has those lines for its generators. In order to show this, one can employ the following equation for the quadric through three lines $\underline{\mathfrak{g}}_1$, $\underline{\mathfrak{g}}_2$, $\underline{\mathfrak{g}}_3$:

$$\begin{aligned}[]
[\bar{\mathfrak{g}}_1\,\bar{\mathfrak{g}}_2\,\bar{\mathfrak{g}}_3] &+ \{<\bar{\mathfrak{g}}_3\,\mathfrak{g}_2> - <\mathfrak{g}_3\,\bar{\mathfrak{g}}_2> + [\mathfrak{g}_2\,\mathfrak{g}_3\,\mathfrak{x}]\}<\bar{\mathfrak{g}}_1\,\mathfrak{x}> \\
&+ \{<\bar{\mathfrak{g}}_1\,\mathfrak{g}_3> - <\mathfrak{g}_1\,\bar{\mathfrak{g}}_3> + [\mathfrak{g}_3\,\mathfrak{g}_1\,\mathfrak{x}]\}<\bar{\mathfrak{g}}_2\,\mathfrak{x}> \\
&+ \{<\bar{\mathfrak{g}}_2\,\mathfrak{g}_2> - <\mathfrak{g}_2\,\bar{\mathfrak{g}}_1> + [\mathfrak{g}_1\,\mathfrak{g}_2\,\mathfrak{x}]\}<\bar{\mathfrak{g}}_3\,\mathfrak{x}> = 0.
\end{aligned} \tag{48.1}$$

For the center of this quadric, one gets the equation:

$$\{<\bar{\mathfrak{g}}_3\,\mathfrak{g}_2> - <\mathfrak{g}_3\,\bar{\mathfrak{g}}_2> + [\mathfrak{g}_2\,\mathfrak{g}_3\,\mathfrak{x}]\}\,\bar{\mathfrak{g}}_1 + <\bar{\mathfrak{g}}_1\,\mathfrak{x}>(\mathfrak{g}_2 \times \mathfrak{g}_3) + \ldots = 0, \tag{48.2}$$

in which the dots mean cyclic permutations of 1, 2, 3.

CHAPTER SIX

SURFACE-CONSTRAINED SPATIAL MOTION PROCESSES

§ 49. Line congruences

Whereas the compulsive motion processes have treated thoroughly, in particular, by engineers, ones with several parameters have still been considered only slightly, despite the fact that they are especially attractive for the geometer. Here, we would like to restrict ourselves to two-parameter ones (= surface-constrained) and first introduce some prefatory facts from the differential geometry of line congruences.

Let $\{\mathfrak{x}: \mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3\}$ be an axis-cross in R_3 that depends upon two real variables u, v with the origin $\mathfrak{x}$ and the perpendicular unit vectors $\mathfrak{t}_j$ on the axes. We set:

(49.1)
$$d\mathfrak{t}_1 = \mathfrak{t}_2\sigma_3 - \mathfrak{t}_3\sigma_2,\ d\mathfrak{t}_2 = \mathfrak{t}_3\sigma_1 - \mathfrak{t}_1\sigma_3,\ d\mathfrak{t}_3 = \mathfrak{t}_1\sigma_2 - \mathfrak{t}_2\sigma_1,$$
$$d\mathfrak{x} = \mathfrak{t}_1\bar{\sigma}_1 + \mathfrak{t}_2\bar{\sigma}_2 + \mathfrak{t}_3\bar{\sigma}_3.$$

In this, the σ mean Pfaffian forms in the u, v. We will assume that the σ_1, σ_2 are linearly independent:

(49.2)
$$[\sigma_1\ \sigma_2] = \Omega \neq 0.$$

The integrability conditions follow from (49.1) by exterior differentiation:

(49.3)
$$\begin{aligned} d\sigma_1 &= -[\sigma_2\sigma_3], & d\bar{\sigma}_1 &= -[\bar{\sigma}_2\sigma_3] - [\sigma_2\bar{\sigma}_3], \\ d\sigma_2 &= -[\sigma_3\sigma_1], & d\bar{\sigma}_2 &= -[\bar{\sigma}_3\sigma_1] - [\sigma_3\bar{\sigma}_1], \\ d\sigma_3 &= -[\sigma_1\sigma_2], & d\bar{\sigma}_3 &= -[\bar{\sigma}_1\sigma_2] - [\sigma_1\bar{\sigma}_2]. \end{aligned}$$

We consider the congruence K of the axes $\mathfrak{x}_3$ with:

(49.4)
$$\underline{\mathfrak{t}}_3 = \mathfrak{t}_3 + \varepsilon\bar{\mathfrak{t}}_3, \qquad \bar{\mathfrak{t}}_3 = \mathfrak{x} \times \mathfrak{t}_3.$$

One will then have:

(49.5)
$$d\underline{\mathfrak{t}}_3 = \mathfrak{t}_1\bar{\sigma}_2 - \mathfrak{t}_2\bar{\sigma}_1 + \bar{\mathfrak{t}}_1\sigma_2 - \bar{\mathfrak{t}}_2\sigma_1.$$

For a point $\mathfrak{y}$ on $\underline{\mathfrak{t}}_3$, one has:

(49.6)
$$\mathfrak{y} = \mathfrak{x} + h\,\mathfrak{t}_3,$$

and from this:

(49.7)
$$d\mathfrak{y} = \mathfrak{t}_1(\bar{\sigma}_1 + h\sigma_2) + \mathfrak{t}_2(\bar{\sigma}_2 - h\sigma_1) + \mathfrak{t}_3(\bar{\sigma}_3 + dh).$$

A *focal point* $\mathfrak{y}$ of K on $\underline{\mathfrak{t}}_3$ will then be defined by the following requirement:

(49.8)
$$\bar{\sigma}_1 + h\sigma_2 = 0, \qquad \bar{\sigma}_2 - h\sigma_1 = 0.$$

By eliminating h, we find the differential equation for the developable surfaces in K from this:

(49.9) $$\sigma_1\bar{\sigma}_1 + \sigma_2\bar{\sigma}_2 = 0,$$

while alternating multiplication of (49.8) yields the following equation for the focal points:

(49.10) $$[\bar{\sigma}_1\bar{\sigma}_2] + \{[\sigma_1\bar{\sigma}_1] + [\sigma_2\bar{\sigma}_2]\}h + [\sigma_1\sigma_2]h^2 = 0.$$

§ 50. Differential invariants of line congruences

For the center $\mathfrak{z}$ of the focal points on $\underline{\mathfrak{r}}_3$, it follows from (49.10) that:

(50.1) $$\mathfrak{z} = \mathfrak{x} + h_0\,\mathfrak{r}_3\,, \qquad h_0 = \frac{1}{2}\frac{[\bar{\sigma}_1\sigma_1] + [\bar{\sigma}_2\sigma_2]}{[\sigma_1\sigma_1]}.$$

We introduce *canonical axes* by two requirements:
First, we take $\mathfrak{z}$ to be the origin; from (50.1), that gives:

(50.2) $$[\bar{\sigma}_1\sigma_1] + [\bar{\sigma}_2\sigma_2] = 0.$$

Second, the axes $\mathfrak{r}_1$, $\mathfrak{r}_2$ shall separate the null directions of (49.9) harmonically. For the moment, if we set:

(50.3) $$\bar{\sigma}_1 = A\sigma_1 + B\sigma_2\,, \qquad \bar{\sigma}_2 = C\sigma_1 + D\sigma_2$$

then (49.9) gives:

(50.4) $$A\sigma_1^2 + (B + C)\sigma_1\sigma_2 + D\sigma_2^2 = 0.$$

Our second requirement thus leads to the condition:

(50.5) $$B + C = 0,$$

or:

(50.6) $$[\bar{\sigma}_1\sigma_1] - [\bar{\sigma}_2\sigma_2] = 0.$$

From (50.2), (50.6), one then has:

(50.7) $$[\bar{\sigma}_1\sigma_1] = 0, \qquad [\bar{\sigma}_2\sigma_2] = 0,$$

or:

(50.8) $$\bar{\sigma}_1 = k_1\sigma_1\,, \qquad \bar{\sigma}_2 = k_2\sigma_2\,.$$

For the half-distance H of the focal point, it now follows from (49.10) that:

(50.9) $$H^2 = -\,k_1\,k_2\,.$$

If we introduce canonical coordinates x_j for a point $\mathfrak{p}$ by the requirement:

(50.10) $$\mathfrak{p} = \mathfrak{z} + x_1\,\mathfrak{p}_1 + x_2\,\mathfrak{p}_2 + x_3\,\mathfrak{p}_3$$

then it follows from (49.7) that the equation of the *focal plane*, which generally contacts the *focal surface* that is described by the focal points at the focal points, is:

(50.11) $$k_2 x_1^2 + k_1 x_2^2 = 0.$$

For the surface element of the spherical image ($\mathfrak{r}_3$) of K, we have, from (49.1):

(50.12) $$\Omega = [\sigma_1\ \sigma_2],$$

and by "dualization" one gets another invariant surface element from this:

(50.13) $$\bar{\Omega} = [\bar{\sigma}_1\sigma_2] + [\sigma_1\bar{\sigma}_2],$$

and in the canonical case:

(50.14) $$\bar{\Omega} = (k_1 + k_2)\,\Omega.$$

For our irrational invariants of motion k_1, k_2 of K, we thus have, in general:

(50.15) $$k_1 + k_2 = \frac{\bar{\Omega}}{\Omega} = \frac{[\bar{\sigma}_1\sigma_2] + [\sigma_1\bar{\sigma}_2]}{[\sigma_1\sigma_2]}$$

and

(50.16) $$k_1\,k_2 = -H^2 = \frac{[\bar{\sigma}_1\bar{\sigma}_2]}{[\sigma_1\sigma_2]} - h_0^2 = \frac{[\bar{\sigma}_1\bar{\sigma}_2]}{[\sigma_1\sigma_2]} - \frac{1}{4}\left\{\frac{[\bar{\sigma}_1\sigma_1] + [\bar{\sigma}_2\sigma_2]}{[\sigma_1\sigma_2]}\right\}^2.$$

