On
the relations between

two general ray systems,

one of which proceeds into the other one by various reflections and refractions in media with arbitrary wave surfaces
and the consequences that yields for

optically-representable ray systems

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At the beginning of the Winter semester from 1881 to 1882, Herrn Prof. Weierstrass posed the following problem in the mathematical seminar:

Prove that a ray system whose rays possess the property of being normal to a surface in an isotropic medium will always retain that property when one comes back to an isotropic medium after various reflections and refractions in media with arbitrary wave surfaces.

In my efforts to go into optical ray systems in more detail, I found, among other things, the following theorem of Herrn Prof. Kummer that he published in the Monatsberichten der Berliner Akademie der Wissenschaften in the year 1860:

Any infinitely-thin optical ray bundle inside of a homogeneous, transparent medium has the property that its two focal planes cut out two curves from the wave surface of light that belongs to that medium, whose center can be chosen to be on the axis of the ray bundle, that intersect in conjugate directions. Any ray bundle that has this property is also actually optically-representable.

Since Kummer did not publish a proof of that theorem, which you will allow me to refer to as Kummer's theorem, I sought to derive a proof of it. That led to the development of the present work, whose content is the following:

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§ 1. Light seeks to travel from the point P with coordinates (x_0, y_0, z_0) to a point Q with the coordinates (x_1, y_1, z_1) in the shortest time. The point P lies in a medium for which one lets the velocity (ρ) of light in the direction (ξ, η, ζ) be:

$$\rho = \varphi(\xi, \eta, \zeta).$$

The point Q lies in a medium for which one lets the velocity of light (ρ_1) in the direction (ξ_1, η_1, ζ_1) be:

$$\rho_1 = \varphi_1(\xi_1, \eta_1, \zeta_1).$$

Let the equation of the separation surface of both media be:

$$f(x, y, z) = 0.$$

The light ray that goes from P to Q meets the separation surface at the point S with the coordinates x, y, z.

If the segment $PS = r$ has the direction (ξ, η, ζ) and the segment $SQ = r_1$ has the direction (ξ_1, η_1, ζ_1) then one will have:

$$\xi = \frac{x - x_0}{r}, \qquad \eta = \frac{y - y_0}{r}, \qquad \zeta = \frac{z - z_0}{r},$$

$$r = +\sqrt{\sum (x - x_0)^2},$$

(1) $$\xi_1 = -\frac{x - x_0}{r_1}, \quad \eta_1 = -\frac{y - y_0}{r_1}, \quad \zeta_1 = -\frac{z - z_0}{r_1},$$

$$r_1 = +\sqrt{\sum (x - x_1)^2},$$

in which, now as well as later, I interpret the simple symbol "Σ" to mean that the summand that is written down is to be added to two other summands that have the same interpretation relative to the y and z axes that the term that is written down has to the x-axis. The time (T) that light needs in order to travel from P from Q is:

(2) $$T = \frac{r}{\rho} + \frac{r_1}{\rho_1}.$$

T is a function of x, y, z whose magnitude must satisfy the condition $f(x, y, z) = 0$. Should T be a minimum, then the following equations would have to be true:

(3) $$\frac{\partial T}{\partial x} - \mu \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial T}{\partial y} - \mu \frac{\partial f}{\partial y} = 0, \qquad \frac{\partial T}{\partial z} - \mu \frac{\partial f}{\partial z} = 0,$$

where μ is an undetermined factor such that these three equations will yield only two condition equations between the determining data of the incident and refracted rays. Now, if, e.g.:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\frac{r}{\rho}\right) + \frac{\partial}{\partial x}\left(\frac{r_1}{\rho_1}\right) \tag{4}$$

then with the use of equations (1) one will get:

$$\frac{\partial}{\partial x}\left(\frac{r}{\rho}\right) = \frac{1}{\rho}\cdot\frac{\partial r}{\partial x} + r\cdot\left\{\frac{\partial(1/\rho)}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial(1/\rho)}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial(1/\rho)}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\right\}$$

or

$$\frac{\partial}{\partial x}\left(\frac{r}{\rho}\right) = \left(\frac{1}{\rho} - \sum\frac{\partial(1/\rho)}{\partial \xi}\xi\right)\cdot\xi + \frac{\partial(1/\rho)}{\partial \xi} = \lambda\cdot A. \tag{5}$$

One obtains similar expressions for:

$$\frac{\partial}{\partial y}\left(\frac{r}{\rho}\right) = \lambda\cdot B, \qquad \frac{\partial}{\partial z}\left(\frac{r}{\rho}\right) = \lambda\cdot C.$$

In the same way, one gets:

$$\frac{\partial}{\partial x}\left(\frac{r_1}{\rho_1}\right) = -\left(\frac{1}{\rho_1} - \sum\frac{\partial(1/\rho_1)}{\partial \xi_1}\xi_1\right)\cdot\xi_1 + \frac{\partial(1/\rho_1)}{\partial \xi_1} = -\lambda_1\cdot A_1, \tag{6}$$

and analogously:

$$\frac{\partial}{\partial y}\left(\frac{r_1}{\rho_1}\right) = -\lambda_1\cdot B_1, \quad \frac{\partial}{\partial z}\left(\frac{r_1}{\rho_1}\right) = -\lambda_1\cdot C_1.$$

If follows from (5) that one has:

$$\lambda\sum A\,d(\rho\,\xi) = \sum\left[\left(\frac{1}{\rho} - \sum\frac{\partial(1/\rho)}{\partial \xi}\right)\xi + \frac{\partial(1/\rho)}{\partial \xi}\right](d\rho\cdot\xi + \rho\cdot d\xi).$$

Since:

$$\sum\xi^2 = 1, \qquad \sum\xi\,d\xi = 0, \qquad \sum\frac{\partial(1/\rho)}{\partial \xi}d\xi = d\left(\frac{1}{\rho}\right),$$

one will have:

$$\lambda\sum A\,d(\rho\,\xi) = \frac{1}{\rho}\cdot d\rho + \rho\cdot d\left(\frac{1}{\rho}\right) = 0. \tag{7}$$

One likewise obtains:

$$\sum A_1\, d(\rho_1\, \xi_1) = 0,$$

from which, it will follow that the quantities A, B, C are proportional to the direction cosines for the normal to the wave surface of the first medium at the point at which the radius vector that points in the direction (ξ, η, ζ) meets it, and the analogous statement will be true for the second medium.

If one defines $\sum \lambda \cdot A\,\rho\,\xi$ with the help of equations (5) then one will obtain:

$$\begin{aligned}\sum \lambda \cdot A\,\rho\,\xi &= \sum\left[\left(\frac{1}{\rho} - \sum\frac{\partial(1/\rho)}{\partial\xi}\cdot\xi\right)\cdot\rho\xi + \frac{\partial(1/\rho)}{\partial\xi}\cdot\rho\xi\right] \\ &= \rho\cdot\left[\left(\frac{1}{\rho} - \sum\frac{\partial(1/\rho)}{\partial\xi}\cdot\xi\right)\cdot\sum\xi^2 + \frac{\partial(1/\rho)}{\partial\xi}\cdot\xi\right] = 1.\end{aligned}$$

Thus:

(8) $$\lambda = \frac{1}{\sum A\rho\xi}, \qquad \text{and likewise} \qquad \lambda_1 = \frac{1}{\sum A_1\rho_1\xi_1}.$$

Thus, we will now have:

$$\frac{\partial}{\partial x}\left(\frac{r}{\rho}\right) = \frac{A}{\sum A\rho\xi}, \qquad \frac{\partial}{\partial x}\left(\frac{r_1}{\rho_1}\right) = \frac{A_1}{\sum A_1\rho_1\xi_1}.$$

Since one gets analogous expressions for the remaining quantities, one can now write equations (3) as:

(9) $$\begin{aligned}\frac{A}{\sum A\rho\xi} - \frac{A_1}{\sum A_1\rho_1\xi_1} &= \mu\,\frac{\partial f}{\partial x}, \\ \frac{B}{\sum A\rho\xi} - \frac{B_1}{\sum A_1\rho_1\xi_1} &= \mu\,\frac{\partial f}{\partial y}, \\ \frac{C}{\sum A\rho\xi} - \frac{C_1}{\sum A_1\rho_1\xi_1} &= \mu\,\frac{\partial f}{\partial z}.\end{aligned}$$

If one multiplies equations (9) by the arbitrary quantities X, Y, Z, resp., and adds them then one can give the final equation the form:

(10) $$\left(\frac{\sum AX}{\sum A\rho\xi} - 1\right) - \left(\frac{\sum A_1X}{\sum A_1\rho_1\xi_1} - 1\right) = \mu\sum\frac{\partial f}{\partial x}X.$$

If one regards X, Y, Z as running coordinates then the equation $\frac{\sum AX}{\sum A\rho\xi} - 1 = 0$, which one can also give the form $\sum A\,(X - \rho\,\xi) = 0$, will be the equation of the tangential plane that is drawn at the point ($\rho\xi$, $\rho\eta$, $\rho\zeta$) to the wave surface in the first medium that is constructed with its center at the starting point.

Similarly, $\frac{\sum A_1 X}{\sum A_1 \rho_1 \xi_1} - 1 = 0$, or $\sum A_1 (X - \rho_1 \xi_1) = 0$, will be the equation of the tangential plane that is drawn at the point ($\rho_1\xi_1$, $\rho_1\eta_1$, $\rho_1\zeta_1$) to the wave surface in the second medium, which is likewise constructed with its center at the starting point.

Furthermore: $\sum \frac{\partial f}{\partial x} X = 0$ is the equation of the plane that goes through the starting point that is parallel to the tangential plane that is drawn through it at the point (x, y, z) of the separation surface at which the incident ray meets it.

Since eq. (10) will then be an identity equation for any system of values for X, Y, Z, this equation expresses the idea that each of the three planes goes through the intersection of the other two. One therefore finds the refracted ray in the following way: Let the wave surfaces of both media be constructed with their centers at the starting point O. One draws a radius vector of the wave surface in the first medium that is parallel to the incident ray, lays a tangential plane through its endpoint on that surface and lays a plane ON through the point O that is parallel to the tangential plane to the separation surface at the point at which the incident ray meets it. Finally, if one lays a tangential plane to the wave surface in the second medium through the intersection N of both planes then the radius vector OM to the contact point M of it will give the direction and velocity of the refracted ray. The tangential planes to the two wave surfaces must lie on the same side of O.