The k_1, k_2 then satisfy the quadratic equation:

(50.17) $$k^2 - \frac{\bar{\Omega}}{\Omega}\,k - H^2 = 0.$$

It finally follows that:

(50.18) $$(k_1 - k_2)^2 = 4H^2 + \left(\frac{\bar{\Omega}}{\Omega}\right)^2.$$

From (49.3), (50.12), (50.13), we emphasize the following relations:

(50.19) $$d\sigma_3 = -\,\Omega, \qquad d\bar{\sigma}_3 = -\,\bar{\Omega}.$$

§ 51. Displacement in congruences

The concept of "displacement" (§ 22) may now be transplanted from surfaces to congruences with no further assumptions. We consider a congruence $K = (\underline{\mathfrak{r}}_3)$ with the associated axis-cross $\underline{\mathfrak{r}}_j$ (j = 1, 2, 3), and set, from (49.1):

$$(51.1) \qquad d\underline{\mathfrak{r}}_1 = \underline{\mathfrak{r}}_2\underline{\sigma}_3 - \underline{\mathfrak{r}}_3\underline{\sigma}_2, \qquad \underline{\mathfrak{r}}_j = \mathfrak{r}_j + \varepsilon\bar{\mathfrak{r}}_j, \qquad \underline{\sigma}_j = \sigma_j + \varepsilon\bar{\sigma}_j.$$

We choose a ruled surface R from K and call the lines $\underline{\mathfrak{r}}_1$ along R *parallel* when one has:

$$(51.2) \qquad \underline{\sigma}_3 = 0, \qquad \sigma_3 = 0, \qquad \bar{\sigma}_3 = 0$$

along R, and then speak of the *displacement* of $\underline{\mathfrak{r}}_1$ along R. In this, $\bar{\sigma}_3 = 0$ means that the path of the intersection points of $\underline{\mathfrak{r}}_1$ and $\underline{\mathfrak{r}}_3$ cuts the generators $\underline{\mathfrak{r}}_3$ of R at right angles. On the other hand, $\sigma_3 = 0$ means that the vectors $\mathfrak{r}_1$ on the spherical image of R run parallel to the unit sphere ($\mathfrak{r}_3$). Displacement in K is defined by these two requirements. By dualization, one gets from § 22 that if the lines $\underline{\mathfrak{r}}_1$ on R are parallel then so are the lines:

$$(51.3) \qquad \underline{\mathfrak{r}}_1^* = \underline{\mathfrak{r}}_1 \cos\underline{\varphi} - \underline{\mathfrak{r}}_3 \sin\underline{\varphi}, \qquad \underline{\varphi} = \text{fixed}.$$

Moreover, from (50.19), we have:

$$(51.4) \qquad \int_G \Omega = -\oint_{dB} \sigma_3, \qquad \int_G \bar{\Omega} = -\oint_{dB} \bar{\sigma}_3.$$

From the last formulas, one infers that:

$$(51.5) \qquad \bar{\Omega} = 0$$

characterizes the *normal congruences*. In general, the last integral in (51.4) measures the "opening" of a transverse perpendicular line to the generators of R during a circuit.

§ 52. The cylindroid of the common perpendiculars

The common perpendicular $\underline{\mathfrak{q}}$ of a line $\underline{\mathfrak{r}}_3$ of the congruence and a neighboring line $\underline{\mathfrak{r}}_3 + d\underline{\mathfrak{r}}_3$ will be given by:

$$(52.1) \qquad \underline{\mathfrak{q}} = \frac{\underline{\mathfrak{r}}_3 \times d\underline{\mathfrak{r}}_3}{\underline{\rho}}, \qquad \underline{\rho} = \rho + \varepsilon\bar{\rho}, \quad \underline{\rho}^2 = \langle d\underline{\mathfrak{r}}_3, d\underline{\mathfrak{r}}_3 \rangle.$$

It explicitly follows from this that:

(52.2) $$\mathfrak{q} = \frac{\mathfrak{r}_3 \times d\mathfrak{r}_3}{\rho}, \qquad \overline{\mathfrak{q}} = \frac{(\overline{\mathfrak{r}}_3 \times d\mathfrak{r}_3) + (\mathfrak{r}_3 \times d\overline{\mathfrak{r}}_3)}{\rho} - \frac{\overline{\rho}}{\rho}\mathfrak{q}.$$

For a point $\mathfrak{x}$ on $\underline{\mathfrak{q}}$, one immediately has:

(52.3) $$\overline{\mathfrak{q}} = \mathfrak{x} \times \mathfrak{q},$$

or

(52.4) $$\frac{\rho\{(\overline{\mathfrak{r}}_3 \times d\mathfrak{r}_3) + (\mathfrak{r}_3 \times d\overline{\mathfrak{r}}_3)\} - \overline{\rho}(\mathfrak{r}_3 \times d\mathfrak{r}_3)}{\rho^2} = \mathfrak{x} \times \frac{\mathfrak{r}_3 \times d\mathfrak{r}_3}{\rho} = \frac{1}{\rho}\{< \mathfrak{r}, d\mathfrak{r}_3 > - < \mathfrak{r}\,\mathfrak{r}_3 > d\mathfrak{r}_3\}.$$

The scalar product with $d\mathfrak{r}_3$ gives:

(52.5) $$[\mathfrak{r}_3,\ d\overline{\mathfrak{r}}_3,\ d\mathfrak{r}_3] = - <\mathfrak{x}\ \mathfrak{r}_3> <d\mathfrak{r}_3, d\mathfrak{r}_3>,$$

or, in canonical coordinates, since $\overline{\mathfrak{r}}_j = 0$:

(52.6) $$[\mathfrak{r}_3,\ \mathfrak{r}_1\overline{\sigma}_2 - \mathfrak{r}_2\overline{\sigma}_1,\ \mathfrak{r}_1\sigma_2 - \mathfrak{r}_2\sigma_1] = - x_3(\sigma_1^2 + \sigma_2^2).$$

From this, it follows that:

(52.7) $$x_3 = \frac{\sigma_1\overline{\sigma}_2 - \sigma_2\overline{\sigma}_1}{\sigma_1^2 + \sigma_2^2} = (k_2 - k_1)\ \frac{\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}.$$

If we set:

(52.8) $$\sigma_1 = \rho \cos \varphi, \qquad \sigma_2 = \rho \sin \varphi$$

then we will get:

(52.9) $$x_3 = \frac{k_2 - k_1}{2} \sin 2\varphi.$$

From (52.2), (52.9), it follows that the common perpendicular for variable φ in the overbarred surface is, in canonical coordinates:

(52.10) $$(x_1^2 + x_2^2)x_3 = (k_2 - k_1)\, x_1\, x_2\,.$$

This *cylindroid* lies between the planes:

(52.11) $$2\,|\,x_3\,| = |\,k_2 - k_1|,$$

and one calls their intersection points with $\underline{\mathfrak{r}}_3$ ($x_1 = x_3 = 0$) its *boundary points*.

§ 53. Isotropic congruences

We have already considered the normal congruences a short while ago. As a second family of special congruences, we look at the "isotropic" ones, which can be characterized by the fact that their focal planes (50.11) are isotropic, so they satisfy the equation:

(53.1) $$x_1^2 + x_2^2 = 0,$$

from which, due to (50.11), it follows that:

(53.2) $$k_2 = k_1 .$$

These congruences will thus also be characterized by the fact that their boundary points (§ 52) coincide, so the cylindroid of the common perpendiculars degenerates into a pencil of lines. From (52.7), one then has:

(53.3) $$\sigma_1\bar{\sigma}_2 - \sigma_2\bar{\sigma}_1 = 0,$$

and thus, from (49.1):

(53.4) $$\langle d\mathfrak{r}_3, d\mathfrak{z}\rangle = 0,$$

if $\mathfrak{z}$ means the center of the focal points. Therefore, the isotropic congruences will also be characterized by the fact that the surface ($\mathfrak{z}$) and the spherical image ($\mathfrak{r}_3$) "correspond to the element (53.4) by orthogonality." From (53.4), two surfaces ($\mathfrak{y}$), ($\mathfrak{y}'$) with:

(53.5) $$\mathfrak{y} = \mathfrak{z} + h\mathfrak{r}_3 , \quad \mathfrak{y}' = \mathfrak{z} - h\mathfrak{r}_3 , \quad h = \text{fixed}$$

have the arc length elements:

(53.6) $$\langle d\mathfrak{y}, d\mathfrak{y}\rangle = \langle d\mathfrak{y}', d\mathfrak{y}'\rangle = \langle d\mathfrak{z}, d\mathfrak{z}\rangle + \langle d\mathfrak{r}_3, d\mathfrak{r}_3\rangle\, h^2,$$

and are thus related to each other in a distance-preserving way in our case. From (53.1), the focal surfaces are isotropic – i.e., enveloping isotropic planes. If we write the equation of two such planes in the form:

(53.7) $$\begin{aligned}(1-s^2)\xi_1 + i(1+s^2)\xi_2 - 2s\xi_3 &= +2i\,w(s),\\ (1-t^2)\xi_1 - i(1+t^2)\xi_2 - 2t\xi_3 &= -2i\,k(t),\end{aligned}$$

in which $s = u + iv$, $t = u - iv$ are complex conjugates and $w(s)$, $k(t)$ are complex conjugate analytic functions, then it follows that the associated isotropic congruence of the real cut lines $(\underline{\mathfrak{r}}_3)$ has the parametric representation:

$$(53.8)\qquad \begin{aligned} r_1 &= \frac{s+t}{1+st}, & \bar{r}_1 &= \frac{w(1-t^2)+k(1-s^2)}{(1+st)^2}, \\ r_2 &= \frac{s-t}{i(1+st)}, & \bar{r}_2 &= \frac{w(1+t^2)-k(1+s^2)}{i(1+st)^2}, \\ r_3 &= \frac{1-st}{1+st}, & \bar{r}_3 &= -\frac{2(wt+ks)}{(1+st)^2}. \end{aligned}$$