Since the principle of fastest arrival time is likewise true for reflections, eq. (9) and eq. (10), which is derived from it, will also be true for them. Now, the first medium is also to be considered as being the second medium: The wave surface of the second medium will then coincide with that of the first one. However, since the directions of the incident and reflected rays lie on opposite sides of the tangential planes to the separation surface (which would be better called the "reflecting surface"), the two tangential planes NL and NM must lie on opposite sides of O, here. If one can lay several tangential planes from the line of intersection N to the wave surface in the second medium on the same side of the point O then the incident ray will split by refraction (reflection, resp.) into a corresponding number of rays.

§ 2. Kummer regarded the determining data of a ray system as given functions of two independent variables in his treatise on general ray systems. If will also be very preferable to think of all of the given or calculated quantities as given (determined, resp.) functions of two independent variables u, v in the present article.

In general, $x_{s,\,s+1}$, $y_{s,\,s+1}$, $z_{s,\,s+1}$ might denote the coordinates of a point of the separation surface between the s^{th} and $(s + 1)^{\text{th}}$ medium, which are coupled by the equation:

$$f_{s,\,s+1}(x_{s,\,s+1}, y_{s,\,s+1}, z_{s,\,s+1}) = 0,$$

and ξ_s, η_s, ζ_s might mean the direction cosines of the rays of the ray system in the s^{th} medium, or – as I would like to say more briefly – the s^{th} ray system. I imagine the wave surfaces of all media as being constructed with their centers at the coordinate origin, and generally let the wave surface in the s^{th} medium be given by the equation:

$$0 = f_s\,(\rho_s\,\xi_s,\ \rho_s\,\eta_s,\ \rho_s\,\zeta_s),$$

where ρ_s refers to the radius vector to the wave surface that is parallel to the ray (ξ_s, η_s, ζ_s), and as such, it will imply the velocity of light in the medium in a unit time. It will be assumed that all surfaces possess no singularities for the components that come under consideration. Furthermore, A_s, B_s, C_s ; A_{s+1}, B_{s+1}, C_{s+1} ; $A_{s,\,s+1}$, $B_{s,\,s+1}$, $C_{s,\,s+1}$ might refer to quantities that are proportional to the direction cosines of the normal to the wave surface of the s^{th} medium, the normal to the wave surface $(s + 1)^{th}$ medium, and the normal to the separation surface of the s^{th} and $(s + 1)^{th}$ medium, respectively. Let the length of a ray in the s^{th} medium be r_s , such that:

(1) $$r_s = \sqrt{\sum (x_{s,s+1} - x_{s-1,s})^2}\ .$$

It will follow from the definition of r_s that:

(2) $$\begin{aligned} x_{s,s+1} &= x_{s-1,s} + r_s\,\xi_s\,, \\ y_{s,s+1} &= y_{s-1,s} + r_s\,\eta_s\,, \\ z_{s,s+1} &= z_{s-1,s} + r_s\,\zeta_s\,. \end{aligned}$$

Let the quantities $x_{s-1,s}$, $y_{s-1,s}$, $z_{s-1,s}$, ξ_s, η_s, ζ_s, r_s be determined already as functions of the two independent variables u, v, and in fact, as single-valued and continuous functions of u, v for a well-defined ray complex that has been taken from the s^{th} ray system. They can then be determined as functions of u, v for the same symbols, but with quantities that are denoted with the next higher index, as well as r_s, as follows.

The equation of a ray of the s^{th} ray system reads:

$$\frac{X - x_{s-1,s}}{\xi_s} = \frac{Y - y_{s-1,s}}{\eta_s} = \frac{Z - z_{s-1,s}}{\zeta_s}.$$

Should this ray meet the separation surface of the s^{th} and $(s + 1)^{th}$ medium at the point ($x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$) then one would need to have:

$$\frac{x_{s,s+1} - x_{s-1,s}}{\xi_s} = \frac{y_{s,s+1} - y_{s-1,s}}{\eta_s} = \frac{z_{s,s+1} - z_{s-1,s}}{\zeta_s}.$$

$x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$ can now be determined as functions of $x_{s-1,s}$, $y_{s-1,s}$, $z_{s-1,s}$, ξ_s, η_s, ζ_s, and also as functions of u, v, from these two equations and the equation $f_{s,s+1}(x_{s,s+1}, y_{s,s+1}, z_{s,s+1}) = 0$ of the separation surface. These functions will be multi-valued functions, in general. However, if one directs one's attention to just the values of $x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$ that correspond to the points at which the rays of the complex considered cut the surface $f_{s,s+1}(\ldots) = 0$ for the first time in the direction (ξ_s, η_s, ζ_s) then one can regard $x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$ as single-valued, continuous functions of u, v. (The component of the surface at which the rays cut the surface the second time after they are refracted is to be considered as being a component of the surface $f_{s+1,s+2}(\dots) = 0$.) Since $x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$ are determined as functions of u, v, r_s will also be defined as a function of u, v. Eq. (9) of the previous paragraph must be valid between the s^{th} and $(s + 1)^{\text{th}}$ ray system, which now reads, in the other notation:

$$\frac{A_s}{\sum A_s \rho_s \xi_s} - \frac{A_{s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}} = \mu\, A_{s,s+1}.$$

The other two equations are analogous.

Since one can set $A_s = \dfrac{\partial f_s}{\partial(\rho_s \xi_s)}$, one can regard A_s, B_s, C_s as being given functions of $\rho_s\ \xi_s$, $\rho_s\ \eta_s$, $\rho_s\ \zeta_s$; likewise, A_{s+1}, B_{s+1}, C_{s+1} are given functions of $\rho_{s+1}\ \xi_{s+1}$, $\rho_{s+1}\ \eta_{s+1}$, $\rho_{s+1}\ \zeta_{s+1}$ ($x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$, resp.). Now, since ρ_s, ξ_s, η_s, ζ_s, $x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$ are known functions of u, v, eq. (3) will yield two relations between the four quantities $x_{s,s+1}$, $y_{s,s+1}$, $z_{s,s+1}$, and u, v when one eliminates the undetermined μ. By means of it and the equations $f_{s+1}(\rho_{s+1}\ \xi_{s+1}, \rho_{s+1}\ \eta_{s+1}, \rho_{s+1}\ \zeta_{s+1}) = 0$ and $\sum \xi_{s+1}^2 = 1$, the four quantities can now be determined as functions of u¸v. These functions will be multi-valued functions, in general. However, if one chooses a well-defined pair of values (u_0, v_0) that correspond to a central ray of the complex to be one that is determined by a system of values of ρ_{s+1}, ξ_{s+1}, η_{s+1}, ζ_{s+1} that belongs to it – which I would like to denoted by ξ_0, η_0, ζ_0 – then one can, within certain limits, represent the differences $\rho_{s+1} - \rho_0$, $\xi_{s+1} - \xi_0$, $\eta_{s+1} - \eta_0$, $\zeta_{s+1} - \zeta_0$ as powers of the differences $u - u_0$, $v - v_0$, and one can extend the domain of validity of these series to all rays of the complex, since the functions ρ_s, etc., are single-valued and continuous for them, and since the surface $f_{s+1} = 0$ possesses no singularities for them, and the quantities ρ_{s+1}, ξ_{s+1}, η_{s+1}, ζ_{s+1} are not infinitely large, moreover. These power series then define the quantities ρ_{s+1}, ξ_{s+1}, η_{s+1}, ζ_{s+1} as functions of u, v. The $(s + 1)^{\text{th}}$ ray system that is obtained from this kind of determination belongs to an arbitrarily-chosen one of the ray systems that emerge from the s^{th} ray system by refraction. If one draws a line through a fixed point that is parallel to each ray of the chosen $(s + 1)^{\text{th}}$ ray system whose length yields the velocity of light in it in a certain system of units then the end points of these lines will define a surface that I would like to call the *wave surface of the* $(s + 1)^{\text{th}}$ *ray system*, and which is a component – or, as one can also say, a *shell* – of the general wave surface of the $(s + 1)^{\text{th}}$ medium. ρ_{s+1} is the radius vector of that shell.

Let the first ray system be determined by the coordinates x_{01}, y_{01}, z_{01} of a point through which a ray goes and by the direction cosines ξ_1, η_1, ζ_1 of that ray. Let these six quantities be given functions of u, v, and indeed, in such a way that they are single-valued and continuous for the chosen ray complex. The points x_{01}, y_{01}, z_{01} define a surface that will be called the *initial surface of the ray system*. Furthermore, let the velocity ρ_1 of the ray with the direction cosines ξ_1, η_1, ζ_1 be determined uniquely by these givens, and thus also as a single-valued, continuous function of u, v. As a consequence of the foregoing discussion, one can now determine all of the quantities that are present as single-valued, continuous functions of u, v from these and the other given quantities, and indeed in such a way that the ray in an arbitrary one of the various ray systems that belongs to a well-defined pair of values (u, v) will correspond to the ray in the first ray system that belongs to the same pair of values (u, v), in such a way that it arises from that ray by various refractions and reflections.

The theorems that will be derived in what follows will also be valid when the functions are multi-valued, since the necessity of the functions being single-valued will not enter into them anywhere. I have assumed that they are single-valued only in order to not make it necessary to modify their statements or corollaries in places at which double-valuedness can possibly arise.

§ 3. Let the n^{th} ray system be the last one. The systems of equations (2) and (3) in the previous paragraph then yield the following equations, which are valid for $s = 1, 2, \ldots, n - 1$:

$$\begin{aligned} x_{s,s+1} &= x_{s-1,s} + \xi_s\, r_s\,, \\ y_{s,s+1} &= y_{s-1,s} + \eta_s\, r_s\,, \\ z_{s,s+1} &= z_{s-1,s} + \zeta_s\, r_s\,, \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{A_s}{\sum A_s \rho_s \xi_s} - \frac{A_{s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}} &= \mu\, A_{s,s+1}\,, \\ \frac{B_s}{\sum A_s \rho_s \xi_s} - \frac{B_{s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}} &= \mu\, B_{s,s+1}\,, \\ \frac{C_s}{\sum A_s \rho_s \xi_s} - \frac{C_{s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}} &= \mu\, C_{s,s+1}\,, \end{aligned} \tag{2}$$

in which the quantities A are defined such that the following equations exist:

$$\begin{aligned} &\sum A_s\, d(\rho_s\, \xi_s) = 0, \\ &\sum A_{s+1}\, d(\rho_{s+1}\, \xi_{s+1}) = 0, \\ &\sum A_{s,s+1}\, dx_{s,s+1} = 0. \end{aligned} \tag{3}$$

If one chooses the initial surface of the last (i.e., n^{th}) ray system to be a surface ($x_{n,n+1}$, $y_{n,n+1}$, $z_{n,n+1}$), instead of the separation surface ($x_{n-1,n}$, $y_{n-1,n}$, $z_{n-1,n}$) of the $(n - 1)^{\text{th}}$ and n^{th} medium, such that up to it the rays of the n^{th} system cover the segment r_n in the n^{th} medium, where r_n is a temporarily arbitrary function of u, v, then the following equations will be true:

$$\begin{aligned} x_{n,n+1} &= x_{n-1,n} + r_n\, \xi_n\,, \\ y_{n,n+1} &= y_{n-1,n} + r_n\, \eta_n\,, \\ z_{n,n+1} &= z_{n-1,n} + r_n\, \zeta_n\,. \end{aligned} \tag{4}$$

If one multiplies eqs. (2) by $dx_{s,s+1}$, $dy_{s,s+1}$, $dz_{s,s+1}$, in turn, and considers the third of eq. (3) then if one brings the second term on the left-hand side to the right-hand side then one will obtain:

(5) $$\frac{\sum A_s dx_{s,s+1}}{\sum A_s \rho_s \xi_s} = \frac{\sum A_{s+1} dx_{s,s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}}.$$

As a result of equations (1), one will have:

(6) $$\sum A_s \, dx_{s,s+1} = \sum A_s \, dx_{s-1,s} + \sum A_s \, d(r_s \, \xi_s).$$

Now, one has:

$$\sum A_s \, d(r_s \, \xi_s) = \sum A_s \, d\left(\frac{r_s}{\rho_s} \cdot \rho_s \xi_s\right) = d\left(\frac{r_s}{\rho_s}\right) \sum A_s \rho_s \xi_s + \frac{r_s}{\rho_s} \sum A_s \, d(\rho_s \xi_s),$$

or, as a result of the first of eqs. (3):

$$\sum A_s \, d(r_s \, \xi_s) = d\left(\frac{r_s}{\rho_s}\right) \sum A_s \rho_s \xi_s .$$

If one substitutes this in eq. (6) and divides by $\sum A_s \, \rho_s \, \xi_s$ then one will obtain:

(7) $$\frac{\sum A_s dx_{s,s+1}}{\sum A_s \rho_s \xi_s} = \frac{\sum A_s dx_{s-1,s}}{\sum A_s \rho_s \xi_s} + d\left(\frac{r_s}{\rho_s}\right).$$

Since eq. (4) have the same form as eq. (1), that will yield, in the same way:

(8) $$\frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n} = \frac{\sum A_n dx_{n-1,n}}{\sum A_n \rho_n \xi_n} + d\left(\frac{r_n}{\rho_n}\right).$$

With consideration given to eq. (7), eq. (5) now reads:

(9) $$\frac{\sum A_s dx_{s-1,s}}{\sum A_s \rho_s \xi_s} + d\left(\frac{r_s}{\rho_s}\right) = \frac{\sum A_{s+1} dx_{s,s+1}}{\sum A_{s+1} \rho_{s+1} \xi_{s+1}}.$$

Since s can assume the values 1, 2, …, n – 1, eq. (7) will represent (n – 1) equations. If one adds them together then one will see that for s = 2, 3, …, n – 1, each term:

$$\frac{\sum A_s dx_{s-1,s}}{\sum A_s \rho_s \xi_s}$$

on the left-hand side will cancel an equal term on the right-hand side, and one will then obtain:

(10) $$\frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + \sum_{s=1}^{n-1} d\left(\frac{r_s}{\rho_s}\right) = \frac{\sum A_n dx_{n-1,n}}{\sum A_n \rho_n \xi_n}.$$

With the use of eq. (8), one can also write eq. (10) as:

(11) $$\frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + \sum_{s=1}^{n} d\left(\frac{r_s}{\rho_s}\right) = \frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n}.$$

Up to now, r_n was an arbitrary function of u, v. Now, if one determines r_s in such a way that:

$$\sum_{s=1}^{n} d\left(\frac{r_s}{\rho_s}\right) = d\sum_{s=1}^{n} \frac{r_s}{\rho_s} = 0,$$

or

(12) $$\sum_{s=1}^{n} \frac{r_s}{\rho_s} = C,$$

and thus equal to a constant, then if eq. (12) is true then eq. (11) will read:

(13) $$\frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} = \frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n}.$$

Since eq. (13) is true for any increase in the variables u, v, and thus for any value of du / dv, it will express two relations that exist between the first and last ray system. However, as was emphasized in § 1, only two relations exist between two ray systems, one of which arises from the other by refraction or reflection, with the exception of the relation that corresponding rays of both systems must cut in a point of the separation surface. However, the latter restriction drops out under multiple refractions, such that in general only two relations will exist between two ray systems, one of which arises from the other by various refractions and reflections, and they will be expressed by eq. (13).

Eq. (12) tells one how the initial surface of the second ray system should be chosen in order for the relations that exist between the ray systems to be expressible in the form of eq. (13).

Eq. (11), as well as eq. (12) and eq. (13), which are derived from them, also tells one how large that n can be. Therefore, n can also be infinitely large – i.e., the light can go through inhomogeneous media. One can then think of an inhomogeneous medium as being decomposed into infinitely many infinitely-small strips and regard each of those pieces as a homogeneous medium.

ρ_s denotes the velocity of light in the s^{th} medium in the direction (ξ_s η_s, ζ_s), in some system of units, and r_s is the length of the path that the light follows in the same direction in the s^{th} medium, so r_s / ρ_s will be the time during which a light ray traverses the s^{th} medium, and therefore $\sum_{s=1}^{n} \frac{r_s}{\rho_s}$ will be the time that light ray needs in order to travel from the surface (x_{01}, y_{01}, z_{01}) to the surface ($x_{n,\,n+1}$, $y_{n,\,n+1}$, $z_{n,\,n+1}$) through the different media. Eq. (12) then says that all rays take the same length of time in order to travel from the surface (x_{01}, y_{01}, z_{01}) to the surface ($x_{n,\,n+1}$, $y_{n,\,n+1}$, $z_{n,\,n+1}$).

If one chooses an arbitrary infinitely-thin ray bundle from the first ray system then it will correspond to a likewise infinitely-thin ray bundle in the last ray system, which emerges from the latter by various refractions and reflections. Now, if one lays a plane through the first ray bundle that is perpendicular to normal of the wave surface in the first system (namely, the normal at that point of the wave surface at which the radius vector that is parallel to the central ray meets the wave surface) that corresponds to the central ray (i.e., the axis) of the bundle and considers this plane to be the initial surface of the rays of the bundle then one will have:

$$\sum A_1\, dx_{01} = 0.$$

Should eq. (13), and with it, eq. (12), be true then, since $\sum A_n\ \rho_n\ \xi_n$ becomes infinitely large only when one has $\rho_n = \infty$, one must have:

$$\sum A_n\, dx_{n,n+1} = 0.$$

The initial surface of the rays of the corresponding bundle of the last ray system must then be chosen to be likewise a plane that is perpendicular to the normal to the wave surface of the last ray system that corresponds to the axis of the bundle, and one can therefore express the relations between the ray systems in the following way:

If a ray system arises from a given ray system by various refractions and reflections, and one assumes that the rays of any infinitely-thin ray bundle of the first system simultaneously go through a plane that is parallel to the tangential plane to the wave surface of the first ray system at the endpoint of the radius vector that is drawn parallel to the axis of the bundle then the rays of the infinitely-thin ray bundle of the second ray system that arise from that bundle will also simultaneously pass through a plane that is parallel to the tangential plane to the wave surface of the second ray system at the end point of the radius vector that is drawn parallel to the axis of the bundle.

This theorem also expresses the relations that were given by eqs. (12) and (13) completely; one then chooses another arbitrary initial surface $(x'_{01}, y'_{01}, z'_{01})$, instead of the plane ($x_{01}$, y_{01}, z_{01}), in the infinitely-thin ray bundle of the first system and sets:

$$x'_{01} = x_{01} - d_1 \cdot \xi_1\,, \qquad y'_{01} = y_{01} - d_1 \cdot \eta_1\,, \qquad z'_{01} = z_{01} - d_1 \cdot \zeta_1.$$

If one likewise chooses another initial surface $(x'_{n.n+1}, y'_{n,n+1}, z'_{n,n+1})$, instead of the plane ($x_{n,n+1}$, $y_{n,n+1}$, $z_{n,n+1}$), in the infinitely-thin ray bundle of the lat system and sets:

$$x'_{n,n+1} = x_{n,n+1} + d_n \cdot \xi_n\,, \qquad y'_{n,n+1} = y_{n,n+1} + d_n \cdot \eta_n\,,$$

$$z'_{n,n+1} = z_{n,n+1} + d_n \cdot \zeta_n$$

then that will yield, in the same way that eq. (7) follows from eq. (1):

$$\frac{\sum A_1\, dx'_{01}}{\sum A_1 \rho_1 \xi_1} + d\left(\frac{d_1}{\rho_1}\right) = \frac{\sum A_1\, dx_{01}}{\sum A_1 \rho_1 \xi_1}$$

and

$$\frac{\sum A_n\, dx'_{n,n+1}}{\sum A_n \rho_n \xi_n} = \frac{\sum A_n\, dx_{n,n+1}}{\sum A_n \rho_n \xi_n} + d\left(\frac{d_n}{\rho_n}\right).$$

Due to the position of the planes, one will have:

$$\sum A_1\, dx_{01} = \sum A_n\, dx_{n,n+1} = 0.$$

As a result, the foregoing equations yield:

$$\frac{\sum A_1\, dx'_{01}}{\sum A_1 \rho_1 \xi_1} + d\left(\frac{d_1}{\rho_1}\right) = \frac{\sum A_n\, dx'_{n,n+1}}{\sum A_n \rho_n \xi_n} - d\left(\frac{d_n}{\rho_n}\right).$$

Now, from the aforementioned theorem, one has: $\sum_{s=1}^{n} \frac{r_s}{\rho_s} = 0.$

Therefore, the last equation will also remain correct when one writes it as:

$$\frac{\sum A_1\, dx'_{01}}{\sum A_1 \rho_1 \xi_1} + \frac{d_1 + r_1}{\rho_1} + \sum_{s=2}^{n-1} \frac{r_s}{\rho_s} + \frac{r_n + d_n}{\rho_n} = \frac{\sum A_n\, dx'_{n,n+1}}{\sum A_n \rho_n \xi_n}.$$

Now, $(d_1 + r_1)$ is equal to the r_1 that corresponds to the surface $(x'_{01}, y'_{01}, z'_{01})$, and $(d_n + r_n)$ is equal to the r_n that corresponds to the surface $(x'_{n.n+1}, y'_{n,n+1}, z'_{n,n+1})$, and therefore this equation is eq. (11), only in a somewhat different notation, from which it follows that the relations that exist between the ray systems can once more be given completely.