§ 54. Further formulas for line congruences

Now, let the general line congruence K be given by the formulas:

$$(54.1)\qquad \mathfrak{r} = \mathfrak{r}(u, v), \qquad \overline{\mathfrak{r}} = \overline{\mathfrak{r}}(u, v), \qquad \langle \mathfrak{r}\ \mathfrak{r}\rangle = 1, \qquad \langle \mathfrak{r}\ \overline{\mathfrak{r}}\rangle = 0.$$

We would like to calculate its invariants of motion from this. For the base point $\mathfrak{p}$ of the perpendicular from the origin O to the line $\underline{\mathfrak{r}}$, we have:

$$(54.2)\qquad \mathfrak{p} = \mathfrak{r} \times \overline{\mathfrak{r}},$$

so it follows from (54.2) that:

$$(54.3)\qquad \langle \mathfrak{p}\ \mathfrak{r}\rangle = 0, \qquad \mathfrak{p} \times \mathfrak{r} = \overline{\mathfrak{r}}.$$

For the point $\mathfrak{y}$ of $\underline{\mathfrak{r}}$ one thus has:

$$(54.4)\qquad \mathfrak{y} = \mathfrak{p} + h\mathfrak{r},$$

and from this:

$$(54.5)\qquad d\mathfrak{y} = d\mathfrak{p} + h\, d\mathfrak{r} + \mathfrak{r}\, dh.$$

For a focal point, one then has:

$$(54.6)\qquad d\mathfrak{p} + h\, d\mathfrak{r} = \lambda\mathfrak{r},$$

or

$$(54.7)\qquad (d\mathfrak{r} \times \overline{\mathfrak{r}}) + (\mathfrak{r} \times d\overline{\mathfrak{r}}) + h\, d\mathfrak{r} = \lambda\mathfrak{r}.$$

Taking the scalar product with $d\mathfrak{r}$ gives:

$$(54.8)\qquad [\mathfrak{r}, d\mathfrak{r}, d\overline{\mathfrak{r}}] = h \langle d\mathfrak{r}, d\mathfrak{r}\rangle.$$

We introduce the following abbreviations:

(54.9)
$$\begin{aligned}&<\mathfrak{r}_u\mathfrak{r}_u>=E,\ <\mathfrak{r}_u\overline{\mathfrak{r}}_u>=P,\\ &<\mathfrak{r}_u\mathfrak{r}_v>=F,\ <\mathfrak{r}_u\overline{\mathfrak{r}}_v>=Q,\ [\mathfrak{r}\,\mathfrak{r}_u\mathfrak{r}_v]=W,\ EG-F^2=W^2,\\ &<\mathfrak{r}_v\mathfrak{r}_u>=F,\ <\mathfrak{r}_v\overline{\mathfrak{r}}_u>=R,\\ &<\mathfrak{r}_v\mathfrak{r}_v>=G,\ <\mathfrak{r}_v\overline{\mathfrak{r}}_v>=S.\end{aligned}$$

It then follows from (54.7) by scalar multiplication with $\mathfrak{r}_u$, $\mathfrak{r}_v$ that:

(54.10)
$$\begin{aligned}&[\mathfrak{r}\,\overline{\mathfrak{r}}_u\,\mathfrak{r}_u]du+[\mathfrak{r}\,\overline{\mathfrak{r}}_v\,\mathfrak{r}_u]du+(E\,du+F\,dv)h=0,\\ &[\mathfrak{r}\,\overline{\mathfrak{r}}_u\,\mathfrak{r}_v]du+[\mathfrak{r}\,\overline{\mathfrak{r}}_v\,\mathfrak{r}_v]du+(F\,du+G\,dv)h=0,\end{aligned}$$

or after multiplying by W:

(54.10)*
$$\begin{aligned}&(FP-ER)du+(FQ-ES)dv+W(E\,du+F\,dv)=0,\\ &(GP-FR)du+(GQ-FS)dv+W(F\,du+G\,dv)=0,\end{aligned}$$

and taking the alternating product of both formulas gives:

(54.11) $$(PS-QR)+W(Q-R)\,h+W^2\,h^2=0$$

for the focal points. The arc length element ρ^2 in the spherical image is:

(54.12) $$\rho^2=E\,du^2+2F\,du\,dv+G\,dv^2,$$

and from this it follows that, dually:

(54.13) $$\rho\overline{\rho}=P\,du^2+(Q+R)\,du\,dv+S\,dv^2.$$

The second form will be made to vanish for the developable surfaces in K, and it follows from (54.11) that the center of the focal points is:

(54.14) $$h_0=\frac{1}{2}\frac{R-Q}{W}.$$

For canonical axes, one has, from (54.11), (54.13):

(54.15) $$Q_0=0,\qquad R_0=0.$$

The common perpendicular $\underline{\mathfrak{q}}$ of $\underline{\mathfrak{r}}$, $\underline{\mathfrak{r}}+d\underline{\mathfrak{r}}$ is:

(54.16) $$\underline{\mathfrak{q}}=\frac{\underline{\mathfrak{r}}\times d\underline{\mathfrak{r}}}{\underline{\rho}},$$

or, in more detail:

(54.17) $$\mathfrak{q} = \frac{\mathfrak{r}\times d\mathfrak{r}}{\rho}, \qquad \overline{\mathfrak{q}} = \frac{(\overline{\mathfrak{r}}\times d\mathfrak{r}) + (\mathfrak{r}\times d\overline{\mathfrak{r}})}{\rho} - \frac{\overline{\rho}}{\rho}\mathfrak{q}.$$

For the intersection point:

(54.18) $$\mathfrak{x} = \mathfrak{p} + h\mathfrak{r}$$

of $\underline{\mathfrak{r}}$ with $\underline{\mathfrak{q}}$, we have:

(54.19) $$\overline{\mathfrak{q}} = \mathfrak{x}\times\mathfrak{q}.$$

By substituting (54.17), (54.18) in (54.19), it follows that:

(54.20) $$h = \frac{[\mathfrak{r}, d\mathfrak{r}, d\overline{\mathfrak{r}}]}{< d\mathfrak{r}, d\mathfrak{r} >}.$$

Multiplication by:

(54.21) $$W = [\mathfrak{r}\ \mathfrak{r}_u\ \mathfrak{r}_v]$$

gives, from (54.9):

(54.22) $$Wh = \frac{\begin{vmatrix} E\,du + F\,dv & P\,du + Q\,dv \\ F\,du + G\,dv & R\,du + S\,dv \end{vmatrix}}{E\,du^2 + 2F\,du\,dv + G\,dv^2},$$

or

(54.23) $$Wh = \frac{(Er - FP)\,du^2 + \{(ES - GP) + F(R - Q)\}\,du\,dv + (FS - GQ)\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}.$$

We have the invariants:

(54.24) $$4H^2 = \frac{(Q+R)^2 - 4PS}{EG - F^2}, \qquad \frac{\overline{\Omega}}{\Omega} = \frac{(GP + ES) - F(Q+R)}{EG - F^2},$$

and k_1, k_2 are roots of equation (50.17):

(54.25) $$k^2 - \frac{\overline{\Omega}}{\Omega}k - H^2 = 0.$$

From this, one has:

(54.26) $$(k_1 - k_2)^2 = 4H^2 + \left(\frac{\overline{\Omega}}{\Omega}\right)^2.$$

For the surface element Ω of the spherical image, one has:

(54.27) $$\Omega = [\mathfrak{r}\ \mathfrak{r}_u\ \mathfrak{r}_v]\,[du, dv],$$

and from this, by dualization, and due to the fact that $[\overline{\mathfrak{r}}\ \ \mathfrak{r}_u\ \mathfrak{r}_v] = 0$, one gets:

$$\bar{\Omega} = \{[\mathfrak{r}\ \bar{\mathfrak{r}}_u\ \mathfrak{r}_v] + [\mathfrak{r}\ \mathfrak{r}_u\ \bar{\mathfrak{r}}_v]\}\,[du, dv]. \tag{54.28}$$

Taking the product of both formulas and dividing by Ω^2 yields:

$$\frac{\bar{\Omega}}{\Omega} = GP - F(Q + R) + ES. \tag{54.29}$$

The condition (51.5) for normal congruences then gives the vanishing of the bilinear invariants of the quadratic forms (54.12), (54.13):

$$GP - F(Q + R) + ES = 0. \tag{54.30}$$

Since this means that the null lines of these forms (54.12), (54.13) are harmonically separated, we see: The normal congruences are thus characterized by saying that their focal planes intersect at right angles.

The isotropic congruences in § 53 were characterized by isotropic focal planes, so the following equations must be compatible with them:

$$\begin{aligned} &E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0, \\ &P\,du^2 + (Q+R)\,du\,dv + S\,dv^2 = 0. \end{aligned} \tag{54.31}$$

A thorough presentation of the differential geometry of line congruences in the German language will appear soon: S. P. FINIKOW, *Theorie der Konguenzen*, Berlin, 1959 (translated from the Russian). [1])

§ 55. Differential equations for surface-constrained motion processes

Let:

$$\underline{\mathfrak{r}}(u,v) = \bar{\underline{\mathfrak{Q}}}(u,v)\,\underline{\mathfrak{r}}'(u,v)\,\underline{\mathfrak{Q}}(u,v)\ , \qquad \underline{\mathfrak{Q}}\tilde{\underline{\mathfrak{Q}}} = 1 \tag{55.1}$$

be a surface-constrained motion process that acts on the lines $\underline{\mathfrak{r}}$. We extend $\underline{\mathfrak{Q}}(u,v)$ to a polar tetrahedron, for which we set $\underline{\mathfrak{Q}} = \underline{\mathfrak{Q}}_0$ and assume that $d\underline{\mathfrak{Q}}_0$, $d\underline{\mathfrak{Q}}_1$, $d\underline{\mathfrak{Q}}_2$ are linearly dependent. One then has differential equations of the form:

$$\begin{aligned} d\mathfrak{Q}_0 &= \quad * \quad + \mathfrak{Q}_1\alpha_1 + \mathfrak{Q}_2\alpha_2 \quad * \quad , \\ d\mathfrak{Q}_1 &= -\mathfrak{Q}_0\alpha_1 \quad * \quad + \mathfrak{Q}_2\beta_3 - \mathfrak{Q}_3\beta_2, \\ d\mathfrak{Q}_2 &= -\mathfrak{Q}_0\alpha_2 - \mathfrak{Q}_1\beta_3 \quad * \quad + \mathfrak{Q}_3\beta_1, \\ d\mathfrak{Q}_3 &= \quad * \quad + \mathfrak{Q}_1\beta_2 - \mathfrak{Q}_2\beta_1 \quad * \quad , \end{aligned} \tag{55.2}$$

and

[1]) Cf., also: R. SAUER, *Projektive Liniengeometrie*, Berlin and Leipzig, 1937; V. HLAVATÝ, *Differentielle Liniengeometrie*, Gronigen, 1945.

(55.3)
$$\begin{array}{llllllllll}
d\bar{\mathfrak{Q}}_0 = & * & +\mathfrak{Q}_1\bar{\alpha}_1 & +\mathfrak{Q}_2\bar{\alpha}_2 & * & * & +\bar{\mathfrak{Q}}_1\alpha_1 & +\bar{\mathfrak{Q}}_2\alpha_2 & * & , \\
d\bar{\mathfrak{Q}}_1 = & -\mathfrak{Q}_0\bar{\alpha}_1 & * & +\mathfrak{Q}_2\bar{\beta}_3 & -\mathfrak{Q}_3\bar{\beta}_2 & -\bar{\mathfrak{Q}}_0\alpha_1 & * & +\bar{\mathfrak{Q}}_2\beta_3 & -\bar{\mathfrak{Q}}_3\beta_2 & , \\
d\bar{\mathfrak{Q}}_2 = & -\mathfrak{Q}_0\bar{\alpha}_2 & -\mathfrak{Q}_1\bar{\beta}_3 & * & +\mathfrak{Q}_3\bar{\beta}_1 & -\bar{\mathfrak{Q}}_0\alpha_2 & -\bar{\mathfrak{Q}}_1\beta_3 & * & +\bar{\mathfrak{Q}}_3\beta_1 & , \\
d\bar{\mathfrak{Q}}_3 = & * & +\mathfrak{Q}_1\bar{\beta}_2 & -\mathfrak{Q}_2\bar{\beta}_1 & * & * & +\bar{\mathfrak{Q}}_1\beta_2 & -\bar{\mathfrak{Q}}_2\beta_1 & * & ,
\end{array}$$

with

(55.4)
$$< \underline{\mathfrak{Q}}_j \, \underline{\mathfrak{Q}}_k > = \delta_{jk}, \qquad [\underline{\mathfrak{Q}}_0 \, \underline{\mathfrak{Q}}_1 \, \underline{\mathfrak{Q}}_2 \, \underline{\mathfrak{Q}}_3] = +1.$$

In this, α, β, $\bar{\alpha}$, $\bar{\beta}$ are real Pfaffian forms in u, v. As in § 9, we construct the unit vectors:

(55.5)
$$\begin{array}{c|cccc}
 & \mathfrak{Q}_0 & \mathfrak{Q}_1 & \mathfrak{Q}_2 & \mathfrak{Q}_3 \\ \hline
\tilde{\mathfrak{Q}}_0 & 1 & \mathfrak{p}_1 & \mathfrak{p}_2 & \mathfrak{p}_3 \\
\tilde{\mathfrak{Q}}_1 & -\mathfrak{p}_1 & 1 & -\mathfrak{p}_3 & \mathfrak{p}_2 \\
\tilde{\mathfrak{Q}}_2 & -\mathfrak{p}_2 & \mathfrak{p}_3 & 1 & -\mathfrak{p}_1 \\
\tilde{\mathfrak{Q}}_3 & -\mathfrak{p}_3 & -\mathfrak{p}_2 & \mathfrak{p}_1 & 1
\end{array} \qquad
\begin{array}{c|cccc}
 & \tilde{\mathfrak{Q}}_0 & \tilde{\mathfrak{Q}}_1 & \tilde{\mathfrak{Q}}_2 & \tilde{\mathfrak{Q}}_3 \\ \hline
\mathfrak{Q}_0 & 1 & -\mathfrak{p}'_1 & -\mathfrak{p}'_2 & -\mathfrak{p}'_3 \\
\mathfrak{Q}_1 & \mathfrak{p}'_1 & 1 & -\mathfrak{p}'_3 & \mathfrak{p}'_2 \\
\mathfrak{Q}_2 & \mathfrak{p}'_2 & \mathfrak{p}'_3 & 1 & -\mathfrak{p}'_1 \\
\mathfrak{Q}_3 & \mathfrak{p}'_3 & -\mathfrak{p}'_2 & \mathfrak{p}'_1 & 1
\end{array}$$

and their dual components; for example:

(55.6)
$$\bar{\mathfrak{p}}_1 = \bar{\tilde{\mathfrak{Q}}}_0 \mathfrak{Q}_1 + \tilde{\mathfrak{Q}}_0 \bar{\mathfrak{Q}}_1 .$$

We then have the following differential equations for the $\mathfrak{p}$, $\mathfrak{p}'$:

(55.7)
$$\begin{array}{lllll}
d\mathfrak{p}_1 = & * & +\mathfrak{p}_2\sigma_3 & -\mathfrak{p}_3\sigma_2, \\
d\mathfrak{p}_2 = & -\mathfrak{p}_1\sigma_3 & * & +\mathfrak{p}_3\sigma_1, \\
d\mathfrak{p}_3 = & +\mathfrak{p}_1\sigma_2 & -\mathfrak{p}_2\sigma_1 & * & ,
\end{array} \qquad
\begin{array}{lllll}
d\mathfrak{p}'_1 = & * & +\mathfrak{p}'_2\sigma'_3 & -\mathfrak{p}'_3\sigma'_2, \\
d\mathfrak{p}'_2 = & -\mathfrak{p}'_1\sigma'_3 & * & +\mathfrak{p}'_3\sigma'_1, \\
d\mathfrak{p}'_3 = & +\mathfrak{p}'_1\sigma'_2 & -\mathfrak{p}'_2\sigma'_1 & * & ,
\end{array}$$

and the dual components to them; e.g.:

(55.8)
$$d\bar{\mathfrak{p}}_1 = \mathfrak{p}_2\bar{\sigma}_3 - \mathfrak{p}_3\bar{\sigma}_2 + \bar{\mathfrak{p}}_2\sigma_3 - \bar{\mathfrak{p}}_2\sigma_3 .$$

Thus, one has:

(55.9)
$$\begin{array}{lll}
\sigma_j = \beta_j - \alpha_j, & \sigma'_j = \beta_j + \alpha_j, & \alpha_3 = 0, \\
\bar{\sigma}_j = \bar{\beta}_j - \bar{\alpha}_j, & \bar{\sigma}'_j = \bar{\beta}_j + \bar{\alpha}_j, & \bar{\alpha}_3 = 0,
\end{array}$$

and

(55.10)
$$\begin{array}{ll}
< \underline{\mathfrak{p}}_j \, \underline{\mathfrak{p}}_k > = \delta_{jk}, & < \underline{\mathfrak{p}}'_j \, \underline{\mathfrak{p}}'_k > = \delta_{jk}, \\
[\underline{\mathfrak{p}}_1 \underline{\mathfrak{p}}_2 \underline{\mathfrak{p}}_3] = +1, & [\underline{\mathfrak{p}}'_1 \underline{\mathfrak{p}}'_2 \underline{\mathfrak{p}}'_3] = +1.
\end{array}$$

Let $\mathfrak{z}$ be the intersection point of the three pair-wise perpendicular axes $\underline{\mathfrak{p}}_j$, so:

(55.11)
$$\underline{\mathfrak{p}}_j = \mathfrak{z} \times \mathfrak{p}_j ,$$

and thus:

$$
\begin{aligned}
&z_1 = <\mathfrak{p}_3\, \bar{\mathfrak{p}}_2> = -<\mathfrak{p}_2\, \bar{\mathfrak{p}}_3>, \\
&z_2 = <\mathfrak{p}_1\, \bar{\mathfrak{p}}_3> = -<\mathfrak{p}_3\, \bar{\mathfrak{p}}_1>, \quad \mathfrak{z} = z_1\mathfrak{p}_1 + z_2\mathfrak{p}_2 + z_3\mathfrak{p}_3. \\
&z_3 = <\mathfrak{p}_2\, \bar{\mathfrak{p}}_1> = -<\mathfrak{p}_1\, \bar{\mathfrak{p}}_2>,
\end{aligned} \tag{55.12}
$$

One then obtains the following relations by differentiating (55.12):

$$
d\mathfrak{z} = \mathfrak{p}_1\bar{\sigma}_1 + \mathfrak{p}_2\bar{\sigma}_2 + \mathfrak{p}_3\bar{\sigma}_3 . \tag{55.13}
$$