§ 4. Eq. (11) of the previous § reads:

$$\frac{\sum A_1 dx_{01}}{\sum A_1 \rho_1 \xi_1} + \sum_{s=1}^{n} d\left(\frac{r_s}{\rho_s}\right) = \frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n}.$$

This equation also says how one might choose the initial surfaces in the first and last ray systems. Now, if the first ray system were composed of rays that start from a luminous point, and one chooses the initial surface to be that luminous point then, since it can possess no extension, one will have: $dx_{01} = dy_{01} = dz_{01} = 0$.

The equation will then read:

$$\frac{\sum A_n dx_{n,n+1}}{\sum A_n \rho_n \xi_n} = d\left(\sum_{s=1}^{n} \frac{r_s}{\rho_s}\right). \tag{1}$$

If one again chooses r_n such that one has:

$$\sum_{s=1}^{n} \frac{r_s}{\rho_s} = C, \tag{2}$$

where C is a constant, then, since the denominator does not become infinite, eq. (1) will read:

$$\sum A_n \, dx_{n,n+1} = 0. \tag{3}$$

The surfaces that are determined by eq. (2) are, as a result of their definition, arranged in such a way that all of the rays that emanate from the luminous points that generate the ray system at the same time will meet each of these surfaces at the same time, or that a light motion that has started from a luminous point at any time has propagated from a well-defined surface in the last medium that is defined by eq. (2) at some point in time. Therefore, these surfaces could be given the name of "wave surfaces" in the *Kirchhoff-Helmholtz* sense. However, in order to avoid confusion with the surfaces that I have been calling "wave surfaces" up to now, I would like to call the surfaces that were defined in eq. (2) "surfaces of equal arrival time," where "equal" is taken to mean "simultaneous."

Eq. (3) now says that the direction cosines of the normals to these surfaces are proportional to A_n, B_n, C_n, and we therefore have the theorem:

1) *For optical ray systems, the rays are inclined with respect to the surfaces of equal arrival time in all directions in the same way as the radius vectors of the wave surface of the ray system (which they are parallel to) are inclined with respect to the wave surface.*

The following theorem then follows from this immediately:

2) *For optical ray systems whose wave surface is a sphere, the rays of the system will be normal to the surfaces of equal arrival time.*

3) *If the rays of an optical system with an "aspherical" wave surface possesses the property that they are normal to a surface and the surfaces that are parallel to it – surfaces that I would like to call "normal surfaces" – then these normal surfaces can never coincide with the surfaces of equal arrival time. However, this must be the case for optical ray systems whose wave surfaces are a sphere.*

4) *The necessary and sufficient condition for a ray system in an optical medium to be representable is that its rays be normals to a surface.*

The first of the stated theorems is an analogue of the *Malus-Dupin* theorem for optical ray systems in media with arbitrary wave surfaces. The fourth one is an extension of that theorem, insofar as the media in which the light must travel before it returns to an isotropic medium can be not just isotropic or crystalline, but media with completely arbitrary wave surfaces.

Eq. (1) (pp. 12) yields that if the n^{th} ray system is to be optically-representable then the quantities $\dfrac{\sum A_n\, dx_{n,n+1}}{\sum A_n\, \rho_n\, \xi_n}$ must be equal to a complete differential of a function W of u, v; however, this condition is also sufficient. If one then assumes, for the sake of simplicity, that the given ray system arises directly from a single refraction or reflection from a ray system whose rays start from a luminous point, and that this luminous point lies at the coordinate origin then one will have $n = 2$, and the condition will read:

$$\frac{\sum A_2\, dx_{23}}{\sum A_2\, \rho_2\, \xi_2} = dW = d\left(\frac{r_1}{\rho_1} + \frac{r_2}{\rho_2}\right).$$

Since x_{23}, y_{23}, z_{23}, ξ_2, η_2, ζ_2 are given functions of the independent variables u, v, and likewise ρ_2 [as a given function of (ξ_2, η_2, ζ_2)], dW will be a given quantity. ρ_1, which gives the velocity of light in air for a certain unit of time, is a known, constant quantity.

It follows from eq. (4) that:

$$\frac{r_1}{\rho_1} + \frac{r_2}{\rho_2} = W, \tag{5}$$

where W can still contain an arbitrary constant. Moreover, one has:

$$\begin{aligned} x_{12} &= r_1\, \xi_1\,, & x_{23} &= r_1\, \xi_1 + r_2\, \xi_2\,, \\ y_{12} &= r_1\, \eta_1\,, & y_{23} &= r_1\, \eta_1 + r_2\, \eta_2\,, \\ z_{12} &= r_1\, \zeta_1\,, & z_{23} &= r_1\, \zeta_1 + r_2\, \zeta_2\,. \end{aligned} \tag{6}$$

If one eliminates the twelve quantities r_1, r_2, r_3, ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2, x_{23}, y_{23}, z_{23} from the seven equations (5) and (6), from the seven given functions of u, v, and from the equation $\sum \xi_1^2 = 1$, and then from u and v, then one will obtain the equation of the separation surface of both media, namely, an equation between x_{12}, y_{12}, z_{12}. Now, since the separation surface of both media can actually be determined, the ray system will also be optically-representable.

§ 5. I shall now alter the notation slightly. Let the coordinates of the points of the initial surface of a given ray system be x', y', z', and let those of its wave surface be $x = \rho\xi$, $y = \rho\eta$, $z = \rho\zeta$, where ρ is the radius vector. If A, B, C are quantities that are proportional to the direction cosines of the normal to the wave surface then the condition for the optical representability of the system will read:

(1) $$\frac{\sum A\,dx'}{\sum Ax} = W,$$

where W is a function of u, v.

If one denotes an increment in the quantities in question by the prefix of d_1 (d_2, resp.) when the ratio $\tau = dv / du$ equals τ_1 (τ_2, resp.) then one must also have:

(2) $$\frac{\sum A\,d_1x'}{\sum Ax} = d_1W \qquad \text{and} \qquad \frac{\sum A\,d_2x'}{\sum Ax} = d_2W.$$

Now, since one has $d_2\,d_1W = d_1\,d_2W$, one will also have:

(3) $$d_1\left(\frac{\sum A\,d_2x'}{\sum Ax}\right) = d_2\left(\frac{\sum A\,d_1x'}{\sum Ax}\right).$$

If one carries out the differentiation and considers that:

(4) $$\sum A\,d_1x = 0 \quad \text{and} \quad \sum A\,d_2x = 0$$

then, since $\sum A\,d_1\,d_2\,x = \sum A\,d_2\,d_1x$, and if one multiplies it by $(\sum Ax)^2$, eq. (3) will read:

(5) $$\sum Ax \sum d_1A\,d_2x' - \sum A\,d_2\,x' \sum d_1\,Ax = \sum Ax \sum d_2\,A\,d_1x' - \sum A\,d_1x' \sum d_2Ax\,.$$

One can write the left-hand side of eq. (5) as:

(6) $$\sum d_1A\,d_2x'\,(Ax + By + Cz) - \sum d_1A \cdot x\,(A\,d_2x' + B\,d_2y' + C\,d_2z').$$

Since the quantities A, B, C are proportional to the direction cosines of the normal to the wave surface, one can set, under the assumption that τ_1 and τ_2 are different from each other:

(7) $$A = d_1y\,d_2z - d_2y\,d_1z, \qquad B = d_1z\,d_2x - d_2z\,d_1x,$$
$$C = d_1x\,d_2y - d_2x\,d_1y\,.$$

If one consolidates the sums in the expression (6) and sets A, B, C equal to the value 7 then one will get:

$$\sum d_1A\,[(d_2x' \cdot y - x\,d_2y')(\,d_1z\,d_2x - d_2z\,d_1x) + (\,d_2x' \cdot z - x\,d_2z')(\,d_1x\,d_2y - d_2x\,d_1y)],$$

or

(8) $$\sum d_1A \cdot \left\{ \begin{array}{l} d_2x[-d_1z(x\,d_2y' - y\,d_2x') - d_1y(z\,d_2x' - x\,d_2z')] \\ +d_1x[d_2z(x\,d_2y' - y\,d_2x') + d_1y(z\,d_2x' - x\,d_2z')]. \end{array} \right\}$$

If one removes the quantity $d_2x \cdot d_1x\ (y\ d_2z' - z\ d_2y')$ from the first summand in the curly brackets and adds it to the second summand then the expression (8) will read:

$$\sum d_1A\ \{d_2x\ [-\sum d_1x\ [y\ d_2z' - z\ d_2y')] + d_1x\ [\sum d_2x\ (y\ d_2z' - z\ d_2y')]\},$$

or

$$(9)\qquad - |\ d_1x\ x\ d_2x'\ | \cdot \sum d_1A\ d_2x + |\ d_2x\ x\ d_2y'\ | \cdot \sum d_1A\ d_1x,$$

if one understands the symbol | … | to mean a determinant whose first row is written out, while the other two arise from the first one by replacing the symbol x with y and z, resp.

The right-hand side of eq. (5) will emerge from the left-hand side when one exchanges d_1 and d_2 with each other. If one then exchanges d_1 and d_2 with each other in the expression (9) then one will obtain the right-hand side of eq. (5) as a result, but with the opposite sign, since one would set A equal to $d_2y\ d_1z - d_1y\ d_2z$, instead of $d_1y\ d_2z - d_2y\ d_1z$ (and thus, the opposite value), and likewise for B and C, and since the expression is homogeneous and linear relative to the quantities A, B, C. If one now brings the right-hand side of eq. (5) over to the left-hand side then it will read:

$$(10)\qquad \begin{aligned} &-|\ d_1x\ x\ d_2x'\ | \cdot \sum d_1A\ d_2x + |\ d_2x\ x\ d_2x'\ | \cdot \sum d_1A\ d_1x \\ &-|\ d_2x\ x\ d_1x'\ | \cdot \sum d_2A\ d_1x + |\ d_1x\ x\ d_1x'\ | \cdot \sum d_2A\ d_2x = 0. \end{aligned}$$

As a result of eq. (4), one will have: $d_2 \sum A\ d_1x = d_1 \sum A\ d_2x = 0$, or:

$$(11)\qquad \sum d_2A\ d_1x = \sum d_1A\ d_2x = -\sum A\ d_1d_2x\ .$$

Eq. (10) will then also read:

$$(12)\qquad \begin{aligned} &\sum A\ d_1d_2x\ (|\ d_2x'\ d_1x\ x\ | + |\ d_1x'\ d_2x\ x\ |) \\ &+ |\ d_2x'\ d_2x\ x\ | \cdot \sum d_1A \cdot d_1x + |\ d_1x'\ d_1x\ x\ | \cdot \sum d_2A \cdot d_2x = 0. \end{aligned}$$

This equation will be true for any two distinct increments.

§ 6. If one denotes the partial differential quotients of the quantities in question with respect to u (v, resp.) by the index 1 (2, resp.) then the equation:

$$(13)\qquad |\ dx'\ dx\ x\ | = 0$$

will read:

$$|\ x_1' + x_2'\tau \quad x_1 + x_2\tau \quad x\ | = 0$$

or

$$(14)\qquad |\ x_1' \quad x_1 \quad x\ | + \tau \big[\,|\ x_1' \quad x_2 \quad x\ | + |\ x_2' \quad x_1 \quad x\ |\,\big] + \tau^2\,|\ x_2' \quad x_2 \quad x\ | = 0.$$

Since this a second-degree equation in τ, there will always be two real or imaginary directions, which can also coincide for equal roots, for which eq. (13) is fulfilled. If one now chooses the increments d_1 and d_2 in such a way that the values of τ : τ_1 and τ_2 that correspond to them are roots of eq. (14) then one will have:

$$(15) \qquad | \, d_1x' \; d_1x \; x \, | = 0 \qquad \text{and} \qquad | \, d_2x' \; d_2x \; x \, | = 0,$$

and therefore if eq. (12) is to be valid, one must have either:

$$(16) \qquad \sum A \, d_1d_2x = -\sum d_1A \, d_2x = 0$$

or

$$(16') \qquad | \, d_2x' \; d_1x \; x \, | + | \, d_1x' \; d_2x \; x \, | = 0.$$

If one adds the quantity:

$$| \, d_2x' \; d_2x \; x \, | + | \, d_1x' \; d_1x \; x \, |,$$

which, from eq. (15), is equal to 0, to equation (16′) then that will give:

$$0 = | \, d_2x' \; d_2x + d_1x \; x \, | + | \, d_1x' \; d_2x + d_1x \; x \, |$$

or

$$(16'') \qquad 0 = | \, d_2x' + d_1x' \; d_2x + d_1x \; x \, | = 4 \begin{vmatrix} \dfrac{d_2x' + d_1x'}{2} & \dfrac{d_2x + d_1x}{2} & x \end{vmatrix}.$$

Now, one will have, e.g.:

$$\frac{d_2x + d_1x}{2} = x_1 + x_2 \left(\frac{\tau_1 + \tau_2}{2} \right) = d_3x,$$

if one lets d_3 denote an increment that corresponds to the value $\tau = \dfrac{\tau_1 + \tau_2}{2}$. Eq. (16′) will then read:

$$| \, d_3x' \; d_3x \; x \, | = 0.$$

However, this equation is eq. (13), and since only two values of τ can satisfy it (if the equation is not fulfilled identically), one must have $\dfrac{\tau_1 + \tau_2}{2} = \tau_1$ or τ_2, which will yield $\tau_1 = \tau_2$, but this case is excluded.

Therefore, eq. (16) must be true:

$$(16) \qquad \sum A \, d_1d_2x = -\sum d_1A \, d_2x = 0.$$

Now, if eq. (16) is also always fulfilled when eq. (15) is true, no matter how little τ_1 and τ_2 differ from each other, then it must likewise be true for the limiting case of $\tau_1 = \tau_2$.

In order to arrive at the geometric interpretations of eqs. (15) and (16), I imagine that the coordinate system has been parallel displaced in such a way that the its origin comes to lie at the point O of the initial surface whose original coordinates were x', y', z', and that O has been constructed as the center of the wave surface of the ray system, moreover. The ray that emanates from the point O will then intersect the wave surface at the point $P(x, y, z)$. Let the ray that emanates from the point $R(dx', dy', dz')$ be RS, and let the radius vector that is parallel to it be OQ, so the point Q will have the coordinates $x + dx$, $y + dy$, $z + dz$.

The equation of the plane that goes through Q, P, and O can then be written:

$$| X \; x + dx \; x | = 0$$

or

$$| X \; dx \; x | = 0, \tag{17}$$

where X, Y, Z are the running coordinates. If the increment d is equal to one of the two increments d_1 or d_2, which satisfy eq. (15), then the coordinates of R will satisfy eq. (17). R, and therefore also the ray RS that goes through R, will then lie in the plane that is drawn through OP and OQ. It then follows from this that the ray RS, which is infinitely close to the ray OP, will cut the first one, so it will lie in a focal plane of the infinitely-thin ray bundle that has the axis OP. Therefore, when eq. (15) are true, the points Q of the wave surface, which are infinitely close to the point P, which might correspond to the increments d_1 and d_2 and be denoted by Q_1 and Q_2, will lie in the directions in which the intersection curves of the focal planes of the bundle will cut the wave surface.

The equations of the tangential planes at the points P and Q_1 read:

$$\sum A\,(X - x) = 0 \qquad \text{and} \qquad \sum (A + d_1A)(X - x - d_1x) = 0.$$

Both equations together will yield the equations of the line of intersection of both planes. If one replaces the second equation with the difference of both equations, while neglecting the second-order infinitesimal quantities, and with consideration given to the equation $\sum A\, d_1x = 0$, then the equations of the line of intersection will also read:

$$\sum A\,(X - x) = 0 \qquad \text{and} \qquad \sum d_1A\,(X - x) = 0.$$

Since these equations will be satisfied when one sets:

$$X = x,\; Y = y,\; Z = z \qquad \text{or} \qquad X = x + d_2x,\; Y = y + d_2y,\; Z = z + d_2z$$

[the last one is true because eq. (18) (pp. 15) is true], the points P and Q_2 will lie on that line of intersection, and therefore the tangents PQ_1 and PQ_2 are conjugate tangents, and as such, will lie in conjugate directions.

Conversely, if eq. (16) is true for the two increments d_1 and d_2 for which eq. (18) (pp. 15) is fulfilled then the condition for the optical representability of the ray system will be fulfilled, which one will deduce from the following:

Since the values τ_1 and τ_2 of $\tau = dv / du$ that correspond to two increments d_1 and d_2 are roots of eq. (14) (pp. 15), one will have:

$$(18) \qquad \tau_1 \tau_2 = \frac{|\,x_1' \;\; x_1 \;\; x\,|}{|\,x_2' \;\; x_2 \;\; x\,|}, \qquad \tau_1 + \tau_2 = -\frac{|\,x_1' \;\; x_2 \;\; x\,| + |\,x_2' \;\; x_1 \;\; x\,|}{|\,x_2' \;\; x_2 \;\; x\,|}.$$

Furthermore, from eq. (16), one will have:

$$0 = \sum d_1A\, d_2x = \sum (A_1 + A_2\, \tau_1)\,(x_1 + x_2\, \tau_2)$$

or

$$(19) \qquad \sum A_1\, x_1 + \tau_1 \sum A_2\, x_1 + \tau_2 \sum A_1\, x_2 + \tau_1 \tau_2 \sum A_2\, x_2 = 0.$$

Now, analogous to eq. (11) (pp. 15), one will have:

$$(20) \qquad \sum A_2\, x_1 = \sum A_1\, x_2 = -\sum A\, x_{12}\,,$$

if the double indices 11, 12, 22 denote the corresponding second partial differential quotients of the quantities in question.

As a result of eq. (20), eq. (19) will read:

$$(21) \qquad -(\tau_1 + \tau_2) \sum A\, x_{12} + \sum A_1\, x_1 + \tau_1 \tau_2 \sum A_2\, x_2 = 0.$$

If one substitutes the values of $\tau_1 \tau_2$ and $\tau_1 + \tau_2$ that follow from eq. (13) in this then one will obtain:

$$(22) \qquad (|\,x_1' \;\; x_2 \;\; x\,| + |\,x_2' \;\; x_1 \;\; x\,|) \cdot \sum A x_{12} + |\,x_2' \;\; x_2 \;\; x\,| \cdot \sum A_1 x_1 + |\,x_2' \;\; x_1 \;\; x\,| \cdot \sum A_2 x_2 = 0.$$

One can derive the condition that $\dfrac{\sum A\, dx'}{\sum A\, x}$ is a complete differential in the following way: One has:

$$\frac{\sum A\, dx'}{\sum A\, x} = \frac{\sum A\, dx_1'}{\sum A\, x} du + \frac{\sum A\, dx_2'}{\sum A\, x} dv\,.$$

Should this be a complete differential then one would need to have:

$$(22)[\text{sic}] \qquad \frac{\partial}{\partial v}\left(\frac{\sum A\, x_1'}{\sum A\, x}\right) = \frac{\partial}{\partial u}\left(\frac{\sum A\, x_2'}{\sum A\, x}\right).$$

This equation is analogous to eq. (3) (pp. 14), and when it is developed, it will give eq. (22), which emerges from eq. (12) (pp. 15) when one replaces the increment d_1 (d_2, resp.) with the partial differential quotient with respect to u (v, resp.), and therefore eq. (22) also expresses the condition for optical representability.

Kummer's theorem is proved completely with that, which now reads:

Any infinitely-thin optical ray bundle inside of a homogeneous, transparent medium has the property that its two focal planes cut out two curves from the wave surface of the light that belongs to this medium, whose center is assumed to lie on the axis of the ray bundle, and they will intersect in conjugate directions. Any ray bundle that has this property is also actually optically-representable.

One can now also assume that the increments d_1 and d_2 in eq. (12) (pp. 15) are such that one has:

$$\sum d_1\, A\, d_1 x = 0 \qquad \text{and} \qquad \sum d_2\, A\, d_2 x = 0.$$

The increments d_1 and d_2 will then correspond to directions that are not mutually-conjugate directions, since each of them will be conjugate to itself, when one excludes the case in which the two directions coincide. In order for eq. (12) to be fulfilled, one must then have:

$$|\, d_2 x' \;\; d_1 x \;\; x \,| + |\, d_1 x' \;\; d_2 x \;\; x \,| = 0.$$

I have, however, found no sufficiently simple and obvious geometric interpretation for this expression that I could give here.