§ 56. Integrability conditions

It follows by exterior differentiation of (55.2) that:

$$
\begin{aligned}
d\alpha_1 &= [\beta_3\alpha_2], \quad & d\alpha_2 &= [\alpha_1\beta_3], \quad & 0 &= +[\beta_1\alpha_2] + [\alpha_1\beta_2], \\
d\beta_1 &= [\beta_3\beta_2], \quad & d\beta_2 &= [\beta_1\beta_3], \quad & d\beta_3 &= -[\alpha_1\alpha_2] - [\beta_1\beta_2],
\end{aligned} \tag{56.1}
$$

corresponding to (55.3):

$$
\begin{aligned}
d\bar{\alpha}_1 &= [\bar{\beta}_3\alpha_2] + [\beta_3\bar{\alpha}_2], \\
d\bar{\alpha}_2 &= [\bar{\alpha}_1\beta_3] + [\alpha_1\bar{\beta}_3], \\
0 &= [\bar{\beta}_1\alpha_2] + [\beta_1\bar{\alpha}_2] + [\bar{\alpha}_1\beta_2] + [\alpha_1\bar{\beta}_2], \\
d\bar{\beta}_1 &= [\bar{\beta}_3\beta_2] + [\beta_3\bar{\beta}_2], \\
d\bar{\beta}_2 &= [\bar{\beta}_1\beta_3] + [\beta_1\bar{\beta}_3], \\
d\bar{\beta}_3 &= -[\bar{\alpha}_1\alpha_2] - [\alpha_1\bar{\alpha}_2] - [\bar{\beta}_1\beta_2] - [\beta_1\bar{\beta}_2].
\end{aligned} \tag{56.2}
$$

From (55.7), we obtain the equivalent conditions:

$$
\begin{aligned}
d\sigma_1 &= -[\sigma_2\sigma_3], \quad & d\bar{\sigma}_1 &= -[\bar{\sigma}_2\sigma_3] - [\sigma_2\bar{\sigma}_3], \\
d\sigma_2 &= -[\sigma_3\sigma_1], \quad & d\bar{\sigma}_2 &= -[\bar{\sigma}_3\sigma_1] - [\sigma_3\bar{\sigma}_1], \\
d\sigma_3 &= -[\sigma_1\sigma_2], \quad & d\bar{\sigma}_3 &= -[\bar{\sigma}_1\sigma_2] - [\sigma_1\bar{\sigma}_2].
\end{aligned} \tag{56.3}
$$

Corresponding formulas are true for the σ'. In particular, it follows from (55.9), (56.3) that:

$$
\begin{aligned}
\sigma_3 = \sigma'_3,\ \bar{\sigma}_3 &= \bar{\sigma}'_3, \\
[\sigma_1\sigma_2] &= [\sigma'_1\sigma'_2], \\
[\bar{\sigma}_1\sigma_2] + [\sigma_1\bar{\sigma}_2] &= [\bar{\sigma}'_1\sigma'_2] + [\sigma'_1\bar{\sigma}'_2].
\end{aligned} \tag{56.4}
$$

If we introduce the surface elements:

$$\begin{aligned} &[\alpha_1\alpha_2] = \Phi, && [\bar{\alpha}_1\alpha_2] + [\alpha_1\bar{\alpha}_2] = \bar{\Phi}, \\ &[\beta_1\beta_2] = \Psi, && [\bar{\beta}_1\beta_2] + [\beta_1\bar{\beta}_2] = \Psi, \\ &[\sigma_1\sigma_2] = \Omega, && [\bar{\sigma}_1\sigma_2] + [\sigma_1\bar{\sigma}_2] = \bar{\Omega}, \\ &[\sigma'_1\sigma'_2] = \Omega', && [\bar{\sigma}'_1\sigma'_2] + [\sigma'_1\bar{\sigma}'_2] = \bar{\Omega}', \end{aligned} \tag{56.5}$$

then, from (56.3), (56.4), we get:

$$\begin{aligned} -d\sigma_3 = -d\sigma'_3 = -d\beta_3 = \Omega = \Omega', \\ -d\bar{\sigma}_3 = -d\bar{\sigma}'_3 = -d\bar{\beta}_3 = \bar{\Omega} = \bar{\Omega}'. \end{aligned} \tag{56.6}$$

Moreover, it follows from (56.1), (56.2), and (56.6) that:

$$\begin{aligned} \Omega = \Phi + \Psi, \\ \bar{\Omega} = \bar{\Phi} + \bar{\Psi}, \end{aligned} \tag{56.7}$$

and from (56.1), (56.5) that:

$$\beta_3 = \frac{d\alpha_1}{\Phi}\alpha_1 + \frac{d\alpha_2}{\Phi}\alpha_2 = \frac{d\beta_1}{\Phi}\beta_1 + \frac{d\beta_2}{\Phi}\beta_2, \tag{56.8}$$

and finally, from (56.3) that:

$$\sigma_3 = \frac{d\sigma_1}{\Phi}\sigma_1 + \frac{d\sigma_2}{\Phi}\sigma_2. \tag{56.9}$$

The Gaussian curvatures of the quadratic forms $\sigma_1^2 + \sigma_2^2$, $\sigma_1'^2 + \sigma_2'^2$ are, from (22.8), equal to 1, the curvature K_0 of $\alpha_1^2 + \alpha_2^2$ is:

$$K_\alpha = \frac{\Omega}{\Phi} = \frac{\Phi + \Psi}{\Phi}, \tag{56.10}$$

and that of $\beta_1^2 + \beta_2^2$ is:

$$K_\beta = \frac{\Omega}{\Psi} = \frac{\Phi + \Psi}{\Psi}. \tag{56.11}$$

§ 57. Guiding and rest conditions

For the direction of advance vector $d\mathfrak{x}$ of a guided point $\mathfrak{x}$, we obtain, by differentiating (41.2), by means of (55.2), and corresponding to (43.1):

$$\tfrac{1}{2}d\mathfrak{x} = -(\mathfrak{p}_1\bar{\alpha}_1 + \mathfrak{p}_2\bar{\alpha}_2 + \mathfrak{p}_1\bar{\alpha}_2 + \mathfrak{p}_2\bar{\alpha}_2) + \{\mathfrak{x} \times (\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2)\}, \tag{57.1}$$

or, due to (55.11):

(57.2) $$\tfrac{1}{2}\,d\mathfrak{x} = -(\mathfrak{p}_1\bar{\alpha}_1 + \mathfrak{p}_2\bar{\alpha}_2) + \{(\mathfrak{x} - \mathfrak{z})\times(\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2)\}.$$

If we introduce the canonical coordinates x_j for $\mathfrak{x}$ by way of:

(57.3) $$\mathfrak{x} = \mathfrak{z} + x_1\mathfrak{p}_1 + x_2\mathfrak{p}_2 + x_3\mathfrak{p}_3$$

then we will get:

(57.4) $$\tfrac{1}{2}\,d\mathfrak{x} = -(-\bar{\alpha}_1 - x_2\alpha_2)\mathfrak{p}_1 + (-\bar{\alpha}_2 + x_3\alpha_1)\mathfrak{p}_2 + (x_1\alpha_2 - x_2\alpha_1)\mathfrak{p}_3.$$

(57.3), (57.4) yield the *guiding conditions:*

(57.5) $$\begin{array}{lllll} dx_1 = & -\bar{\sigma}_1' & * & +x_2\sigma_3' & -x_3\sigma_2', \\ dx_2 = & -\bar{\sigma}_2' & -x_1\sigma_3' & * & +x_3\sigma_1', \\ dx_3 = & -\bar{\sigma}_3' & +x_1\sigma_2' & -x_2\sigma_1' & * \;\;, \end{array}$$

and correspondingly for the point:

(57.6) $$\mathfrak{x}' = \mathfrak{z}' + x_1'\mathfrak{p}_1' + x_2'\mathfrak{p}_2' + x_3'\mathfrak{p}_3',$$

one gets the *rest conditions:*

(57.7) $$\begin{array}{lllll} dx_1' = & -\bar{\sigma}_1 & * & +x_2'\sigma_3 & -x_3'\sigma_2, \\ dx_2' = & -\bar{\sigma}_2 & -x_1'\sigma_3 & * & +x_3'\sigma_1, \\ dx_3' = & -\bar{\sigma}_3 & +x_1'\sigma_2 & -x_2'\sigma_1 & * \;\;. \end{array}$$

Guiding and rest conditions for *lines* with the canonical coordinates r_j , r_j' are produced by dualizing our formulas (26.10), (26.12):

(57.8) $$\begin{array}{llll} dr_1 = & * & +r_2\sigma_3' & -r_3\sigma_2', \\ dr_2 = & -r_1\sigma_3' & * & +r_3\sigma_1', \\ dr_3 = & +r_1\sigma_2' & -r_2\sigma_1' & * \;\;, \end{array}$$

(57.9) $$\begin{array}{lllllll} d\bar{r}_1 = & * & +r_2\sigma_3' & -r_3\sigma_2' & * & +\bar{r}_2\sigma_3' & -\bar{r}_3\sigma_2', \\ d\bar{r}_2 = & -r_1\sigma_1' & * & +r_3\sigma_1' & -\bar{r}_1\sigma_3' & * & +\bar{r}_3\sigma_1', \\ d\bar{r}_3 = & +r_1\sigma_2' & -r_2\sigma_1' & * & +\bar{r}_1\sigma_2' & -\bar{r}_2\sigma_1' & * \;\;, \end{array}$$

and

(57.10) $$\begin{array}{llll} dr_1' = & * & +r_2'\sigma_3 & -r_3'\sigma_2, \\ dr_2' = & -r_1'\sigma_3 & * & +r_3'\sigma_1, \\ dr_3' = & +r_1'\sigma_2 & -r_2'\sigma_1 & * \;\;, \end{array}$$