Even though the normal surfaces of optical ray systems with "aspherical" wave surfaces (i.e., the normal surfaces to the so-called "irregular" ray systems) then lose the property of likewise being wave surfaces in the *Kirchhoff-Helmholtz* sense, it still seems worth investigating whether there are any irregular ray systems, at all, whose rays are normals to a surface.

§ 7. Since it will prove to be more convenient for this part of the examination to give the arbitrary variables u, v a well-defined geometric meaning, they might be conferred that meaning from the outset.

If one lets U denote the curves of the wave surface for which u varies and v is a constant that changes from curve to curve and lets V denote the ones for which v varies and u has a constant value, and one assumes that the curves V are the curves in which the spheres that are concentric with the wave surface cut that surface and that the curves U intersect the curves V at right angles then one can set:

$$u = \frac{1}{\rho},$$

and the expression for the rectangular intersection of both curves will be:

$$0 = \sum x_1\, x_2 = \sum (\rho\, \xi)_1 \cdot (\rho\, \xi)_2 ,$$

moreover.

One first has:

$$(a) \qquad \sum \xi^2 = 1, \qquad \sum \xi\, \xi_1 = 0, \qquad \sum \xi\, \xi_2 = 0.$$

It follows from the first condition equation for u, v that:

$$(b) \qquad \rho_1 = -\rho_2 , \qquad \rho_{11} = 2\rho_2 , \qquad \rho_2 = \rho_{11} - \rho_{22} = 0.$$

It follows from the second one, with consideration given to eq. (a) and (b), that:

$$0 = \sum (-\rho_2\, \xi + \rho\, \xi_1) \cdot \rho\, \xi_2 = -\rho_2 \sum \xi\, \xi_1 + \rho^2 \sum \xi_1\, \xi_2 ,$$

or

$$(c) \qquad \sum \xi_1\, \xi_2 = 0.$$

I now set, to abbreviate:

$$(d) \qquad \sum \xi_1^2 = E, \qquad \sum \xi_2^2 = G,$$

from which, it will follow that:

$$(e) \qquad \begin{array}{ll} \sum \xi_1 \xi_{11} = \tfrac{1}{2} E_1, & \sum \xi_1 \xi_{12} = \tfrac{1}{2} E_2, \\ \sum \xi_2 \xi_{12} = \tfrac{1}{2} G_1, & \sum \xi_2 \xi_{22} = \tfrac{1}{2} G_2. \end{array}$$

Eq. (a), with consideration given to eqs. (c) and (d), will yield:

$$\begin{array}{lll} 0 = (\sum \xi\, \xi_1)_2 = \sum \xi_2\, \xi_1 + \sum \xi\, \xi_{12} & \text{or} & \sum \xi\, \xi_{12} = 0, \\ 0 = (\sum \xi\, \xi_1)_1 = \sum \xi_1^2 + \sum \xi\, \xi_{11} & \text{or} & \sum \xi\, \xi_{11} = -E, \\ 0 = (\sum \xi\, \xi_2)_2 = \sum \xi_2^2 + \sum \xi\, \xi_{22} & \text{or} & \sum \xi\, \xi_{22} = -G. \end{array}$$

Thus, one will have:

$$(f) \qquad \sum \xi\, \xi_{12} = 0, \quad \sum \xi\, \xi_{11} = -E, \quad \sum \xi\, \xi_{12} = -G.$$

Eq. (c), with consideration given to eq. (e), yields:

$$\begin{array}{lll} 0 = (\sum \xi_1\, \xi_2)_1 = \sum \xi_{11}\, \xi_2 + \sum \xi_1\, \xi_{21} & \text{or} & \sum \xi_2\, \xi_{11} = -\tfrac{1}{2} E_2 , \\ 0 = (\sum \xi_1\, \xi_2)_2 = \sum \xi_{12}\, \xi_2 + \sum \xi_1\, \xi_{22} & \text{or} & \sum \xi_1\, \xi_{22} = -\tfrac{1}{2} G_1 . \end{array}$$

One then has:

(*g*) $$\sum \xi_2\, \xi_{11} = -\tfrac{1}{2} E_2\,, \qquad \sum \xi_1\, \xi_{22} = -\tfrac{1}{2} G_1\,.$$

Some expressions can now be derived that will find an application later on. First, let it be remarked that the cases in which E or G are identically equal to 0 will be excluded; the equation $E = 0$ will then represent a conic surface, and the equation $G = 0$ will represent a curve with double curvature. Let the ratio of the direction cosines of the normal to the wave surface to the quantities A, B, C be chosen such that:

(α) $$\sum A\,\rho\,\xi = 1.$$

Moreover, it follows from $\sum A\, d(\rho\,\xi) = 0$ and eq. (*b*) that:

(β) $$\sum A\,(\rho\,\xi)_1 = \sum A\,(-\rho^2\,\xi + \rho\,\xi_1) = 0,$$

(γ) $$\sum A\,(\rho\,\xi)_2 = \sum A\,\rho\,\xi_2 = 0.$$

If one sets:

(δ) $$A = \lambda\xi + \mu\xi_1 + \nu\xi_2\,, \quad B = \lambda\eta + \mu\eta_1 + \nu\eta_2\,, \quad C = \lambda\zeta + \mu\zeta_1 + \nu\zeta_2$$

then it will follow from eqs. (γ), (δ), (*a*), and (*c*) that:

$$0 = \sum A\rho\xi_2 = \rho\lambda \sum \xi\xi_2 + \rho\mu \sum \xi_1\,\xi_2 + \rho\nu \sum \xi_2^2\,, \quad \text{or} \qquad \rho\cdot\nu \sum \xi_2^2 = 0,$$

so:

(ε) $$\nu = 0.$$

Moreover, it follows from eqs. (α), (δ), (ε), (*a*), and (*d*) that:

$$1 = \sum A\rho\xi = \rho\lambda \sum \xi^2 + \rho\mu \sum \xi_1\,\xi_2, \qquad \text{or} \qquad 1 = \rho\lambda$$

so:

(ζ) $$\lambda = \frac{1}{\rho}\,,$$

and finally, it follows from eqs. (β), (δ), (ε), (ζ), (*a*), and (*d*) that:

$$0 = \sum A\,(-\rho^2\,\xi + \rho\xi_1) = \sum \left(\frac{1}{\rho}\xi + \mu\xi_1\right)(-\rho^2\xi + \rho\xi_1)$$

or

$$0 = -\rho \sum \xi^2 + \sum \xi\xi_1 - \mu\rho^2 \sum \xi\xi_1 + \mu\rho \sum \xi_1^2 = -\rho + \mu\rho \sum \xi_1^2\,,$$

or

$$\mu = \frac{1}{E}\,,$$

and therefore eq. (δ) will read:

(1) $$A = \frac{1}{\rho}\xi + \frac{1}{E}\xi_1, \qquad B = \frac{1}{\rho}\eta + \frac{1}{E}\eta_1, \qquad C = \frac{1}{\rho}\zeta + \frac{1}{E}\zeta_1,$$

when the following equation is true:

(2) $$\sum A\,\rho\,\xi = 1.$$

The equations of two planes that are drawn through the radius vector to the wave surface, which points in the direction (ξ, η, ζ), through each of which, a radius vector that points in the direction ($\xi + d_1\xi$, $\eta + d_1\eta$, $\zeta + d_1\zeta$) [($\xi + d_2\xi$, $\eta + d_2\eta$, $\zeta + d_2\zeta$), resp.] goes, read:

$$|\, X\ \xi\ \xi + d_1\xi\,| = 0 \qquad \text{or} \qquad |\, X\ \xi\ d_1\xi\,| = 0 \ \text{ and } \ |\, X\ \xi\ d_2\xi\,| = 0.$$

Should both planes be perpendicular to each other, then one would need to have:

$$\sum (\eta\, d_1\zeta - \xi\, d_1\eta)\,(\eta\, d_2\zeta - \zeta\, d_2\eta) = 0,$$

or

$$0 = \sum \eta^2\, d_1\zeta\, d_2\zeta + \sum \zeta^2\, d_1\eta\, d_2\eta - \sum \eta\, d_2\eta \cdot \zeta\, d_1\zeta - \sum \zeta\, d_2\zeta \cdot \eta\, d_1\eta\,.$$

If one writes down another suitable summand under the $\sum$ sign, instead of the ones that are written, and, at the same time, adds and subtracts the quantities:

$$\sum \xi^2\, d_1\xi\, d_2\xi = \sum \xi\, d_2\xi \cdot \xi\, d_1\xi$$

then the equation will read:

$$0 = \sum \xi^2\, d_1\xi\, d_2\xi + \sum \xi^2\, d_1\eta\, d_2\eta + \sum \xi^2\, d_1\zeta\, d_2\zeta$$
$$- \sum \xi\, d_2\xi \cdot \xi\, d_1\xi - \sum \xi\, d_2\xi \cdot \eta\, d_1\eta - \sum \xi\, d_2\xi \cdot \zeta\, d_1\zeta$$

or

$$\sum \xi^2 \cdot \sum d_1\xi\, d_2\xi - \sum \xi\, d_2\xi \cdot \sum \xi\, d_1\xi = 0.$$

If one now sets $d_1\xi = \xi_1 + \xi_2\,\tau_1$, $d_2\xi = \xi_1 + \xi_2\,\tau_2$, and similarly for $d_1\eta$, $d_1\zeta$, $d_2\eta$, $d_2\eta$, then, since $\sum \xi^2 = 1$, one will have:

$$\sum (\xi_1 + \xi_2\,\tau_1)(\xi_1 + \xi_2\,\tau_2) - \sum \xi\,(\xi_1 + \xi_2\,\tau_2) \cdot \sum \xi\,(\xi_1 + \xi_2\,\tau_1) = 0.$$

If one resolves the brackets and considers eqs. (*a*), (*c*), (*d*) (pp. 19 and 20) then one will get:

(3) $$E + G\,\tau_1\,\tau_2 = 0.$$

Equation (3) is then the condition for two directions that emanate from a point of the wave surface (for which the corresponding values of τ are denoted by τ_1 and τ_2) and describe two planes that are laid through that point and the center of the wave surface to be perpendicular to each other.