(57.11)
$$\begin{array}{lcccccc}
d\bar{r}_1' = & * & +r_2'\sigma_3 & -r_3'\sigma_2 & * & +\bar{r}_2'\sigma_3 & -\bar{r}_3'\sigma_2, \\
d\bar{r}_2' = & -r_1'\sigma_1 & * & +r_3'\sigma_1 & -\bar{r}_1'\sigma_3 & * & +\bar{r}_3'\sigma_1, \\
d\bar{r}_3' = & +r_1'\sigma_2 & -r_2'\sigma_1 & * & +\bar{r}_1'\sigma_2 & -\bar{r}_2'\sigma_1 & * \quad .
\end{array}$$

For the *plane:*

(57.12)
$$u_0 + u_1\, x_1 + u_2\, x_2 + u_3\, x_3 = 0,$$

one ultimately finds the guiding conditions:

(57.13)
$$\begin{array}{lccc}
du_0 = & +u_1\bar{\sigma}_3' & +u_2\bar{\sigma}_2' & +u_3\bar{\sigma}_3', \\
du_1 = & * & +u_2\sigma_3' & -u_3\sigma_2', \\
du_2 = & -u_1\sigma_3' & * & +u_3\sigma_1', \\
du_3 = & +u_1\sigma_2' & -u_2\sigma_1' & *
\end{array}$$

and the rest conditions:

(57.14)
$$\begin{array}{lccc}
du_0' = & +u_1'\bar{\sigma}_3 & +u_2'\bar{\sigma}_2 & +u_3'\bar{\sigma}_3, \\
du_1' = & * & +u_2'\sigma_3' & -u_3'\sigma_2, \\
du_2' = & -u_1'\sigma_3 & * & +u_3'\sigma_1, \\
du_3' = & +u_1'\sigma_2 & -u_2'\sigma_1 & * \quad .
\end{array}$$

§ 58. Canonical axes

If we rotate our associated tetrahedron:

(58.1)
$$\begin{array}{ll}
\mathfrak{Q}_0^* = \mathfrak{Q}_0, & \mathfrak{Q}_3^* = \mathfrak{Q}_3, \\
\mathfrak{Q}_1^* = \mathfrak{Q}_1 \cos\varphi - \mathfrak{Q}_2 \sin\varphi, & \mathfrak{Q}_2^* = \mathfrak{Q}_1 \sin\varphi + \mathfrak{Q}_2 \cos\varphi, \\
\bar{\mathfrak{Q}}_1^* = \bar{\mathfrak{Q}}_1 \cos\varphi - \bar{\mathfrak{Q}}_2 \sin\varphi - \mathfrak{Q}_2^*\bar{\varphi}, & \\
\bar{\mathfrak{Q}}_2^* = \bar{\mathfrak{Q}}_1 \sin\varphi + \bar{\mathfrak{Q}}_2 \cos\varphi + \mathfrak{Q}_1^*\bar{\varphi} &
\end{array}$$

then the Pfaffian forms α, $\bar{\alpha}$ change as follows:

(58.2)
$$\begin{aligned}
\alpha_1^* &= \alpha_1 \cos\varphi - \alpha_2 \sin\varphi, \\
\alpha_2^* &= \alpha_1 \sin\varphi + \alpha_2 \cos\varphi, \\
\bar{\alpha}_1^* &= \bar{\alpha}_1 \cos\varphi - \bar{\alpha}_2 \sin\varphi - \alpha_2^*\bar{\varphi}, \\
\bar{\alpha}_2^* &= \bar{\alpha}_1 \sin\varphi + \bar{\alpha}_2 \cos\varphi + \alpha_1^*\bar{\varphi}.
\end{aligned}$$

Likewise, the β, $\bar{\beta}$ transform as:

(58.3)
$$\begin{aligned}\beta_1^* &= \beta_1 \cos\varphi - \beta_2 \sin\varphi,\\ \beta_2^* &= \beta_1 \sin\varphi + \beta_2 \cos\varphi,\\ \bar{\beta}_1^* &= \bar{\beta}_1 \cos\varphi - \bar{\beta}_2 \sin\varphi - \beta_2^*\bar{\varphi},\\ \bar{\beta}_2^* &= \bar{\beta}_1 \sin\varphi + \bar{\beta}_2 \cos\varphi + \beta_1^*\bar{\varphi}.\end{aligned}$$

On the other hand, for β_3, $\bar{\beta}_3$:

(58.4) $$\beta_3^* = \beta_3 - d\varphi, \qquad \bar{\beta}_3^* = \bar{\beta}_3 - d\bar{\varphi}.$$

Under the assumption that:

(58.5) $$[\alpha_1\ \alpha_2] \neq 0,$$

if we set down the relations:

(58.6) $$\bar{\alpha}_1 = A\alpha_1 + B\alpha_2, \qquad \bar{\alpha}_2 = C\alpha_1 + D\alpha_2$$

then we get:

(58.7)
$$\begin{aligned}A^* &= A\cos^2\varphi - (B+C)\cos\varphi\sin\varphi + D\sin^2\varphi,\\ D^* &= A\sin^2\varphi + (B+C)\cos\varphi\sin\varphi + D\cos\varphi,\\ B^* &= (A-D)\cos\varphi\sin\varphi + B\cos^2\varphi - C\sin^2\varphi - \bar{\varphi},\\ C^* &= (A-D)\cos\varphi\sin\varphi - B\sin^2\varphi + C\cos\varphi + \bar{\varphi}.\end{aligned}$$

From this, it follows that:

(58.8)
$$\begin{aligned}A^* + D^* &= A + D,\\ B^* - C^* &= B - C - 2\bar{\varphi},\\ A^* - D^* &= (A-D)\cos 2\varphi - (B+C)\sin 2\varphi,\\ B^* + C^* &= (A-D)\sin 2\varphi + (B+C)\cos 2\varphi.\end{aligned}$$

From (58.8), one can choose (and generally in essentially one way) φ, $\bar{\varphi}$ such that:

(58.9) $$B^* = C^* = 0.$$

We then call the axis *canonical*. This uniqueness breaks down only in the case:

(58.10) $$A - D = 0, \qquad B + C = 0.$$

We call the values of A^*, D^* that belong to the canonical axes:

(58.11) $$A^* = L_1, \qquad D^* = L_2\,.$$

They are the roots of the quadratic equation:

(58.12) $$L^2 - (A + D)\,L + AD - \left\{\tfrac{1}{2}(B+C)\right\}^2 = 0,$$

or

$$(58.13)\qquad L^2 - \frac{[\bar{\alpha}_1\alpha_2]+[\alpha_1\bar{\alpha}_2]}{[\alpha_1\alpha_2]}L + \frac{4[\bar{\alpha}_1\alpha_2][\alpha_1\bar{\alpha}_2]-\{[\bar{\alpha}_1\alpha_1]-[\bar{\alpha}_2\alpha_2]\}^2}{[\alpha_1\alpha_2]^2}.$$

We have the two rational invariants:

$$(58.14)\qquad \begin{aligned} L_1 + L_2 &= A + D = \frac{[\bar{\alpha}_1\alpha_2]+[\alpha_1\bar{\alpha}_2]}{[\alpha_1\alpha_2]}, \\ L_1L_2 &= AD - \left\{\tfrac{1}{2}(B+C)\right\}^2 = \frac{4[\bar{\alpha}_1\alpha_2][\alpha_1\bar{\alpha}_2]-\{[\alpha_1\bar{\alpha}_1]-[\alpha_2\bar{\alpha}_2]\}^2}{4[\alpha_1\alpha_2]^2}. \end{aligned}$$

From this, one has:

$$(58.15)\qquad (L_1 - L_2)^2 = (A - D)^2 + (B + C)^2.$$

For the directions of the canonical axes, it follows from $(58.8)_4$ that:

$$(58.16)\qquad 2(A - D)\ \alpha_1\ \alpha_2 - (B + C)\ (\alpha_1^2 - \alpha_2^2) = 0,$$

or

$$(58.17)\qquad 2\{[\bar{\alpha}_1\alpha_2]-[\alpha_1\bar{\alpha}_2]\}\alpha_1\alpha_2 - \{[\alpha_1\bar{\alpha}_1]-[\alpha_2\bar{\alpha}_2]\}(\alpha_1^2 - \alpha_2^2) = 0.$$

For the center of the boundary points, we have:

$$(58.18)\qquad \mathfrak{m} = \mathfrak{z} + h\,\mathfrak{p}_3, \qquad h = \frac{C-B}{2} = \frac{1}{2}\frac{[\bar{\alpha}_1\alpha_1]+[\bar{\alpha}_2\alpha_2]}{[\alpha_1\alpha_2]},$$

if $\mathfrak{z}$ means the axis intersection point.

§ 59. Cylindroid of the screw axes

We consider two neighboring positions $\underline{\mathfrak{Q}}$ and $\underline{\mathfrak{Q}} + d\underline{\mathfrak{Q}}$ of our motion process (55.1). For the axis $\underline{\mathfrak{p}}$ of the infinitely small screw from $\underline{\mathfrak{Q}}$ to $\underline{\mathfrak{Q}} + d\underline{\mathfrak{Q}}$, one then has:

$$(59.1)\qquad \underline{\mathfrak{p}} = \frac{1}{\rho}\tilde{\underline{\mathfrak{Q}}}\,d\underline{\mathfrak{Q}}, \qquad \underline{\rho}^2 = \langle d\underline{\mathfrak{Q}},\ d\underline{\mathfrak{Q}}\rangle = \underline{\alpha}_1^2 + \underline{\alpha}_2^2,$$

or, more completely, taking (55.2), (55.5) into account:

$$(59.2)\qquad \mathfrak{p} = \frac{1}{\rho}\tilde{\mathfrak{Q}}\,d\mathfrak{Q} = \frac{1}{\rho}\tilde{\mathfrak{Q}}(\mathfrak{Q}_1\alpha_1 + \mathfrak{Q}_2\alpha_2) = \frac{1}{\rho}(\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2),$$

$$(59.3)\qquad \bar{\mathfrak{p}} = \frac{1}{\rho}(\mathfrak{p}_1\bar{\alpha}_1 + \mathfrak{p}_2\bar{\alpha}_2 + \bar{\mathfrak{p}}_1\alpha_1 + \bar{\mathfrak{p}}_2\alpha_2) - \frac{\bar{\rho}}{\rho}\mathfrak{p},$$

(59.4) $$\rho^2 = \alpha_1^2 + \alpha_2^2, \qquad \rho\bar{\rho} = \alpha_1\bar{\alpha}_1 + \alpha_2\bar{\alpha}_2.$$

If we locate the origin at the intersection point of the axes $\underline{\mathfrak{p}}_j$ then, since $\bar{\mathfrak{p}}_j = 0$, (59.3) simplifies to:

(59.5) $$\bar{\mathfrak{p}} = \frac{1}{\rho}\left\{\mathfrak{p}_1\left(\bar{\alpha}_1 - \frac{\bar{\rho}}{\rho}\alpha_1\right) + \mathfrak{p}_2\left(\bar{\alpha}_2 - \frac{\bar{\rho}}{\rho}\alpha_2\right)\right\}.$$

If we make the intersection point of $\underline{\mathfrak{p}}$ with $\underline{\mathfrak{p}}_3$ be:

(59.6) $$\mathfrak{x} = \mathfrak{p}_3 \, x_3$$

then due to (59.2), (59.6), we get:

(59.7) $$\bar{\mathfrak{p}} = \frac{1}{\rho}\mathfrak{p}_3 \times (\mathfrak{p}_1 \, \alpha_1 + \mathfrak{p}_2 \, \alpha_2) \, x_3 = \left(\mathfrak{p}_2\frac{\alpha_1}{\rho} - \mathfrak{p}_1\frac{\alpha_2}{\rho}\right)x_3.$$

A comparison of (59.5), (59.7) then gives:

(59.8) $$x_3\alpha_1\rho = \bar{\alpha}_2\rho - \alpha_2\bar{\rho}, \qquad x_3\alpha_2\rho = \alpha_1\bar{\rho} - \bar{\alpha}_1\rho,$$

or, due to (59.4):

(59.9) $$x_3 = \frac{\alpha_1\bar{\alpha}_2 - \alpha_2\bar{\alpha}_1}{\alpha_1^2 + \alpha_2^2}.$$

If we employ the canonical coordinates of § 58 then we get:

(59.10) $$x_3 = (L_2 - L_2)\frac{\alpha_1\alpha_2}{\alpha_1^2 + \alpha_2^2}.$$

If we now set:

(59.11) $$x_1 : x_2 = \alpha_1 : \alpha_2$$

then we find that the locus of the screw axes is the cylindroid with the equation:

(59.12) $$(x_1^2 + x_2^2)x_3 = (L_2 - L_2) \, x_1 x_2.$$

On the other hand, if one sets:

(59.13) $$\alpha_1 : \alpha_2 = \cos\varphi : \sin\varphi$$

then one gets:

(59.14) $$x_3 = \frac{L_2 - L_1}{2}\sin 2\varphi.$$

For the "boundary point" on $\mathfrak{p}_3$, we then have:

(59.15) $$x_1 = x_2 = 0, \qquad x_3 = \pm\frac{L_2 - L_1}{2}.$$

The line $\underline{\mathfrak{p}}_3$ is then the double line of the cylindroid (59.12) of the screw axes, and (59.15) yields a geometric interpretation for the invariants L_1, L_2, together with:

(59.16)
$$\frac{\bar{\rho}}{\rho} = \frac{L_1\alpha_1^2 + L_2\alpha_2^2}{\alpha_1^2 + \alpha_2^2}.$$

§ 60. Path congruences of lines

For a two-parameter motion process that acts on lines $\underline{\mathfrak{p}}$, we have:

(60.1)
$$\begin{aligned} d\mathfrak{r} &= 2\mathfrak{r}\times(\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2), \\ d\bar{\mathfrak{r}} &= 2\bar{\mathfrak{r}}\times(\mathfrak{p}_1\alpha_1 + \mathfrak{p}_2\alpha_2) + 2\mathfrak{r}\times(\mathfrak{p}_1\bar{\alpha}_1 + \mathfrak{p}_2\bar{\alpha}_2). \end{aligned}$$

If we take:

(60.2) $\alpha_1 = du,\qquad \alpha_2 = dv,\qquad \bar{\alpha}_1 = L_1\, du,\qquad \bar{\alpha}_2 = L_2\, dv$

for the place in question then we get:

(60.3) $\mathfrak{r}_u = 2\mathfrak{r}\times\mathfrak{p}_1,\qquad \mathfrak{r}_v = 2\mathfrak{r}\times\mathfrak{p}_2,$

and

(60.4)
$$\begin{aligned} \bar{\mathfrak{r}}_u &= 2(\bar{\mathfrak{r}}\times\mathfrak{p}_1) + 2(\mathfrak{r}\times\mathfrak{p}_1)L_1, \\ \bar{\mathfrak{r}}_v &= 2(\bar{\mathfrak{r}}\times\mathfrak{p}_2) + 2(\mathfrak{r}\times\mathfrak{p}_2)L_2. \end{aligned}$$

From (60.3), (60.4), (54.9), it follows that:

(60.5)
$$\begin{aligned} &\tfrac14 E = r_2^2 + r_3^2, \quad \tfrac14 F = -r_1 r_2, \quad \tfrac14 G = r_3^2 + r_1^2, \\ &\tfrac14 P = L_1(r_2^2 + r_3^2) - r_1\bar{r}_2, \quad \tfrac14 Q = -L_2 r_1 r_2 - r_2\bar{r}_1, \\ &\tfrac14 R = -L_1 r_1 r_2 - r_1\bar{r}_2, \quad \tfrac14 S = L_2(r_3^2 + r_1^2) - r_2\bar{r}_2. \end{aligned}$$

If we seek – e.g., the *normal lines* – in moving bodies – i.e., the lines $\underline{\mathfrak{r}}$ that fulfill the condition (54.27) for a normal congruence with their neighbors $\underline{\mathfrak{r}} + d\underline{\mathfrak{r}}$ – then by substituting (60.5) in the condition (54.27), we find the following condition:

(60.6)
$$(L_1 + L_2)\, r_3 + \bar{r}_3 = 0.$$

In any event, the normal lines thus define a thread (a linear complex) with the axis $\underline{\mathfrak{p}}_3$. It degenerates only when:

(60.7)
$$L_1 + L_2 = 0,$$

and indeed for the line in question of the axis $\underline{\mathfrak{p}}_3$.

Correspondingly, by substituting the value (60.5) in the equation:

$$(60.8) \qquad (k_1 - k_2)^2 = 4H^2 + \left(\frac{\bar{\Omega}}{\Omega}\right)^2 = 0$$

one obtains the condition for a ray $\underline{\mathfrak{r}}$ of a congruence with a neighbor $\underline{\mathfrak{r}} + d\underline{\mathfrak{r}}$ to belong to an isotropic line congruence.

§ 61. Surface elements

Starting from the formula (57.4), by forming the alternating vector product, we calculate the vectorial surface element:

$$(61.1) \qquad \begin{aligned} \tfrac{1}{8}[d\mathfrak{x}\times d\mathfrak{x}] = {} & \{[\alpha_2\bar{\alpha}_2]\,x_1 - [\alpha_1\bar{\alpha}_2]\,x_2 + [\alpha_1\alpha_2]\,x_3x_1\}\,\mathfrak{p}_1 \\ & + \{[\bar{\alpha}_1\alpha_2]\,x_1 + [\alpha_1\bar{\alpha}_1]\,x_2 + [\alpha_1\alpha_2]\,x_2x_3\}\,\mathfrak{p}_2 \\ & + \{[\bar{\alpha}_1\bar{\alpha}_2] + ([\alpha_1\bar{\alpha}_2] + [\alpha_2\bar{\alpha}_2])\,x_3 + [\alpha_1\alpha_2]\,x_3x_3\}\,\mathfrak{p}_3. \end{aligned}$$

If we set (under the assumption that $[\alpha_1\ \alpha_2] \neq 0$):

$$(61.2) \qquad \bar{\alpha}_1 = A\alpha_1 + B\alpha_2, \qquad \bar{\alpha}_2 = C\alpha_1 + D\alpha_2$$

then we get:

$$(61.3) \qquad \begin{aligned} \frac{1}{8}\frac{[d\mathfrak{x}\times d\mathfrak{x}]}{[\alpha_1\alpha_2]} = & - \{- C\,x_1 - D\,x_2 + x_3\,x_1\}\mathfrak{p}_1 + \{A\,x_1 + B\,x_2 + x_2\,x_3\}\mathfrak{p}_2 \\ & + \{AD - BC) + (B - C)\,x_3 + x_3^2\,\}\mathfrak{p}_2\,. \end{aligned}$$

The singular points of the path surface thus satisfy the condition:

$$(61.4) \qquad [d\mathfrak{x} \times d\mathfrak{x}] = 0,$$

or, from (61.3), the equations:

$$(61.5) \qquad \begin{aligned} -Cx_1 - Cx_2 + x_3x_1 &= 0, \\ +Ax_1 + Bx_2 + x_2x_3 &= 0, \\ (AD - BC) + (B - C)x_3 + x_3^2 &= 0. \end{aligned}$$

In them, the last one is a consequence of the first two when one does not have $x_1 = x_2 = 0$ simultaneously. The position of the singular points of the path surface thus consists at each moment of two real or conjugate-imaginary lines in the planes x_3 = fixed that satisfy the third equation.