From eq. (16) of § 6, (pp. 16), the condition for two directions that emanate from a point of the wave surface (for which the corresponding values of τ might be denoted by τ_1 and τ_2) to be conjugate directions reads:

$$0 = \sum A\, d_1 d_1 x = \sum A\, [x_{11} + x_{12}\,(\tau_1 + \tau_2) + x_{23}\,\tau_1\,\tau_2]$$

or

$$\sum A\, x_{11} + (\tau_1 + \tau_2) \sum A\, x_{12} + \tau_2 \sum A\, x_{22} = 0. \tag{4}$$

Now, as a result of eq. (1) and eq. (*b*) (pp. 20), one will have:

$$\sum A\, x_{11} = \sum A\, (\rho\xi)_1 = \sum\left(\frac{1}{\rho}\xi + \frac{1}{E}\xi_1\right)(2\rho^3\,\xi - 2\rho^2\,\xi_1 + \rho\,\xi_{11}).$$

If one employs eq. (*a*), (*d*), (*f*), and (*e*) then one will get:

$$\sum A\, x_{11} = 2\rho^2 - E - 2\rho^2 + \tfrac{1}{2}\frac{\rho}{E}E_1 = \tfrac{1}{2}E\,(-\,2E^2 + \rho\,E_1),$$

$$\sum A\, x_{12} = \sum A\, (\rho\xi)_{12} = \sum\left(\frac{1}{\rho}\xi + \frac{1}{E}\xi_1\right)(-\rho^2\,\xi_2 + \rho\,\xi_{12}).$$

If one employs eq. (*a*), (*f*), and (*e*) then one will get:

$$\sum A\, x_{12} = \frac{1}{2E}\rho E_2\,,$$

$$\sum A\, x_{22} = \sum A\, (\rho\xi)_{22} = \sum\left(\frac{1}{\rho}\xi + \frac{1}{E}\xi_1\right)\rho\xi_{22}\,,$$

or, when one employs eqs. (*f*) and (*g*):

$$\sum A\, x_{22} = -\,G - \frac{\rho}{2E}G_2 = \frac{-1}{2E}(2EG + \rho G_1).$$

If one substitutes the calculated expressions into eq. (4) and multiplies it by $2E$ then it will read:

$$(-\,2E^2 + \rho E_1) + \rho\,E_2\,(\tau_1 + \tau_2) - (2EG + \rho\,G_1)\,\tau_1\,\tau_2 = 0. \tag{5}$$

§ 8. If one temporarily excludes the case in which eq. (2) is identical to eq. (4) then one can determine τ_1 and τ_2 from these equations uniquely as roots of a quadratic equation. Therefore, in general, there will be only one pair of mutually perpendicular planes that go through a radius vector of the wave surface and cut the wave surface in curves that intersect in conjugate directions. If one determines these directions for each radius vector then one will obtain two new families of curves *P* and *Q* on the wave surface in such a way that, in general, only one curve *P* and one curve *Q* will go through each point of the surface that intersect in conjugate directions in such a way that the two planes that are laid through the starting point and one of the directions will be perpendicular to each other. Now, if surfaces can be determined in such a way that their lines of curvature correspond to the families of curves *P* and *Q* that are defined on a given surface in such a way that the normals along a line of curvature are parallel to the radius vectors that are drawn from the corresponding points of the corresponding curves *P* or *Q* of the given surface then the normals to a surface thus-determined will define an optically-representable ray system with the given surface as its wave surface, so *Kummer's* theorem will be fulfilled by an infinitely thin ray bundle of this system as a result of the definition of the lines of curvature.

One will obtain surfaces with the aforementioned special property, e.g., when one sets:

$$x' = a + \kappa A, \quad y' = b + \kappa B, \quad z' = c + \kappa C, \tag{6}$$

where x', y', z' are the coordinates of a point on the surface, a, b, c, κ are constants. and A, B, C are the quantities that are defined by eq. (1) (pp. 21). One will then have:

$$\sum \xi\, dx' = \kappa \sum \xi \cdot d\left(\frac{1}{\rho}\xi + \frac{1}{E}\xi_1\right) \tag{1}$$

$$= \kappa \left\{ d\left(\frac{1}{\rho}\right)\sum \xi^2 + \frac{1}{\rho}\sum \xi\, d\xi + d\left(\frac{1}{E}\right)\sum \xi\,\xi_1 + \frac{1}{E}\sum \xi\,(\xi_{11}du + \xi_{12}dv)\right\}$$

or

$$\sum \xi\, dx' = k \left\{ du + \frac{du}{E}\sum \xi\xi_{11} + \frac{dv}{E}\sum \xi\xi_{12}\right\} = 0,$$

due to eq. (*f*), (pp. 20).

Due to eq. (2), (pp. 21): $\sum A\,\xi = \sum A\,\rho\,\xi = 1$, so one has:

$$\frac{\sum A\, dx'}{\sum Ax} = \kappa \sum A\, dA = d(\kappa \cdot \sum A^2), \tag{2}$$

so this is equal to a complete differential, and therefore, from eq. (1) of § 5, the condition for optical representability is fulfilled.

As a result of eq. (2) and eq. (6), one has:

$$\sum\left(\frac{x'-a}{\kappa}\rho\xi\right) = 1 \qquad \text{or} \qquad \sum \xi\,(x'-a) = \frac{\kappa}{\rho} = \rho'. \tag{7}$$

If one then thinks of the wave surface S of a medium as being constructed around an arbitrary, fixed point D whose coordinates are a, b, c, and a segment $\rho' = DE'$ as being removed from a radius vector $\rho = DE$ of D, which points in an arbitrary direction (ξ, η, ζ), such that $\rho \cdot \rho' = k$, then the points E' will define an inverse surface S' to the surface S. If one erects a plane through E' that is perpendicular to DE' then, as a result of eq. (7), the point (x',y', z') will lie on this plane, and indeed on the point of intersection E'' of the plane with the line DE'', which is parallel to the normal EG to the surface S at the point E. If the points E'' define the surface S'' then, as is obvious from the foregoing, the normal $E''G''$ to the surface S'' at the point E'' will be parallel to the radius vector DE of the surface S, so the plane $E'E''$ will also be a tangential plane to the surface S'', and all planes $E'E''$ will envelop the surface S''. Therefore, the surface S' is the base point surface of the surface S'', and therefore S'' will be the first negative base point surface to the surface S'. With that, we have the theorem:

The normals to the first, negative base point surface of an inverse surface to the wave surface of a medium define an optically-representable ray system in that medium.

For $\kappa = 0$, such a ray system will go to a ray system whose rays all cross at a point. For a ray system of this kind, the spheres that are constructed around the crossing point will be normal surfaces, and the wave surfaces that are constructed around that point as a center, in the various units of length, will be surfaces of equal arrival time.

§ 9. As a result of the meaning of the equations, the surfaces for which eq. (2) in § 7 is identical to eq. (4) or (5) of § 7 have the property that any pair of mutually-perpendicular planes that go through the starting point will cut the surface in curves that intersect in conjugate directions.

For these surfaces, the following equations must be true:

(8) $$E_2 = 0$$

and

$$\frac{-2E^2 + \rho E_1}{E} = \frac{-2EG - \rho G_1}{G},$$

or

$$\frac{2\rho}{\sqrt{EG}} \cdot \frac{GE_1 + EG_1}{2\sqrt{EG}} = 0,$$

or

(9) $$(\sqrt{E} \cdot \sqrt{G})_1 = 0.$$

The form of the surfaces that satisfy the condition $E_2 = 0$ shall next be ascertained.

One can write the equations of the osculating planes of a curve U as:

$$| X - \rho\xi \, (\rho\xi)_1 \, (\rho\xi)_{11} | = 0$$

or

$$| X - \rho\xi \quad -\rho^2\xi + \rho\xi_1 \quad 2\rho^3\xi - 2\rho^2\xi_1 + \rho\xi_{11} | = 0.$$

If one splits the determinant into two pieces by separating the third column, one of which will vanish due to the constant ratio of the terms in the same places in two columns, and then divides by ρ^2 then one will get:

$$| X - \rho\xi \quad -\rho\xi + \xi_1 \quad \xi_{11} | = 0.$$

If one adds ξ_2 times the first row to η_2 times the second row (ζ_2 times the third row, resp.) then one will get:

$$\begin{vmatrix} \sum\xi_2 X - \rho\sum\xi\xi_2 & -\rho\sum\xi\xi_2 + \sum\xi_1\xi_2 & \sum\xi_2\xi_{11} \\ y - \rho\eta & -\rho\eta + \eta_1 & \eta_{11} \\ x - \rho\xi & -\rho\zeta + \zeta_1 & \zeta_{11} \end{vmatrix} = 0.$$

As a result of eq. (*a*), (*c*), (*g*), (pp. 19, 20), and eq. (8), the sums in the first row, with the exception of the first row, will be equal to 0, and therefore when the determinant is simultaneously multiplied and divided by $\sqrt{G}$ and expanded, it will read:

$$[\sqrt{G}\ (-\rho\eta + \eta_1)\ \zeta_{11} - (-\rho\zeta + \zeta_1)\ \eta_{11}] \cdot \sum\left(\frac{\xi_2}{\sqrt{G}} X\right) = 0.$$

However, the expression in the square brackets can indeed vanish for special pairs of values of u, v, but not for all of them, especially since eq. (8) says that this expression, as calculation would show, is a function of u alone, multiplied by ξ_2. Therefore, the equation of the osculating plane of a curve will read:

$$\sum\left(\frac{\xi_2}{\sqrt{G}} X\right) = 0. \tag{10}$$

As a result of eqs. (*f*) and (*e*) of § 7, with consideration given to eq. (8), one will have:

$$\sum \xi\xi_{12} = 0 \quad \text{and} \quad \sum \xi_1\,\xi_{12} = 0.$$

Furthermore, from eqs. (*a*) and (*c*) of § 7, one has:

$$\sum \xi\xi_2 = 0 \quad \text{and} \quad \sum \xi_1\,\xi_2 = 0,$$

from which, it follows that:

$$\frac{\xi_{12}}{\xi_2} = \frac{\eta_{12}}{\eta_2} = \frac{\zeta_{12}}{\zeta_2} = \nu,$$

where ν is a proportionality factor that can be a function of u, v. Now, if, e.g.:

$$\frac{\xi_{12}}{\xi_2} = \frac{\partial}{\partial u}[\ln(\xi_2)] \qquad \text{or} \qquad \xi_2 = \varphi_\xi(v) \cdot e^{\int \nu\, du},$$

then similar statements will be true for η_2 and ζ_2. Therefore:

$$\sqrt{G} = \sqrt{\sum \xi_2^2} = e^{\int \nu\, du} \sqrt{\sum \left[\varphi_\xi(v)\right]^2} \qquad \text{or} \qquad \sqrt{G} = e^{\int \nu\, du}\, \varphi(v),$$

from which, it follows that:

$$\text{(11)} \qquad \frac{\xi_2}{\sqrt{G}} = \frac{\varphi_\xi(v)}{\varphi(v)} = f_\xi(v), \quad \frac{\eta_2}{\sqrt{G}} = f_\eta(v), \quad \frac{\zeta_2}{\sqrt{G}} = f_\zeta(v).$$

Since all of the planes that are defined by eq. (10) go through the starting point (viz., the center of the wave surface), and since, due to eq. (11), the direction cosines of the normal to such a plane are functions of only v, and thus independent of u, it will then follow that all osculating planes to a curve U will coincide; i.e., that all curves U will be planar curves whose planes go through the starting point.