The complex of the normals of the surface element (61.3) yields:

$$
\begin{aligned}
r_1 &= -Cx_1 - Dx_2 + x_3x_1, \\
r_2 &= +Ax_1 + Bx_2 + x_2x_3, \\
r_3 &= (AD - BC) + (B - C)x_2 + x_3^2, \\
\bar{r}_1 &= +(AD - BC)x_2 - Cx_2x_3 - Ax_3x_1, \\
\bar{r}_2 &= -(AD - BC)x_1 - Dx_2x_3 - Bx_3x_1, \\
\bar{r}_3 &= Ax_1^2 + (B + C)x_1x_2 + Dx_2^2.
\end{aligned}
\tag{61.6}
$$

For canonical axes (§ 58), formulas (61.5) simplify to:

$$L_1\, x_1 + x_2\, x_3 = 0, \qquad L_2\, x_2 - x_3\, x_1 = 0, \qquad L_1\, L_2 + x_3^2 = 0, \tag{61.7}$$

and formulas (61.6) simplify to:

$$
\begin{aligned}
r_1 &= -L_2x_2 + x_3x_1, & \bar{r}_1 &= +L_1L_2x_2 - L_1x_3x_1, \\
r_2 &= +L_1x_1 + x_2x_3, & \bar{r}_2 &= -L_1L_2x_1 - L_2x_2x_3, \\
r_3 &= +L_1x_2 + x_3^2, & \bar{r}_3 &= +L_1x_1^2 + L_2x_2^2.
\end{aligned}
\tag{61.8}
$$

§ 62. Special motion processes

Three types of surface-constrained motion processes are especially noteworthy. First, the ones for which:

$$\bar{\Phi} = [\bar{\alpha}_1\alpha_2] + [\alpha_1\bar{\alpha}_2] \tag{62.1}$$

vanishes, so in the canonical coordinates of § 58, one will have:

$$L_1 + L_2 = 0. \tag{62.2}$$

From (60.7), these motion processes are characterized by the degeneracy of the thread of normal lines.

Second, one considers the motion processes for which the cylindroid of screw axes (§ 59) degenerates into a pencil of lines. From (59.9), that gives the condition:

$$\alpha_1\bar{\alpha}_2 - \alpha_2\bar{\alpha}_1 = 0, \tag{62.3}$$

or, in canonical coordinates:

$$L_1 - L_2 = 0. \tag{62.4}$$

Finally, the intersection of (62.2), (62.4) gives:

$$L_1 = L_2 = 0. \tag{62.5}$$

In general, the motion processes with:

(62.6) $$\bar{\alpha}_1 = \bar{\alpha}_2 = 0$$

will be generated in such a way that a rigidly moving surface F "rolls" on a surface F' that is isometric to it in such a way that for every pair of values u, v the point $\mathfrak{x}(u, v)$ of F coincides with the point $\mathfrak{x}'(u, v)$ that corresponds to it under the isometry $F \leftrightarrow F'$, where F and F' contact each other at this point, and the line elements $d\mathfrak{x}(u, v)$, $d\mathfrak{x}'(u, v)$ coincide. This rolling motion (62.5) has been treated many times by geometers; in particular, by L. BIANCHI: cf., L. BIANCHI, Opera VII, "Problemi di rotolamento," Roma, 1957.

We now cast our gaze on motion processes that satisfy the condition:

(62.7) $$\bar{\Omega} = 0.$$

From (56.6), one then has:

(62.8) $$\bar{\Omega} = \bar{\Omega}' = - d\bar{\beta}_3 = - d\bar{\sigma}_3 = - d\bar{\sigma}'_3 = 0,$$

and due to (58.4) we can, by a suitable choice of $\bar{\varphi}$, arrange that:

(62.9) $$\bar{\beta}_3 = 0.$$

From (55.9), one therefore also has:

(62.10) $$\bar{\beta}_3 = \bar{\sigma}_3 = \bar{\sigma}'_3 = 0.$$

Because of (55.13) – in fact, because of:

(62.11) $$d\mathfrak{z} = \mathfrak{p}_1\bar{\sigma}_1 + \mathfrak{p}_2\bar{\sigma}_2, \qquad d\mathfrak{p}_3 = \mathfrak{p}_1\,\sigma_2 - \mathfrak{p}_2\,\sigma_1,$$

the intersection point $\mathfrak{z}$ of the axes then describes a surface F with $\underline{\mathfrak{p}}_3$ as its surface normals. Correspondingly, $\mathfrak{z}'$ describes a surface F' with $\underline{\mathfrak{p}}'_3$ as its surface normals:

(62.12) $$d\mathfrak{z}' = \mathfrak{p}'_1\bar{\sigma}'_1 + \mathfrak{p}'_2\bar{\sigma}'_2, \qquad d\mathfrak{p}'_3 = \mathfrak{p}'_1\,\sigma'_2 - \mathfrak{p}'_2\,\sigma'_1.$$

Due to (56.6), one has:

(62.13) $$\Omega = [\sigma_1\,\sigma_2] = \Omega' = [\sigma'_1\sigma'_2],$$

and that means: The spherical images $(\mathfrak{p}_3)$, $(\mathfrak{p}'_3)$ of the surfaces F, F' are related to each other in an area-preserving way.

However, if the surfaces $\mathfrak{x}(u, v)$, $\mathfrak{x}'(u, v)$ are given with the curvatures:

(62.14) $$K \neq 0, \qquad K' \neq 0,$$

and, according to the condition (62.13), their spherical images $\mathfrak{p}_3(u, v)$, $\mathfrak{p}'_3(u,v)$ are also related to each other in an area-preserving way, then our motion process can be constructed. Starting from a point u_0, v_0, one covers the sphere ($\mathfrak{p}_3$) completely in a neighborhood of this point with curves C that begin at u_0, v_0, and displaces a vector $\mathfrak{p}_1$ that contacts the sphere ($\mathfrak{p}_3$) at u_0, v_0 along C. One likewise constructs the vectors $\mathfrak{p}_1'$. Our motion process:

(62.15) $$\{\mathfrak{z}';\ \mathfrak{p}_1',\ \mathfrak{p}_2',\ \mathfrak{p}_3'\} \rightarrow \{\mathfrak{z};\ \mathfrak{p}_1,\ \mathfrak{p}_2,\ \mathfrak{p}_3\}$$

then takes the axis-crosses thus obtained to each other.

In the special case [cf., (55.9)]:

(62.16) $$\bar{\sigma}_1' - \bar{\sigma}_1 = 0, \qquad \bar{\sigma}_2' - \bar{\sigma}_2 = 0,$$

and therefore [(55.6), (55.9)]:

(62.17) $$L_1 = 0, \qquad L_2 = 0,$$

the surfaces F, F' are related to each other in a length-preserving way, and our motion process consists of the "rolling" of F on F'.

Corresponding to the case $\bar{\Omega} = 0$, one can also generally derive the motion process $\underline{\mathfrak{p}}'_j(u,v) \rightarrow \underline{\mathfrak{p}}_j(u,v)$ that belongs to any pair of mutually related congruences $\underline{\mathfrak{p}}_3(u,v)$, $\underline{\mathfrak{p}}'_3(u,v)$ with:

(62.18) $$\Omega = \Omega' \neq 0, \qquad \bar{\Omega} = \Omega'.$$

§ 63. Outlook

Several question are connected with the consideration of the surface-constrained motion process (= *SMP*) that have not be treated up to now. For instance, there is the derivation of the *SMP*'s with:

(63.1) $$L_1 + L_2 = 0,$$

and the *SMP*'s with:

(63.2) $$L_1 - L_2 = 0.$$

All of the integral theorems about *SMP*'s up to now break down then. Whereas we have only set down the first derivatives here in the small, there seem to be questions relating to the appearance of second derivatives (such as the determination of the curvature for the path surfaces) that are connected with a great expense of computation.

Furthermore, one can ask about non-trivial *SMP*'s for which a great number of points move in a planar domain or a great number of lines of the boundary system move on a fixed thread or run through the normals of a surface (or isotropic congruences).

Even the simplest algebraic *SMP*'s merit attention. They correspond to algebraic surfaces in the six-dimensional space of q_j, $\bar{q}_j$ (j = 0, 1, 2, 3) with:

(63.3) $$\sum q_j^2 = 1, \qquad \sum q_j \bar{q}_j = 0.$$

One can also seek to convert the theorem that L. BIANCHI found on the two-dimensional rolling of length-preserving surfaces to the general case of *SMP*'s with $\bar{\Omega} = 0$.

Finally, one can still look at the two-parameter motion processes that lead back to one-parameter ones:

$$\underline{\mathfrak{Q}}(u,v) = \underline{\mathfrak{Q}}_1(u) \cdot \underline{\mathfrak{Q}}_2(v) . \tag{63.4}$$

The line congruences for which either:

$$\underline{\mathfrak{r}}(u,v) = \underline{\mathfrak{r}}_1(u)\underline{\mathfrak{r}}_2(v) \tag{63.5}$$

or

$$\underline{\mathfrak{r}}(u,v) = (\underline{\mathfrak{r}}_1(u) \times \underline{\mathfrak{r}}_2(v)) : \sqrt{1 - < \underline{\mathfrak{r}}_1(u), \underline{\mathfrak{r}}_2(v) >^2} \tag{63.6}$$

have already been examined. Substantial investigations on the line congruences are contained in volume VI of the complete works of L. BIANCHI (Opera VI, "Congruenze di rette e di sfere e loro deformazioni," Roma, 1957), along with the book of FINIKOW that was cited in § 54.

I am perfectly aware that this present pamphlet, which will indeed be my last one, exhibits many deficiencies and oversights. Hopefully, this situation might induce younger geometers to take up this classical subject anew!