Now, the direction cosines of the arc element of the curve V that starts at the point $(\rho\xi, \rho\eta, \rho\zeta)$ are proportional to the $(\rho\xi)_2$, $(\rho\eta)_2$, $(\rho\zeta)_2$, or to the ξ_2, η_2, ζ_2, so they will be equal to the direction cosines of the normal to the plane of the curve U that goes through the point $(\rho\xi, \rho\eta, \rho\zeta)$, and thus the arc element of the curve V that starts from a point of a plane curve U will be perpendicular to the plane of that curve. It follows from this that:

If one lays three successive, infinitely-close curves U in the planes I, II, III, which must then likewise be infinitely-close to each other and all go through the starting point, (although generally they will not intersect in the same line), and then rotates plane I around the line of intersection (I II) until it coincides with plane II then the curve U that lies in plane I will describe the part of the surface that lies between planes I and plane II under the rotation, such that when planes I and II coincide, it will also coincide with the curve U that lies in plane II. Likewise, the part of the surface that lies between planes II and III will be described by the curve U that lies in plane II when one rotates plane II around the line of intersection (II III) until it falls upon plane III, and so forth.

Thus, the surface will have the character of a surface of rotation between any two successive, infinitely-close curves U. The totality of lines of intersection of two successive, infinitely-close planes in which the curves U lie defines a conic surface. The planes themselves are the tangential planes to that conic surface, and the rotation of a plane around the intersection with the following one until it coincides with it is the same as when a plane rolls in the cone without slipping.

Thus, if an arbitrary planar curve is given, and the plane of that curve rolls on an arbitrary cone without slipping then the given curve will describe a surface $E_2 = 0$, and one can think of every surface $E_2 = 0$ as arising in that way.

In addition to eq. (8), $E_2 = 0$, eq. (9), $(\sqrt{E}\cdot\sqrt{G})_1 = 0$, must also be fulfilled. When that equation is integrated, that will give $\sqrt{E}\cdot\sqrt{G} = f(v)$. Since $du = d(1 / \rho)$, one can write it as:

$$\sqrt{E}\,du\cdot\sqrt{G}\,dv = d\frac{1}{\rho}\cdot f(v)\cdot dv. \tag{12}$$

Let a circle around the starting point O with radius 1 be described in the plane of the curve U that goes through the point $C(\rho\xi, \rho\eta, \rho\zeta)$. It will cut the radius vector OC at B. Let OA be the line around which the plane must be rotated infinitely little in order for the curve to describe the part of the surface between that curve and the following infinitely-close curve U. After that rotation, B might lie at B_2 and C, at C_2. The infinitely-small angle of rotation ($d\varphi$) is independent of $u = 1 / \rho$, so it will be equal to $\varphi(v)\,dr$: C_2C will then be equal to the arc element of the curve V that goes through the point C, and therefore:

$$BB_2 = \sqrt{\sum \xi_2^2}\,dv = \sqrt{G}\;dv.$$

If one denotes the angle AOB by ϑ then one will have, on the other hand: $BB_2 = \sin\vartheta\,d\varphi$, so:

$$\sqrt{G}\;dv = \sin\vartheta\cdot d\varphi = \sin\vartheta\cdot\varphi(v)\,dv.$$

Let the arc element of the curve U that starts at C be CC_1, so when the radius vector OC_1 cuts the circle at B_1:

$$BB_1 = \sqrt{\sum \xi_1^2}\;du = \sqrt{E}\,du\,;$$

on the other hand, one has:

$$BB_1 = \frac{\partial\vartheta}{\partial u}\,du, \qquad \text{so} \qquad \sqrt{E}\,du = \frac{\partial\vartheta}{\partial u}\,du.$$

Thus, eq. (12) reads:

$$\frac{\partial\vartheta}{\partial u}\,du\sin\vartheta\cdot\varphi(v)\,dv = d\frac{1}{\rho}\cdot f(v)\,dv.$$

If one regards v as constant then one can set $\frac{\partial\vartheta}{\partial u}du = d\vartheta$, and one will then have a differential equation that is true for any curve U such that the constants in the differential equation can have different values for different curves U. The differential equation reads:

$$d\frac{1}{\rho} = b\sin\vartheta\,d\vartheta,$$

or, when integrated:

$$\frac{1}{\rho} = c - b \cos \vartheta \qquad \text{or} \qquad \rho = \frac{\frac{1}{c}}{1 - \frac{b}{c}\cos\vartheta}.$$

If one sets: $b / c = + e$, $1 / c = \pm a(1 - c^2)$, where the sign is chosen such that e and a are positive, then one will have:

$$\rho = \frac{+a(1-c^2)}{1+e\cos\vartheta}. \tag{13}$$

This is the equation of a conic section in polar coordinates whose starting point coincides with a focal point and whose axis coincides with the principal axis of the conic section. a and e, which are constant relative to the same curve, can have different values for different curves U, since they, like b and d, can also be functions of v. However, since it emerges from the way that the surfaces that satisfy the condition $E_2 = 0$ come about that all curves U exhibit the same form of one and the same curve, and can differ only in relation to their position with respect to the instantaneous rotational axis OA that lies in their plane, which is not the case here, since the instantaneous rotational axis must always coincide with the principal axis of the conic section, so it will follow that the principal axis of the conic section is always the instantaneous rotation axis, so the surface is a surface of rotation that arises by the rotation of a conic section around its principal axis.

The remark that was made at the beginning of this paragraph yields the theorem:

Any surface that arises by the rotation of a conic section around its principal axis will be cut by every pair of mutually-perpendicular planes that go through one of the two focal points of the surface in curves that intersect in conjugate directions, and only those surfaces will possess that property.

This, and what was said in § 8, immediately yields the following theorem:

Should the normals to an arbitrary surface define an optical ray system in a medium, and only them, then the wave surface of the medium would have to be a surface that arises by rotating a conic section around its principal axis, and a focal point of this surface must be the so-called center of the surface as a wave surface.

Of all these surfaces, the sphere is the only surface for which the center of the surface coincides with the so-called center of the surface as wave surface.

§ 10. If we return to the surfaces $E_2 = 0$ that were determined in § 9 then it will follow immediately from their manner of definition itself – namely, that the normals at the points of a planar curve U that lies in that plane will then define the constant angle 0 with the plane of the curve – that the curves U will therefore define one family of lines of curvature, and accordingly, the curve V will define the other family, and that as a result of this the families of curves will always intersect in conjugate directions. Moreover, since when one lays tangents planes through the starting point and either of the two points of the surface on a line of curvature, they will be perpendicular to each other, it will follow that all curves P and Q (whose meaning was given in § 8, pp. 24) will coincide with the lines of curvature for the surfaces $E_2 = 0$. Thus, if an optical ray system has a surface $E_2 = 0$ for its wave surface and normal surfaces, in addition, then the lines of curvature of each normal surface will correspond to lines of curvature of the wave surface. Now, a line of curvature S_1 of a normal surface will correspond to a planar line of curvature S of the wave surface. Since the normals along the curve S, two of which follow in sequence and intersect infinitely-close to each other, will be parallel to the radius vectors that are drawn to the corresponding points of the curve S, which all lie in the plane of that curve, it will follow that all normals along the curve S_1, and with them, the curve S_1, must lie in a plane that is parallel to the plane of the curve S. That will imply: One family of lines of curvature of the normal surfaces are plane curves. The normals along such a line of curvature will lie in its plane, and therefore the arc elements that emanate from such a line of curvature will be perpendicular to the plane of the former. The surface will then have the character of a surface of rotation between two successive, infinitely-close planar lines of curvature. In order for mutually-corresponding, planar lines of curvature of both surfaces – viz., the wave surface and one normal surface – to always lie in parallel planes, it is necessary for the instantaneous rotational axes to have the same direction in corresponding planes. Since the restriction that all instantaneous rotational axes must go through a fixed point falls away for the normal surfaces, the totality of all of them will define a developable surface. One deduces the following theorem from the foregoing:

In a medium whose wave surface is generated by an arbitrary planar curve in such a way that its plane rolls on an arbitrary cone without slipping, an optical ray system will only be defined by the normals to a surface that is generated by an arbitrary planar curve in such a way that its plane rolls without slipping on a developable surface that is such that the contact edge of any tangential plane to it is parallel to the contact edge of the tangential plane of the cone that the wave surface is based upon that is parallel to that tangential plane.

The contact edge is then the current instantaneous rotational axis, and two points in both curves, which generate the normal (wave, resp.) surface, will correspond when, for parallel position of the planes of both curves, the normal that is drawn to the first curve at a point is parallel to the radius vector that is drawn to the point of the second curve from the vertex of the cone.

The developable surface can also be a conic surface, and that cone must then be entirely equal to the cone of the wave surface.

If the cone that the wave surface is based upon shrinks to a line then the wave surface will be a surface of rotation whose rotational axis is the line. The developable surface that the normal surfaces are based upon will be a cylindrical surface whose sides have the same direction as the rotational axis of the wave surface. It will then follow that:

In a medium whose wave surface is a surface of rotation, an optical ray system will be defined by the normals to any surface that is generated by an arbitrary plane curve in *such a way that the plane of the curve rolls without slipping on a cylinder whose sides have the same direction as the rotational axis of the wave surface.*

Since the cylinder can likewise degenerate into merely a line, it will then follow that:

In a medium whose wave surface is a surface of rotation, the normals to any surface of rotation whose rotational axis has the same direction as the rotational axis of the wave surface define an optical ray system.

The surfaces that enter into this paragraph were treated more thoroughly by Monge in his book *Application de l'analyse à la géométrie.*